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Gerard Valentí-Rojas ,[1,*] Aneirin J. Baker ,[1] Alessio Celi ,[2] and Patrik Öhberg[1]
[1]*SUPA, Institute of Photonics and Quantum Sciences, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom*
[2]*Departament de Física, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain*

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We unveil a duality that extends the notions of both flux attachment and statistical transmutation in spacetime dimensions beyond (2+1)D. Thus, a quantum system in arbitrary dimensions can experience a modification of its statistical properties if coupled to a certain gauge field. For instance, a bosonic quantum fluid can feature composite fermionic (or anyonic) excitations when coupled to a statistical gauge field. We compute the explicit form of the aforementioned synthetic gauge fields in $\mathrm{D} \leqslant 3+1$. We introduce a bosonic liquid and its composite dual in (1+1)D as proof of principle. We recover well-known results, resolve old controversies, and suggest a microscopic mechanism for the emergence of such a gauge field. We also outline potential directions for experimental realizations in ultracold atom platforms.

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I. INTRODUCTION

Flux attachment [1] is a physical mechanism describing how charged particles capture magnetic flux quanta and become composite entities, often featuring exotic properties [2]. In two spatial dimensions, this constitutes a well-established picture to intuitively understand the low-energy effective description of some topologically ordered phases of matter [3,4]. The appearance of a Chern-Simons field is found responsible for this phenomenon and is intrinsically related to the fractionalization of quantum numbers [5,6]. All the above can be encapsulated as part of a Bose-Fermi correspondence in which dynamical gauge fields play a pivotal role [7]. The situation is much different in other dimensions, where the previous concepts become ill defined. For instance, there is no Chern-Simons term in even spacetime dimensions, so the existence of pointlike anyons [8], as well as their interpretation as composites, seems no longer valid. It is then natural to ask whether an analogous mechanism to flux attachment can be found in all generality. This is a subtle question, especially in spatially one-dimensional systems, where the notion of a magnetic field is not even defined so, strictly speaking, there is no flux to attach. Yet, Bose-Fermi correspondences [9–12] seem almost inevitable and many instances of linear anyons have been proposed [13].

In this paper, we introduce a composite particle duality understood as a correspondence between theories with different quantum statistics. This is the physical statement that gauge dressing a set of fields can effectively transmute their commutation relations. Matter becomes dyonically charged by construction and, in a gauge transformed or dual picture, is seen as composites. This is a familiar notion that we now disclose as generic across dimensions under the same mechanism (see Table I). As an illustration, we study a nonrelativistic model in one spatial dimension. We verify the presence of fractional statistics and resolve an old controversy [14–16]. In this context, and based on recent experimental work in ultracold atoms [17,18], we suggest a microscopic origin for such a statistical gauge field emerging in interacting quantum many-body systems. More broadly, evidence from the aforementioned duality enables us to make the following general observations:

(i) The composite particle duality naturally generalizes those of conventional flux attachment and statistical transmutation. It survives in any dimension, its origin is geometric, it is physically enforced by topological terms, and satisfies an order-disorder operator structure [19].

(ii) Bose-Fermi correspondences can be seen as statistical transmutation. Hence, they can be probed in experiments by gauge-coupling quantum matter and tuning the coupling constant of the statistical gauge field. This will correspond to a physical interpolation between faces of the duality or, equivalently, changing the statistical parameter. We also expect the presence of exotic gauge-charge composite quasiparticles.

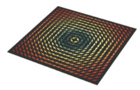
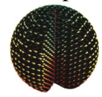
II. COMPOSITE PARTICLE DUALITY

We establish that the minimal coupling of charged matter to a $U(1)$ statistical gauge field a_μ in $\mathrm{D} = d + 1$ spacetime dimensions is equivalent to the formation of electric-magnetic entities identified as gauge-charge composites. In some instances, the latter may be regarded as anyons. This correspondence is summarized as

$$\mathcal{H}_{\mathrm{B}} = \sum_{i=1}^{N} \frac{\boldsymbol{\pi}_i^2}{2m} + \mathcal{H}_{\mathrm{int}} \longleftrightarrow \tilde{\mathcal{H}}_{\mathrm{C}} = \sum_{i=1}^{N} \frac{\tilde{\boldsymbol{p}}_i^2}{2m} + \tilde{\mathcal{H}}_{\mathrm{int}}, \quad (1)$$

*Corresponding author: gv16@hw.ac.uk

TABLE I. Main features of the electromagnetic response for matter coupled to statistical gauge fields. Blue (red) coloring at the bottom row denotes high (low) intensity of the vector field. Observe a jump discontinuity at the origin for the kink, a singular point for the vortex, and a singular Dirac string on the positive z axis for the monopole.

