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ON THE PROBABILISTIC CAUCHY THEORY OF THE CUBIC NONLINEAR SCHRÖDINGER EQUATION ON \mathbb{R}^d , $d \geq 3$

ÁRPÁD BÉNYI, TADAHIRO OH, AND OANA POCOVNICU

ABSTRACT. We consider the Cauchy problem of the cubic nonlinear Schrödinger equation (NLS) : $i\partial_t u + \Delta u = \pm |u|^2 u$ on \mathbb{R}^d , $d \geq 3$, with random initial data and prove almost sure well-posedness results below the scaling-critical regularity $s_{\text{crit}} = \frac{d-2}{2}$. More precisely, given a function on \mathbb{R}^d , we introduce a randomization adapted to the Wiener decomposition, and, intrinsically, to the so-called modulation spaces. Our goal in this paper is three-fold. (i) We prove almost sure local well-posedness of the cubic NLS below the scaling-critical regularity along with small data global existence and scattering. (ii) We implement a probabilistic perturbation argument and prove ‘conditional’ almost sure global well-posedness for $d = 4$ in the defocusing case, assuming an a priori energy bound on the critical Sobolev norm of the nonlinear part of a solution; when $d \neq 4$, we show that conditional almost sure global well-posedness in the defocusing case also holds under an additional assumption of global well-posedness of solutions to the defocusing cubic NLS with deterministic initial data in the critical Sobolev regularity. (iii) Lastly, we prove global well-posedness and scattering with a large probability for initial data randomized on dilated cubes.

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1. INTRODUCTION

1.1. Background. In this paper, we consider the Cauchy problem of the cubic nonlinear Schrödinger equation (NLS) on \mathbb{R}^d , $d \geq 3$:

$$(1.1) \quad \begin{cases} i\partial_t u + \Delta u = \pm \mathcal{N}(u), \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^d), \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

where $\mathcal{N}(u) := |u|^2 u$. The cubic NLS (1.1) has been studied extensively from both the theoretical and applied points of view. Our main focus is to study well-posedness of (1.1) with *random* and *rough* initial data.

It is well known that the cubic NLS (1.1) enjoys the dilation symmetry. More precisely, if $u(t, x)$ is a solution to (1.1) on \mathbb{R}^d with an initial condition u_0 , then

$$(1.2) \quad u_\mu(t, x) := \mu^{-1} u(\mu^{-2} t, \mu^{-1} x)$$

is also a solution to (1.1) with the μ -scaled initial condition $u_{0,\mu}(x) := \mu^{-1} u_0(\mu^{-1} x)$. Associated to this dilation symmetry, there is the so-called scaling-critical Sobolev index $s_{\text{crit}} := \frac{d-2}{2}$ such that the homogeneous $\dot{H}^{s_{\text{crit}}}$ -norm is invariant under this dilation symmetry. In general, we have

$$(1.3) \quad \|u_{0,\mu}\|_{\dot{H}^s(\mathbb{R}^d)} = \mu^{\frac{d-2}{2}-s} \|u_0\|_{\dot{H}^s(\mathbb{R}^d)}.$$

If an initial condition u_0 is in $H^s(\mathbb{R}^d)$, we say that the Cauchy problem (1.1) is subcritical, critical, or supercritical, depending on whether $s > s_{\text{crit}}$, $s = s_{\text{crit}}$, or $s < s_{\text{crit}}$, respectively.

Let us first discuss the (sub)critical regime. In this case, (1.1) is known to be locally well-posed. See Cazenave-Weissler [17] for local well-posedness of (1.1) in the critical Sobolev spaces. As is well known, the conservation laws play an important role in discussing long time behavior of solutions. There are three known conservation laws for the cubic NLS (1.1):

$$\text{Mass: } M[u](t) := \int_{\mathbb{R}^d} |u(t, x)|^2 dx,$$

$$\text{Momentum: } P[u](t) := \text{Im} \int_{\mathbb{R}^d} u(t, x) \overline{\nabla u(t, x)} dx,$$

$$\text{Hamiltonian: } H[u](t) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx \pm \frac{1}{4} \int_{\mathbb{R}^d} |u(t, x)|^4 dx.$$

The Hamiltonian is also referred to as the energy. In view of the conservation of the energy, the cubic NLS is called energy-subcritical when $d \leq 3$ ($s_{\text{crit}} < 1$), energy-critical when $d = 4$ ($s_{\text{crit}} = 1$), and energy-supercritical when $d \geq 5$ ($s_{\text{crit}} > 1$), respectively.

In the following, let us discuss the known results on the global-in-time behavior of solutions to the defocusing NLS, corresponding to the $+$ sign in (1.1), in high dimensions $d \geq 3$. When $d = 4$, the Hamiltonian is invariant under the scaling (1.2) and plays a crucial role in the global well-posedness theory. Indeed, Ryckman-Vişan [53] proved global well-posedness and scattering for the defocusing cubic NLS on \mathbb{R}^4 . See also Vişan [60]. When $d \neq 4$, there is no known scaling invariant positive conservation law for (1.1) in high dimensions $d \geq 3$. This makes it difficult to study the global-in-time behavior of solutions, in particular, in the scaling-critical regularity. There are, however, ‘conditional’ global well-posedness and scattering results as we describe below. When $d = 3$ ($s_{\text{crit}} = \frac{1}{2}$), Kenig-Merle [35] applied

the concentration compactness and rigidity method developed in their previous paper [34] and proved that if $u \in L_t^\infty \dot{H}_x^{\frac{1}{2}}(I \times \mathbb{R}^3)$, where I is a maximal interval of existence, then u exists globally in time and scatters. For $d \geq 5$, the cubic NLS is supercritical with respect to any known conservation law. Nonetheless, motivated by a similar result of Kenig-Merle [36] on radial solutions to the energy-supercritical nonlinear wave equation (NLW) on \mathbb{R}^3 , Killip-Vişan [39] proved that if $u \in L_t^\infty \dot{H}_x^{s_{\text{crit}}}(I \times \mathbb{R}^d)$, where I is a maximal interval of existence, then u exists globally in time and scatters. Note that the results in [35] and [39] are *conditional* in the sense that they assume an *a priori* control on the critical Sobolev norm. The question of global well-posedness and scattering without any *a priori* assumption remains a challenging open problem for $d = 3$ and $d \geq 5$.

So far, we have discussed well-posedness in the (sub)critical regularity. In particular, the cubic NLS (1.1) is locally well-posed in the (sub)critical regularity, i.e. $s \geq s_{\text{crit}}$. In the supercritical regime, i.e. $s < s_{\text{crit}}$, on the contrary, (1.1) is known to be ill-posed. See [1, 11, 16, 18]. In the following, however, we consider the Cauchy problem (1.1) with initial data in $H^s(\mathbb{R}^d)$, $s < s_{\text{crit}}$ in a probabilistic manner. More precisely, given a function $\phi \in H^s(\mathbb{R}^d)$ with $s < s_{\text{crit}}$, we introduce a randomization ϕ^ω and prove almost sure well-posedness of (1.1).

In studying the Gibbs measure for the defocusing (Wick ordered) cubic NLS on \mathbb{T}^2 , Bourgain [6] considered random initial data of the form:

$$(1.4) \quad u_0^\omega(x) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1 + |n|^2}} e^{in \cdot x},$$

where $\{g_n\}_{n \in \mathbb{Z}^2}$ is a sequence of independent standard complex-valued Gaussian random variables. The function (1.4) represents a typical element in the support of the Gibbs measure, more precisely, in the support of the Gaussian free field on \mathbb{T}^2 associated to this Gibbs measure, and is critical with respect to the scaling. With a combination of deterministic PDE techniques and probabilistic arguments, Bourgain showed that the (Wick ordered) cubic NLS on \mathbb{T}^2 is well-posed almost surely with respect to the random initial data (1.4). In the context of the cubic NLW on a three-dimensional compact Riemannian manifold M , Burq-Tzvetkov [14] considered the Cauchy problem with a more general class of random initial data. Given an eigenfunction expansion $u_0(x) = \sum_{n=1}^\infty c_n e_n(x) \in H^s(M)$ of an initial condition,¹ where $\{e_n\}_{n=1}^\infty$ is an orthonormal basis of $L^2(M)$ consisting of the eigenfunctions of the Laplace-Beltrami operator, they introduced a randomization u_0^ω by

$$(1.5) \quad u_0^\omega(x) = \sum_{n=1}^\infty g_n(\omega) c_n e_n(x).$$

Here, $\{g_n\}_{n=1}^\infty$ is a sequence of independent mean-zero random variables with a uniform bound on the fourth moments. Then, they proved almost sure local well-posedness with random initial data of the form (1.5) for $s \geq \frac{1}{4}$. Since the scaling-critical Sobolev index for this problem is $s_{\text{crit}} = \frac{1}{2}$, this result allows us to take initial data below the critical regularity and still construct solutions upon randomization of the initial data. We point out that the randomized function u_0^ω in (1.5) has the same Sobolev regularity as the original function u_0 and is not smoother, almost surely.

¹For NLW, one needs to specify $(u, \partial_t u)|_{t=0}$ as an initial condition. For simplicity of presentation, we only discuss $u|_{t=0}$.

However, it enjoys a better integrability, which allows one to prove improvements of Strichartz estimates. (See Lemmata 2.2 and 2.3 below.) Such an improvement on integrability for random Fourier series is known as Paley-Zygmund's theorem [49]. See also Kahane [32] and Ayache-Tzvetkov [2]. There are several works on Cauchy problems of evolution equations with random data that followed these results, including some on almost sure global well-posedness: [7, 9, 10, 12, 13, 15, 20–23, 42–45, 51, 52, 58].

1.2. Randomization adapted to the Wiener decomposition and modulation spaces. Many of the results mentioned above are on compact domains, where there is a countable basis of eigenfunctions of the Laplacian and thus there is a natural way to introduce a randomization. On \mathbb{R}^d , there is no countable basis of $L^2(\mathbb{R}^d)$ consisting of eigenfunctions of the Laplacian. Randomizations have been introduced with respect to some other countable bases of $L^2(\mathbb{R}^d)$, for example, a countable basis of eigenfunctions of the Laplacian with a confining potential such as the harmonic oscillator $\Delta - |x|^2$, leading to a careful study of properties of eigenfunctions. In this paper, our goal is to introduce a simple and natural randomization for functions on \mathbb{R}^d . For this purpose, we first review some basic notions related to the so-called modulation spaces of time-frequency analysis [28].

The modulation spaces were introduced by Feichtinger [24] in the early eighties. The groundwork theory regarding these spaces of time-frequency analysis was then established in joint collaboration with Gröchenig [25, 26]. The modulation spaces arise from a uniform partition of the frequency space, commonly known as the *Wiener decomposition* [61]: $\mathbb{R}^d = \bigcup_{n \in \mathbb{Z}^d} Q_n$, where Q_n is the unit cube centered at $n \in \mathbb{Z}^d$. The Wiener decomposition of \mathbb{R}^d induces a natural uniform decomposition of the frequency space of a signal via the (nonsmooth) frequency-uniform decomposition operators $\mathcal{F}^{-1} \chi_{Q_n} \mathcal{F}$. Here, $\mathcal{F}u = \widehat{u}$ denotes the Fourier transform of a distribution u . The drawback of this approach is the roughness of the characteristic functions χ_{Q_n} , but this issue can easily be fixed by smoothing them out appropriately. We have the following definition of the (weighted) modulation spaces $M_s^{p,q}$. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$(1.6) \quad \text{supp } \psi \subset [-1, 1]^d \quad \text{and} \quad \sum_{n \in \mathbb{Z}^d} \psi(\xi - n) \equiv 1 \quad \text{for any } \xi \in \mathbb{R}^d.$$

Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$; $M_s^{p,q}$ consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ for which the (quasi) norm

$$(1.7) \quad \|u\|_{M_s^{p,q}(\mathbb{R}^d)} := \|\langle n \rangle^s \|\psi(D - n)u\|_{L_x^p(\mathbb{R}^d)}\|_{\ell_n^q(\mathbb{Z}^d)}$$

is finite. Note that $\psi(D - n)u(x) = \int_{\mathbb{R}^d} \psi(\xi - n) \widehat{u}(\xi) e^{2\pi i x \cdot \xi} d\xi$ is just a Fourier multiplier operator with symbol χ_{Q_n} conveniently smoothed.

It is worthwhile to compare the definition (1.7) with that of the Besov spaces. Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\text{supp } \varphi_0 \subset \{|\xi| \leq 2\}$, $\text{supp } \varphi \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$, and $\varphi_0(\xi) + \sum_{j=1}^{\infty} \varphi(2^{-j}\xi) \equiv 1$. With $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, we define the (inhomogeneous) Besov spaces $B_s^{p,q}$ via the norm

$$(1.8) \quad \|u\|_{B_s^{p,q}(\mathbb{R}^d)} = \|2^{js} \|\varphi_j(D)u\|_{L^p(\mathbb{R}^d)}\|_{\ell_j^q(\mathbb{Z}_{\geq 0})}.$$

There are several known embeddings between Besov, Sobolev, and modulation spaces. See, for example, Okoudjou [47], Toft [59], Sugimoto-Tomita [55], and Kobayashi-Sugimoto [40].

Now, given a function ϕ on \mathbb{R}^d , we have

$$\phi = \sum_{n \in \mathbb{Z}^d} \psi(D - n)\phi,$$

where $\psi(D - n)$ is defined above. This decomposition leads to a randomization of ϕ that is very natural from the perspective of time-frequency analysis associated to modulation spaces. Let $\{g_n\}_{n \in \mathbb{Z}^d}$ be a sequence of independent mean zero complex-valued random variables on a probability space (Ω, \mathcal{F}, P) , where the real and imaginary parts of g_n are independent and endowed with probability distributions $\mu_n^{(1)}$ and $\mu_n^{(2)}$. Then, we can define the *Wiener randomization* of ϕ by

$$(1.9) \quad \phi^\omega := \sum_{n \in \mathbb{Z}^d} g_n(\omega)\psi(D - n)\phi.$$

Almost simultaneously with our first paper [4], Lührmann-Mendelson [42] also considered a randomization of the form (1.9) (with cubes Q_n being substituted by appropriately localized balls) in the study of NLW on \mathbb{R}^3 . See Remark 1.6 below. For a similar randomization used in the study of the Navier-Stokes equations, see the work of Zhang and Fang [63]. We would like to stress again, however, that our reason for considering the randomization of the form (1.9) comes from its connection to time-frequency analysis. See also our previous papers [3] and [4].

In the sequel, we make the following assumption on the distributions $\mu_n^{(j)}$: there exists $c > 0$ such that

$$(1.10) \quad \left| \int_{\mathbb{R}} e^{\gamma x} d\mu_n^{(j)}(x) \right| \leq e^{c\gamma^2}$$

for all $\gamma \in \mathbb{R}$, $n \in \mathbb{Z}^d$, $j = 1, 2$. Note that (1.10) is satisfied by standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions.

It is easy to see that, if $\phi \in H^s(\mathbb{R}^d)$, then the randomized function ϕ^ω is almost surely in $H^s(\mathbb{R}^d)$. While there is no smoothing upon randomization in terms of differentiability in general, this randomization *behaves better under integrability*; if $\phi \in L^2(\mathbb{R}^d)$, then the randomized function ϕ^ω is almost surely in $L^p(\mathbb{R}^d)$ for any finite $p \geq 2$. As a result of this enhanced integrability, we have improvements of the Strichartz estimates. See Lemmata 2.2 and 2.3. These improved Strichartz estimates play an essential role in proving probabilistic well-posedness results, which we describe below.

1.3. Main results. Recall that the scaling-critical Sobolev index for the cubic NLS on \mathbb{R}^d is $s_{\text{crit}} = \frac{d-2}{2}$. In the following, we take $\phi \in H^s(\mathbb{R}^d) \setminus H^{s_{\text{crit}}}(\mathbb{R}^d)$ for some range of $s < s_{\text{crit}}$, that is, below the critical regularity. Then, we consider the well-posedness problem of (1.1) with respect to the randomized initial data ϕ^ω defined in (1.9).

For $d \geq 3$, define s_d by

$$(1.11) \quad s_d := \frac{d-1}{d+1} \cdot s_{\text{crit}} = \frac{d-1}{d+1} \cdot \frac{d-2}{2}.$$

Note that $s_d < s_{\text{crit}}$ and $\frac{s_d}{s_{\text{crit}}} \rightarrow 1$ as $d \rightarrow \infty$. Throughout the paper, we use $S(t) = e^{it\Delta}$ to denote the linear propagator of the Schrödinger group.

We are now ready to state our main results.

Theorem 1.1 (Almost sure local well-posedness). *Let $d \geq 3$ and $s > s_d$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.9), satisfying (1.10). Then, the cubic NLS (1.1) on \mathbb{R}^d is almost surely locally well-posed with respect to the randomization ϕ^ω as initial data. More precisely, there exist $C, c, \gamma > 0$ such that for each $0 < T \ll 1$, there exists a set $\Omega_T \subset \Omega$ with the following properties:*

- (i) $P(\Omega_T^c) < C \exp\left(-\frac{c}{T^\gamma \|\phi\|_{H^s}^2}\right)$.
- (ii) For each $\omega \in \Omega_T$, there exists a (unique) solution u to (1.1) with $u|_{t=0} = \phi^\omega$ in the class

$$S(t)\phi^\omega + C([-T, T] : H^{\frac{d-2}{2}}(\mathbb{R}^d)) \subset C([-T, T] : H^s(\mathbb{R}^d)).$$

We prove Theorem 1.1 by considering the equation satisfied by the nonlinear part of a solution u . Namely, let $z(t) = z^\omega(t) := S(t)\phi^\omega$ and $v(t) := u(t) - S(t)\phi^\omega$ be the linear and nonlinear parts of u , respectively. Then, (1.1) is equivalent to the following perturbed NLS:

$$(1.12) \quad \begin{cases} i\partial_t v + \Delta v = \pm |v + z|^2(v + z), \\ v|_{t=0} = 0. \end{cases}$$

We reduce our analysis to the Cauchy problem (1.12) for v , viewing z as a random forcing term. Note that such a point of view is common in the study of stochastic PDEs. As a result, the uniqueness in Theorem 1.1 refers to uniqueness of the nonlinear part $v(t) = u(t) - S(t)\phi^\omega$ of a solution u .

The proof of Theorem 1.1 is based on the fixed point argument involving the variants of the $X^{s,b}$ -spaces adapted to the U^p - and V^p -spaces introduced by Koch, Tataru, and their collaborators [29, 30, 41]. See Section 3 for the basic definitions and properties of these function spaces. The main ingredient is the local-in-time improvement of the Strichartz estimates (Lemma 2.2) and the refinement of the bilinear Strichartz estimate (Lemma 3.5 (ii)). We point out that, although ϕ and its randomization ϕ^ω have a supercritical Sobolev regularity, the randomization essentially makes the problem subcritical, at least locally in time, and therefore, one can also prove Theorem 1.1 only with the classical subcritical $X^{s,b}$ -spaces, $b > \frac{1}{2}$. See [4] for the result when $d = 4$.

Next, we turn our attention to the global-in-time behavior of the solutions constructed in Theorem 1.1. The key nonlinear estimate in the proof of Theorem 1.1 combined with the global-in-time improvement of the Strichartz estimates (Lemma 2.3) yields the following result on small data global well-posedness and scattering.

Theorem 1.2 (Probabilistic small data global well-posedness and scattering). *Let $d \geq 3$ and $s \in (s_d, s_{\text{crit}}]$, where s_d is as in (1.11). Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.9), satisfying (1.10). Then, there exist $C, c > 0$ such that for each $0 < \varepsilon \ll 1$, there exists a set $\Omega_\varepsilon \subset \Omega$ with the following properties:*

- (i) $P(\Omega_\varepsilon^c) \leq C \exp\left(-\frac{c}{\varepsilon^2 \|\phi\|_{H^s}^2}\right) \rightarrow 0$ as $\varepsilon \rightarrow 0$.
- (ii) For each $\omega \in \Omega_\varepsilon$, there exists a (unique) global-in-time solution u to (1.1) with

$$u|_{t=0} = \varepsilon\phi^\omega$$

in the class

$$\varepsilon S(t)\phi^\omega + C(\mathbb{R} : H^{\frac{d-2}{2}}(\mathbb{R}^d)) \subset C(\mathbb{R} : H^s(\mathbb{R}^d)).$$

(iii) We have scattering for each $\omega \in \Omega_\varepsilon$. More precisely, for each $\omega \in \Omega_\varepsilon$, there exists $v_+^\omega \in H^{\frac{d-2}{2}}(\mathbb{R}^d)$ such that

$$\|u(t) - S(t)(\varepsilon\phi^\omega + v_+^\omega)\|_{H^{\frac{d-2}{2}}(\mathbb{R}^d)} \rightarrow 0$$

as $t \rightarrow \infty$. A similar statement holds for $t \rightarrow -\infty$.

In general, a local well-posedness result in a critical space is often accompanied by small data global well-posedness and scattering. In this sense, Theorem 1.2 is an expected consequence of Theorem 1.1, since, in our construction, the nonlinear part v lies in the critical space $H^{\frac{d-2}{2}}(\mathbb{R}^d)$. The next natural question is probabilistic global well-posedness for large data. In order to state our result, we need to make several hypotheses. The first hypothesis is on a probabilistic a priori energy bound on the nonlinear part v .

Hypothesis (A). *Given any $T, \varepsilon > 0$, there exist $R = R(T, \varepsilon)$ and $\Omega_{T, \varepsilon} \subset \Omega$ such that*

- (i) $P(\Omega_{T, \varepsilon}^c) < \varepsilon$, and
- (ii) if $v = v^\omega$ is the solution to (1.12) for $\omega \in \Omega_{T, \varepsilon}$, then the following a priori energy estimate holds:

$$(1.13) \quad \|v(t)\|_{L^\infty([0, T]; H^{\frac{d-2}{2}}(\mathbb{R}^d))} \leq R(T, \varepsilon).$$

Note that Hypothesis (A) does *not* refer to existence of a solution $v = v^\omega$ on $[0, T]$ for given $\omega \in \Omega_{T, \varepsilon}$. It only hypothesizes the *a priori* energy bound (1.13), just like the usual conservation laws. It may be possible to prove (1.13) independently from the argument presented in this paper. Such a probabilistic a priori energy estimate is known, for example, for the cubic NLW. See Burq-Tzvetkov [15]. We point out that the upper bound $R(T, \varepsilon)$ in [15] tends to ∞ as $T \rightarrow \infty$. See also [50].

