Corrigendum to the paper “Morse boundaries of proper geodesic metric spaces”

Citation for published version:
Cordes, M, Sisto, A & Zbinden, S 2023, 'Corrigendum to the paper "Morse boundaries of proper geodesic metric spaces"', Groups, Geometry, and Dynamics.

Link:
Link to publication record in Heriot-Watt Research Portal

Document Version:
Peer reviewed version

Published In:
Groups, Geometry, and Dynamics

General rights
Copyright for the publications made accessible via Heriot-Watt Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy
Heriot-Watt University has made every reasonable effort to ensure that the content in Heriot-Watt Research Portal complies with UK legislation. If you believe that the public display of this file breaches copyright please contact open.access@hw.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.
CORRIGENDUM

MATTHEW CORDES, ALESSANDRO SISTO, AND STEFANIE ZBINDEN

Abstract. We introduce refined Morse gauges to correct the proof and statement of Lemma 2.10 in Morse boundaries of proper geodesic metric spaces written by the first author.

1. Introduction

We adopt the notation of [Cor17]. In Lemma 2.10 of [Cor17] the first author claimed that if a sequence of $N$-Morse geodesic rays converged uniformly on compact sets to a geodesic ray, then that ray would be $N$-Morse. This does not follow and we provide a counterexample below (Example 2.3). This corrigendum will do three things: We first introduce the notion of refined Morse gauge and (quasi-)geodesics and use them to prove a corrected version of Lemma 2.10 of [Cor17]. We then reprove Corollary 3.2 of [Cor17] without using Lemma 2.10 of the same paper so that one can define the Morse boundary without reference to refined Morse gauges. Finally we show that for any proper geodesic metric space, the Morse boundary defined with Morse gauges and the refined Morse boundary, defined by refined Morse gauges only, are homeomorphic. A key step in this is Lemma 2.4 which associate to any Morse gauge $N$ a refined Morse gauge $\hat{N}$ such that all $N$-Morse geodesics are $\hat{N}$-Morse; essentially the lemma says that one can always replace Morse gauges with refined Morse gauges.

A major consequence of the incorrectness of Lemma 2.10 is that the $N$-Morse strata, $\partial^N_M X$, are not necessarily compact. Many papers which use the Morse boundary rely on Lemma 2.10 either directly or by using Theorem 3.14 of [CH17]. Thankfully, by Theorem 2.10 anywhere where these results are called upon, unless relying specifically on Morse gauges that are not refined, one may simply substitute Morse gauges for refined Morse gauges and the same conclusion will follow. For instance, Theorem 3.14 of [CH17] holds if one considers only refined Morse gauges.

2. Refined Morse boundary

We now introduce the notion of refined Morse gauge and geodesic:

Definition 2.1. A refined Morse gauge is a Morse gauge $N : \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with the following additional properties:

1. $N$ is non-decreasing
2. $N$ is continuous in the second coordinate

We denote the collection of refined Morse gauges $\hat{M}$.

To correct the proof of Lemma 2.10, we do not actually need that a Morse gauge $N$ is non-decreasing, but it is generally useful. In particular, given a Morse gauge $N$ one has an associated constant $\delta_N$ (to be thought of as a hyperbolicity constant)
which is useful in several ways. If $N \leq N'$ are non-decreasing Morse gauges, then $\delta_N \leq \delta_N'$, which does not hold in general otherwise (unless one redefines $\delta_N$ as proposed in Remark 2.7.1). Furthermore, it is an intermediate step in the proof of Lemma 2.4. We now define a refined Morse quasi-geodesic.

**Definition 2.2.** Let $X$ be a metric space, $N$ a refined Morse gauge, and $I$ a closed interval of $\mathbb{R}$. The quasi-geodesic $\gamma : I \rightarrow X$ is a refined $N$-Morse geodesic if for any $(\lambda, \epsilon)$-quasi-geodesic $\sigma$ with endpoints on $\gamma$, we have the image of $\sigma$ is contained in the closed $N(\lambda, \epsilon)$-neighborhood of $\gamma$.

There are two differences between the definition of Morse geodesics and refined-Morse geodesics. The first is that we require the use of a refined Morse gauge. The second is that we ask for the quasi-geodesic $\sigma$ to be in the closed neighborhood of $\gamma$.