Spatial dimension	$d=1$	$d=2$	$d=3$
Flux attachment	$a_x \propto n$	$b \propto n$	$\nabla\cdot\boldsymbol{b} \propto n$
Topological quantization	No	Yes	Yes
Topological soliton	Kink	Vortex	Monopole
Statistical field			

where $\boldsymbol{\pi}_i = \boldsymbol{p}_i - \boldsymbol{a}(\mathbf{x}_i)$, N is the number of particles, and interactions are short ranged. We postulate that the statistical gauge potential is a topologically nontrivial pure gauge configuration

$$\boldsymbol{a}(\mathbf{x}_i) = \nabla_{\mathbf{x}_i}\Phi(\mathbf{x}_1,\dots,\mathbf{x}_N), \qquad (2)$$

where $\boldsymbol{a}(\mathbf{x}_i) \equiv \boldsymbol{a}(\mathbf{x}_i;\mathbf{x}_1,\dots,\mathbf{x}_N)$ refers to the gauge potential being evaluated at the location of particle $\mathbf{x}_i$, although it might be a function of the position of all the particles in the system and, as we will find later, also of matter density $|\Psi_{\mathrm{B}}(\mathbf{x}_1,\dots,\mathbf{x}_N)|^2$. We identify a many-body Hamiltonian in the bare (B) basis, and another corresponding to the composite (C) basis. Both sides of the duality are related by a large gauge transformation which removes/introduces a minimally coupled statistical gauge field. In doing so, it connects homotopically distinct states with the same physical properties. This transformation corresponds to the naive generalized continuum version of the well-known Jordan-Wigner transformation which, in second-quantized language, reads

$$\hat{\Psi}_{\mathrm{C}}(\mathbf{x};\Gamma_{\mathbf{x}}) = \hat{\mathcal{W}}^{\dagger}(\mathbf{x};\Gamma_{\mathbf{x}})\hat{\Psi}_{\mathrm{B}}(\mathbf{x}). \qquad (3)$$

The operator $\hat{\mathcal{W}}(\mathbf{x};\Gamma_{\mathbf{x}}) = \exp[i\hbar^{-1}\hat{\Phi}(\mathbf{x};\Gamma_{\mathbf{x}})]$ is identified as a disorder operator [20,21], $\hat{\Phi}$ is a $(\mathrm{D}-1)$-dimensional Jordan-Wigner brane [22], and $\Gamma_{\mathbf{x}}$ is a reference contour centered at $\mathbf{x}$ in the sense of Ref. [19] but physically identified as a Dirac string or open 't Hooft line. Correspondence (3) can be understood as an operator identity valid regardless of the underlying Hamiltonian. Considering the bare species $\hat{\Psi}_{\mathrm{B}}$ to be a bosonic (fermionic) field, and hence satisfying ordinary equal-time (anti)commutation relations, then $\hat{\Psi}_{\mathrm{C}}$ constitutes a composite field obeying generalized commutation relations, except for at the point $\mathbf{x}=\mathbf{x}'$, where relations reduce to those of bare species due to the cancellation of branes. The density operator is $\hat{n}(\mathbf{x}) = \hat{\Psi}^{\dagger}_{\mathrm{B}}(\mathbf{x})\hat{\Psi}_{\mathrm{B}}(\mathbf{x}) = \hat{\Psi}^{\dagger}_{\mathrm{C}}(\mathbf{x};\Gamma_{\mathbf{x}})\hat{\Psi}_{\mathrm{C}}(\mathbf{x};\Gamma_{\mathbf{x}})$, so all possible local interaction terms which are functions of the density are identical on both sides of the duality.