The next hypothesis is on global existence and space-time bounds of solutions to the cubic NLS (1.1) with deterministic initial data belonging to the critical space $H^{\frac{d-2}{2}}(\mathbb{R}^d)$.

Hypothesis (B). *Given any $w_0 \in H^{\frac{d-2}{2}}(\mathbb{R}^d)$, there exists a global solution w to the defocusing cubic NLS (1.1) with $w|_{t=0} = w_0$. Moreover, there exists a function $C : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ which is nondecreasing in each argument such that*

$$(1.14) \quad \|w\|_{L_{t,x}^{d+2}([0, T] \times \mathbb{R}^d)} \leq C(\|w_0\|_{H^{\frac{d-2}{2}}(\mathbb{R}^d)}, T)$$

for any $T > 0$.

Note that when $d = 4$, Hypothesis (B) is known to be true for any $T > 0$ thanks to the global well-posedness result by Ryckman-Vişan [53] and Vişan [60]. For other dimensions $d \geq 3$ with $d \neq 4$, it is not known whether Hypothesis (B) holds. Let us compare (1.14) and the results in [35] and [39]. Assuming that $w \in L_t^\infty \dot{H}_x^{s_{\text{crit}}}(I_* \times \mathbb{R}^d)$, where I_* is a maximal interval of existence, it was shown in [35] and [39] that $I_* = \mathbb{R}$ and

$$(1.15) \quad \|w\|_{L_{t,x}^{d+2}(\mathbb{R} \times \mathbb{R}^d)} \leq C\left(\|w\|_{L_t^\infty \dot{H}_x^{\frac{d-2}{2}}(\mathbb{R} \times \mathbb{R}^d)}\right).$$

We point out that Hypothesis (B) is not directly comparable to the results in [35, 39] in the following sense. On the one hand, by assuming that $w \in L_t^\infty \dot{H}_x^{s_{\text{crit}}}(I_* \times \mathbb{R}^d)$,

the results in [35, 39] yield the global-in-time bound (1.15), while Hypothesis (B) assumes the bound (1.14) only for each *finite* time $T > 0$ and does not assume a global-in-time bound. On the other hand, (1.14) is much stronger than (1.15) in the sense that the right-hand side of (1.14) depends only on the size of an initial condition w_0 , while the right-hand side of (1.15) depends on the global-in-time $L_t^\infty \dot{H}_x^{\frac{d-2}{2}}$ -bound of the solution w . Hypothesis (B), just like Hypothesis (A), is of independent interest from Theorem 1.3 below and is closely related to the fundamental open problem of global well-posedness and scattering for the defocusing cubic NLS (1.1) for $d = 3$ and $d \geq 5$.

We now state our third theorem on almost sure global well-posedness of the cubic NLS under Hypotheses (A) and (B). We restrict ourselves to the defocusing NLS in the next theorem.

Theorem 1.3 (Conditional almost sure global well-posedness). *Let $d \geq 3$ and $s \in (s_d, s_{\text{crit}}]$, where s_d is as in (1.11). Assume Hypothesis (A). Furthermore, assume Hypothesis (B) if $d \neq 4$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.9), satisfying (1.10). Then, the defocusing cubic NLS (1.1) on \mathbb{R}^d is almost surely globally well-posed with respect to the randomization ϕ^ω as initial data. More precisely, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) = 1$ such that, for each $\omega \in \Sigma$, there exists a (unique) global-in-time solution u to (1.1) with $u|_{t=0} = \phi^\omega$ in the class*

$$S(t)\phi^\omega + C(\mathbb{R} : H^{\frac{d-2}{2}}(\mathbb{R}^d)) \subset C(\mathbb{R} : H^s(\mathbb{R}^d)).$$

The main tool in the proof of Theorem 1.3 is a perturbation lemma for the cubic NLS (Lemma 7.1). Assuming a control on the critical norm (Hypothesis (A)), we iteratively apply the perturbation lemma in the *probabilistic* setting to show that a solution can be extended to a time depending only on the critical norm. Such a perturbative approach was previously used by Tao-Viřan-Zhang [57] and Killip-Viřan with the second and third authors [37]. The novelty of Theorem 1.3 is an application of such a technique in the probabilistic setting. While there is no invariant measure for the nonlinear evolution in our setting, we exploit the quasi-invariance property of the distribution of the linear solution $S(t)\phi^\omega$. See Remark 8.2. Our implementation of the proof of Theorem 1.3 is sufficiently general that it can be easily applied to other equations. See [50] in the context of the energy-critical NLW on \mathbb{R}^d , $d = 4, 5$, where both Hypotheses (A) and (B) are satisfied.

When $d \neq 4$, the conditional almost sure global well-posedness in Theorem 1.3 has a flavor analogous to the deterministic conditional global well-posedness in the critical Sobolev spaces by Kenig-Merle [35] and Killip-Viřan [39]. In the following, let us discuss the situation when $d = 4$. In this case, we only assume Hypothesis (A) for Theorem 1.3. While it would be interesting to remove this assumption, we do not know how to prove the validity of Hypothesis (A) at this point. This is mainly due to the lack of conservation of $H[v](t)$, i.e. the Hamiltonian evaluated at the nonlinear part v of a solution. In the context of the energy-critical defocusing cubic NLW on \mathbb{R}^4 , however, one can prove an analogue of Hypothesis (A) by establishing a probabilistic a priori bound on the energy $\mathcal{E}[v]$ of the nonlinear part v of a solution, where the energy $\mathcal{E}[v]$ is defined by

$$\mathcal{E}[v](t) = \frac{1}{2} \int_{\mathbb{R}^4} |\partial_t v(t, x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^4} |\nabla v(t, x)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^4} |v(t, x)|^4 dx.$$

As a consequence, the third author [50] successfully implemented a probabilistic perturbation argument and proved almost sure global well-posedness of the energy-critical defocusing cubic NLW on \mathbb{R}^4 with randomized initial data below the scaling-critical regularity.² We point out that the first term in the energy $\mathcal{E}[v]$ involving the time derivative plays an essential role in establishing a probabilistic a priori bound on the energy for NLW. It seems substantially harder to verify Hypothesis (A) for NLS, even when $d = 4$.

While Theorem 1.3 provides only conditional almost sure global existence, our last theorem (Theorem 1.4) below presents a way to construct global-in-time solutions below the scaling-critical regularity with a large probability. The main idea is to use the scaling (1.2) of the equation for random initial data below the scaling criticality. For example, suppose that we have a solution u to (1.1) on a short time interval with a deterministic initial condition $u_0 \in H^s(\mathbb{R}^d)$, $s < s_{\text{crit}}$. In view of (1.2) and (1.3), by taking $\mu \rightarrow 0$, we see that the H^s -norm of the scaled initial condition goes to 0. Thus, one might think that the problem can be reduced to small data theory. This, of course, does not work in the usual deterministic setting, since we do not know how to construct solutions depending only on the H^s -norm of the initial data, $s < s_{\text{crit}}$. Even in the probabilistic setting, this naive idea does not work if we simply apply the scaling to the randomized function ϕ^ω defined in (1.9). This is due to the fact that we need to use (sub)critical space-time norms controlling the random linear term $z^\omega(t) = S(t)\phi^\omega$, which do not become small even if we take $\mu \ll 1$.

To resolve this issue, we consider a randomization based on a partition of the frequency space by *dilated* cubes. Given $\mu > 0$, define ψ^μ by

$$(1.16) \quad \psi^\mu(\xi) = \psi(\mu^{-1}\xi).$$

Then, we can write a function ϕ on \mathbb{R}^d as

$$\phi = \sum_{n \in \mathbb{Z}^d} \psi^\mu(D - \mu n)\phi.$$

Now, we introduce the randomization $\phi^{\omega, \mu}$ of ϕ on dilated cubes of scale μ by

$$(1.17) \quad \phi^{\omega, \mu} := \sum_{n \in \mathbb{Z}^d} g_n(\omega)\psi^\mu(D - \mu n)\phi,$$

where $\{g_n\}_{n \in \mathbb{Z}^d}$ is a sequence of independent mean zero complex-valued random variables, satisfying (1.10) as before. Then, we have the following global well-posedness of (1.1) with a large probability.

Theorem 1.4. *Let $d \geq 3$ and $\phi \in H^s(\mathbb{R}^d)$, for some $s \in (s_d, s_{\text{crit}})$, where s_d is as in (1.11). Then, given the randomization $\phi^{\omega, \mu}$ on dilated cubes of scale $\mu \ll 1$ defined in (1.17), satisfying (1.10), the cubic NLS (1.1) on \mathbb{R}^d is globally well-posed with a large probability. More precisely, for each $0 < \varepsilon \ll 1$, there exists a small dilation scale $\mu_0 = \mu_0(\varepsilon, \|\phi\|_{H^s}) > 0$ such that for each $\mu \in (0, \mu_0)$, there exists a set $\Omega_\mu \subset \Omega$ with the following properties:*

- (i) $P(\Omega_\mu^c) < \varepsilon$.

²In [50], the third author also proved almost sure global well-posedness of the energy-critical defocusing NLW on \mathbb{R}^5 . This result was recently extended to dimension 3 by the second and third authors [46].

- (ii) If $\phi^{\omega,\mu}$ is the randomization on dilated cubes defined in (1.17), satisfying (1.10), then, for each $\omega \in \Omega_\mu$, there exists a (unique) global-in-time solution u to (1.1) with $u|_{t=0} = \phi^{\omega,\mu}$ in the class

$$S(t)\phi^\omega + C(\mathbb{R} : H^{\frac{d-2}{2}}(\mathbb{R}^d)) \subset C(\mathbb{R} : H^s(\mathbb{R}^d)).$$

Moreover, for each $\omega \in \Omega_\mu$, scattering holds in the sense that there exists $v_+^\omega \in H^{\frac{d-2}{2}}(\mathbb{R}^d)$ such that

$$\|u(t) - S(t)(\phi^{\omega,\mu} + v_+^\omega)\|_{H^{\frac{d-2}{2}}(\mathbb{R}^d)} \rightarrow 0$$

as $t \rightarrow \infty$. A similar statement holds for $t \rightarrow -\infty$.

We conclude this introduction with several remarks.

Remark 1.5. In probabilistic well-posedness results [6, 7, 20, 44] for NLS on \mathbb{T}^d , random initial data are assumed to be of the following specific form:

$$(1.18) \quad u_0^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{(1 + |n|^2)^{\frac{\alpha}{2}}} e^{in \cdot x},$$

where $\{g_n\}_{n \in \mathbb{Z}^d}$ is a sequence of independent complex-valued standard Gaussian random variables. The expression (1.18) has a close connection to the study of invariant measures and hence it is of importance. At the same time, due to the lack of a full range of Strichartz estimates on \mathbb{T}^d , one could not handle a general randomization of a given function as in (1.5). In this paper, we consider NLS on \mathbb{R}^d and thus we do not encounter this issue thanks to a full range of the Strichartz estimates. For NLW, finite speed of propagation allows us to use a full range of Strichartz estimates even on compact domains, at least locally in time. Thus, one does not encounter such an issue.

Remark 1.6. In a recent preprint, Lührmann-Mendelson [42] considered the defocusing NLW on \mathbb{R}^3 with randomized initial data, essentially given by (1.9), below the critical regularity and proved almost sure global well-posedness in the energy-subcritical case, following the method developed in [20], namely an adaptation of Bourgain's high-low method [8] in the probabilistic setting. As Bourgain's high-low method is a subcritical tool, their global result misses the energy-critical case.³

The third author [50] recently proved almost sure global well-posedness of the energy-critical defocusing NLW on \mathbb{R}^d , $d = 4, 5$, with randomized initial data below the critical regularity. The argument is based on an application of a perturbation lemma as in Theorem 1.3 along with a probabilistic a priori control on the energy, which is not available for the cubic NLS (1.1).

This paper is organized as follows. In Section 2, we state some probabilistic lemmata. In Section 3, we go over the basic definitions and properties of function spaces involving the U^p - and V^p -spaces. We prove the key nonlinear estimates in Section 4 and then use them to prove Theorems 1.1 and 1.2 in Section 5. We divide the proof of Theorem 1.3 into three sections. In Sections 6 and 7, we discuss the Cauchy theory for the defocusing cubic NLS with a deterministic perturbation. We implement these results in the probabilistic setting and prove Theorem 1.3 in Section 8. In Section 9, we show how Theorem 1.4 follows from the arguments in

³In [46], the second and third authors recently proved almost sure global well-posedness of the energy-critical defocusing quintic NLW on \mathbb{R}^3 .

Sections 4 and 5, once we consider a randomization on dilated cubes. In Appendix A, we state and prove some additional properties of the function spaces defined in Section 3.

Lastly, note that we present the proofs of these results only for positive times in view of the time reversibility of (1.1).

2. PROBABILISTIC LEMMATA

In this section, we summarize the probabilistic lemmata used in this paper. In particular, the probabilistic Strichartz estimates (Lemmata 2.2 and 2.3) play an essential role. First, we recall the usual Strichartz estimates on \mathbb{R}^d for the readers' convenience. We say that a pair (q, r) is Schrödinger admissible if it satisfies

$$(2.1) \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

with $2 \leq q, r \leq \infty$ and $(q, r, d) \neq (2, \infty, 2)$. Then, the following Strichartz estimates are known to hold.

Lemma 2.1 ([27, 33, 54, 62]). *Let (q, r) be Schrödinger admissible. Then, we have*

$$(2.2) \quad \|S(t)\phi\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\phi\|_{L_x^2(\mathbb{R}^d)}.$$

In particular, when $q = r$, we have $q = r = \frac{2(d+2)}{d}$. By applying Sobolev inequality and (2.2), we also have

$$(2.3) \quad \|S(t)\phi\|_{L_{t,x}^p(\mathbb{R} \times \mathbb{R}^d)} \lesssim \||\nabla|^{\frac{d}{2} - \frac{d+2}{p}}\phi\|_{L_x^2(\mathbb{R}^d)}$$

for $p \geq \frac{2(d+2)}{d}$. Recall that the derivative loss in (2.3) depends only on the size of the frequency support and not its location. Namely, if $\widehat{\phi}$ is supported on a cube Q of side length N , then we have

$$(2.4) \quad \|S(t)\phi\|_{L_{t,x}^p(\mathbb{R} \times \mathbb{R}^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}} \|\phi\|_{L_x^2(\mathbb{R}^d)},$$

regardless of the center of the cube Q .

Next, we present improvements of the Strichartz estimates under the Wiener randomization (1.9) and where, throughout, we assume (1.10). See [4] for the proofs.

Lemma 2.2 (Improved local-in-time Strichartz estimate). *Given $\phi \in L^2(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.9), satisfying (1.10). Then, given finite $q, r \geq 2$, there exist $C, c > 0$ such that*

$$P\left(\|S(t)\phi^\omega\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} > \lambda\right) \leq C \exp\left(-c \frac{\lambda^2}{T^{\frac{2}{q}} \|\phi\|_{L^2}^2}\right)$$

for all $T > 0$ and $\lambda > 0$. In particular, with $\lambda = T^\theta$, we have

$$(2.5) \quad \|S(t)\phi^\omega\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \lesssim T^\theta$$

outside a set of probability

$$\leq C \exp\left(-c \frac{1}{T^{2(\frac{1}{q} - \theta)} \|\phi\|_{L^2}^2}\right).$$

Note that this probability can be made arbitrarily small by letting $T \rightarrow 0$ as long as $\theta < \frac{1}{q}$.

The next lemma states an improvement of the Strichartz estimates in the global-in-time setting.

Lemma 2.3 (Improved global-in-time Strichartz estimate). *Given $\phi \in L^2(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.9), satisfying (1.10). Given a Schrödinger admissible pair (q, r) with $q, r < \infty$, let $\tilde{r} \geq r$. Then, there exist $C, c > 0$ such that*

$$P\left(\|S(t)\phi^\omega\|_{L_t^q L_x^{\tilde{r}}(\mathbb{R} \times \mathbb{R}^d)} > \lambda\right) \leq Ce^{-c\lambda^2\|\phi\|_{L^2}^{-2}}.$$

In particular, given any small $\varepsilon > 0$, we have

$$\|S(t)\phi^\omega\|_{L_t^q L_x^{\tilde{r}}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{2}} \|\phi\|_{L^2(\mathbb{R}^d)}$$

outside a set of probability $< \varepsilon$.

Recall that the diagonal Strichartz admissible index is given by $p = \frac{2(d+2)}{d}$. In the diagonal case $q = \tilde{r}$, it is easy to see that the condition of Lemma 2.3 is satisfied if $q = \tilde{r} \geq p = \frac{2(d+2)}{d}$. In the following, we apply Lemma 2.3 in this setting.

We also need the following lemma on the control of the size of H^s -norm of ϕ^ω .

Lemma 2.4. *Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.9), satisfying (1.10). Then, we have*

$$(2.6) \quad P\left(\|\phi^\omega\|_{H^s(\mathbb{R}^d)} > \lambda\right) \leq Ce^{-c\lambda^2\|\phi\|_{H^s}^{-2}}.$$

We conclude this section by introducing some notation involving Strichartz and space-time Lebesgue spaces. In the sequel, given an interval $I \subset \mathbb{R}$, we often use $L_t^q L_x^r(I)$ to denote $L_t^q L_x^r(I \times \mathbb{R}^d)$. We also define the $\dot{S}^{\text{crit}}(I)$ -norm in the usual manner by setting

$$\|u\|_{\dot{S}^{\text{crit}}(I)} := \sup \left\{ \|\ |\nabla|^{\frac{d-2}{2}} u \|_{L_t^q L_x^r(I \times \mathbb{R}^d)} \right\},$$

where the supremum is taken over all Schrödinger admissible pairs (q, r) .

3. FUNCTION SPACES AND THEIR PROPERTIES

In this section, we go over the basic definitions and properties of the U^p - and V^p -spaces, developed by Tataru, Koch, and their collaborators [29, 30, 41]. These spaces have been very effective in establishing well-posedness of various dispersive PDEs in critical regularities. See Hadac-Herr-Koch [29] and Herr-Tataru-Tzvetkov [30] for detailed proofs.

Let H be a separable Hilbert space over \mathbb{C} . In particular, it will be either $H^s(\mathbb{R}^d)$ or \mathbb{C} . Let \mathcal{Z} be the collection of finite partitions $\{t_k\}_{k=0}^K$ of \mathbb{R} : $-\infty < t_0 < \dots < t_K \leq \infty$. If $t_K = \infty$, we use the convention $u(t_K) := 0$ for all functions $u : \mathbb{R} \rightarrow H$. We use χ_I to denote the sharp characteristic function of a set $I \subset \mathbb{R}$.

Definition 3.1. Let $1 \leq p < \infty$.

(i) A U^p -atom is defined by a step function $a : \mathbb{R} \rightarrow H$ of the form

$$a = \sum_{k=1}^K \phi_{k-1} \chi_{[t_{k-1}, t_k)},$$

where $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset H$ with $\sum_{k=0}^{K-1} \|\phi_k\|_H^p = 1$. Then, we define the atomic space $U^p(\mathbb{R}; H)$ to be the collection of functions $u : \mathbb{R} \rightarrow H$ of the form

$$(3.1) \quad u = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{where } a_j\text{'s are } U^p\text{-atoms and } \{\lambda_j\}_{j \in \mathbb{N}} \in \ell^1(\mathbb{N}; \mathbb{C}),$$

with the norm

$$\|u\|_{U^p(\mathbb{R}; H)} := \inf \left\{ \|\lambda\|_{\ell^1} : (3.1) \text{ holds with } \lambda = \{\lambda_j\}_{j \in \mathbb{N}} \text{ and some } U^p\text{-atoms } a_j \right\}.$$

(ii) We define the space $V^p(\mathbb{R}; H)$ of functions of bounded p -variation to be the collection of functions $u : \mathbb{R} \rightarrow H$ with $\|u\|_{V^p(\mathbb{R}; H)} < \infty$, where the V^p -norm is defined by

$$\|u\|_{V^p(\mathbb{R}; H)} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^K \|u(t_k) - u(t_{k-1})\|_H^p \right)^{\frac{1}{p}}.$$

We also define $V_{\text{rc}}^p(\mathbb{R}; H)$ to be the closed subspace of all right-continuous functions in $V^p(\mathbb{R}; H)$ such that $\lim_{t \rightarrow -\infty} u(t) = 0$.

(iii) Let $s \in \mathbb{R}$. We define $U_{\Delta}^p H^s$ (and $V_{\Delta}^p H^s$, respectively) to be the spaces of all functions $u : \mathbb{R} \rightarrow H^s(\mathbb{T}^d)$ such that the following $U_{\Delta}^p H^s$ -norm (and $V_{\Delta}^p H^s$ -norm, respectively) is finite:

$$\|u\|_{U_{\Delta}^p H^s} := \|S(-t)u\|_{U^p(\mathbb{R}; H^s)} \quad \text{and} \quad \|u\|_{V_{\Delta}^p H^s} := \|S(-t)u\|_{V^p(\mathbb{R}; H^s)},$$

where $S(t) = e^{it\Delta}$ denotes the linear propagator for (1.1). We use $V_{\text{rc}, \Delta}^p H^s$ to denote the subspace of right-continuous functions in $V_{\Delta}^p H^s$.

Remark 3.2. Note that the spaces $U^p(\mathbb{R}; H)$, $V^p(\mathbb{R}; H)$, and $V_{\text{rc}}^p(\mathbb{R}; H)$ are Banach spaces. The closed subspace of continuous functions in $U^p(\mathbb{R}; H)$ is also a Banach space. Moreover, we have the following embeddings:

$$U^p(\mathbb{R}; H) \hookrightarrow V_{\text{rc}}^p(\mathbb{R}; H) \hookrightarrow U^q(\mathbb{R}; H) \hookrightarrow L^{\infty}(\mathbb{R}; H)$$

for $1 \leq p < q < \infty$. Similar embeddings hold for $U_{\Delta}^p H^s$ and $V_{\Delta}^p H^s$.