We now present the counterexample to Lemma 2.10:

**Example 2.3.** Consider the space $X$ formed by taking the hyperbolic plane $\mathbb{H}^2$ and gluing in a line segment connecting two points $p, q \in \mathbb{H}^2$ of length $d(p, q)$. Thus in $X$ there are two geodesics between $p$ and $q$, one in $\mathbb{H}^2$ and the other along the line segment. Let $\alpha$ be a geodesic ray with basepoint $p$ passing through $q$ whose image is in $\mathbb{H}^2$ and let $\{\alpha_i\}$ be a collection of geodesic rays with image in $\mathbb{H}^2$ and basepoint $p$ converging to $\alpha$. Since $X$ is hyperbolic, every geodesic is uniformly $N$-Morse for some $N$. And because of the extra segment we glued on, $N(1, 0) \neq 0$. Define

$$N'(\lambda, \epsilon) = \begin{cases} 0 & \lambda = 1, \epsilon = 0 \\ N(\lambda, \epsilon) & \text{else} \end{cases}$$

Since $X$ is uniquely geodesic except for geodesics passing through $p$ and $q$, we know that the $\alpha_i$ are $N'$-Morse. Since $\alpha$ is not $N'$-Morse we have our counterexample.

The Morse gauge $N'$ is not a refined Morse gauge because it is not continuous in the second coordinate. To see this, we note that $N'(1, \epsilon) = N(1, \epsilon)$ must be at least $\frac{1}{2}d(p, q)$ since a geodesic is a $(1, \epsilon)$-quasi-geodesic. So $N'$ is discontinuous at 0.

The following lemma can be used in several contexts to harmlessly pass from Morse gauges to refined Morse gauges.

**Lemma 2.4.** For every Morse gauge $N$, there exists a refined Morse gauge $\tilde{N}$ with the property that every $N$-Morse geodesic is also an $\tilde{N}$-Morse geodesic.

**Proof.** We will first construct a non-decreasing Morse gauge from $N$. This is done by setting

$$N'(\lambda, \epsilon) = \inf_{\lambda', \epsilon'} \{N(\lambda', \epsilon') \mid \lambda' \geq \lambda, \epsilon' \geq \epsilon\} + 1$$

We see that by construction $N'$ is non-decreasing.

To arrange that $N'$ is continuous in the second coordinate, we first note that since we have arranged that $N'$ is non-decreasing, then if we fix a $\lambda \in \mathbb{R}_{\geq 1}$, the function $N'(\lambda, \epsilon) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is non-decreasing and thus (Riemann) integrable. We then set

$$\tilde{N}(\lambda, \epsilon) = \int_{\epsilon}^{\epsilon+1} N'(\lambda, t) \ dt.$$
A standard calculation using that \( \hat{N}(\lambda, \cdot) \) is bounded on compact intervals shows that \( \hat{N} \) is continuous in the second coordinate. We also note that, since \( N' \) is non-decreasing, we have \( N' \leq \hat{N} \).

Let \( \alpha \) be a \( N \)-Morse geodesic and let \( \sigma \) be a \((\lambda, \epsilon)\)-geodesic with endpoints on \( \alpha \). We note that \( \sigma \) is also an \( N(\lambda', \epsilon') \)-quasi geodesic for any \( \lambda' \geq \lambda \) and \( \epsilon' \geq \epsilon \). Thus by construction, we know that \( \sigma \) must be in the \( N'(\lambda, \epsilon) \)-neighborhood of \( \alpha \) and since \( N' \leq \hat{N} \), we conclude that \( \sigma \) is in the \( \hat{N}(\lambda, \epsilon) \)-neighborhood of \( \alpha \). \( \square \)

We now state and prove the corrected version of Lemma 2.10 from [Cor17].

**Lemma 2.5** (Lemma 2.10 redux). Let \( X \) be a geodesic metric space and suppose that \( \{ \gamma_i : \mathbb{R}_{\geq 0} \to X \} \) is a sequence of refined \( N \)-Morse geodesics that converge uniformly on compact sets to a geodesic ray \( \gamma \). Then \( \gamma \) is refined \( N \)-Morse.

**Proof.** Let \( \eta > 0 \). Let \( \sigma \) be a \((\lambda, \epsilon)\)-quasi-geodesic with endpoints \( \gamma(s) \) and \( \gamma(t) \) on \( \gamma \). Since the \( \gamma_i \) converge uniformly on compact sets and are refined \( N \)-Morse, there exists an \( I \in \mathbb{N} \) such that for any \( i \geq I \), we have \( d(\gamma_i(t), \gamma(t)) \leq \eta \) on \( \gamma|_{[s,t]} \). It follows that \( \gamma \) is \( N' \)-Morse where \( N'(\lambda, \epsilon) = N(\lambda, \epsilon + \eta) + \eta \). Since this is true for all \( \eta > 0 \) and \( N \) is continuous in the second coordinate, we can conclude that \( \sigma \) is in the closed \( N(\lambda, \epsilon) \)-neighborhood of \( \gamma|_{[s,t]} \). It follows that \( \gamma \) is refined \( N \)-Morse. \( \square \)

We wish to reprove Corollary 3.2 from [Cor17] to avoid using Lemma 2.10 from the same paper so that we may define the Morse boundary without reference to refined Morse geodesics.