The explicit form of the statistical gauge field might be guessed, but it can also be derived on the grounds of topological field theory. Hence, we introduce a $U(1)$ topological gauge field $\hat{\mathcal{A}}$ as a new gauge connection obtained from solving a topological current constraint satisfying $\partial_\mu \hat{\mathcal{J}}^\mu(\mathbf{x}) = 0$ by construction. A particularly simple form is

$$\hat{\mathcal{J}}^\mu = \frac{\kappa}{2\pi}\epsilon^{\mu\nu\lambda\alpha\beta\dots}\,\partial_\nu\,\hat{\mathcal{A}}_{\lambda\alpha\beta\dots}, \qquad (4)$$

where $\hat{\mathcal{A}}$ is an antisymmetric gauge-dependent $(\mathrm{D}-2)$-form field that transforms as $\hat{\mathcal{A}}_{\mu\nu\lambda\alpha\dots} \to \hat{\mathcal{A}}_{\mu\nu\lambda\alpha\dots} + \partial_{[\mu}\hat{\xi}_{\nu\lambda\alpha\dots]}$, and κ is a real constant whose value may or may not be constrained through Dirac quantization. This general relation reduces to a scalar field ($\hat{\Phi}$) in (1+1)D, a Chern-Simons field ($\hat{A}_\mu$) in (2+1)D, and a Kalb-Ramond field ($\hat{B}_{\mu\nu}$) in (3+1)D or higher ($\hat{C}_{\mu\nu\lambda\dots}$). A relation such as that in Eq. (4) has been identified and discussed extensively in previous work on functional bosonization [23–33] and is nothing but a manifestation of the Hodge duality $\hat{\mathcal{F}} \equiv d\hat{\mathcal{A}} = \star\hat{\mathcal{J}}$ mapping p forms in D dimensions to $(\mathrm{D}-p)$ forms through the Hodge star operator ($\star$). We can write the $(\mathrm{D}-2)$-form field as $\hat{\mathcal{A}} = d\hat{\alpha}$ at the expense of violating the corresponding Bianchi identity $d(d\hat{\alpha}) \neq 0$ due to the nontrivial topology of $\hat{\alpha}$, hence the name for $\hat{\mathcal{A}}$. The current constraint can also be found as an equation of motion of a parent effective gauge action when minimally coupled to a matter source. In the following, we restrict ourselves to spatial dimensions $d=1,2,3$. An educated guess for an action is then $S[a,\mathcal{A}] = S_{\text{topo.}} - \int d^{\mathrm{D}}\mathbf{x}\,\mathcal{J}^\mu a_\mu$, where

$$S_{\text{topo.}} = \int d^{\mathrm{D}}\mathbf{x}\left(\frac{\kappa}{2\pi}\epsilon^{\mu\nu\lambda\dots\alpha\beta}\mathcal{A}_{\mu\nu\lambda\dots}\partial_\alpha a_\beta + \mathcal{L}[\mathcal{A}]\right) \qquad (5)$$

is a combination of background field (BF) and another topological term changing with spatial dimension. Upon elimination of the topological gauge field, one recovers Chern-Simons terms for odd D dimensions or θ terms for even. These terms will, in general, give restricted dynamics to the statistical gauge field when coupled to matter via a Gauss's law constraint. The statistical gauge field (a) is physically coupled to matter while the topological gauge field ($\mathcal{A}$) is auxiliary. The Euler-Lagrange equations with respect to both define, respectively, the current bosonization rule and a local condition expressing the topological gauge field in terms of the statistical one. While we have no apparent evidence to expect the action (5) to be unique, we claim that the terms involved should induce torsion or helicity [34] (e.g., a twist) in the gauge connection and not merely curvature. Upon using the equations of motion, we recover the relations

$$\mathcal{J}^\mu_{(1+1)\mathrm{D}} = \frac{\kappa}{2\pi}\epsilon^{\mu\nu}\partial_\nu\Phi = \frac{\kappa}{2\pi}\epsilon^{\mu\nu}a_\nu, \qquad (6)$$

$$\mathcal{J}^\mu_{(2+1)\mathrm{D}} = \frac{\kappa}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = \frac{\kappa}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda, \qquad (7)$$

$$\mathcal{J}^\mu_{(3+1)\mathrm{D}} = \frac{\kappa}{2\pi}\epsilon^{\mu\nu\lambda\alpha}\partial_\nu B_{\lambda\alpha} = \frac{\kappa}{2\pi}\epsilon^{\mu\nu\lambda\alpha}\partial_\nu\partial_\lambda a_\alpha. \qquad (8)$$