Next, we state a transference principle and an interpolation result.

Lemma 3.3. (i) (Transference principle) *Suppose that we have*

$$\|T(S(t)\phi_1, \dots, S(t)\phi_k)\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^d)} \lesssim \prod_{j=1}^k \|\phi_j\|_{L_x^2}$$

for some $1 \leq p, q \leq \infty$. Then, we have

$$\|T(u_1, \dots, u_k)\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^d)} \lesssim \prod_{j=1}^k \|u_j\|_{U_{\Delta}^p L_x^2}.$$

(ii) (Interpolation) *Let E be a Banach space. Suppose that $T : U^{p_1} \times \dots \times U^{p_k} \rightarrow E$ is a bounded k -linear operator such that*

$$\|T(u_1, \dots, u_k)\|_E \leq C_1 \prod_{j=1}^k \|u_j\|_{U^{p_j}}$$

for some $p_1, \dots, p_k > 2$. Moreover, assume that there exists $C_2 \in (0, C_1]$ such that

$$\|T(u_1, \dots, u_k)\|_E \leq C_2 \prod_{j=1}^k \|u_j\|_{U^2}.$$

Then, we have

$$\|T(u_1, \dots, u_k)\|_E \leq C_2 \left(\ln \frac{C_1}{C_2} + 1 \right)^k \prod_{j=1}^k \|u_j\|_{V^2}$$

for $u_j \in V_{rc}^2$, $j = 1, \dots, k$.

A transference principle as above has been commonly used in the Fourier restriction norm method. See [29, Proposition 2.19] for the proof of Lemma 3.3 (i). The proof of the interpolation result follows from extending the trilinear result in [30] to a general k -linear case. See also [29, Proposition 2.20].

Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be an even, smooth cutoff function supported on $[-\frac{8}{5}, \frac{8}{5}]$ such that $\eta \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$. Given a dyadic number $N \geq 1$, we set $\eta_1(\xi) = \eta(|\xi|)$ and

$$\eta_N(\xi) = \eta\left(\frac{|\xi|}{N}\right) - \eta\left(\frac{2|\xi|}{N}\right)$$

for $N \geq 2$. Then, we define the Littlewood-Paley projection operator \mathbf{P}_N as the Fourier multiplier operator with symbol η_N . Moreover, we define $\mathbf{P}_{\leq N}$ and $\mathbf{P}_{\geq N}$ by $\mathbf{P}_{\leq N} = \sum_{1 \leq M \leq N} \mathbf{P}_M$ and $\mathbf{P}_{\geq N} = \sum_{M \geq N} \mathbf{P}_M$.

Definition 3.4. (i) Let $s \in \mathbb{R}$. We define $X^s(\mathbb{R})$ to be the space of all tempered distributions $u : \mathbb{R} \rightarrow H^s(\mathbb{R}^d)$ such that $\|u\|_{X^s(\mathbb{R})} < \infty$, where the X^s -norm is defined by

$$\|u\|_{X^s(\mathbb{R})} := \left(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{2s} \|\mathbf{P}_N u\|_{U_{\Delta}^2 L^2}^2 \right)^{\frac{1}{2}}.$$

(ii) Let $s \in \mathbb{R}$. We define $Y^s(\mathbb{R})$ to be the space of all tempered distributions $u : \mathbb{R} \rightarrow H^s(\mathbb{R}^d)$ such that for every $N \in \mathbb{N}$, the map $t \mapsto \mathbf{P}_N u(t)$ is in $V_{rc, \Delta}^2 H^s$ and $\|u\|_{Y^s(\mathbb{R})} < \infty$, where the Y^s -norm is defined by

$$\|u\|_{Y^s(\mathbb{R})} := \left(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{2s} \|\mathbf{P}_N u\|_{V_{\Delta}^2 L^2}^2 \right)^{\frac{1}{2}}.$$

Recall the following embeddings:

$$(3.2) \quad U_{\Delta}^2 H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V_{\Delta}^2 H^s \hookrightarrow U_{\Delta}^p H^s$$

for $p > 2$.

Given an interval $I \subset \mathbb{R}$, we define the local-in-time versions $X^s(I)$ and $Y^s(I)$ of these spaces as restriction norms. For example, we define the $X^s(I)$ -norm by

$$\|u\|_{X^s(I)} = \inf \{ \|v\|_{X^s(\mathbb{R})} : v|_I = u \}.$$

We also define the norm for the nonhomogeneous term:

$$(3.3) \quad \|F\|_{N^s(I)} = \left\| \int_{t_0}^t S(t-t') F(t') dt' \right\|_{X^s(I)}.$$

In the following, we will perform our analysis in $X^s(I) \cap C(I; H^s)$, that is, in a Banach subspace of continuous functions in $X^s(I)$. See Appendix A for additional properties of the $X^s(I)$ -spaces.

We conclude this section by presenting some basic estimates involving these function spaces.

Lemma 3.5. (i) (Linear estimates) *Let $s \geq 0$ and $0 < T \leq \infty$. Then, we have*

$$\|S(t)\phi\|_{X^s([0,T])} \leq \|\phi\|_{H^s},$$

$$\|F\|_{N^s([0,T])} \leq \sup_{\substack{v \in Y^{-s}([0,T]) \\ \|v\|_{Y^{-s}}=1}} \left| \int_0^T \int_{\mathbb{R}^d} F(t,x) \overline{v(t,x)} dx dt \right|$$

for all $\phi \in H^s(\mathbb{R}^d)$ and $F \in L^1([0,T]; H^s(\mathbb{R}^d))$.

(ii) (Strichartz estimates) *Let (q, r) be Schrödinger admissible with $q > 2$ and $p \geq \frac{2(d+2)}{d}$. Then, for $0 < T \leq \infty$ and $N_1 \leq N_2$, we have*

$$(3.4) \quad \|u\|_{L_t^q L_x^r([0,T] \times \mathbb{R}^d)} \lesssim \|u\|_{Y^0([0,T])},$$

$$(3.5) \quad \|u\|_{L_{t,x}^p([0,T] \times \mathbb{R}^d)} \lesssim \|\nabla\|^{\frac{d}{2} - \frac{d+2}{p}} u\|_{Y^0([0,T])},$$

$$(3.6) \quad \|\mathbf{P}_{N_1} u_1 \mathbf{P}_{N_2} u_2\|_{L_{t,x}^2([0,T] \times \mathbb{R}^d)} \\ \lesssim N_1^{\frac{d-2}{2}} \left(\frac{N_1}{N_2} \right)^{\frac{1}{2}-} \|\mathbf{P}_{N_1} u_1\|_{Y^0([0,T])} \|\mathbf{P}_{N_2} u_2\|_{Y^0([0,T])}.$$

Note that there is a slight loss of regularity in (3.6) since we use the Y^0 -norm on the right-hand side instead of the X^0 -norm. In view of (3.2), we may replace the Y^0 -norms on the right-hand sides of (3.4), (3.5), and (3.6) by the X^0 -norm in the following.

Proof. In the following, we briefly discuss the proof of (ii). See [29, 30] for the proof of (i). The first estimate (3.4) follows from the Strichartz estimate (2.2), Lemma 3.3 (i), and (3.2):

$$\|u\|_{L_t^q L_x^r} \lesssim \|u\|_{U_{\Delta}^q L^2} \lesssim \|u\|_{Y^0}$$

for $q > 2$. The second estimate (3.5) follows from (2.3) in a similar manner. It remains to prove (3.6). On the one hand, the following bilinear refinement of the Strichartz estimate by Bourgain [8] and Ozawa-Tsutsumi [48]:

$$\|\mathbf{P}_{N_1} S(t)\phi_1 \mathbf{P}_{N_2} S(t)\phi_2\|_{L_{t,x}^2} \lesssim N_1^{\frac{d-2}{2}} \left(\frac{N_1}{N_2} \right)^{\frac{1}{2}} \|\mathbf{P}_{N_1} \phi_1\|_{L^2} \|\mathbf{P}_{N_2} \phi_2\|_{L^2}$$

and Lemma 3.3 (i) yield

$$(3.7) \quad \|\mathbf{P}_{N_1} u_1 \mathbf{P}_{N_2} u_2\|_{L_{t,x}^2} \lesssim N_1^{\frac{d-2}{2}} \left(\frac{N_1}{N_2} \right)^{\frac{1}{2}} \|\mathbf{P}_{N_1} u_1\|_{U_{\Delta}^2 L^2} \|\mathbf{P}_{N_2} u_2\|_{U_{\Delta}^2 L^2}.$$

On the other hand, by Bernstein's inequality and noting that $(4, \frac{2d}{d-1})$ is Strichartz admissible, we have

$$\|\mathbf{P}_{N_j} S(t)\phi_j\|_{L_{t,x}^4} \lesssim N_j^{\frac{d-2}{4}} \|\mathbf{P}_{N_j} S(t)\phi_j\|_{L_t^4 L_x^{\frac{2d}{d-1}}} \lesssim N_j^{\frac{d-2}{4}} \|\mathbf{P}_{N_j} \phi_j\|_{L^2}.$$

Then, by Cauchy-Schwarz' inequality and Lemma 3.3 (i), we obtain

$$(3.8) \quad \|\mathbf{P}_{N_1} u_1 \mathbf{P}_{N_2} u_2\|_{L^2_{t,x}} \lesssim N_1^{\frac{d-2}{4}} N_2^{\frac{d-2}{4}} \|\mathbf{P}_{N_1} u_1\|_{U^4_{\Delta} L^2} \|\mathbf{P}_{N_2} u_2\|_{U^4_{\Delta} L^2}.$$

Hence, by Lemma 3.3 (ii), with (3.7) and (3.8), we have

$$(3.9) \quad \begin{aligned} \|\mathbf{P}_{N_1} u_1 \mathbf{P}_{N_2} u_2\|_{L^2_{t,x}} \\ \lesssim N_1^{\frac{d-2}{2}} \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \left(\ln\left(\frac{N_2}{N_1}\right) + 1\right)^2 \|\mathbf{P}_{N_1} u_1\|_{V^2_{\Delta} L^2} \|\mathbf{P}_{N_2} u_2\|_{V^2_{\Delta} L^2}. \end{aligned}$$

Finally, (3.6) follows from (3.2) and (3.9). \square

Similar to the usual Strichartz estimate (2.3), the derivative loss in (3.5) depends only on the size of the spatial frequency support and not its location. Namely, if the spatial frequency support of $\widehat{u}(t, \xi)$ is contained in a cube of side length N for all $t \in \mathbb{R}$, then we have

$$(3.10) \quad \|u\|_{L^p_{t,x}([0,T] \times \mathbb{R}^d)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}} \|u\|_{Y^0([0,T])}.$$

This is a direct consequence of (2.4).

Lastly, we recall Schur's test for the readers' convenience.

Lemma 3.6 (Schur's test). *Suppose that we have*

$$\sup_m \sum_n |K_{m,n}| + \sup_n \sum_m |K_{m,n}| < \infty$$

for some $K_{m,n} \in \mathbb{C}$, $m, n \in \mathbb{Z}$. Then, we have

$$\sum_{m,n} K_{m,n} a_m b_n \lesssim \|a_m\|_{\ell^2_m} \|b_n\|_{\ell^2_n}$$

for any ℓ^2 -sequences $\{a_m\}_{m \in \mathbb{Z}}$ and $\{b_n\}_{n \in \mathbb{Z}}$.

4. PROBABILISTIC NONLINEAR ESTIMATES

In this section, we prove the key nonlinear estimates in the critical regularity $s_{\text{crit}} = \frac{d-2}{2}$. In the next section, we use them to prove Theorems 1.1 and 1.2. Given $z(t) = S(t)\phi^\omega$, define Γ by

$$(4.1) \quad \Gamma v(t) = \mp i \int_0^t S(t-t') \mathcal{N}(v+z)(t') dt',$$

where $\mathcal{N}(v+z) = |v+z|^2(v+z)$. Then, we have the following nonlinear estimates.

Proposition 4.1. *Given $d \geq 3$, let $s \in (s_d, s_{\text{crit}}]$, where s_d is defined in (1.11). Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.9), satisfying (1.10).*

(i) *Let $0 < T \leq 1$. Then, there exists $0 < \theta \ll 1$ such that we have*

$$(4.2) \quad \|\Gamma v\|_{X^{\frac{d-2}{2}}([0,T])} \leq C_1 \left(\|v\|_{X^{\frac{d-2}{2}}([0,T])}^3 + T^\theta R^3 \right),$$

$$(4.3) \quad \begin{aligned} \|\Gamma v_1 - \Gamma v_2\|_{X^{\frac{d-2}{2}}([0,T])} \\ \leq C_2 \left(\sum_{j=1}^2 \|v_j\|_{X^{\frac{d-2}{2}}([0,T])}^2 + T^\theta R^2 \right) \|v_1 - v_2\|_{X^{\frac{d-2}{2}}([0,T])} \end{aligned}$$

for all $v, v_1, v_2 \in X^{\frac{d-2}{2}}([0, T])$ and $R > 0$, outside a set of probability $\leq C \exp(-c \frac{R^2}{\|\phi\|_{H^s}^2})$.

(ii) Given $0 < \varepsilon \ll 1$, define $\tilde{\Gamma}$ by

$$(4.4) \quad \tilde{\Gamma}v(t) = \mp i \int_0^t S(t-t') \mathcal{N}(v + \varepsilon z)(t') dt'.$$

Then, we have

$$(4.5) \quad \|\tilde{\Gamma}v\|_{X^{\frac{d-2}{2}}(\mathbb{R})} \leq C_3 (\|v\|_{X^{\frac{d-2}{2}}(\mathbb{R})}^3 + R^3),$$

$$(4.6) \quad \|\tilde{\Gamma}v_1 - \tilde{\Gamma}v_2\|_{X^{\frac{d-2}{2}}(\mathbb{R})} \leq C_4 \left(\sum_{j=1}^2 \|v_j\|_{X^{\frac{d-2}{2}}(\mathbb{R})}^2 + R^2 \right) \|v_1 - v_2\|_{X^{\frac{d-2}{2}}(\mathbb{R})}$$

for all $v, v_1, v_2 \in X^{\frac{d-2}{2}}(\mathbb{R})$ and $R > 0$, outside a set of probability $\leq C \exp(-c \frac{R^2}{\varepsilon^2 \|\phi\|_{H^s}^2})$.

Proof. (i) Let $0 < T \leq 1$. We only prove (4.2) since (4.3) follows in a similar manner. Given $N \geq 1$, define Γ_N by

$$(4.7) \quad \Gamma_N v(t) = \mp i \int_0^t S(t-t') \mathbf{P}_{\leq N} \mathcal{N}(v + z)(t') dt'.$$

By Bernstein's and Hölder's inequalities, we have

$$(4.8) \quad \begin{aligned} \|\mathbf{P}_{\leq N} \mathcal{N}(v + z)\|_{L_t^1([0, T]; H_x^{\frac{d-2}{2}})} &\lesssim N^{\frac{d-2}{2}} \|\mathcal{N}(v + z)\|_{L_t^1 L_x^2} \\ &\lesssim N^{\frac{d-2}{2}} \|v\|_{L_t^3([0, T]; L_x^6)}^3 + N^{\frac{d-2}{2}} \|z\|_{L_t^3([0, T]; L_x^6)}. \end{aligned}$$

On the one hand, it follows from Lemma 2.2 that the second term on the right-hand side of (4.8) is finite almost surely. On the other hand, noting that $(3, \frac{6d}{3d-4})$ is Strichartz admissible, it follows from Sobolev's inequality and (3.4) in Lemma 3.5 that

$$(4.9) \quad \|v\|_{L_t^3([0, T]; L_x^6)} \lesssim \|\langle \nabla \rangle^{\frac{d-2}{3}} v\|_{L_t^3([0, T]; L_x^{\frac{6d}{3d-4}})} \lesssim \|v\|_{X^{\frac{d-2}{3}}([0, T])} < \infty.$$

Therefore, by Lemma 3.5 (i), we have

$$(4.10) \quad \|\Gamma_N v(t)\|_{X^{\frac{d-2}{2}}} \lesssim \sup_{\substack{v_4 \in Y^0([0, T]) \\ \|v_4\|_{Y^0} = 1}} \left| \int_0^T \int_{\mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} \mathcal{N}(v + z)(t, x) \overline{v_4(t, x)} dx dt \right|,$$

almost surely, where $v_4 = \mathbf{P}_{\leq N} v_4$. In the following, we estimate the right-hand side of (4.10), independently of the cutoff size $N \geq 1$, by performing a case-by-case analysis of expressions of the form:

$$(4.11) \quad \left| \int_0^T \int_{\mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} (w_1 w_2 w_3) v_4 dx dt \right|,$$

where $\|v_4\|_{Y^0([0, T])} \leq 1$ and $w_j = v$ or z , $j = 1, 2, 3$. As a result, by taking $N \rightarrow \infty$, the same estimates hold for Γv without any cutoff, thus yielding (4.2).

Before proceeding further, let us simplify some of the notation. In the following, we drop the complex conjugate sign. We also denote $X^s([0, T])$ and $Y^s([0, T])$ by X^s and Y^s since T is fixed. Similarly, it is understood that the time integration in $L_{t,x}^p$ is over $[0, T]$. Lastly, in most of the cases, we dyadically decompose $w_j = v_j$ or z_j , $j = 1, 2, 3$, and v_4 such that their spatial frequency supports are $\{|\xi_j| \sim N_j\}$ for some dyadic $N_j \geq 1$ but still denote them as $w_j = v_j$ or z_j , $j = 1, 2, 3$, and v_4 .

Note that, if we can afford a small derivative loss in the largest frequency, there is no difficulty in summing over the dyadic blocks N_j , $j = 1, \dots, 4$.

Case (1): vvv case. In this case, we do not need to perform dyadic decompositions and we divide the frequency spaces into $\{|\xi_1| \geq |\xi_2|, |\xi_3|\}$, $\{|\xi_2| \geq |\xi_1|, |\xi_3|\}$, and $\{|\xi_3| \geq |\xi_1|, |\xi_2|\}$. Without loss of generality, assume that $|\xi_1| \geq |\xi_2|, |\xi_3|$. By the $L_{t,x}^{\frac{2(d+2)}{d}} L_{t,x}^{d+2} L_{t,x}^{d+2} L_{t,x}^{\frac{2(d+2)}{d}}$ -Hölder's inequality, (3.5) in Lemma 3.5, and (3.2), we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} v_1 v_2 v_3 v_4 dx dt \right| &\leq \| \langle \nabla \rangle^{\frac{d-2}{2}} v_1 \|_{L_{t,x}^{\frac{2(d+2)}{d}}} \| v_2 \|_{L_{t,x}^{d+2}} \| v_3 \|_{L_{t,x}^{d+2}} \| v_4 \|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\ &\lesssim \prod_{j=1}^3 \| v_j \|_{Y^{\frac{d-2}{2}}} \| v_4 \|_{Y^0} \lesssim \prod_{j=1}^3 \| v_j \|_{X^{\frac{d-2}{2}}}. \end{aligned}$$

Case (2): zzz case. Without loss of generality, assume $N_3 \geq N_2 \geq N_1$.

- *Subcase (2.a):* $N_2 \sim N_3$. By the $L_{t,x}^{d+2} L_{t,x}^4 L_{t,x}^4 L_{t,x}^{\frac{2(d+2)}{d}}$ -Hölder's inequality, we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^d} z_1 z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| &\lesssim \| z_1 \|_{L_{t,x}^{d+2}} \| \langle \nabla \rangle^{\frac{d-2}{4}} z_2 \|_{L_{t,x}^4} \| \langle \nabla \rangle^{\frac{d-2}{4}} z_3 \|_{L_{t,x}^4} \| v_4 \|_{L_{t,x}^{\frac{2(d+2)}{d}}}. \end{aligned}$$

Hence, by Lemmata 2.2 and 3.5, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R^3$ outside a set of probability

$$\leq C \exp \left(-c \frac{R^2}{T^{\frac{2}{d+2}} - \|\phi\|_{L^2}^2} \right) + C \exp \left(-c \frac{R^2}{T^{\frac{1}{2}} - \|\phi\|_{H^{\frac{d-2}{4}+}}^2} \right)$$

as long as $s > \frac{d-2}{4}$. Note that s needs to be strictly greater than $\frac{d-2}{4}$ due to the summations over dyadic blocks. See [4] for more details. Similar comments apply in the following.

- *Subcase (2.b):* $N_3 \sim N_4 \gg N_1, N_2$.

◦ *Subsubcase (2.b.i):* $N_1, N_2 \ll N_3^{\frac{1}{d-1}}$. For small $\alpha > 0$, it follows from Cauchy-Schwarz' inequality and Lemma 3.5 that

$$\begin{aligned} \| z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 \|_{L_{t,x}^2} &\lesssim N_3^{\frac{d-2}{2}} \| z_2 \|_{L_{t,x}^4}^\alpha \| z_3 \|_{L_{t,x}^4}^\alpha \| z_2 z_3 \|_{L_{t,x}^2}^{1-\alpha} \\ (4.12) \quad &\lesssim N_2^{\frac{d-1}{2} - s - \frac{d-1}{2} \alpha -} N_3^{\frac{d-3}{2} - s + \frac{1}{2} \alpha +} \prod_{j=2}^3 (\| \langle \nabla \rangle^s z_j \|_{L_{t,x}^4}^\alpha \| \mathbf{P}_{N_j} \phi^\omega \|_{H^s}^{1-\alpha}). \end{aligned}$$

Then, by (4.12) and the bilinear estimate (3.6) in Lemma 3.5, we have

$$\begin{aligned}
 \left| \int_0^T \int_{\mathbb{R}^d} z_1 z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| &\lesssim \|z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3\|_{L_{t,x}^2} \|z_1 v_4\|_{L_{t,x}^2} \\
 &\lesssim N_1^{\frac{d-1}{2}-s-} N_2^{\frac{d-1}{2}-s-\frac{d-1}{2}\alpha-} N_3^{\frac{d-4}{2}-s+\frac{1}{2}\alpha+} \\
 &\quad \times \|\mathbf{P}_{N_1} \phi^\omega\|_{H^s} \prod_{j=2}^3 (\|\langle \nabla \rangle^s z_j\|_{L_{t,x}^4}^\alpha \|\mathbf{P}_{N_j} \phi^\omega\|_{H^s}^{1-\alpha}) \|v_4\|_{Y^0} \\
 &\lesssim N_3^{\frac{d-2}{2}-\frac{d+1}{d-1}s+} \|\mathbf{P}_{N_1} \phi^\omega\|_{H^s} \prod_{j=2}^3 (\|\langle \nabla \rangle^s z_j\|_{L_{t,x}^4}^\alpha \|\mathbf{P}_{N_j} \phi^\omega\|_{H^s}^{1-\alpha}) \|v_4\|_{Y^0}.
 \end{aligned}$$

Hence, by Lemmata 2.2 and 2.4, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R^3$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{1}{2}} - \|\phi\|_{H^s}^2}\right) + C \exp\left(-c \frac{R^2}{\|\phi\|_{H^s}^2}\right)$$

as long as

$$(4.13) \quad s > \frac{d-1}{d+1} \cdot \frac{d-2}{2} = s_d$$

and $\alpha < 1 - \frac{2}{d-1}s$.