**Lemma 2.6.** Let \( X \) be a geodesic metric space and let \( \alpha : [0, A] \to X \) be an \( N \)-Morse geodesic and \( \beta : [0, A] \to X \) be any geodesic. Also assume that \( \alpha(0) = \beta(0) \). Then for \( t \in [0, A - d(\alpha(A), \beta(A))] \) we have \( d(\alpha(A), (t), \beta(t)) < 4N(3, 0) \).

**Proof.** Let \( \alpha(x) \) be the closest point on \( \alpha \) to \( \beta(b) \). By the triangle inequality \( x \geq A - d(\alpha(A), \beta(A)) \). It follows as in the proof of CASE 1 of Proposition 2.4 in [Cor17], that the concatenation \( \sigma = \beta([0, A]) \cup [\beta(A), \alpha(x)] \) is a \((3, 0)\)-quasi-geodesic. Since \( \alpha \) is \( N \)-Morse, we know that \( \sigma \) is in the \( N(3, 0) \)-neighborhood of \( \alpha([0, y]) \) and by Lemma 2.1 of [Cor17] we can bound the Hausdorff distance between \( \alpha([0, y]) \) and \( \sigma \) by \( 2N(3, 0) \). By a standard argument we may conclude that for all \( t \in [0, A - d(\alpha(A), \beta(A))] \), we have \( d(\alpha(A), (t), \beta(t)) < 4N(3, 0) \). \( \square \)

We now reprove Corollary 3.2 from [Cor17]:

**Lemma 2.7** (Corollary 3.2 redux). Let \( N \) and \( N' \) be Morse gauges such that every \( N \)-Morse geodesic is also \( N' \)-Morse. Then the natural inclusion \( i : \partial_M^n X_p \to \partial_M^n X_p \) is continuous. In particular if \( N \leq N' \) then this condition is satisfied.

**Proof.** Let \( U \) be an open set in \( \partial_M^n X_p \). We wish to show that \( i^{-1}(U) \) is open. Let \( x \in i^{-1}(U) \) and \( \alpha_x \) a geodesic ray representing \( x \). Since \( i \) is an inclusion and \( U \) is open in \( \partial_M^n X_p \), then there exists an \( j \in \mathbb{N} \) so that \( V_j^n(\alpha_x) \subset U \). Let \( k = j + 3N \). Applying Lemma 2.6 thinking of \( \alpha_x \) as an \( N' \)-Morse geodesic, we know that for any \( y \in V_k^n(\alpha_x) \) and any geodesic \( \alpha_y \) representing \( y \), that \( d(\alpha_x(t), \alpha_y(t)) < 4N'(3, 0) \) for \( t \in [0, j] \). So \( i(V_k^n(\alpha_x)) \subset V_j^n(\alpha_x) \subset U \). Since we can do this for any \( x \in i^{-1}(U) \) we can conclude that \( i^{-1}(U) \) is open. \( \square \)

**Remark 2.7.1.** We note that one can also prove this using Corollary 2.5 from [Cor17], but we prove Lemma 2.6 because it is a useful lemma of independent
interest. In fact, in future research on the Morse boundary one might wish to redefine $\delta_N$ to be $4N(3,0)$.

**Definition 2.8.** Let $X$ be a proper geodesic metric space and let $p \in X$. Let $\alpha: [0, \infty) \to X$ be an $N$-Morse geodesic ray with $\alpha(0) = p$ and for each positive integer $n$, let $V_n(\alpha)$ be the set of geodesic rays $\gamma$ such that $d(\alpha(t), \gamma(t)) < \delta_N$ for all $t < n$. Then by Lemma 3.1 of [Cor17] this forms a fundamental system of neighborhoods of $[\alpha]$ in $\partial^N_M X_p$. We topologize $\partial^N_M X_p$ with the topology arising from this fundamental system of neighborhoods.

Let $\mathcal{M}$ be the set of all Morse gauges with the usual partial order. The Morse boundary is

$$\partial_M X_p = \lim_{\mathcal{M}} \partial^N_M X_p$$

with the induced direct limit topology.

**Definition 2.9.** The refined Morse boundary, denoted $\partial^c_M X_p$, is defined the same way as the Morse boundary in [Cor17] but rather than considering all Morse gauges $\mathcal{M}$, we consider only refined Morse gauges $\hat{\mathcal{M}}$.

**Theorem 2.10.** Let $X$ be a proper geodesic metric space. Then the inclusion $\partial^c_M X \hookrightarrow \partial_M X$ induces a homeomorphism.

**Proof.** This follows from Lemma 2.4, Lemma 2.7 and the universal property of direct limits. $\Box$

**References**