From here, we can read the corresponding flux attachment Gauss's laws in vector notation as

$$[(1+1)\mathrm{D}]: \quad \frac{2\pi}{\kappa}n(t,x) = a_x(t,x), \qquad (9)$$

$$[(2+1)\mathrm{D}]: \quad \frac{2\pi}{\kappa}n(t,\boldsymbol{x}) = b(t,\boldsymbol{x}), \qquad (10)$$

$$[(3+1)\mathrm{D}]: \quad \frac{2\pi}{\kappa}n(t,\boldsymbol{x}) = \nabla\cdot\boldsymbol{b}(t,\boldsymbol{x}), \qquad (11)$$

where the "electric" charge density is $\mathcal{J}^0 = n(t,\boldsymbol{x})$ and "magnetic" field is $\boldsymbol{b}(t,\boldsymbol{x}) \equiv b^i = \epsilon^{ijk}\partial_j a_k$ [35]. The correspondence in Eq. (1) supplemented by the statistical gauge fields found in Eqs. (9)–(11) can be understood as a physical mechanism for flux attachment and statistical transmutation. They constitute the main result of this work. The (2+1)D constraint corresponds to the usual description of flux attachment [1]. Equation (11) in (3+1)D can be immediately identified as the magnetic monopole law, but with $n(t, \boldsymbol{x})$ being an electric charge density, not the usual magnetic source. In other words, an entity carrying electric charge also becomes a source magnetic field, i.e., it forms a composite reminiscent of a dyon [36–38] and undergoes an inverse Witten effect. This is indeed the consequence of flux attachment in higher dimensions.

III. A SIMPLE DUAL MODEL

Hereafter, for illustrative purposes, we will be interested in the (1+1)D case, for which the time component of the topological current in Eq. (4) reduces to the usual bosonization relation $\hat{n}(\mathrm{x}) = \gamma^{-1}\,\partial_x\,\hat{\Phi}(\mathrm{x})$, where $\gamma = 2\pi/\kappa$. By simple integration, we see that $\hat{\Phi}(\mathrm{x}) = \gamma\int_{-\infty}^{x} dx'\,\hat{n}(\mathrm{x}')$. This appears naturally for a density-dependent gauge potential of the form

$$\hat{a}_x(\mathrm{x}) = \gamma\int_{-\infty}^{\infty} dx'\,\partial_x\,\Theta(x-x')\,\hat{n}(\mathrm{x}') = \gamma\,\hat{n}(\mathrm{x}), \tag{12}$$

where $\Theta(x)$ denotes a Heaviside step function, which plays the role of a kink. We observe that the gauge potential in (1+1)D is a pure gauge one, $\hat{a}_x = \partial_x\hat{\Phi}$. Provided there is no singularity to wind around, we expect no topological quantization in this model as opposed to its higher-dimensional relatives. Additionally, we notice that for a system of point particles, in first-quantized language, Eq. (12) nontrivially reduces to $a_x(\mathrm{x}_i) = \partial_{x_i}\Phi = \gamma\sum_{j\neq i}\mathrm{sgn}(j-i)\delta(x_i - x_j)$ with a string $\Phi = \gamma\sum_{j<l}\Theta(x_j - x_l)$. This resolves the long-standing tension between Refs. [14–16] as it systematically corrects the sign error made in Ref. [14] and identified in Ref. [15].

In order to understand the implications of the density-dependent gauge field (12) and the duality, we should study its behavior in the presence of dynamical matter. The discussion that follows can be taken as an extension of Kundu's results [16]. We consider a (1+1)D nonrelativistic weakly interacting Bose gas [39] minimally coupled to a statistical gauge field with action $S[\hat{\Psi}, \hat{a}_\mu] = \int dt\,dx\,\mathcal{L}_{\mathrm{B}}$ and Lagrangian density

$$\mathcal{L}_{\mathrm{B}} = i\hbar\,\hat{\Psi}^\dagger D_t\hat{\Psi} - \frac{\hbar^2}{2m}(D_x\hat{\Psi})^\dagger(D_x\hat{\Psi}) - \frac{g}{2}\hat{\Psi}^\dagger\hat{\Psi}^\dagger\hat{\Psi}\hat{\Psi}. \tag{13}$$