◦ Subsubcase (2.b.ii): $N_2 \gtrsim N_3^{\frac{1}{d-1}} \gg N_1$. By Hölder's inequality and the bilinear estimate (3.6) in Lemma 3.5, we have

$$\begin{aligned}
 \left| \int_0^T \int_{\mathbb{R}^d} z_1 z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| &\lesssim \|z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^{\frac{d-2}{2}} z_3\|_{L_{t,x}^4} \|z_1 v_4\|_{L_{t,x}^2} \\
 &\lesssim N_1^{\frac{d-1}{2}-s-} N_2^{-s} N_3^{\frac{d-3}{2}-s+} \|\mathbf{P}_{N_1} \phi^\omega\|_{H^s} \prod_{j=2}^3 \|\langle \nabla \rangle^s z_j\|_{L_{t,x}^4} \|v_4\|_{Y^0} \\
 &\lesssim N_3^{\frac{d-2}{2}-\frac{d+1}{d-1}s+} \|\mathbf{P}_{N_1} \phi^\omega\|_{H^s} \prod_{j=2}^3 \|\langle \nabla \rangle^s z_j\|_{L_{t,x}^4} \|v_4\|_{Y^0}.
 \end{aligned}$$

Hence, by Lemmata 2.2 and 2.4, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R^3$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{1}{2}} - \|\phi\|_{H^s}^2}\right) + C \exp\left(-c \frac{R^2}{\|\phi\|_{H^s}^2}\right)$$

as long as (4.13) is satisfied.

◦ Subsubcase (2.b.iii): $N_1, N_2 \gtrsim N_3^{\frac{1}{d-1}}$. By the $L_{t,x}^{\frac{6(d+2)}{d+4}} L_{t,x}^{\frac{6(d+2)}{d+4}} L_{t,x}^{\frac{6(d+2)}{d+4}} L_{t,x}^{\frac{2(d+2)}{d}}$ -Hölder's inequality and (3.5) in Lemma 3.5, we have

$$\left| \int_0^T \int_{\mathbb{R}^d} z_1 z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| \lesssim N_3^{\frac{d-2}{2}-\frac{d+1}{d-1}s} \prod_{j=1}^3 \|\langle \nabla \rangle^s z_j\|_{L_{t,x}^{\frac{6(d+2)}{d+4}}} \|v_4\|_{Y^0}.$$

Hence, by Lemma 2.2, the contribution to (4.10) in this case is at most $\lesssim T^{0+}R^3$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{d+4}{3(d+2)} - \|\phi\|_{H^s}^2}}\right)$$

as long as (4.13) is satisfied.

Case (3): vvz case. Without loss of generality, assume $N_1 \geq N_2$.

• *Subcase (3.a): $N_1 \gtrsim N_3$.* In the following, we apply dyadic decompositions only to v_1 , v_2 , and z_3 . In this case, we have $N_1 \sim \max(N_2, N_3, |\xi_4|)$, where ξ_4 is the spatial frequency of v_4 . Then, by Hölder's inequality, (3.6), and (3.5), we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} v_1 v_2 z_3 v_4 dx dt \right| \\ & \lesssim \sum_{N_1 \gtrsim N_2, N_3} \|\langle \nabla \rangle^{\frac{d-2}{2}} \mathbf{P}_{N_1} v_1 \mathbf{P}_{N_2} v_2\|_{L_{t,x}^2} \|\mathbf{P}_{N_3} z_3\|_{L_{t,x}^{d+2}} \|v_4\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\ & \lesssim \sum_{N_1 \geq N_2} \left(\frac{N_2}{N_1}\right)^{\frac{1}{2}-} \prod_{j=1}^2 \|\mathbf{P}_{N_j} v_j\|_{X^{\frac{d-2}{2}}} \sum_{N_3} \|\mathbf{P}_{N_3} z_3\|_{L_{t,x}^{d+2}} \|v_4\|_{Y^0}. \end{aligned}$$

By Lemma 3.6 and summing over N_3 with a slight loss of derivative,

$$\lesssim \prod_{j=1}^2 \|v_j\|_{X^{\frac{d-2}{2}}} \|\langle \nabla \rangle^{0+} z_3\|_{L_{t,x}^{d+2}} \|v_4\|_{Y^0}.$$

Hence, by Lemma 2.2, the contribution to (4.10) in this case is at most $\lesssim T^{0+}R \prod_{j=1}^2 \|v_j\|_{X^{\frac{d-2}{2}}}$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{2}{d+2} - \|\phi\|_{H^{0+}}^2}}\right)$$

as long as $s > 0$.

• *Subcase (3.b): $N_3 \sim N_4 \gg N_1 \geq N_2$.*

◦ *Subsubcase (3.b.i): $N_1 \gtrsim N_3^{\frac{1}{d-1}}$.* By Hölder's inequality followed by (3.5) and (3.6) in Lemma 3.5, we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^d} v_1 v_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| \lesssim \|v_1\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|\langle \nabla \rangle^{\frac{d-2}{2}} z_3\|_{L_{t,x}^{d+2}} \|v_2 v_4\|_{L_{t,x}^2} \\ & \lesssim N_1^{-\frac{d-2}{2}} N_2^{\frac{1}{2}-} N_3^{\frac{d-3}{2}-s+} \|v_1\|_{X^{\frac{d-2}{2}}} \|v_2\|_{X^{\frac{d-2}{2}}} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^{d+2}} \|v_4\|_{Y^0} \\ & \lesssim N_3^{\frac{d-3}{d-1} \frac{d-2}{2} - s+} \|v_1\|_{X^{\frac{d-2}{2}}} \|v_2\|_{X^{\frac{d-2}{2}}} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^{d+2}} \|v_4\|_{Y^0}. \end{aligned}$$

Hence, by Lemma 2.2, the contribution to (4.10) in this case is at most $\lesssim T^{0+}R \prod_{j=1}^2 \|v_j\|_{X^{\frac{d-2}{2}}}$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{2}{d+2} - \|\phi\|_{H^s}^2}}\right)$$

as long as

$$(4.14) \quad s > \frac{d-3}{d-1} \cdot \frac{d-2}{2}.$$

Note that the condition (4.14) is less restrictive than (4.13).

◦ Subsubcase (3.b.ii): $N_2 \leq N_1 \ll N_3^{\frac{1}{d-1}}$. For small $\alpha > 0$, it follows from Hölder's inequality and Lemma 3.5 that

$$(4.15) \quad \begin{aligned} \|v_1 \langle \nabla \rangle^{\frac{d-2}{2}} z_3\|_{L_{t,x}^2} &\lesssim N_3^{\frac{d-2}{2}} \|v_1\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^\alpha \|z_3\|_{L_{t,x}^{d+2}}^\alpha \|v_1 z_3\|_{L_{t,x}^2}^{1-\alpha} \\ &\lesssim N_1^{\frac{1}{2} - \frac{d-1}{2}\alpha} N_3^{\frac{d-3}{2} - s + \frac{1}{2}\alpha} \|v_1\|_{X^{\frac{d-2}{2}}} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^{d+2}}^\alpha \|\mathbf{P}_{N_3} \phi^\omega\|_{H^s}^{1-\alpha}. \end{aligned}$$

Then, by (4.15) and (3.6) in Lemma 3.5, we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^d} v_1 v_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| &\lesssim \|v_1 \langle \nabla \rangle^{\frac{d-2}{2}} z_3\|_{L_{t,x}^2} \|v_2 v_4\|_{L_{t,x}^2} \\ &\lesssim N_1^{\frac{1}{2} - \frac{d-1}{2}\alpha} N_2^{\frac{1}{2}} N_3^{\frac{d-4}{2} - s + \frac{1}{2}\alpha} \|v_1\|_{X^{\frac{d-2}{2}}} \|v_2\|_{X^{\frac{d-2}{2}}} \\ &\quad \times \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^{d+2}}^\alpha \|\mathbf{P}_{N_3} \phi^\omega\|_{H^s}^{1-\alpha} \|v_4\|_{Y^0}. \end{aligned}$$

Hence, by Lemmata 2.2 and 2.4, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R \prod_{j=1}^2 \|v_j\|_{X^{\frac{d-2}{2}}}$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{2}{d+2}} - \|\phi\|_{H^s}^2}\right) + C \exp\left(-c \frac{R^2}{\|\phi\|_{H^s}^2}\right)$$

as long as (4.14) is satisfied and $\alpha < \frac{1}{d-1}$.

Case (4): vzz case. Without loss of generality, assume $N_3 \geq N_2$.

• *Subcase (4.a):* $N_1 \gtrsim N_3$. By the $L_{t,x}^{\frac{2(d+2)}{d}} L_{t,x}^{d+2} L_{t,x}^{d+2} L_{t,x}^{\frac{2(d+2)}{d}}$ -Hölder's inequality and (3.5) in Lemma 3.5, we have

$$\left| \int_0^T \int_{\mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} v_1 z_2 z_3 v_4 dx dt \right| \lesssim \|v_1\|_{X^{\frac{d-2}{2}}} \|z_2\|_{L_{t,x}^{d+2}} \|z_3\|_{L_{t,x}^{d+2}} \|v_4\|_{Y^0}.$$

Hence, by Lemma 2.2, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R^2 \|v_1\|_{X^{\frac{d-2}{2}}}$ outside a set of probability

$$(4.16) \quad \leq C \exp\left(-c \frac{R^2}{T^{\frac{2}{d+2}} - \|\phi\|_{H^{0+}}^2}\right)$$

as long as $s > 0$. As before, we have $\|\phi\|_{H^{0+}}$ instead of $\|\phi\|_{L^2}$ in (4.16), allowing us to sum over N_2 and N_3 . If $N_3 \gtrsim \max(N_1, N_4)$, then this also allows us to sum over N_1 and N_4 . Otherwise, we have $N_1 \sim N_4 \gg N_3$. In this case, we can use Cauchy-Schwarz' inequality to sum over $N_1 \sim N_4$.

• *Subcase (4.b)*: $N_3 \gg N_1$. First, suppose that $N_2 \sim N_3$. Note that we must have $N_3 \gtrsim N_4$ in this case. Then, by the $L_{t,x}^{d+2} L_{t,x}^4 L_{t,x}^4 L_{t,x}^{\frac{2(d+2)}{d}}$ -Hölder's inequality with (3.5) in Lemma 3.5, we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^d} v_1 z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| \\ & \lesssim \|v_1\|_{X^{\frac{d-2}{2}}} \|\langle \nabla \rangle^{\frac{d-2}{4}} z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^{\frac{d-2}{4}} z_3\|_{L_{t,x}^4} \|v_4\|_{Y^0} \\ & \lesssim N_3^{\frac{d-2}{2}-2s} \|v_1\|_{X^{\frac{d-2}{2}}} \|\langle \nabla \rangle^s z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^4} \|v_4\|_{Y^0}. \end{aligned}$$

Hence, by Lemma 2.2, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R^2 \|v_1\|_{X^{\frac{d-2}{2}}}$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{1}{2}} - \|\phi\|_{H^s}^2}\right)$$

as long as $s > \frac{d-2}{4}$.

Hence, it remains to consider the case $N_3 \sim N_4 \gg N_1, N_2$.

◦ *Subsubcase (4.b.i)*: $N_1, N_2 \ll N_3^{\frac{1}{d-1}}$. By (4.12) and (3.6) in Lemma 3.5, we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^d} v_1 z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| \lesssim \|z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3\|_{L_{t,x}^2} \|v_1 v_4\|_{L_{t,x}^2} \\ & \lesssim N_1^{\frac{1}{2}-} N_2^{\frac{d-1}{2}-s-\frac{d-1}{2}\alpha-} N_3^{\frac{d-4}{2}-s+\frac{1}{2}\alpha+} \|v_1\|_{X^{\frac{d-2}{2}}} \\ & \quad \times \prod_{j=2}^3 (\|\langle \nabla \rangle^s z_j\|_{L_{t,x}^4}^\alpha \|\mathbf{P}_{N_j} \phi^\omega\|_{H^s}^{1-\alpha}) \|v_4\|_{Y^0}. \end{aligned}$$

Hence, by Lemmata 2.2 and 2.4, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R^2 \|v_1\|_{X^{\frac{d-2}{2}}}$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{1}{2}} - \|\phi\|_{H^s}^2}\right) + C \exp\left(-c \frac{R^2}{\|\phi\|_{H^s}^2}\right)$$

as long as

$$(4.17) \quad s > \frac{(d-2)^2}{2d} = \frac{d-2}{d} \cdot \frac{d-2}{2}$$

and $\alpha < 1 - \frac{2}{d-1}s$. Note that the condition (4.17) is less restrictive than (4.13) and thus does not add a further constraint.

◦ *Subsubcase (4.b.ii)*: $N_1 \ll N_3^{\frac{1}{d-1}} \lesssim N_2$. By Hölder's inequality and (3.6) in Lemma 3.5, we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^d} v_1 z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| \lesssim \|z_2\|_{L_{t,x}^4} \|\langle \nabla \rangle^{\frac{d-2}{2}} z_3\|_{L_{t,x}^4} \|v_1 v_4\|_{L_{t,x}^2} \\ & \lesssim N_1^{\frac{1}{2}-} N_2^{-s} N_3^{\frac{d-3}{2}-s+} \|v_1\|_{X^{\frac{d-2}{2}}} \prod_{j=2}^3 \|\langle \nabla \rangle^s z_j\|_{L_{t,x}^4} \|v_4\|_{Y^0}. \end{aligned}$$

Hence, by Lemma 2.2, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R^2 \|v_1\|_{X^{\frac{d-2}{2}}}$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{1}{2}} - \|\phi\|_{H^s}^2}\right)$$

as long as (4.17) is satisfied.

◦ Subsubcase (4.b.iii): $N_2 \ll N_3^{\frac{1}{d-1}} \lesssim N_1$. By Hölder's inequality and Lemma 3.5, we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^d} v_1 z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| &\lesssim \|v_1\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|\langle \nabla \rangle^{\frac{d-2}{2}} z_3\|_{L_{t,x}^{d+2}} \|z_2 v_4\|_{L_{t,x}^2} \\ &\lesssim N_1^{-\frac{d-2}{2}} N_2^{\frac{d-1}{2}-s} N_3^{\frac{d-3}{2}-s+} \|v_1\|_{X^{\frac{d-2}{2}}} \|\mathbf{P}_{N_2} \phi^\omega\|_{H^s} \|\langle \nabla \rangle^s z_3\|_{L_{t,x}^{d+2}} \|v_4\|_{Y^0}. \end{aligned}$$

Hence, by Lemmata 2.4 and 2.2, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R^2 \|v_1\|_{X^{\frac{d-2}{2}}}$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{\|\phi\|_{H^s}^2}\right) + C \exp\left(-c \frac{R^2}{T^{\frac{2}{d+2}} - \|\phi\|_{H^s}^2}\right)$$

as long as (4.17) is satisfied.

◦ Subsubcase (4.b.iv): $N_1, N_2 \gtrsim N_3^{\frac{1}{d-1}}$. By the $L_{t,x}^{\frac{2(d+2)}{d}} L_{t,x}^{d+2} L_{t,x}^{d+2} L_{t,x}^{\frac{2(d+2)}{d}}$ -Hölder's inequality and (3.5) in Lemma 3.5, we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^d} v_1 z_2 \langle \nabla \rangle^{\frac{d-2}{2}} z_3 v_4 dx dt \right| &\lesssim \|v_1\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|z_2\|_{L_{t,x}^{d+2}} \|\langle \nabla \rangle^{\frac{d-2}{2}} z_3\|_{L_{t,x}^{d+2}} \|v_4\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\ &\lesssim N_1^{-\frac{d-2}{2}} N_2^{-s} N_3^{\frac{d-2}{2}-s} \|v_1\|_{X^{\frac{d-2}{2}}} \prod_{j=2}^3 \|\langle \nabla \rangle^s z_j\|_{L_{t,x}^{d+2}} \|v_4\|_{Y^0}. \end{aligned}$$

Hence, by Lemma 2.2, the contribution to (4.10) in this case is at most $\lesssim T^{0+} R^2 \|v_1\|_{X^{\frac{d-2}{2}}}$ outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{T^{\frac{2}{d+2}} - \|\phi\|_{H^s}^2}\right)$$

as long as (4.17) is satisfied.

Putting together Cases (1) - (4) above, the conclusion of part (i) follows, provided that (4.13) is satisfied.

(ii) First, define $\tilde{\Gamma}_N$ by

$$\tilde{\Gamma}_N v(t) = \mp i \int_0^t S(t-t') \mathbf{P}_{\leq N} \mathcal{N}(v + \varepsilon z)(t') dt'$$

for $N \geq 1$. As before, we have

$$(4.18) \quad \|\mathbf{P}_{\leq N} \mathcal{N}(v + \varepsilon z)\|_{L_t^1(\mathbb{R}; H_x^{\frac{d-2}{2}})} \lesssim N^{\frac{d-2}{2}} \|v\|_{L_t^3(\mathbb{R}; L_x^6)} + \varepsilon^3 N^{\frac{d-2}{2}} \|z\|_{L_t^3(\mathbb{R}; L_x^6)}.$$

By a computation similar to (4.9), we see that the first term is finite. Noting that $(3, \frac{6d}{3d-4})$ is Strichartz admissible and $6 \geq \frac{6d}{3d-4}$, it follows from Lemma 2.3 that the

second term on the right-hand side of (4.18) is finite almost surely. Hence, we can apply Lemma 3.5 (i) to $\tilde{\Gamma}_N v$ for each finite $N \geq 1$, almost surely.

The rest of the proof for this part follows in a similar manner to the proof of part (i) by changing the time interval from $[0, T]$ to \mathbb{R} and replacing z by εz . By applying Lemma 2.3 instead of Lemma 2.2 in the above computation, we see that the contribution to (4.10), where $[0, T]$ is replaced by \mathbb{R} , is given by

$$\text{Case (2): } R^3, \quad \text{Case (3): } R \prod_{j=1}^2 \|v_j\|_{X^{\frac{d-2}{2}}(\mathbb{R})}, \quad \text{Case (4): } R^2 \|v_1\|_{X^{\frac{d-2}{2}}(\mathbb{R})}$$

outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{\varepsilon^2 \|\phi\|_{H^s}^2}\right)$$

in all cases as long as $s > s_d$. \square

5. PROOFS OF THEOREMS 1.1 AND 1.2

In this section, we establish the almost sure local well-posedness (Theorem 1.1) and probabilistic small data global theory (Theorem 1.2). First, we present the proof of Theorem 1.1. Given C_1 and C_2 as in (4.2) and (4.3), let $\eta_1 > 0$ be sufficiently small such that

$$(5.1) \quad C_1 \eta_1^2 \leq \frac{1}{2} \quad \text{and} \quad 2C_2 \eta_1^2 \leq \frac{1}{4}.$$

Also, given $R \gg 1$, choose $T = T(R)$ such that

$$T^\theta = \min\left(\frac{\eta_1}{2C_1 R^3}, \frac{1}{4C_2 R^2}\right).$$

Then, it follows from Proposition 4.1 that Γ is a contraction on the ball B_{η_1} defined by

$$B_{\eta_1} := \{u \in X^{\frac{d-2}{2}}([0, T]) \cap C([0, T]; H^{\frac{d-2}{2}}) : \|u\|_{X^{\frac{d-2}{2}}([0, T])} \leq \eta_1\}$$

outside a set of probability

$$\leq C \exp\left(-c \frac{R^2}{\|\phi\|_{H^s}^2}\right) \sim C \exp\left(-c \frac{1}{T^\gamma \|\phi\|_{H^s}^2}\right)$$

for some $\gamma > 0$. This proves Theorem 1.1.

Next, we prove Theorem 1.2. Let $\eta_2 > 0$ be sufficiently small such that

$$(5.2) \quad 2C_3 \eta_2^2 \leq 1 \quad \text{and} \quad 3C_4 \eta_2^2 \leq \frac{1}{2},$$

where C_3 and C_4 are as in (4.5) and (4.6). Then, by Proposition 4.1 with $R = \eta_2$ and ϕ^ω replaced by $\varepsilon \phi^\omega$, we have

$$(5.3) \quad \|\tilde{\Gamma} v\|_{X^{\frac{d-2}{2}}(\mathbb{R})} \leq 2C_3 \eta_2^3 \leq \eta_2,$$

$$(5.4) \quad \|\tilde{\Gamma} v_1 - \tilde{\Gamma} v_2\|_{X^{\frac{d-2}{2}}(\mathbb{R})} \leq 3C_4 \eta_2^2 \|v_1 - v_2\|_{X^{\frac{d-2}{2}}(\mathbb{R})} \leq \frac{1}{2} \|v_1 - v_2\|_{X^{\frac{d-2}{2}}(\mathbb{R})}$$

outside a set of probability $\leq C \exp\left(-c \frac{\eta_2^2}{\varepsilon^2 \|\phi\|_{H^s}^2}\right)$. Noting that η_2 is an absolute constant, we conclude that there exists a set $\Omega_\varepsilon \subset \Omega$ such that (i) $\tilde{\Gamma} = \tilde{\Gamma}^\omega$ is a contraction on the ball B_{η_2} defined by

$$B_{\eta_2} := \{u \in X^{\frac{d-2}{2}}(\mathbb{R}) \cap C(\mathbb{R}; H^{\frac{d-2}{2}}) : \|u\|_{X^{\frac{d-2}{2}}(\mathbb{R})} \leq \eta_2\}$$

for $\omega \in \Omega_\varepsilon$, and (ii) $P(\Omega_\varepsilon^c) \leq C \exp\left(-\frac{c}{\varepsilon^2 \|\phi\|_{H^s}^2}\right)$. This proves global existence for (1.1) with initial data $\varepsilon\phi^\omega$ if $\omega \in \Omega_\varepsilon$.