The form of the gauge field is given by the topological constraint $\hat{\mathcal{J}}^\mu = \hat{J}^\mu = \gamma^{-1}\epsilon^{\mu\nu}\hat{a}_\nu$ in Eq. (6), where $\hat{J}^\mu = (\hat{n}, \hat{J}_x)$, which reads

$$\hat{J}^\mu(\mathrm{x}) = \left(\hat{\Psi}^\dagger\hat{\Psi},\, \frac{\hbar}{2mi}[\hat{\Psi}^\dagger D_x\,\hat{\Psi} - (D_x\,\hat{\Psi})^\dagger\,\hat{\Psi}]\right), \tag{14}$$

is a conserved current. The gauge covariant derivative is given by $D_\mu = \partial_\mu - i\hbar^{-1}\hat{a}_\mu$ and the commutation relations are

$$[\hat{\Psi}(x), \hat{\Psi}(x')] = [\hat{\Psi}^\dagger(x), \hat{\Psi}^\dagger(x')] = 0, \tag{15}$$

$$[\hat{\Psi}(x), \hat{\Psi}^\dagger(x')] = \delta(x-x'). \tag{16}$$

This model admits a composite dual description of the form $\tilde{S}[\hat{\Psi}_{\mathrm{C}}] = \int dt\,dx\,\tilde{\mathcal{L}}_{\mathrm{C}}$ with

$$\tilde{\mathcal{L}}_{\mathrm{C}} = i\hbar\,\hat{\Psi}_{\mathrm{C}}^\dagger\,\partial_t\hat{\Psi}_{\mathrm{C}} - \frac{\hbar^2}{2m}\partial_x\hat{\Psi}_{\mathrm{C}}^\dagger\,\partial_x\hat{\Psi}_{\mathrm{C}} - \frac{g}{2}\hat{\Psi}_{\mathrm{C}}^\dagger\hat{\Psi}_{\mathrm{C}}^\dagger\hat{\Psi}_{\mathrm{C}}\hat{\Psi}_{\mathrm{C}}, \tag{17}$$

where the composite field obeys the algebra

$$\hat{\Psi}_{\mathrm{C}}^\dagger(x)\hat{\Psi}_{\mathrm{C}}^\dagger(x') - e^{\frac{i}{\hbar}\gamma\,\mathrm{sgn}(x-x')}\hat{\Psi}_{\mathrm{C}}^\dagger(x')\hat{\Psi}_{\mathrm{C}}^\dagger(x) = 0, \tag{18}$$

$$\hat{\Psi}_{\mathrm{C}}(x)\hat{\Psi}_{\mathrm{C}}^\dagger(x') - e^{-\frac{i}{\hbar}\gamma\,\mathrm{sgn}(x-x')}\hat{\Psi}_{\mathrm{C}}^\dagger(x')\hat{\Psi}_{\mathrm{C}}(x) = \delta(x-x'), \tag{19}$$

and the coupling γ constitutes also the statistical angle. The composite dual action is found via the Jordan-Wigner transformation [Eq. (3)], which reads as

$$\hat{\Psi}(\mathrm{x}) = e^{\frac{i}{\hbar}\gamma\int_{-\infty}^{\infty}dx'\,\Theta(x-x')\hat{n}(\mathrm{x}')}\hat{\Psi}_{\mathrm{C}}(\mathrm{x}), \tag{20}$$

and acts as a large gauge transformation and allows interpolation between both faces of the duality. This constitutes a statistical transmutation. Notice that the Jordan-Wigner string is nothing but the aforementioned scalar (kink) field $\hat{\Phi}(\mathrm{x})$. A crucial rewriting of the kinetic term as

$$\tilde{H}_{\mathrm{kin}} \sim \partial_x\hat{\Psi}_{\mathrm{C}}^\dagger\,\partial_x\hat{\Psi}_{\mathrm{C}} = (D_x\,\hat{\Psi})^\dagger(D_x\,\hat{\Psi}) \tag{21}$$

$$= \partial_x\hat{\Psi}^\dagger\partial_x\hat{\Psi} - \frac{2\gamma m}{\hbar^2} : \hat{n}\,\hat{j}_x : + \frac{\gamma^2}{\hbar^2}\,\hat{\Psi}^\dagger\hat{n}^2\,\hat{\Psi}, \tag{22}$$

with $:\bullet:$ denoting normal ordering and current density

$$\hat{j}_x = \frac{\hbar}{2mi}[\hat{\Psi}^\dagger\partial_x\,\hat{\Psi} - (\partial_x\,\hat{\Psi})^\dagger\,\hat{\Psi}], \tag{23}$$

shows that minimal coupling to a density-dependent gauge potential is nothing but a density-current nonlinearity and a three-body term. Thus, modifying the coupling of the interactions in Eq. (22) is equivalent to tuning the statistics in the dual description. The most dramatic effects are produced by the term involving the current density, which breaks parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$). This leads to asymmetric dynamics and chiral solitons [17,18,40]. We see that the composite dual model is nothing but an anyonic Lieb-Liniger model [41–43]. It is worth stressing that Girardeau's Bose-Fermi correspondence [44–46] is a particular case of the above prescription.

IV. MACROSCOPIC ORIGIN

The reader might wonder whether the statistical gauge field in Eq. (12) has really anything to do with conventional flux attachment. Let us consider the local law $\boldsymbol{\nabla}\times\hat{\boldsymbol{a}}(\boldsymbol{x}) = \gamma\,\hat{n}(\boldsymbol{x})$ on a punctured two-dimensional disk with $r = \epsilon$ inner and $r = R$ outer radius, respectively. When taking the limit $\epsilon \to R$ the disk approaches an annulus. This implies $|R-\epsilon| \approx 0$ and $\partial_r\,\hat{\boldsymbol{a}} \approx \boldsymbol{0}$ in $\epsilon \leqslant r \leqslant R$. The magnetic field in polar coordinates (r, φ) becomes

$$\boldsymbol{\nabla}\times\hat{\boldsymbol{a}}(\boldsymbol{x})|_{\epsilon\to R} = \left[\frac{1}{r}\,\hat{a}_\varphi(r,\varphi) - \frac{1}{r}\partial_\varphi\,\hat{a}_r(r,\varphi)\right]\Bigg|_{r\to R}. \tag{24}$$

In this limit, the flux attachment expression at $r = R$ reads

$$\hat{a}'_\varphi(\varphi) = \frac{1}{R}[\hat{a}_\varphi(\varphi) + \partial_\varphi\,\hat{\xi}(\varphi)] = \gamma\,\hat{n}(\varphi), \tag{25}$$

where $\hat{\xi}(\varphi) = -\hat{a}_r(R,\varphi)$ becomes just a memory of the higher-dimensional space and is absorbed in a new gauge potential, yielding Eq. (9) as a reduced expression for flux attachment. A similar logic can be applied in dimensionally reducing the parent gauge action in Eq. (5). This is written schematically as $S_{2+1} \to S_{1+1}$, with identification

$$S_{\rm BF}[A,a] + S_{\rm CS}[A] \longrightarrow S_{\rm Ax.}[\Phi,a] + S_\chi[\Phi], \quad (26)$$

$$\mathcal{L}_{1+1} = \frac{\kappa}{2\pi}\Phi\,\epsilon^{\mu\nu}\partial_\mu a_\nu + \frac{\kappa}{2\pi}\epsilon^{01}\partial_0\Phi\,\partial_1\Phi, \quad (27)$$

where $\mu,\nu = \{t,x\}$. We notice that $\mathcal{L}_{\rm Ax.}$ in Eq. (26) constitutes both an axion and a BF term. It can also be understood, upon integration by parts, as a many-body Aharonov-Bohm twist effect [47]. The axion contribution alone leads to a decoupling from the gauge field, but the introduction of $\mathcal{L}_\chi$ provides the axion with chiral dynamics. As expected, these contributions give a vanishing Hamiltonian, since they are first order in time derivatives, and are universal in that they do not depend on a specific matter model. Rather, they introduce constraints on the system, and fix the form of the statistical gauge field to be linear in density.