Fix $\omega \in \Omega_\varepsilon$ and let $v = v(\varepsilon, \omega)$ be the global-in-time solution with $v|_{t=0} = \varepsilon\phi^\omega$ constructed above. In order to prove scattering, we need to show that there exists $v_+^\omega \in H^{\frac{d-2}{2}}(\mathbb{R}^d)$ such that

$$(5.5) \quad S(-t)v(t) = \mp i \int_0^t S(-t')\mathcal{N}(v + \varepsilon z)(t')dt' \rightarrow v_+^\omega$$

in $H^{\frac{d-2}{2}}(\mathbb{R}^d)$ as $t \rightarrow \infty$. With $w(t) = S(-t)v(t)$, define $I(t_1, t_2)$ and $\tilde{I}(t_1, t_2)$ by

$$\begin{aligned} I(t_1, t_2) &:= S(t_2)(w(t_2) - w(t_1)), \\ \tilde{I}(t_1, t_2) &:= \mp i \int_0^{t_2} S(t_2 - t')\chi_{[t_1, \infty)}(t')\mathcal{N}(v + \varepsilon z)(t')dt'. \end{aligned}$$

Then, for $0 < t_1 \leq t_2 < \infty$, we have

$$I(t_1, t_2) = \mp i S(t_2) \int_{t_1}^{t_2} S(-t')\mathcal{N}(v + \varepsilon z)(t')dt' = \tilde{I}(t_1, t_2).$$

Also, note that $\tilde{I}(t_1, t_2) = 0$ if $t_1 > t_2$. In the following, we view $\tilde{I}(t_1, t_2)$ as a function of t_2 and estimate its $X^{\frac{d-2}{2}}([0, \infty))$ -norm. We now revisit the computation in the proof of Proposition 4.1 for $\tilde{I}(t_1, t_2)$. In Case (1), we proceed slightly differently. By Lemma 3.5 (i), Hölder's inequality, and (3.5), we have

$$(5.6) \quad \begin{aligned} \|\tilde{I}(t_1, \cdot)\|_{X^{\frac{d-2}{2}}(\mathbb{R}_+)} &\lesssim \sup_{\substack{v_4 \in Y^0(\mathbb{R}_+) \\ \|v_4\|_{Y^0} = 1}} \left| \int_0^\infty \int_{\mathbb{R}^d} \chi_{[t_1, \infty)}(t) \langle \nabla \rangle^{\frac{d-2}{2}} v \bar{v} v v_4 dx dt \right| \\ &\leq \|\langle \nabla \rangle^{\frac{d-2}{2}} v\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t_1, \infty))} \|v\|_{L_{t,x}^{d+2}([t_1, \infty))}^2. \end{aligned}$$

By (3.5) in Lemma 3.5, we have

$$\|\langle \nabla \rangle^{\frac{d-2}{2}} v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R})} + \|v\|_{L_{t,x}^{d+2}(\mathbb{R})} \lesssim \|v\|_{X^{\frac{d-2}{2}}(\mathbb{R})} \leq \eta_2.$$

Then, by the monotone convergence theorem, (5.6) tends to 0 as $t_1 \rightarrow \infty$.

In Cases (2), (3), and (4), we had at least one factor of z . We multiply the cutoff function $\chi_{[t_1, \infty)}$ only on the (εz) -factors but not on the v -factors. Note that $\|v\|_{X^{\frac{d-2}{2}}(\mathbb{R})} \leq \eta_2$. As in the proof of Proposition 4.1, we estimate at least a small portion of these z -factors in $\|\langle \nabla \rangle^s \varepsilon z^\omega\|_{L_{t,x}^q([t_1, \infty))}$, $q = 4, \frac{6(d+2)}{d+4}$, or $d+2$, in each case. Recall that we have $\|\langle \nabla \rangle^s \varepsilon z^\omega\|_{L_{t,x}^q(\mathbb{R})} \leq \eta_2$ for $\omega \in \Omega_\varepsilon$. See Lemma 2.3. Hence, again by the monotone convergence theorem, we have $\|\langle \nabla \rangle^s \varepsilon z^\omega\|_{L_{t,x}^q([t_1, \infty))} \rightarrow 0$ as $t_1 \rightarrow \infty$ and thus the contribution from Cases (2), (3), and (4) tends to 0 as $t_1 \rightarrow \infty$. Therefore, we have

$$\lim_{t_1 \rightarrow \infty} \|\tilde{I}(t_1, t_2)\|_{X^{\frac{d-2}{2}}([0, \infty))} = 0.$$

In conclusion, we obtain

$$\begin{aligned} \lim_{t_1 \rightarrow \infty} \sup_{t_2 > t_1} \|w(t_2) - w(t_1)\|_{H^{\frac{d-2}{2}}} &= \lim_{t_1 \rightarrow \infty} \sup_{t_2 > t_1} \|I(t_1, t_2)\|_{H^{\frac{d-2}{2}}} \\ &= \lim_{t_1 \rightarrow \infty} \|\tilde{I}(t_1, t_2)\|_{L_{t_2}^\infty([0, \infty); H^{\frac{d-2}{2}})} \lesssim \lim_{t_1 \rightarrow \infty} \|\tilde{I}(t_1, t_2)\|_{X^{\frac{d-2}{2}}([0, \infty))} = 0. \end{aligned}$$

This proves (5.5) and scattering of $w^\omega(t) = \varepsilon S(t)\phi^\omega + v^\omega(t)$, which completes the proof of Theorem 1.2.

6. LOCAL WELL-POSEDNESS OF NLS WITH A DETERMINISTIC PERTURBATION

In this and the next sections, we consider the following Cauchy problem of the defocusing NLS with a perturbation:

$$(6.1) \quad \begin{cases} i\partial_t v + \Delta v = |v + f|^2(v + f), \\ v|_{t=t_0} = v_0, \end{cases}$$

where f is a given *deterministic* function. Assuming some suitable conditions on f , we prove local well-posedness of (6.1) in this section (Proposition 6.3) and long time existence under further assumptions in Section 7 (Proposition 7.2). Then, we show, in Section 8, that the conditions imposed on f for long time existence are satisfied with a large probability by setting $f(t) = z(t) = S(t)\phi^\omega$. This yields Theorem 1.3.

Our main goal is to prove long time existence of solutions to the perturbed NLS (6.1) by iteratively applying a perturbation lemma (Lemma 7.1). For this purpose, we first prove a “variant” local well-posedness of (6.1). As in the usual critical regularity theory, we first introduce an auxiliary scaling-invariant norm which is weaker than the $X^{\frac{d-2}{2}}$ -norm. Given an interval $I \subset \mathbb{R}$, we introduce the Z -norm by

$$(6.2) \quad \|u\|_{Z(I)} := \left(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{d-2} \|\mathbf{P}_N u\|_{L_{t,x}^4(I \times \mathbb{R}^d)}^4 \right)^{\frac{1}{4}}.$$

By the Littlewood-Paley theory and (3.5) in Lemma 3.5, we have

$$\|u\|_{Z(I)} \lesssim \|\langle \nabla \rangle^{\frac{d-2}{4}} u\|_{L_{t,x}^4(I \times \mathbb{R}^d)} \lesssim \|u\|_{X^{\frac{d-2}{2}}(I)}.$$

Given $\theta \in (0, 1)$, we define the Z_θ -norm by

$$\|u\|_{Z_\theta(I)} := \|u\|_{Z(I)}^\theta \|u\|_{X^{\frac{d-2}{2}}(I)}^{1-\theta}.$$

Note that the Z_θ -norm is weaker than the $X^{\frac{d-2}{2}}$ -norm:

$$(6.3) \quad \|u\|_{Z_\theta(I)} \leq C_0 \|u\|_{X^{\frac{d-2}{2}}(I)}$$

for some $C_0 > 0$ independent of I .

First, we present the bilinear Strichartz estimate adapted to the Z_θ -norm.

Lemma 6.1. *Let $N_1 \leq N_2$. Then, we have*

$$(6.4) \quad \|\mathbf{P}_{N_1} u_1 \mathbf{P}_{N_2} u_2\|_{L_{t,x}^2(I \times \mathbb{R}^d)} \lesssim \left(\frac{N_1}{N_2} \right)^{\frac{1}{2}(1-\theta)-} \|\mathbf{P}_{N_1} u_1\|_{Z_\theta(I)} \|\mathbf{P}_{N_2} u_2\|_{Y^0(I)}.$$

Proof. Given a cube R of side length N_1 centered at $\xi_0 \in N_1 \mathbb{Z}^d$, let $\mathbf{P}_R = \psi\left(\frac{D-\xi_0}{N_1}\right)$ denote a smooth projection onto R on the frequency side. Here, ψ denotes the smooth projection onto the unit cube defined in (1.6). By (3.10), we have

$$\begin{aligned} \|\mathbf{P}_{N_1} u_1 \mathbf{P}_R \mathbf{P}_{N_2} u_2\|_{L_{t,x}^2(I)} &\leq \|\mathbf{P}_{N_1} u_1\|_{L_{t,x}^4} \|\mathbf{P}_R \mathbf{P}_{N_2} u_2\|_{L_{t,x}^4} \\ &\lesssim N_1^{\frac{d-2}{4}} \|\mathbf{P}_{N_1} u_1\|_{L_{t,x}^4(I)} \|\mathbf{P}_R \mathbf{P}_{N_2} u_2\|_{Y^0(I)}. \end{aligned}$$

Then, by almost orthogonality, we have

$$\begin{aligned} \|\mathbf{P}_{N_1} u_1 \mathbf{P}_{N_2} u_2\|_{L_{t,x}^2(I)} &\sim \left(\sum_R \|\mathbf{P}_{N_1} u_1 \mathbf{P}_R \mathbf{P}_{N_2} u_2\|_{L_{t,x}^2(I)}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|\mathbf{P}_{N_1} u_1\|_{Z(I)} \left(\sum_R \|\mathbf{P}_R \mathbf{P}_{N_2} u_2\|_{Y^0(I)}^2 \right)^{\frac{1}{2}} \lesssim \|\mathbf{P}_{N_1} u_1\|_{Z(I)} \|\mathbf{P}_{N_2} u_2\|_{Y^0(I)}. \end{aligned}$$

Then, (6.4) follows from interpolating this with (3.6). \square

Next, we state the key nonlinear estimate. Given $I \subset \mathbb{R}$, we define the W^s -norm by

$$(6.5) \quad \|f\|_{W^s(I)} := \max \left(\|\langle \nabla \rangle^s f\|_{L_{t,x}^4(I)}, \|\langle \nabla \rangle^s f\|_{L_{t,x}^{d+2}(I)}, \|\langle \nabla \rangle^s f\|_{L_{t,x}^{\frac{6(d+2)}{d+4}}(I)} \right).$$

As in the proof of Proposition 4.1, different space-time norms of f appear in the estimate but they are all controlled by this W^s -norm. The following lemma is analogous to Proposition 4.1 but with one important difference. All the terms on the right-hand side have (i) two factors of the Z_θ -norm of v_j , which is weaker than the X^s -norm, or (ii) the W^s -norm of f , which can be made small by shrinking the interval I .

Lemma 6.2. *Let $d \geq 3$ and $\theta \in (0, 1)$. Suppose that $s, \alpha \in \mathbb{R}$ satisfy*

$$(6.6) \quad s \in (s_d, s_{\text{crit}}] \quad \text{and} \quad 0 < \alpha < 1 - \frac{2}{d-1}s,$$

where s_d is as in (1.11). Then, given any interval $I = [t_0, t_1] \subset \mathbb{R}$, we have

$$\begin{aligned} \left\| \prod_{j=1}^3 (v_j + f)^* \right\|_{N^{\frac{d-2}{2}}(I)} &\lesssim \sum_{j=1}^3 \|v_j\|_{X^{\frac{d-2}{2}}(I)} \prod_{\substack{k=1 \\ k \neq j}}^3 \|v_k\|_{Z_\theta(I)} \\ &+ \prod_{\substack{j,k=1 \\ j \neq k}}^3 \|v_j\|_{X^{\frac{d-2}{2}}(I)} \|v_k\|_{X^{\frac{d-2}{2}}(I)} \left(\|f\|_{W^s(I)} + \|f\|_{Y^s(I)}^{1-\alpha} \|f\|_{W^s(I)}^\alpha \right) \\ &+ \sum_{j=1}^3 \|v_j\|_{X^{\frac{d-2}{2}}(I)} \left(\|f\|_{Y^s(I)} \|f\|_{W^s(I)} + \|f\|_{Y^s(I)}^{2-2\alpha} \|f\|_{W^s(I)}^{2\alpha} + \|f\|_{W^s(I)}^2 \right) \\ &+ \|f\|_{Y^s(I)} \|f\|_{W^s(I)}^2 + \|f\|_{Y^s(I)}^{3-2\alpha} \|f\|_{W^s(I)}^{2\alpha} + \|f\|_{W^s(I)}^3 \end{aligned}$$

for all $f \in W^s(I) \cap Y^s(I)$ and $v_j \in X^{\frac{d-2}{2}}(I)$, $j = 1, 2, 3$, where $(v_j + f)^* = v_j + f$ or $\bar{v}_j + \bar{f}$.

We first state and prove the following local well-posedness result for the perturbed NLS (6.1), assuming Lemma 6.2. The proof of Lemma 6.2 is presented at the end of this section.

Proposition 6.3 (Local well-posedness of the perturbed NLS). *Given $d \geq 3$, let $s \in (s_d, s_{\text{crit}}]$, where s_d is defined in (1.11). Let $\theta \in (\frac{1}{2}, 1)$ and $\alpha \in \mathbb{R}$ satisfy (6.6). Suppose that*

$$\|v_0\|_{H^{\frac{d-2}{2}}} \leq R \quad \text{and} \quad \|f\|_{Y^s(I)} \leq M$$

for some $R, M \geq 1$. Then, there exists small $\eta_0 = \eta_0(R, M) > 0$ such that if

$$\|S(t-t_0)v_0\|_{Z_\theta(I)} \leq \eta \quad \text{and} \quad \|f\|_{W^s(I)} \leq \eta^{\frac{4-\alpha}{\alpha}}$$

for some $\eta \leq \eta_0$ and some time interval $I = [t_0, t_1] \subset \mathbb{R}$, then there exists a unique solution $v \in X^{\frac{d-2}{2}}(I) \cap C(I; H^{\frac{d-2}{2}}(\mathbb{R}^d))$ to (6.1) with $v(t_0) = v_0$. Moreover, we have

$$(6.7) \quad \|v - S(t-t_0)v_0\|_{X^{\frac{d-2}{2}}(I)} \lesssim \eta^{3-2\theta}.$$

Proof. For $\theta \in (\frac{1}{2}, 1)$, we show that the map Γ defined by

$$(6.8) \quad \Gamma v(t) := S(t-t_0)v_0 - i \int_{t_0}^t S(t-t')\mathcal{N}(v+f)(t')dt'$$

is a contraction on

$$B_{R,M,\eta} = \{v \in X^{\frac{d-2}{2}}(I) \cap C(I; H^{\frac{d-2}{2}}) : \|v\|_{X^{\frac{d-2}{2}}(I)} \leq 2\tilde{R}, \|v\|_{Z_\theta(I)} \leq 2\eta\},$$

where $\tilde{R} := \max(R, M)$. Now, choose

$$(6.9) \quad \eta_0 \ll \tilde{R}^{-\frac{1}{2\theta-1}}.$$

In particular, we have $\eta_0 \ll \tilde{R}^{-1} \leq 1$. Fix $\eta \leq \eta_0$ in the following. Noting that $\frac{4-\alpha}{\alpha} > 3$, Lemma 6.2 with Lemma 3.5 yields

$$(6.10) \quad \begin{aligned} \|\Gamma v\|_{X^{\frac{d-2}{2}}(I)} &\leq \|S(t-t_0)v_0\|_{X^{\frac{d-2}{2}}(I)} + \|\Gamma v - S(t-t_0)v_0\|_{X^{\frac{d-2}{2}}(I)} \\ &\leq \|v_0\|_{H^{\frac{d-2}{2}}} + C\eta^2\tilde{R} \leq 2\tilde{R} \end{aligned}$$

and

$$\|\Gamma v_1 - \Gamma v_2\|_{X^{\frac{d-2}{2}}(I)} \leq \frac{1}{2}\|v_1 - v_2\|_{X^{\frac{d-2}{2}}(I)}$$

for $v, v_1, v_2 \in B_{R,M,\eta}$. Moreover, we have

$$\begin{aligned} \|\Gamma v\|_{Z_\theta(I)} &\leq (\|S(t-t_0)v_0\|_{Z(I)} + C\eta^2\tilde{R})^\theta (\|S(t-t_0)v_0\|_{X^{\frac{d-2}{2}}(I)} + C\eta^2\tilde{R})^{1-\theta} \\ &\leq \eta + C\eta^{2\theta}\tilde{R} + C\eta^{2-\theta}\tilde{R}^{1-\theta} + C\eta^2\tilde{R} \leq 2\eta \end{aligned}$$

for $v \in B_{R,M,\eta}$. Hence, Γ is a contraction on $B_{R,M,\eta}$. The estimate (6.7) follows from (6.9) and (6.10). \square

We conclude this section by presenting the proof of Lemma 6.2. Some cases follow directly from the proof of Proposition 4.1. However, due to the use of the Z_θ -norm, we need to make modifications in several cases.

Proof of Lemma 6.2. As in the proof of Proposition 4.1, we need to estimate the right-hand side of (4.10) by performing a case-by-case analysis of expressions of the form:

$$(6.11) \quad \left| \iint_{I \times \mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} (w_1 w_2 w_3) v_4 dx dt \right|,$$

where $\|v_4\|_{Y^0(I)} \leq 1$ and $w_j = v$ or f , $j = 1, 2, 3$. Before proceeding further, let us simplify some of the notation. In the following, as before, we drop the complex conjugate sign and denote $X^s(I)$ and $Y^s(I)$ by X^s and Y^s . Lastly, we dyadically decompose w_j , $j = 1, 2, 3$, and v_4 such that their spatial frequency supports are $\{|\xi_j| \sim N_j\}$ for some dyadic $N_j \geq 1$ but still denote them as $w_j = v_j$ or f_j , $j = 1, 2, 3$, and v_4 if there is no confusion.

Case (1): vvv case. Without loss of generality, assume that $N_1 \geq N_2, N_3$.

• *Subcase (1.a):* $N_1 \sim N_4$. By Lemma 6.1, we have

$$\begin{aligned} & \left| \iint_{I \times \mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} v_1 v_2 v_3 v_4 dx dt \right| \\ & \lesssim \sum_{N_1 \sim N_4 \gtrsim N_2, N_3} N_1^{\frac{d-2}{2}} \|\mathbf{P}_{N_1} v_1 \mathbf{P}_{N_3} v_3\|_{L_{t,x}^2} \|\mathbf{P}_{N_2} v_2 \mathbf{P}_{N_4} v_4\|_{L_{t,x}^2} \\ & \lesssim \sum_{N_1, \dots, N_4} \left(\frac{N_2 N_3}{N_1 N_4} \right)^{\frac{1}{2}(1-\theta)-} \|\mathbf{P}_{N_1} v_1\|_{Y^{\frac{d-2}{2}}} \|\mathbf{P}_{N_2} v_2\|_{Z_\theta} \|\mathbf{P}_{N_3} v_3\|_{Z_\theta} \|\mathbf{P}_{N_4} v_4\|_{Y^0}. \end{aligned}$$

By first summing over $N_2, N_3 \leq N_1$ and then applying Cauchy-Schwarz' inequality in summing over $N_1 \sim N_4$, we have

$$\lesssim \|v_1\|_{X^{\frac{d-2}{2}}(I)} \|v_2\|_{Z_\theta(I)} \|v_3\|_{Z_\theta(I)}.$$

• *Subcase (1.b):* $N_1 \sim N_2 \gg N_4$. By Lemma 6.1 and (3.5) in Lemma 3.5, we have

$$\begin{aligned} & \left| \iint_{I \times \mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} v_1 v_2 v_3 v_4 dx dt \right| \\ & \lesssim \sum_{N_1 \sim N_2 \geq N_3, N_4} N_1^{\frac{d-2}{2}} \|\mathbf{P}_{N_1} v_1 \mathbf{P}_{N_3} v_3\|_{L_{t,x}^2} \|\mathbf{P}_{N_2} v_2 \mathbf{P}_{N_4} v_4\|_{L_{t,x}^2} \\ & \lesssim \sum_{N_1 \sim N_2 \geq N_3, N_4} \left(\frac{N_3}{N_1} \right)^{\frac{1}{2}(1-\theta)-} \|\mathbf{P}_{N_1} v_1\|_{Y^{\frac{d-2}{2}}} \|\mathbf{P}_{N_3} v_3\|_{Z_\theta} \|\mathbf{P}_{N_2} v_2\|_{L_{t,x}^4} \|\mathbf{P}_{N_4} v_4\|_{L_{t,x}^4} \\ & \lesssim \sum_{N_1 \sim N_2 \geq N_3, N_4} \left(\frac{N_3}{N_1} \right)^{\frac{1}{2}(1-\theta)-} \left(\frac{N_4}{N_2} \right)^{\frac{d-2}{4}} \|\mathbf{P}_{N_1} v_1\|_{Y^{\frac{d-2}{2}}} \|\mathbf{P}_{N_3} v_3\|_{Z_\theta} \\ & \quad \times N_2^{\frac{d-2}{4}} \|\mathbf{P}_{N_2} v_2\|_{L_{t,x}^4} \|\mathbf{P}_{N_4} v_4\|_{Y^0}. \end{aligned}$$

Summing over N_3 and taking a supremum in N_2 ,

$$\lesssim \|v_2\|_{Z_\theta} \|v_3\|_{Z_\theta} \sum_{N_1 \gg N_4} \left(\frac{N_4}{N_1} \right)^{\frac{d-2}{4}} \|\mathbf{P}_{N_1} v_1\|_{Y^{\frac{d-2}{2}}} \|\mathbf{P}_{N_4} v_4\|_{Y^0}.$$

By Lemma 3.6, we have

$$\lesssim \|v_1\|_{Y^{\frac{d-2}{2}}} \|v_2\|_{Z_\theta} \|v_3\|_{Z_\theta} \|v_4\|_{Y^0} \lesssim \|v_1\|_{X^{\frac{d-2}{2}}(I)} \|v_2\|_{Z_\theta(I)} \|v_3\|_{Z_\theta(I)}.$$

In the following, the desired estimates follow from the corresponding cases in the proof of Proposition 4.1. Hence, we just state the results.