V. MICROSCOPIC EMERGENCE OF A STATISTICAL GAUGE FIELD

The previous discussion gives a macroscopic or phenomenological description of matter. However, it does not provide an intuitive explanation for how these statistical gauge fields could effectively be generated. It is evident from Eq. (22) that the minimal coupling to a linear-in-density gauge field [Eq. (12)] can be seen as a combination of two- and three-body contact interactions and an exotic parity-breaking term. The latter is a chiral interaction $\mathcal{H}_{\rm int} \sim: (\mathbf{k}\,\hat{n}^2):$ since it is momentum dependent in $\mathbf{k}$ space. We wonder whether it can arise from more conventional interaction terms. Hence, we consider a bosonic field theory with two-body interactions and a pseudospin-1/2 degree of freedom $\alpha = \{\uparrow,\downarrow\}$. We focus only on central, spin-preserving interactions

$$\sum_{\alpha,\beta}\int d\boldsymbol{x}\,d\boldsymbol{x}'\,\hat{\Psi}^\dagger_\alpha(\mathbf{x})\hat{\Psi}^\dagger_\beta(\mathbf{x}')U^{\alpha\beta}(\boldsymbol{x}-\boldsymbol{x}')\hat{\Psi}_\beta(\mathbf{x}')\hat{\Psi}_\alpha(\mathbf{x}). \quad (28)$$

Upon rotation on the Bloch sphere and Fourier transforming we obtain the effective interaction in a new spin basis $\sigma = \{+,-\}$, as

$$\sum_{\sigma,\tau,\sigma',\tau'}\int_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}\tilde{\chi}^{\sigma\tau\sigma'\tau'}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}}\hat{b}^\dagger_{\mathbf{k}_1+\mathbf{q},\sigma}\,\hat{b}^\dagger_{\mathbf{k}_2-\mathbf{q},\tau}\,\hat{b}_{\mathbf{k}_2,\sigma'}\,\hat{b}_{\mathbf{k}_1,\tau'}. \quad (29)$$

We introduce $\int_{\mathbf{k}} \equiv \frac{1}{\sqrt{\rm vol.}}\int d^d\mathbf{k}$ and define the screening function in momentum space as

$$\tilde{\chi}^{\sigma\tau\sigma'\tau'}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} = \sum_{\alpha,\beta}\tilde{U}^{\alpha\beta}_{\mathbf{q}}\,\hat{\eta}^{\dagger\,\alpha\sigma}_{\mathbf{k}_1+\mathbf{q}}\,\hat{\eta}^{\dagger\,\beta\tau}_{\mathbf{k}_2-\mathbf{q}}\,\hat{\eta}^{\tau'\beta}_{\mathbf{k}_2}\,\hat{\eta}^{\sigma'\alpha}_{\mathbf{k}_1}. \quad (30)$$

Function $\tilde{\chi}$ can be interpreted as describing dressed interactions, while $\tilde{U}$ describes bare ones. The change of spin basis is nothing but a qubit $\mathbf{k}$-dependent rotation parametrized by

$$\hat{\eta}^{\sigma\alpha}_{\mathbf{k}} = \hat{\mathcal{R}}_{\hat{\mathbf{n}}_\sigma}(\theta_{\mathbf{k}}) = \exp\left(-i\,\theta_{\mathbf{k}}\frac{\hat{\mathbf{n}}_\sigma\cdot\vec{\boldsymbol{\sigma}}}{2}\right). \quad (31)$$

A particular instance of the above scheme has been realized in a Raman-coupled Bose-Einstein condensate with internal atomic structure [17,18]. As an instructive example, we consider the lowest-energy spin branch ($\sigma = \tau = \sigma' = \tau'$) and disregard the rest. In this context, ($\pm$) states are dressed states.

We consider ultra-short-range bare interactions $U^{\alpha\beta}(\mathbf{x}) = g^{\alpha\beta}\,\delta(\mathbf{x})$, where $\delta(\mathbf{x})$ is the Dirac delta function. At low orders in $\mathbf{k} \approx \mathbf{k}_0 + \delta\mathbf{k}$ expansion, with $\delta\mathbf{k} \ll 1$, we verify that the effective interaction kernel acquires the form

$$\tilde{\chi}^{\sigma\sigma\sigma\sigma}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q}} \sim O(\delta\mathbf{k}_i^0) + O(\delta\mathbf{k}_i^1) \sim (1+\delta\mathbf{k}_i)\,\delta_{\delta\mathbf{k}_1+\delta\mathbf{k}_2,\delta\mathbf{q}}. \quad (32)$$

Thus, emergent longer-range interactions now appear as a consequence of atomic light dressing [48] and allow for the generation of the chiral interaction.