Case (2): fff case. Without loss of generality, assume $N_3 \geq N_2 \geq N_1$.

- *Subcase (2.a):* $N_2 \sim N_3$. The contribution to (6.11) in this case is at most

$$\lesssim \|f\|_{L_{t,x}^{d+2}}^2 \|\langle \nabla \rangle^{\frac{d-2}{4}} f\|_{L_{t,x}^4}^2 \leq \|f\|_{W^s(I)}^3$$

as long as $s > \frac{d-2}{4}$.

- *Subcase (2.b):* $N_3 \sim N_4 \gg N_1, N_2$.

◦ *Subsubcase (2.b.i):* $N_1, N_2 \ll N_3^{\frac{1}{d-1}}$. The contribution to (6.11) in this case is at most

$$\lesssim \|f\|_{Y^s}^{3-2\alpha} \|\langle \nabla \rangle^s f\|_{L_{t,x}^{2\alpha}}^{2\alpha} \leq \|f\|_{Y^s(I)}^{3-2\alpha} \|f\|_{W^s(I)}^{2\alpha}$$

as long as (4.13) is satisfied and $\alpha < 1 - \frac{2}{d-1}s$.

◦ *Subsubcase (2.b.ii):* $N_2 \gtrsim N_3^{\frac{1}{d-1}}$. The contribution to (6.11) in this case is at most

$$\lesssim \|f\|_{Y^s(I)} \|\langle \nabla \rangle^s f\|_{L_{t,x}^4}^2 \leq \|f\|_{Y^s(I)} \|f\|_{W^s(I)}^2$$

as long as (4.13) is satisfied.

◦ *Subsubcase (2.b.iii):* $N_1, N_2 \gtrsim N_3^{\frac{1}{d-1}}$. The contribution to (6.11) in this case is at most

$$\lesssim \|\langle \nabla \rangle^s f\|_{L_{t,x}^{\frac{6(d+2)}{d+4}}}^3 \leq \|f\|_{W^s(I)}^3$$

as long as (4.13) is satisfied.

Case (3): vvf case. Without loss of generality, assume $N_1 \geq N_2$.

- *Subcase (3.a):* $N_1 \gtrsim N_3$. The contribution to (6.11) in this case is at most

$$\lesssim \|v\|_{X^{\frac{d-2}{2}}}^2 \|\langle \nabla \rangle^s f\|_{L_{t,x}^{d+2}} \leq \|v\|_{X^{\frac{d-2}{2}}(I)}^2 \|f\|_{W^s(I)}$$

as long as $s > 0$.

- *Subcase (3.b):* $N_3 \sim N_4 \gg N_1 \geq N_2$.

◦ *Subsubcase (3.b.i):* $N_1 \gtrsim N_3^{\frac{1}{d-1}}$. The contribution to (6.11) in this case is at most

$$\lesssim \|v\|_{X^{\frac{d-2}{2}}}^2 \|\langle \nabla \rangle^s f\|_{L_{t,x}^{d+2}} \leq \|v\|_{X^{\frac{d-2}{2}}(I)}^2 \|f\|_{W^s(I)}$$

as long as (4.14) is satisfied.

◦ *Subsubcase (3.b.ii):* $N_2 \leq N_1 \ll N_3^{\frac{1}{d-1}}$. The contribution to (6.11) in this case is at most

$$\lesssim \|v\|_{X^{\frac{d-2}{2}}}^2 \|f\|_{Y^s}^{1-\alpha} \|\langle \nabla \rangle^s f\|_{L_{t,x}^{d+2}}^\alpha \leq \|v\|_{X^{\frac{d-2}{2}}(I)}^2 \|f\|_{Y^s(I)}^{1-\alpha} \|f\|_{W^s(I)}^\alpha$$

as long as (4.14) is satisfied.

Case (4): vff case. Without loss of generality, assume $N_3 \geq N_2$.

- *Subcase (4.a):* $N_1 \gtrsim N_3$. The contribution to (6.11) in this case is at most

$$\lesssim \|v\|_{X^{\frac{d-2}{2}}} \|f\|_{L_{t,x}^{d+2}}^2 \leq \|v\|_{X^{\frac{d-2}{2}}(I)} \|f\|_{W^s(I)}^2$$

as long as $s > 0$.

• *Subcase (4.b):* $N_3 \gg N_1$. First, suppose that $N_2 \sim N_3$. Then, the contribution to (6.11) in this case is at most

$$\lesssim \|v\|_{X^{\frac{d-2}{2}}} \|\langle \nabla \rangle^s f\|_{L_{t,x}^4}^2 \leq \|v\|_{X^{\frac{d-2}{2}}(I)} \|f\|_{W^s(I)}^2$$

as long as $s > \frac{d-2}{4}$.

Hence, it remains to consider the case $N_3 \sim N_4 \gg N_1, N_2$.

◦ Subsubcase (4.b.i): $N_1, N_2 \ll N_3^{\frac{1}{d-1}}$. The contribution to (6.11) in this case is at most

$$\lesssim \|v\|_{X^{\frac{d-2}{2}}} \|f\|_{Y^s}^{2-2\alpha} \|\langle \nabla \rangle^s f\|_{L_{t,x}^4}^{2\alpha} \leq \|v\|_{X^{\frac{d-2}{2}}(I)} \|f\|_{Y^s(I)}^{2-2\alpha} \|f\|_{W^s(I)}^{2\alpha}$$

as long as (4.17) is satisfied and $\alpha < 1 - \frac{2}{d-1}s$.

◦ Subsubcase (4.b.ii): $N_1 \ll N_3^{\frac{1}{d-1}} \lesssim N_2$. The contribution to (6.11) in this case is at most

$$\lesssim \|v\|_{X^{\frac{d-2}{2}}} \|\langle \nabla \rangle^s f\|_{L_{t,x}^4}^2 \leq \|v\|_{X^{\frac{d-2}{2}}(I)} \|f\|_{W^s(I)}^2$$

as long as (4.17) is satisfied.

◦ Subsubcase (4.b.iii): $N_2 \ll N_3^{\frac{1}{d-1}} \lesssim N_1$. The contribution to (6.11) in this case is at most

$$\lesssim \|v\|_{X^{\frac{d-2}{2}}} \|f\|_{Y^s} \|\langle \nabla \rangle^s f\|_{L_{t,x}^{d+2}} \leq \|v\|_{X^{\frac{d-2}{2}}(I)} \|f\|_{Y^s(I)} \|f\|_{W^s(I)}$$

as long as (4.17) is satisfied.

◦ Subsubcase (4.b.iv): $N_1, N_2 \gtrsim N_3^{\frac{1}{d-1}}$. The contribution to (6.11) in this case is at most

$$\lesssim \|v\|_{X^{\frac{d-2}{2}}} \|\langle \nabla \rangle^s f\|_{L_{t,x}^{d+2}}^2 \leq \|v\|_{X^{\frac{d-2}{2}}(I)} \|f\|_{W^s(I)}^2$$

as long as (4.17) is satisfied. \square

7. LONG TIME EXISTENCE OF SOLUTIONS TO THE PERTURBED NLS

The main goal of this section is to establish long time existence of solutions to the perturbed NLS (6.1) under some assumptions. See Proposition 7.2. We achieve this goal by iteratively applying the perturbation lemma (Lemma 7.1) for the energy-critical NLS.

We first state the perturbation lemma for the energy-critical cubic NLS involving the $X^{\frac{d-2}{2}}$ - and the Z -norms. See [19, 38, 56, 57] for perturbation and stability results on usual Strichartz and Lebesgue spaces. In the context of the cubic NLS on $\mathbb{R} \times \mathbb{T}^3$, Ionescu-Pausader [31] proved a perturbation lemma involving the critical $X^{s_{\text{crit}}}$ -norm. Our proof essentially follows their argument and is included for the sake of completeness.

Lemma 7.1 (Perturbation lemma). *Let $d \geq 3$ and let I be a compact interval with $|I| \leq 1$. Suppose that $v \in C(I; H^{\frac{d-2}{2}}(\mathbb{R}^d))$ satisfies the following perturbed NLS:*

$$(7.1) \quad i\partial_t v + \Delta v = |v|^2 v + e,$$

satisfying

$$(7.2) \quad \|v\|_{Z(I)} + \|v\|_{L^\infty(I; H^{\frac{d-2}{2}}(\mathbb{R}^d))} \leq R$$

for some $R \geq 1$. Then, there exists $\varepsilon_0 = \varepsilon_0(R) > 0$ such that if we have

$$(7.3) \quad \|u_0 - v(t_0)\|_{H^{\frac{d-2}{2}}(\mathbb{R}^d)} + \|e\|_{N^{\frac{d-2}{2}}(I)} \leq \varepsilon$$

for some $u_0 \in H^{\frac{d-2}{2}}(\mathbb{R}^d)$, some $t_0 \in I$, and some $\varepsilon < \varepsilon_0$, then there exists a solution $u \in X^{\frac{d-2}{2}}(I) \cap C(I; H^{\frac{d-2}{2}}(\mathbb{R}^d))$ to the defocusing cubic NLS (1.1) with $u(t_0) = u_0$ such that

$$(7.4) \quad \|u\|_{X^{\frac{d-2}{2}}(I)} + \|v\|_{X^{\frac{d-2}{2}}(I)} \leq C(R),$$

$$(7.5) \quad \|u - v\|_{X^{\frac{d-2}{2}}(I)} \leq C(R)\varepsilon,$$

where $C(R)$ is a nondecreasing function of R .

Proof. Without loss of generality, we assume $t_0 = \min I$. Given small $\varepsilon_1 = \varepsilon_1(R) > 0$ (to be chosen later), we divide the interval I into subintervals $I_j = [t_j, t_{j+1}]$ such that $I = \bigcup_{j=0}^L I_j$. By choosing $L \sim \left(\frac{R}{\varepsilon_1}\right)^4$, we can guarantee that

$$(7.6) \quad \|v\|_{Z(I_j)} \leq \varepsilon_1$$

for $j = 0, \dots, L$. By assumption, we also have

$$(7.7) \quad \|e\|_{N^{\frac{d-2}{2}}(I_j)} \leq \varepsilon < \varepsilon_0$$

for $j = 0, \dots, L$.

Step 1. Let $\theta \in (\frac{1}{2}, 1)$. We first claim that there exist $\eta_0 = \eta_0(R) > 0$ and $\varepsilon_0 = \varepsilon_0(R) > 0$ such that if

$$(7.8) \quad \|S(t - t_*)v(t_*)\|_{Z_\theta(J)} \leq \eta_0 \quad \text{and} \quad \|e\|_{N^{\frac{d-2}{2}}(J)} \leq \varepsilon_0$$

for some t_* in a subinterval $J \subset I$, then there exists a unique solution v to (7.1) on J , satisfying

$$(7.9) \quad \|v - S(t - t_*)v(t_*)\|_{X^{\frac{d-2}{2}}(J)} \leq C\|S(t - t_*)v(t_*)\|_{Z_\theta(J)}^{3-2\theta} + 2\|e\|_{N^{\frac{d-2}{2}}(J)}.$$

We choose $\eta_0 = \eta_0(R)$ and $\varepsilon_0 = \varepsilon_0(R)$ such that

$$(7.10) \quad \eta_0 \ll R^{-\frac{1}{2\theta-1}} \quad \text{and} \quad \varepsilon_0 \ll R^{-\frac{2}{2\theta-1}}.$$

In the following, we set

$$\eta := \|S(t - t_*)v(t_*)\|_{Z_\theta(J)} \leq \eta_0 \quad \text{and} \quad \varepsilon := \|e\|_{N^{\frac{d-2}{2}}(J)} \leq \varepsilon_0.$$

Then, proceeding as in the proof of Proposition 6.3, we show that the map Γ defined by

$$(7.11) \quad \Gamma v(t) := S(t - t_*)v(t_*) - i \int_{t_*}^t S(t - t')\mathcal{N}(v)(t')dt' - i \int_{t_*}^t S(t - t')e(t')dt'$$

is a contraction on

$$B_{R, \eta, \varepsilon} = \left\{ v \in X^{\frac{d-2}{2}}(J) \cap C(J; H^{\frac{d-2}{2}}) : \|v\|_{X^{\frac{d-2}{2}}(J)} \leq 2R, \|v\|_{Z_\theta(J)} \leq 2(\eta + \varepsilon^{\frac{\theta}{2} + \frac{1}{4}}) \right\}.$$

Indeed, by Lemma 6.2 (with $f = 0$), we have

$$(7.12) \quad \begin{aligned} \|\Gamma v\|_{X^{\frac{d-2}{2}}(J)} &\leq \|v(t_*)\|_{H^{\frac{d-2}{2}}} + C(\eta + \varepsilon^{\frac{\theta}{2} + \frac{1}{4}})^2 R + \varepsilon \\ &\leq \|v(t_*)\|_{H^{\frac{d-2}{2}}} + C\eta^2 R + 2\varepsilon \leq 2R \end{aligned}$$

and

$$\|\Gamma v_1 - \Gamma v_2\|_{X^{\frac{d-2}{2}}(J)} \leq C(\eta + \varepsilon^{\frac{\theta}{2} + \frac{1}{4}})R \|v_1 - v_2\|_{X^{\frac{d-2}{2}}(J)} \leq \frac{1}{2} \|v_1 - v_2\|_{X^{\frac{d-2}{2}}(J)}$$

for $v, v_1, v_2 \in B_{R, \eta, \varepsilon}$. Moreover, we have

$$\begin{aligned} \|\Gamma v\|_{Z_\theta(J)} &\leq (\|S(t-t_*)v(t_*)\|_{Z(J)} + C\eta^2 R + \varepsilon)^\theta \\ &\quad \times (\|S(t-t_*)v(t_*)\|_{X^{\frac{d-2}{2}}(J)} + C\eta^2 R + \varepsilon)^{1-\theta} \\ &\leq \eta + C\eta^{2\theta} R + C\eta^{2-\theta} R^{1-\theta} + C\eta^\theta \varepsilon^{1-\theta} + C\varepsilon^\theta R^{1-\theta} \leq 2(\eta + \varepsilon^{\frac{\theta}{2} + \frac{1}{4}}) \end{aligned}$$

for $v \in B_{R, \eta, \varepsilon}$. Hence, Γ is a contraction on B_{R, η_1} . The estimate (7.9) follows from (7.10) and (7.12).

Step 2. Next, we claim that, given $\varepsilon_2 > 0$, we can choose $\varepsilon_j = \varepsilon_j(R, \varepsilon_2)$, $j = 0, 1$, in (7.7) and (7.6) sufficiently small such that we have

$$(7.13) \quad \|S(t-t_j)v(t_j)\|_{Z_\theta(I_j)} \leq \varepsilon_2 \quad \text{and} \quad \|v\|_{Z_\theta(I_j)} \leq \varepsilon_2.$$

Without loss of generality, assume $\varepsilon_2 \leq \frac{\eta_0}{2}$, where $\eta_0 = \eta_0(R)$ is as in Step 1. Let $h(\tau) = \|S(t-t_j)v(t_j)\|_{Z_\theta([t_j, t_j+\tau])}$. Note that h is continuous and $h(0) = 0$. Thus, we have $h(\tau) \leq 2\varepsilon_2 \leq \eta_0$ for small $\tau > 0$. Then, from the Duhamel formula (7.11) with (7.6), (7.7), and (7.9), we have

$$\begin{aligned} h(\tau) &\leq \|S(t-t_j)v(t_j)\|_{X^{\frac{d-2}{2}}([t_j, t_j+\tau])}^{1-\theta} \|S(t-t_j)v(t_j)\|_{Z([t_j, t_j+\tau])}^\theta \\ &\leq CR^{1-\theta} (\varepsilon_1 + \varepsilon_2^{3-2\theta} + \|e\|_{N^{\frac{d-2}{2}}(I_j)})^\theta \\ (7.14) \quad &\leq CR^{1-\theta} \varepsilon_2^{\theta(3-2\theta)} + CR^{1-\theta} (\varepsilon_1 + \|e\|_{N^{\frac{d-2}{2}}(I_j)})^\theta. \end{aligned}$$

From (7.10) with $\varepsilon_2 \leq \frac{\eta_0}{2}$, we have

$$(7.15) \quad CR^{1-\theta} \varepsilon_2^{\theta(3-2\theta)} \leq C(R\eta_0^{(2\theta-1)})^{1-\theta} \varepsilon_2 \ll \varepsilon_2.$$

Hence, it follows from (7.14) and (7.15) that

$$(7.16) \quad h(\tau) \leq \frac{1}{2}\varepsilon_2 + CR^{1-\theta}(\varepsilon_1 + \varepsilon_0)^\theta \leq \varepsilon_2$$

by choosing $\varepsilon_j = \varepsilon_j(R, \varepsilon_2) > 0$ sufficiently small, $j = 0, 1$. Then, by the continuity argument, we see that (7.16) holds for all $\tau \leq t_{j+1} - t_j$. From Step 1 and (7.6), we have

$$(7.17) \quad \|v\|_{Z_\theta(I_j)} = \|v\|_{Z(I_j)}^\theta \|v\|_{X^{\frac{d-2}{2}}(I_j)}^{1-\theta} \leq C\varepsilon_1^\theta R^{1-\theta}.$$

Therefore, (7.13) follows from (7.16) and (7.17), by choosing $\varepsilon_1 = \varepsilon_1(R, \varepsilon_2)$ smaller if necessary.

Step 3. Given $\varepsilon_2 = \varepsilon_2(R) > 0$ (to be chosen later), it follows from Step 2 that (7.13) holds as long as $\varepsilon_j = \varepsilon_j(R) > 0$, $j = 0, 1$, are sufficiently small. From Step 1 with (7.7), (7.13), and (7.12), we have

$$(7.18) \quad \|v\|_{X^{\frac{d-2}{2}}(I_j)} \leq 2R$$

as long as $\varepsilon_j = \varepsilon_j(R) > 0$, $j = 0, 1, 2$, are sufficiently small.

Let u be a solution to the defocusing cubic NLS (1.1) with initial data $u(t_j)$ given at $t = t_j$ such that

$$(7.19) \quad \|u(t_j) - v(t_j)\|_{H^{\frac{d-2}{2}}} \leq \varepsilon < \varepsilon_0.$$

Let $J_j = [t_j, t_j + \tau] \subset I_j$ be the maximal time interval such that

$$(7.20) \quad \|u - v\|_{Z_\theta(J_j)} \leq 6C_0\varepsilon,$$

where C_0 is as in (6.3). Such an interval exists and is nonempty, since $\tau \mapsto \|u - v\|_{Z_\theta(t_j, t_j + \tau)}$ is finite and continuous (see Lemma A.8), at least on the interval of local existence of u , and vanishes for $\tau = 0$.

Let $w := u - v$. By Lemma 6.2 (with $f = 0$) with (7.7), (7.13), (7.18), (7.19), and (7.20), we have

$$\begin{aligned} \|w\|_{X^{\frac{d-2}{2}}(J_j)} &\leq \|u(t_j) - v(t_j)\|_{H^{\frac{d-2}{2}}} + C_1 \left\{ \|v\|_{X^{\frac{d-2}{2}}(J_j)} \|v\|_{Z_\theta(J_j)} \|w\|_{X^{\frac{d-2}{2}}(J_j)} \right. \\ &\quad + \|v\|_{X^{\frac{d-2}{2}}(J_j)} \|w\|_{X^{\frac{d-2}{2}}(J_j)} \|w\|_{Z_\theta(J_j)} \\ &\quad \left. + \|w\|_{X^{\frac{d-2}{2}}(J_j)} \|w\|_{Z_\theta(J_j)}^2 \right\} + \|e\|_{N^{\frac{d-2}{2}}(J_j)} \\ &\leq 2\varepsilon + C_2(\varepsilon_0 + \varepsilon_2)R \|w\|_{X^{\frac{d-2}{2}}(J_j)}. \end{aligned}$$

Taking $\varepsilon_j = \varepsilon_j(R) > 0$ sufficiently small, $j = 0, 2$, such that $(\varepsilon_0 + \varepsilon_2)R \ll 1$, we obtain

$$(7.21) \quad \|w\|_{X^{\frac{d-2}{2}}(J_j)} \leq 4\varepsilon.$$

Hence, from (6.3), we have

$$(7.22) \quad \|w\|_{Z_\theta(J_j)} \leq C_0 \|w\|_{X^{\frac{d-2}{2}}(J_j)} \leq 4C_0\varepsilon.$$

From (7.18) and (7.21), we have $\|u\|_{X^{\frac{d-2}{2}}(J_j)} \leq 3R < \infty$. Then, from (3.4), we have $\|u\|_{S^{\frac{d-2}{2}}(J_j)} < \infty$. In particular, this implies that u can be extended to some larger interval $J' \supset \overline{J_j}$. Therefore, in view of (7.20) and (7.22), we can apply the continuity argument and conclude that $J_j = I_j$.

Step 4. By (7.3), we have $\|u(t_0) - v(t_0)\|_{H^{\frac{d-2}{2}}} \leq \varepsilon$ for some $\varepsilon < \varepsilon_0$. Then, by Step 3, we have $\|w\|_{X^{\frac{d-2}{2}}(I_0)} \leq 4\varepsilon$ on $I_0 = [t_0, t_1]$. In particular, this yields

$$\|u(t_1) - v(t_1)\|_{H^{\frac{d-2}{2}}} \leq 4C\varepsilon.$$

Then, we can apply Step 3 on the interval I_1 by choosing ε_0 (and hence ε) even smaller. We argue recursively for each interval I_j , $j = 2, \dots, L$. Note that, at each step, we make ε_0 smaller by a factor of $(4C)^{-1}$. Since $L \sim \left(\frac{R}{\varepsilon_1}\right)^4$ and $\varepsilon_1 = \varepsilon_1(R)$, there are a finite number of iterative steps depending only on R . This allows us to choose new $\varepsilon_0 = \varepsilon_0(R) > 0$ such that, by Lemma A.4, we have

$$\begin{aligned} \|u\|_{X^{\frac{d-2}{2}}(I)} + \|v\|_{X^{\frac{d-2}{2}}(I)} &\lesssim LR \lesssim C(R), \\ \|u - v\|_{X^{\frac{d-2}{2}}(I)} &\lesssim L\varepsilon \lesssim C(R)\varepsilon. \end{aligned}$$

This completes the proof of Lemma 7.1. \square

In the remaining part of this section, we consider long time existence of solutions to the perturbed NLS (6.1) under several assumptions. Given $T > 0$, we assume that there exist $\beta, C, M > 0$ such that

$$(7.23) \quad \|f\|_{W^s(I)} \leq C|I|^\beta \quad \text{and} \quad \|f\|_{Y^s([0,T])} \leq M$$

for any interval $I \subset [0, T]$. Then, Proposition 6.3 guarantees existence of a solution to the perturbed NLS (6.1), at least for a short time.

Proposition 7.2. *Let $d \geq 3$. Let $s \in (s_d, s_{\text{crit}}]$, where s_d is defined in (1.11). Given $T > 0$, assume the following conditions (i) - (iii):*

- (i) *Hypothesis (B) holds if $d \neq 4$.*
- (ii) *$f \in Y^s([0, T]) \cap W^s([0, T])$ satisfies (7.23).*
- (iii) *Given a solution v to (6.1), the following a priori bound holds:*

$$(7.24) \quad \|v\|_{L^\infty([0,T]; H^{\frac{d-2}{2}}(\mathbb{R}^d))} \leq R$$

for some $R > 0$.

Then, there exists $\tau = \tau(R, M, T, s, \beta) > 0$ such that, given any $t_0 \in [0, T]$, the solution v to (6.1) exists on $[t_0, t_0 + \tau] \cap [0, T]$. In particular, condition (iii) guarantees existence of v on the entire interval $[0, T]$.

Remark 7.3. We point out that the first condition in (7.23) can be weakened as follows. Let $\tau = \tau(R, M, T, s, \beta) > 0$ be as in Proposition 7.2. Then, it follows from the proof of Proposition 7.2 (see (7.27) and (7.28) below) that if we assume that

$$\|f\|_{W^s([t_0, t_0 + \tau_*])} \leq C|\tau_*|^\beta$$

for some $\tau_* \leq \tau$ instead of the first condition in (7.23), then the conclusion of Proposition 7.2 still holds on $[t_0, t_0 + \tau_*] \cap [0, T]$. Indeed, we use this version of Proposition 7.2 in Section 8.

Proof. By setting $e = |v + f|^2(v + f) - |v|^2v$, (6.1) reduces to (7.1). In the following, we iteratively apply Lemma 7.1 on short intervals and show that there exists $\tau = \tau(R, M, T, s, \beta) > 0$ such that (7.1) is well-posed on $[t_0, t_0 + \tau] \cap [0, T]$ for any $t_0 \in [0, T]$.

Let w be the global solution to the defocusing cubic NLS (1.1) with $w(t_0) = v(t_0) = v_0$. By (7.24), we have $\|w(t_0)\|_{H^{\frac{d-2}{2}}} \leq R$. Then, by Hypothesis (B), we have

$$\|w\|_{L_{t,x}^{d+2}([0,T])} \leq C(R, T) < \infty.$$

By the standard argument, this implies that $\| |\nabla|^{\frac{d-2}{2}} w \|_{L_t^q L_x^r([0,T])} \leq C'(R, T) < \infty$ for all Schrödinger admissible pairs (q, r) . In particular, we have $\|w\|_{Z([0,T])} \leq C''(R, T) < \infty$ and

$$(7.25) \quad \begin{aligned} \|w\|_{X^{\frac{d-2}{2}}([0,T])} &\leq \|v_0\|_{H^{\frac{d-2}{2}}} + \| |\nabla|^{\frac{d-2}{2}} w \|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,T])} \|w\|_{L_{t,x}^{d+2}([0,T])}^2 \\ &\leq C'''(R, T) < \infty. \end{aligned}$$

Let $\theta \in (\frac{1}{2}, 1)$. Given small $\eta > 0$ (to be chosen later), we divide the interval $[t_0, T]$ into $J = J(R, T, \eta)$ many subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|w\|_{Z_\theta(I_j)} \sim \eta.$$

In the following, we fix the value of θ and suppress dependence of various constants such as τ and η on θ .

Fix $\tau > 0$ (to be chosen later in terms of R, M, T, s , and β) and write $[t_0, t_0 + \tau] = \bigcup_{j=0}^{J'} ([t_0, t_0 + \tau] \cap I_j)$ for some $J' \leq J - 1$, where $[t_0, t_0 + \tau] \cap I_j \neq \emptyset$ for $0 \leq j \leq J'$ and $[t_0, t_0 + \tau] \cap I_j = \emptyset$ for $j \geq J'$.

Since the nonlinear evolution w is small on each I_j , it follows that the linear evolution $S(t - t_j)w(t_j)$ is also small on each I_j . Indeed, from the Duhamel formula, we have

$$S(t - t_j)w(t_j) = w(t) + i \int_{t_j}^t S(t - t')|w|^2 w(t') dt'.$$

Then, from Case (1) in the proof of Lemma 6.2 with (7.25), we have

$$\|S(t - t_j)w(t_j)\|_{Z_\theta(I_j)} \leq \|w\|_{Z_\theta(I_j)} + C\|w\|_{X^{\frac{d-2}{2}}(I_j)}\|w\|_{Z_\theta(I_j)}^2 \leq \eta + C(R, T)\eta^2.$$

By taking $\eta = \eta(R, T) > 0$ sufficiently small, we have

$$(7.26) \quad \|S(t - t_j)w(t_j)\|_{Z_\theta(I_j)} \leq 2\eta$$

for all $j = 0, \dots, J - 1$.

Now, we estimate v on the first interval I_0 . Let $\eta_0 = \eta_0(R, M)$ be as in Proposition 6.3. Then, by Lemma 3.5 (i), (7.24), and Proposition 6.3, we have

$$\begin{aligned} \|v\|_{X^{\frac{d-2}{2}}(I_0)} &\leq \|S(t - t_0)v(t_0)\|_{X^{\frac{d-2}{2}}(I_0)} + \|v - S(t - t_0)v(t_0)\|_{X^{\frac{d-2}{2}}(I_0)} \\ &\leq R + C\eta^{3-2\theta} \leq 2R, \end{aligned}$$

as long as $2\eta < \eta_0$ and $\tau = \tau(\eta, \alpha, \beta) = \tau(R, M, T, \alpha, \beta) > 0$ is sufficiently small so that

$$(7.27) \quad \|f\|_{W^s([t_0, t_0 + \tau])} \leq C\tau^\beta \leq \eta^{\frac{4-\alpha}{\alpha}},$$

where $\alpha = \alpha(s)$ satisfies (6.6).

Next, we estimate the error term. By Lemma 6.2 with (7.23), we have

$$(7.28) \quad \|e\|_{N^{\frac{d-2}{2}}(I_0)} \leq C(R, M)\tau^{\alpha\beta}.$$

Given $\varepsilon > 0$, we can choose $\tau = \tau(R, M, T, \varepsilon, \alpha, \beta) > 0$ sufficiently small so that

$$\|e\|_{N^{\frac{d-2}{2}}(I_0)} \leq \varepsilon.$$

In particular, for $\varepsilon < \varepsilon_0$ with $\varepsilon_0 = \varepsilon_0(R) > 0$ dictated by Lemma 7.1, the condition (7.3) is satisfied on I_0 .

Therefore, all the conditions of Lemma 7.1 are satisfied on the first interval I_0 , provided that $\tau = \tau(R, M, T, \varepsilon, \alpha, \beta) > 0$ is chosen sufficiently small. Hence, we obtain

$$(7.29) \quad \|w - v\|_{X^{\frac{d-2}{2}}(I_0)} \leq C_0(R)\varepsilon.$$

In particular, we have

$$(7.30) \quad \|w(t_1) - v(t_1)\|_{H^{\frac{d-2}{2}}} \leq C_1(R)\varepsilon.$$

Then, from (7.26) and Lemma 3.5 (i) with (7.30), we have

$$\begin{aligned} \|S(t - t_1)v(t_1)\|_{Z_\theta(I_1)} &\leq \|S(t - t_1)w(t_1)\|_{Z_\theta(I_1)} + \|S(t - t_1)(w(t_1) - v(t_1))\|_{Z_\theta(I_1)} \\ &\leq 2\eta + C'_1(R)\varepsilon \leq 3\eta \end{aligned}$$

by choosing $\varepsilon = \varepsilon(R, \eta) > 0$ sufficiently small.

Proceeding as before, it follows from Proposition 6.3 with (7.24) and (7.26) that

$$\|v\|_{X^{\frac{d-2}{2}}(I_1)} \leq R + C\eta^{3-2\theta} \leq 2R,$$

as long as $3\eta \leq \eta_0$ and $\tau > 0$ is sufficiently small so that (7.27) is satisfied. Similarly, it follows from Lemma 6.2 with (7.23) that

$$(7.31) \quad \|e\|_{N^{\frac{d-2}{2}}(I_1)} \leq C(R, M)\tau^{\alpha\beta} \leq \varepsilon$$

by choosing $\tau = \tau(R, M, T, \varepsilon, \alpha, \beta) > 0$ sufficiently small. Therefore, all the conditions of Lemma 7.1 are satisfied on the second interval I_1 , provided that $\tau = \tau(R, M, T, \varepsilon, \alpha, \beta)$ is chosen sufficiently small and that $(C_1(R) + 1)\varepsilon < \varepsilon_0$. Hence, by Lemma 7.1, we obtain

$$\|w - v\|_{X^{\frac{d-2}{2}}(I_1)} \leq C_0(R)(C_1(R) + 1)\varepsilon.$$

In particular, we have

$$\|w(t_2) - v(t_2)\|_{H^{\frac{d-2}{2}}} \leq C_2(R)\varepsilon.$$

By choosing $\eta = \eta(R, M, T) > 0$ and $\tau = \tau(R, M, T, \varepsilon, \alpha, \beta) > 0$ sufficiently small, we can argue inductively and obtain

$$(7.32) \quad \|w(t_j) - v(t_j)\|_{H^{\frac{d-2}{2}}} \leq C_j(R)\varepsilon$$

for all $0 \leq j \leq J'$, as long as (i) $(J' + 2)\eta \leq \eta_0$ and (ii) $\varepsilon = \varepsilon(R, \eta, J)$ is sufficiently small such that $(C_j(R) + 1)\varepsilon < \varepsilon_0$, $j = 1, \dots, J'$. Recalling that $J' + 1 \leq J = J(R, T, \eta)$, we see that this can be achieved by choosing $\eta = \eta(R, M, T) > 0$, $\varepsilon = \varepsilon(R, M, T) > 0$, and $\tau = \tau(R, M, T, \alpha, \beta) = \tau(R, M, T, s, \beta) > 0$ sufficiently small. This guarantees existence of the solution v to (7.1) on $[t_0, t_0 + \tau]$.

Under the conditions (i) - (iii), we can apply the above local argument on time intervals of length $\tau = \tau(R, M, T, s, \beta) > 0$, thus extending the solution v to (6.1) on the entire interval $[0, T]$. \square

8. PROOF OF THEOREM 1.3

In this section, we prove the following ‘‘almost’’ almost sure global existence result.

Proposition 8.1. *Let $d \geq 3$ and $s \in (s_d, s_{\text{crit}}]$. Assume Hypothesis (A). Furthermore, assume Hypothesis (B) if $d \neq 4$. Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.9), satisfying (1.10). Then, given any $T, \varepsilon > 0$, there exists a set $\tilde{\Omega}_{T, \varepsilon} \subset \Omega$ such that:*

- (i) $P(\tilde{\Omega}_{T, \varepsilon}^c) < \varepsilon$.
- (ii) For each $\omega \in \tilde{\Omega}_{T, \varepsilon}$, there exists a (unique) solution u to (1.1) on $[0, T]$ with $u|_{t=0} = \phi^\omega$.

It is easy to see that ‘‘almost’’ almost sure global existence implies almost sure global existence. See [20]. For completeness, we first show how Theorem 1.3 follows as an immediate consequence of Proposition 8.1.

Given small $\varepsilon > 0$, let $T_j = 2^j$ and $\varepsilon_j = 2^{-j}\varepsilon$, $j \in \mathbb{N}$. For each j , we apply Proposition 8.1 and construct $\tilde{\Omega}_{T_j, \varepsilon_j}$. Then, let $\Omega_\varepsilon = \bigcap_{j=1}^{\infty} \tilde{\Omega}_{T_j, \varepsilon_j}$. Note that (i) $P(\Omega_\varepsilon^c) < \varepsilon$, and (ii) for each $\omega \in \Omega_\varepsilon$, we have a global solution u to (1.1) with $u|_{t=0} = \phi^\omega$. Now, let $\Sigma = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$. Then, we have $P(\Sigma^c) = 0$. Moreover, for each

$\omega \in \Sigma$, we have a global solution u to (1.1) with $u|_{t=0} = \phi^\omega$. This proves Theorem 1.3.

The rest of this section is devoted to the proof of Proposition 8.1.

Proof of Proposition 8.1. Given $T, \varepsilon > 0$, set

$$(8.1) \quad M = M(T, \varepsilon) \sim \|\phi\|_{H^s} \left(\log \frac{1}{\varepsilon} \right)^{\frac{1}{2}}.$$

Defining $\Omega_1 = \Omega_1(T, \varepsilon)$ by

$$\Omega_1 := \left\{ \omega \in \Omega : \|S(t)\phi^\omega\|_{Y^s([0, T])} \leq M \right\},$$

it follows from Lemma 3.5 (i) and Lemma 2.4 that

$$(8.2) \quad P(\Omega_1^c) < \frac{\varepsilon}{3}.$$

Given $T, \varepsilon > 0$, let $R = R(T, \frac{\varepsilon}{3})$ and let M be as in (1.13) and (8.1), respectively. With $\tau = \tau(R, M, T)$ as in Proposition 7.2, write

$$[0, T] = \bigcup_{j=0}^{\lfloor \frac{T}{\tau_*} \rfloor} [j\tau_*, (j+1)\tau_*] \cap [0, T]$$

for some $\tau_* \leq \tau$ (to be chosen later). Now, define Ω_2 by

$$\Omega_2 = \left\{ \omega \in \Omega : \|S(t)\phi^\omega\|_{W^s([j\tau_*, (j+1)\tau_*])} \leq C\tau_*^{\frac{1}{2(d+2)}}, j = 0, \dots, \lfloor \frac{T}{\tau_*} \rfloor \right\},$$

where C is as in (7.23). Then, by Lemma 2.2, we have

$$P(\Omega_2^c) \leq \sum_{j=0}^{\lfloor \frac{T}{\tau_*} \rfloor} P\left(\|S(t)\phi^\omega\|_{W^s([j\tau_*, (j+1)\tau_*])} \geq C\tau_*^{\frac{1}{2(d+2)}} \right) \lesssim \frac{T}{\tau_*} \exp\left(-\frac{c}{\tau_*^{\frac{1}{d+2}} \|\phi\|_{H^s}^2} \right).$$

By making $\tau_* = \tau_*(\|\phi\|_{H^s})$ smaller, if necessary, we have

$$\lesssim \frac{T}{\tau_*} \cdot \tau_* \exp\left(-\frac{c}{2\tau_*^{\frac{1}{d+2}} \|\phi\|_{H^s}^2} \right) = T \exp\left(-\frac{1}{2\tau_*^{\frac{1}{d+2}} \|\phi\|_{H^s}^2} \right).$$

Hence, by choosing $\tau_* = \tau_*(T, \varepsilon, \|\phi\|_{H^s})$ sufficiently small, we have

$$(8.3) \quad P(\Omega_2^c) < \frac{\varepsilon}{3}.$$

Finally, set $\tilde{\Omega}_{T, \varepsilon} = \Omega_{T, \frac{\varepsilon}{3}} \cap \Omega_1 \cap \Omega_2$, where $\Omega_{T, \frac{\varepsilon}{3}}$ is as in Hypothesis (A) with ε replaced by $\frac{\varepsilon}{3}$. Then, from (8.2) and (8.3), we have

$$P(\tilde{\Omega}_{T, \varepsilon}^c) < \varepsilon.$$

Moreover, for $\omega \in \tilde{\Omega}_{T, \varepsilon}$, we can iteratively apply Proposition 7.2 and Remark 7.3 and construct the solution $v = v^\omega$ on each $[j\tau_*, (j+1)\tau_*]$, $j = 0, \dots, \lfloor \frac{T}{\tau_*} \rfloor - 1$, and $[\lfloor \frac{T}{\tau_*} \rfloor \tau_*, T]$. This completes the proof of Proposition 8.1. \square

Remark 8.2. It is worthwhile to mention that the proof of Proposition 8.1 strongly depends on the quasi-invariance property of the distribution of the linear solution $S(t)\phi^\omega$. More precisely, in the proof above, we exploited the fact that the distribution of $\|S(t)\phi^\omega\|_{W^s([t_0, t_0 + \tau_*])}$ depends basically only on the length τ_* of the interval, but is independent of t_0 .

9. PROBABILISTIC GLOBAL EXISTENCE VIA RANDOMIZATION ON DILATED CUBES

In this section, we present the proof of Theorem 1.4. The main idea is to exploit the dilation symmetry of the cubic NLS (1.1). For a function $\phi = \phi(x)$, we define its scaling ϕ_μ by

$$\phi_\mu(x) := \mu^{-1}\phi(\mu^{-1}x),$$

while for a function $f = f(t, x)$, we define its scaling f_μ by

$$f_\mu(t, x) := \mu^{-1}f(\mu^{-2}t, \mu^{-1}x).$$

Then, given $\phi \in H^s(\mathbb{R}^d)$, we have

$$(9.1) \quad \|\phi_\mu\|_{\dot{H}^s(\mathbb{R}^d)} = \mu^{\frac{d-2}{2}-s}\|\phi\|_{\dot{H}^s(\mathbb{R}^d)}.$$

If $s < s_{\text{crit}} = \frac{d-2}{2}$, that is, if ϕ is supercritical with respect to the scaling symmetry, then we can make the H^s -norm of the scaled function ϕ_μ small by taking $\mu \ll 1$. The issue is that the Strichartz estimates we employ in proving probabilistic well-posedness are (sub)critical and do not become small even if we take $\mu \ll 1$. It is for this reason that we consider the randomization $\phi^{\omega, \mu}$ on *dilated* cubes.