Alternatively, one can view the density-dependent gauge field arises in the mean-field limit if a Pauli coupling term $\propto\hbar\Omega\,\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{n}}$ is present in the Hamiltonian. For weak enough particle interactions we can expand in a perturbation series and project onto one of the eigenstates of the system relying on the adiabatic theorem. The result is an effective Hamiltonian with emergent Berry connection terms arising perturbatively. The first-order contribution $\boldsymbol{\mathcal{A}}^{(1)}_\sigma \propto |\Psi_\sigma|^2$ is linear in mean-field density and can be computed in closed form. Hence, topological terms and the corresponding statistical gauge fields, such as the density-dependent one discussed here [40] or Chern-Simons [49] are not fictitious in the sense of a convenient computational trick, but can instead dynamically arise from effective microscopic interparticle interactions or as interacting Berry phase effects.

VI. LATTICE VERSION

The previous ideas work equally well on the lattice, where they manifest as dynamical complex tunneling rates, being the general prescription of the form

$$\hat{H} = -J\sum_{j,\,\mu=1}^{d}\left(\hat{c}^\dagger_j e^{i\hat{a}_\mu(j)}\hat{c}_{j+\hat{e}_\mu} + \text{H.c.}\right) + \hat{H}_{\rm int}, \quad (33)$$

where interactions are local, $\hat{c}$ is a bosonic (or fermionic) annihilation operator, j denotes the lattice site, and μ sums over nearest neighbors in d spatial dimensions. Here, the statistical gauge field

$$\hat{a}_\mu(j) \equiv \hat{a}(j;j+\hat{e}_\mu) = \frac{1}{\hbar}\int_{\boldsymbol{x}_j+\hat{e}_j}^{\boldsymbol{x}_j} d\boldsymbol{x}\cdot\hat{\boldsymbol{a}}(t,\boldsymbol{x}) \quad (34)$$

is the operator-valued Peierls phase, defined on the links of the lattice, and accumulated in a tunneling event. The explicit form will once again depend on dimensionality, being $\hat{a}_x(j) = \gamma\,\hat{n}_j$ in $d=1$. In $d=2$ it is the solution, in any suitable gauge, of $(\hat{b}_j - \gamma\,\hat{n}_j) = 0$ with $\hat{b}_j = \epsilon^{ab}\Delta_a\,\hat{a}_b(j)$ being the "magnetic" field associated with plaquette j and Δ_a the lattice directional derivative. In $d=3$ it is the solution of $(\Delta_a\,\hat{b}^a_j - \gamma\,\hat{n}_j) = 0$. It is worth observing that matter fields (i.e., electric charges) live on the direct lattice, while fluxes (i.e., magnetic charges) live on the dual lattice. Also, notice that recovering the continuum limit must be considered at all orders [50] in order to preserve the compactness of the statistical parameter $\gamma \in [0,2\pi)$. Now, for the particular case of Hubbard-like interactions for $\hat{H}_{\rm int}$, such a model reduces to an anyon-Hubbard model [51] and has been extensively studied in 1d [52], but remains largely unexplored in other dimensions.

VII. CONCLUSIONS

The composite particle duality establishes statistical transmutation as a generic physical phenomenon in nonrelativistic quantum many-body systems across dimensions. Hence, we advocate the existence of anyons understood as composites, in dimensions other than $d = 2$, although not necessarily as pointlike objects. We support the assertion made by several authors [19,21,53] of an underlying order-disorder scheme in Bose-Fermi correspondences, as it constitutes the basis of our effective transmutation mechanism identifying bare/composite theories. The whole picture is embodied by a generalized Jordan-Wigner mapping playing the role of a gauge transformation with topological features. From our construction, we expect new and old families of topological quantum liquids and solids to be found by merely coupling conventional quantum matter to suitable statistical gauge fields a_μ in the corresponding dimension. Our scheme is valid regardless of the bare species being bosonic or fermionic. This work also opens the door to the experimental realization of Bose-Fermi dualities, a range of anyonic models [e.g. Eq. (33)], and a different class of topological gauge theories that have not yet been studied in the context of quantum simulators, with the recent exception of Ref. [17]. Yet, density-dependent gauge potentials have already been experimentally implemented [17,54–57] in ultracold atomic platforms, so we are hopeful that such realizations can come to fruition in the near future.

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