Fix $\phi \in H^s(\mathbb{R}^d)$ with $s \in (s_d, s_{\text{crit}})$, where $s_{\text{crit}} = \frac{d-2}{2}$ and s_d is as in (1.11). Let $\phi^{\omega, \mu}$ be its randomization on dilated cubes of scale μ as in (1.17). Instead of considering (1.1) with $u_0 = \phi^{\omega, \mu}$, we consider the scaled Cauchy problem

$$(9.2) \quad \begin{cases} i\partial_t u_\mu + \Delta u_\mu = \pm |u_\mu|^2 u_\mu, \\ u_\mu|_{t=0} = u_{0, \mu} = (\phi^{\omega, \mu})_\mu, \end{cases}$$

where u_μ is as in (1.2) and $(\phi^{\omega, \mu})_\mu(x) := \mu^{-1}\phi^{\omega, \mu}(\mu^{-1}x)$ is the scaled randomization. For notational simplicity, we denote $(\phi^{\omega, \mu})_\mu$ by $\phi_\mu^{\omega, \mu}$ in the following. Denoting the linear and nonlinear part of u_μ by $z_\mu(t) = z_\mu^\omega(t) := S(t)\phi_\mu^{\omega, \mu}$ and $v_\mu(t) := u_\mu(t) - S(t)\phi_\mu^{\omega, \mu}$ as before, we reduce (9.2) to

$$(9.3) \quad \begin{cases} i\partial_t v_\mu + \Delta v_\mu = \pm |v_\mu + z_\mu|^2(v_\mu + z_\mu), \\ v_\mu|_{t=0} = 0. \end{cases}$$

Note that if u satisfies (1.1) with initial data $u(0) = \phi^{\omega, \mu}$, then u_μ , z_μ , and v_μ are indeed the scalings of u , $z := S(t)\phi^{\omega, \mu}$, and $v := u - z$, respectively. For u_μ this simply follows from the scaling symmetry of (1.1). For z_μ and v_μ , this follows from the following observation:

$$(9.4) \quad \mathcal{F}_x \left[(S(t)\phi^{\omega, \mu})_\mu \right] (\xi) = \mu^{d-1} e^{-i\frac{t}{\mu^2}|\mu\xi|^2} \widehat{\phi^{\omega, \mu}}(\mu\xi) = e^{-it|\xi|^2} \widehat{\phi_\mu^{\omega, \mu}}(\xi) = \widehat{z_\mu}(t, \xi).$$

Define Γ_μ by

$$(9.5) \quad \Gamma_\mu v_\mu(t) = \mp i \int_0^t S(t-t') \mathcal{N}(v_\mu + z_\mu)(t') dt'.$$

In the following, we show that there exists $\mu_0 = \mu_0(\varepsilon, \|\phi\|_{H^s}) > 0$ such that, for $\mu \in (0, \mu_0)$, the estimates (4.5) and (4.6) in Proposition 4.1 (with $\tilde{\Gamma}$ replaced by Γ_μ) hold with $R = \eta_2$ outside a set of probability $< \varepsilon$, where η_2 is as in (5.2). In view of (1.16), it is easy to see that

$$\psi(D-n)\phi_\mu = (\psi^\mu(D-\mu n)\phi)_\mu.$$

Hence, we have

$$(9.6) \quad \phi_\mu^{\omega, \mu} = (\phi^{\omega, \mu})_\mu = \sum_{n \in \mathbb{Z}^d} g_n(\omega) \psi(D - n) \phi_\mu.$$

Given η_2 as in (5.2) and $\mu > 0$, define $\Omega_{1, \mu}$ by

$$\Omega_{1, \mu} = \left\{ \omega \in \Omega : \|S(t)\phi_\mu^{\omega, \mu}\|_{L_t^q W_x^{s, q}(\mathbb{R} \times \mathbb{R}^d)} \leq \eta_2, q = 4, \frac{6(d+2)}{d+4}, d+2 \right\}.$$

We also define $\Omega_{2, \mu}$ by

$$\Omega_{2, \mu} = \left\{ \omega \in \Omega : \|\phi_\mu^{\omega, \mu}\|_{H^s(\mathbb{R}^d)} \leq \eta_2 \right\}.$$

Now, let $\Omega_\mu = \Omega_{1, \mu} \cap \Omega_{2, \mu}$. Noting that $4, \frac{6(d+2)}{d+4}$, and $d+2$ are larger than the diagonal Strichartz admissible index $\frac{2(d+2)}{d}$, it follows from Lemma 2.3 and Lemma 2.4 with (9.6) and (9.1) that

$$P(\Omega_\mu^c) \leq C \exp\left(-c \frac{\eta_2^2}{\|\phi_\mu\|_{H^s}^2}\right) \leq C \exp\left(-c \frac{\eta_2^2}{\mu^{d-2-2s} \|\phi\|_{H^s}^2}\right)$$

for $\mu \leq 1$. Then, by setting

$$(9.7) \quad \mu_0 \sim \left(\frac{\eta_2}{\|\phi\|_{H^s} (\log \frac{1}{\varepsilon})^{\frac{1}{2}}} \right)^{\frac{1}{\frac{d-2}{2}-s}},$$

we have

$$(9.8) \quad P(\Omega_\mu^c) < \varepsilon$$

for $\mu \in (0, \mu_0)$. Note that $\mu_0 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Recall that $q = 4, \frac{6(d+2)}{d+4}$, and $d+2$ are the only relevant values of the space-time Lebesgue indices controlling the random forcing term in the proof of Proposition 4.1. Hence, the estimates (4.5) and (4.6) in Proposition 4.1 (with $\tilde{\Gamma}$ replaced by Γ_μ) hold with $R = \eta_2$ for each $\omega \in \Omega_\mu$. Then, by repeating the proof of Theorem 1.2 in Section 5, we see that, for each $\omega \in \Omega_\mu$, there exists a global solution u_μ to (9.2) with $u_\mu|_{t=0} = \phi_\mu^{\omega, \mu}$ which scatters both forward and backward in time. By undoing the scaling, we obtain a global solution u to (1.1) with $u|_{t=0} = \phi^{\omega, \mu}$ for each $\omega \in \Omega_\mu$. Moreover, scattering for u_μ implies scattering for u . Indeed, as in Theorem 1.2, there exists $v_{+, \mu} \in H^{\frac{d-2}{2}}(\mathbb{R}^d)$ such that

$$\lim_{t \rightarrow \infty} \|u_\mu(t) - S(t)(\phi_\mu^{\omega, \mu} + v_{+, \mu})\|_{H^{\frac{d-2}{2}}} = 0.$$

Then, a computation analogous to (9.4) yields

$$S(t)(\phi_\mu^{\omega, \mu} + v_{+, \mu}) = (S(t)(\phi^{\omega, \mu} + v_+))_\mu,$$

where $v_+ := (v_{+, \mu})_{\mu^{-1}} \in H^{\frac{d-2}{2}}(\mathbb{R}^d)$. Then, by (9.1), it follows that

$$\lim_{t \rightarrow \infty} \|u - S(t)(\phi^{\omega, \mu} + v_+)\|_{H^{\frac{d-2}{2}}} = 0.$$

This proves that u scatters forward in time. Scattering of u as $t \rightarrow -\infty$ can be proved analogously. This completes the proof of Theorem 1.4.

APPENDIX A. ON THE PROPERTIES OF THE U^p - AND X^s -SPACES

In this appendix, we prove some additional properties of the U^p - and X^s -spaces. In the following, all intervals are half open intervals of the form $[a, b)$ and p denotes a number such that $1 \leq p < \infty$.

Lemma A.1. *Let $u = \sum_{j=1}^{\infty} \lambda_j a_j$, where $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}; \mathbb{C})$ and the a_j 's are U^p -atoms. Given an interval $I \subset \mathbb{R}$, we can write $u \cdot \chi_I$ as $u \cdot \chi_I = \sum_{j=1}^{\infty} \tilde{\lambda}_j \tilde{a}_j$ for some $\{\tilde{\lambda}_j\}_{j=1}^{\infty} \in \ell^1$ and some sequence $\{\tilde{a}_j\}_{j=1}^{\infty}$ of U^p -atoms such that*

$$(A.1) \quad \sum_{j=1}^{\infty} |\tilde{\lambda}_j| \leq \sum_{j=1}^{\infty} |\lambda_j|.$$

As a consequence, we have

$$(A.2) \quad \|u \cdot \chi_I\|_{U^p(\mathbb{R})} \leq \|u\|_{U^p(\mathbb{R})}$$

for any $u \in U^p(\mathbb{R})$ and any $I \subset \mathbb{R}$.

Proof. With $a_j = \sum_{k=1}^{K_j} \phi_{k-1}^j \chi_{[t_{k-1}^j, t_k^j)}$, we have

$$u \cdot \chi_I = \sum_{j=1}^{\infty} \lambda_j a_j \chi_I = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{K_j} \phi_{k-1}^j \chi_{[t_{k-1}^j, t_k^j) \cap I}.$$

Then, setting $\tilde{\lambda}_j$ and \tilde{a}_j as

$$(A.3) \quad \begin{aligned} \tilde{\lambda}_j &= \left(\sum_{k \in A_j(I)} \|\phi_k^j\|_H^p \right)^{\frac{1}{p}} \lambda_j, \\ \tilde{a}_j &= \frac{1}{\left(\sum_{k \in A_j(I)} \|\phi_k^j\|_H^p \right)^{\frac{1}{p}}} \sum_{k \in A_j(I)} \phi_{k-1}^j \chi_{[t_{k-1}^j, t_k^j) \cap I}, \end{aligned}$$

where $A_j(I)$ is defined by

$$A_j(I) = \{k \in \{1, \dots, K_j\} : [t_{k-1}^j, t_k^j) \cap I \neq \emptyset\},$$

we have $u \cdot \chi_I = \sum_{j=1}^{\infty} \tilde{\lambda}_j \tilde{a}_j$. Moreover, noting that

$$\left(\sum_{k \in A_j(I)} \|\phi_k^j\|_H^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{K_j} \|\phi_k^j\|_H^p \right)^{\frac{1}{p}} = 1,$$

we obtain (A.1) from (A.3). Finally, by (A.1), we have

$$\|u \cdot \chi_I\|_{U^p(\mathbb{R})} \leq \sum_{j=1}^{\infty} |\lambda_j|$$

for any representation $u = \sum_{j=1}^{\infty} \lambda_j a_j$ with $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N}; \mathbb{C})$ and U^p -atoms a_j 's. Hence, by taking an infimum over all such representations of u , we obtain (A.2). \square

Given an interval $I \subset \mathbb{R}$, we define the local-in-time U^p -norm in the usual manner as a restriction norm:

$$\|u\|_{U^p(I)} = \inf_v \{ \|v\|_{U^p(\mathbb{R})} : v|_I = u \}.$$

Remark A.2. The infimum is achieved by $v = u \cdot \chi_I$ in view of Lemma A.1. In the following, however, we may use other extensions, depending on our purpose.

The next lemma states the subadditivity of the local-in-time U^p -norm over intervals.

Lemma A.3. *Given an interval $I \subset \mathbb{R}$, let $I = \bigcup_{j=1}^{\infty} I_j$ be a partition of I . Then, we have*

$$(A.4) \quad \|u\|_{U^p(I)} \leq \sum_{j=1}^{\infty} \|u\|_{U^p(I_j)}.$$

Proof. Given $\varepsilon > 0$, it follows from the definition of the local-in-time U^p -norm that there exists $v_j \in U^p(\mathbb{R})$ such that $v_j|_{I_j} = u$ and

$$(A.5) \quad \|v_j\|_{U^p(\mathbb{R})} \leq \|u\|_{U^p(I_j)} + \frac{\varepsilon}{2^j}$$

for each $j \in \mathbb{N}$. Then, by (A.2) and (A.5), we have

$$(A.6) \quad \begin{aligned} \|u\|_{U^p(I)} &\leq \left\| \sum_{j=1}^{\infty} v_j \cdot \chi_{I_j} \right\|_{U^p(\mathbb{R})} \leq \sum_{j=1}^{\infty} \|v_j \cdot \chi_{I_j}\|_{U^p(\mathbb{R})} \leq \sum_{j=1}^{\infty} \|v_j\|_{U^p(\mathbb{R})} \\ &\leq \sum_{j=1}^{\infty} \|u\|_{U^p(I_j)} + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (A.4) follows from (A.6). \square

As a corollary, we immediately obtain the following subadditivity property of the local-in-time X^s -norm over intervals.

Lemma A.4. *Let $s \in \mathbb{R}$. Given an interval $I \subset \mathbb{R}$, let $I = \bigcup_{j=1}^{\infty} I_j$ be a partition of I . Then, we have*

$$\|u\|_{X^s(I)} \leq \sum_{j=1}^{\infty} \|u\|_{X^s(I_j)}.$$

We say that u on $[a, b)$ is a regulated function if both left and right limits exist at every point (including one-sided limits at the endpoints⁴). Given a regulated function u on $[a, b)$ and a partition $\mathcal{P} = \{\tau_1, \dots, \tau_n\}$ of $[a, b)$: $a < \tau_1 < \dots < \tau_n < b$, we define a step function $u_{\mathcal{P}}$ by

$$u_{\mathcal{P}}(t) = \begin{cases} u(t), & \text{if } t = \tau_j, \\ u(\tau_j+), & \text{if } \tau_j < t < \tau_{j+1}, \end{cases}$$

where we set $\tau_0 = a$ and $\tau_{n+1} = b$. In particular, if u is right-continuous, we have $u_{\mathcal{P}}(t) = u(\tau_j)$ for $\tau_j \leq t < \tau_{j+1}$. Note that the mapping $\mathcal{P} : u \mapsto u_{\mathcal{P}}$ is linear.

Lemma A.5. *Let $u \in U^p(\mathbb{R})$.*

(i) *For any partition \mathcal{P} of \mathbb{R} , we have*

$$(A.7) \quad \|u_{\mathcal{P}}\|_{U^p(\mathbb{R})} \leq \|u\|_{U^p(\mathbb{R})}.$$

(ii) *Given $\varepsilon > 0$, there exists a partition \mathcal{P} of \mathbb{R} such that*

$$(A.8) \quad \|u - u_{\mathcal{P}}\|_{U^p(\mathbb{R})} < \varepsilon.$$

⁴We allow $a = -\infty$ and/or $b = \infty$.

Proof. (i) We first claim that, given a U^p -atom a , we have $\|a_{\mathcal{P}}\|_{U^p(\mathbb{R})} \leq 1$ for any partition \mathcal{P} . Given a U^p -atom $a = \sum_{k=1}^K \phi_{k-1} \chi_{[t_{k-1}, t_k]}$ and a partition $\mathcal{P} = \{\tau_1, \dots, \tau_n\}$ of \mathbb{R} , we have

$$(A.9) \quad a_{\mathcal{P}} = \sum_{j=1}^n a(\tau_j) \chi_{[\tau_j, \tau_{j+1})},$$

where $\tau_{n+1} = \infty$. Note that we have

$$a(\tau_j) = \begin{cases} \phi_{k-1}, & \text{if } \tau_j \in [t_{k-1}, t_k) \text{ for some } k, \\ 0, & \text{otherwise.} \end{cases}$$

We can simplify the expression in (A.9) by concatenating neighboring intervals $[\tau_j, \tau_{j+1})$ and $[\tau_{j+1}, \tau_{j+2})$ if $a(\tau_j) = a(\tau_{j+1})$ and obtain

$$a_{\mathcal{P}} = \sum_{\ell=1}^L a(\tau_{j_\ell}) \chi_{[\tau_{j_\ell}, \tau_{j_{\ell+1}})}$$

for some subpartition $\{\tau_{j_\ell}\}_{\ell=1}^L$ of \mathcal{P} , where $a(\tau_{j_\ell}) = \phi_{k-1}$ for some k or $a(\tau_{j_\ell}) = 0$. Note that, given $k \in \{1, \dots, K\}$, there exists at most one $\ell \in \{1, \dots, L\}$ such that $a(\tau_{j_\ell}) = \phi_{k-1}$ (unless $\phi_{k-1} = \phi_{k'-1}$ for some $k' \neq k$). In particular, we have

$$\lambda := \left(\sum_{\ell=1}^L \|a(\tau_{j_\ell})\|_H^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^K \|\phi_{k-1}\|_H^p \right)^{\frac{1}{p}} = 1.$$

If $\lambda = 0$, then $a_{\mathcal{P}} = 0$. Otherwise, we have $a_{\mathcal{P}} = \lambda b$, where b is a U^p -atom given by

$$b = \sum_{\ell=1}^L \frac{a(\tau_{j_\ell})}{\lambda} \chi_{[\tau_{j_\ell}, \tau_{j_{\ell+1}})}.$$

Hence, $\|a_{\mathcal{P}}\|_{U^p(\mathbb{R})} \leq 1$.

Given $u \in U^p(\mathbb{R})$, write $u = \sum_{j=1}^{\infty} \lambda_j a_j$ for some $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$ and some sequence of $\{a_j\}_{j=1}^{\infty}$ of U^p -atoms. Then, we have

$$(A.10) \quad \|u_{\mathcal{P}}\|_{U^p(\mathbb{R})} = \left\| \sum_{j=1}^{\infty} \lambda_j \cdot (a_j)_{\mathcal{P}} \right\|_{U^p(\mathbb{R})} \leq \sum_{j=1}^{\infty} |\lambda_j| \| (a_j)_{\mathcal{P}} \|_{U^p(\mathbb{R})} \leq \sum_{j=1}^{\infty} |\lambda_j|.$$

Therefore, we obtain (A.7), since (A.10) holds for any $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$ and any sequence of $\{a_j\}_{j=1}^{\infty}$ of U^p -atoms such that $u = \sum_{j=1}^{\infty} \lambda_j a_j$.

(ii) Fix a representation $u = \sum_{j=1}^{\infty} \lambda_j a_j$ for some $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$ and some sequence $\{a_j\}_{j=1}^{\infty}$ of U^p -atoms. Then, by setting $u_J = \sum_{j=1}^J \lambda_j a_j$ for sufficiently large J , we have

$$(A.11) \quad \|u - u_J\|_{U^p(\mathbb{R})} \leq \sum_{j=J+1}^{\infty} |\lambda_j| < \frac{\varepsilon}{2}.$$

Note that u_J is a step function with finitely many jump discontinuities. Now, we define a partition \mathcal{P} by setting $\mathcal{P} = \{t \in \mathbb{R} : u_J \text{ is discontinuous at } t\}$. Then, by right-continuity of u_J , we have $u_J - (u_J)_{\mathcal{P}} = 0$. Hence, from (A.11) and part (i),

we obtain

$$\begin{aligned} \|u - u_{\mathcal{P}}\|_{U^p(\mathbb{R})} &\leq \|u - u_J\|_{U^p(\mathbb{R})} + \|u_J - (u_J)_{\mathcal{P}}\|_{U^p(\mathbb{R})} + \|(u - u_J)_{\mathcal{P}}\|_{U^p(\mathbb{R})} \\ &< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Note that any refinement \mathcal{P}' of the partition \mathcal{P} also yields (A.8). \square

Lemma A.6. *Let $I = [a, b) \subset \mathbb{R}$ be an interval. Given $u \in U^p(I) \cap C(I; H)$, the mapping $t \in I \mapsto \|u\|_{U^p([a, t])}$ is continuous.*

Remark A.7. It follows from the proof that we need the (left-)continuity of u only in proving left-continuity of the mapping $t \in I \mapsto \|u\|_{U^p([a, t])}$.

Proof.

• *Part 1: Left-continuity.* Suppose that the mapping $t \mapsto \|u\|_{U^p([a, t])}$ is not left-continuous at $t = t_* \in (a, b)$. Then, there exist $\varepsilon > 0$ and a sequence $\delta_n \in (0, t_* - a)$, $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, such that

$$(A.12) \quad \|u\|_{U^p([a, t_* - \delta_n])} < \|u\|_{U^p([a, t_*])} - \varepsilon.$$

By definition, for any $\delta \in [0, t_* - a)$, there exists $v_\delta \in U^p(\mathbb{R})$ such that $v_\delta|_{[a, t_* - \delta)} = u$ and

$$(A.13) \quad \|v_\delta\|_{U^p(\mathbb{R})} \leq \|u\|_{U^p([a, t_* - \delta])} + \frac{\varepsilon}{4}.$$

Moreover, in view of Lemma A.1, we can assume that $v_\delta = v_\delta \cdot \chi_{[a, t_* - \delta)}$. In particular, we have

$$(A.14) \quad v_\delta = \sum_{j=1}^{\infty} \lambda_j^\delta a_j^\delta = \sum_{j=1}^{\infty} \lambda_j^\delta \sum_{k=1}^{K_j^\delta} \phi_{k-1}^{\delta, j} \chi_{[t_{k-1}^{\delta, j}, t_k^{\delta, j})},$$

where $t_{K_j^\delta}^{\delta, j} \leq t_* - \delta$. Now, we define an extension \tilde{v}_δ of v_δ onto \mathbb{R} by setting $t_{K_j^\delta}^{\delta, j} = \infty$ in (A.14) if $t_{K_j^\delta}^{\delta, j} = t_* - \delta$. By continuity of u and $v_\delta|_{[a, t_* - \delta)} = u$, we have $\tilde{v}_\delta(t) = u(t_* - \delta)$ for $t \in [t_* - \delta, \infty)$. By construction, we have

$$(A.15) \quad \|v_\delta\|_{U^p(\mathbb{R})} = \|\tilde{v}_\delta\|_{U^p(\mathbb{R})}.$$

Let \tilde{v} be the extension of $u \cdot \chi_{[a, t_*)}$ constructed as above with $\delta = 0$. Then, by definition of the U^p -norm and Lemma A.5 (ii), there exists a partition \mathcal{P} of \mathbb{R} such that

$$(A.16) \quad \|u\|_{U^p([a, t_*])} \leq \|\tilde{v}\|_{U^p(\mathbb{R})} \leq \|\tilde{v}_{\mathcal{P}}\|_{U^p(\mathbb{R})} + \frac{\varepsilon}{8}.$$

Since (A.16) holds for any refinement \mathcal{P}' of \mathcal{P} , we can assume that $t_* \in \mathcal{P}$.

By uniform continuity of u , there exists $\delta_0 > 0$ such that

$$(A.17) \quad \|u(t_1) - u(t_2)\|_H < \frac{\varepsilon}{8 \cdot (\#\mathcal{P} + 1)}$$

for any $t_1, t_2 \in (t_* - \delta_0, t_*]$. Since $\tilde{v}_\delta = \tilde{v}$ on $(-\infty, t_* - \delta]$, $\tilde{v}(t) = u(t_*)$ for $t \geq t_*$, and $\tilde{v}_\delta(t) = u(t_* - \delta)$ for $t \geq t_* - \delta$, we have

$$\begin{aligned} \tilde{v}_\mathcal{P} - (\tilde{v}_\delta)_\mathcal{P} &= \sum_{\substack{\tau_j \in \mathcal{P} \\ \tau_j > t_* - \delta}} (u(\tau_j) - u(t_* - \delta)) \chi_{[\tau_j, \tau_{j+1})} \\ &= \sum_{\substack{\tau_j \in \mathcal{P} \\ \tau_j \in (t_* - \delta, t_*)}} (u(\tau_j) - u(t_* - \delta)) \chi_{[\tau_j, \tau_{j+1})} + (u(t_*) - u(t_* - \delta)) \chi_{[t_*, \infty)}. \end{aligned}$$

Then, from (A.17), we have

$$\begin{aligned} \|\tilde{v}_\mathcal{P} - (\tilde{v}_\delta)_\mathcal{P}\|_{U^p(\mathbb{R})} &\leq \sum_{\substack{\tau_j \in \mathcal{P} \\ \tau_j \in (t_* - \delta, t_*)}} \|u(\tau_j) - u(t_* - \delta)\|_H + \|u(t_*) - u(t_* - \delta)\|_H \\ (A.18) \quad &< \frac{\varepsilon}{8} \end{aligned}$$

for any $\delta \in (0, \delta_0)$.

Finally, from (A.16), (A.18), Lemma A.5 (i), (A.15), (A.13), and (A.12), we have

$$\begin{aligned} \|u\|_{U^p([a, t_*])} &\leq \|(\tilde{v}_{\delta_n})_\mathcal{P}\|_{U^p(\mathbb{R})} + \frac{\varepsilon}{4} \leq \|\tilde{v}_{\delta_n}\|_{U^p(\mathbb{R})} + \frac{\varepsilon}{4} \leq \|u\|_{U^p([a, t_* - \delta_n])} + \frac{\varepsilon}{2} \\ &\leq \|u\|_{U^p([a, t_*])} - \frac{\varepsilon}{2} \end{aligned}$$

for sufficiently large n such that $\delta_n < \delta_0$. This is a contradiction. Therefore, the mapping $t \mapsto \|u\|_{U^p([a, t])}$ is left-continuous at $t = t_*$.

• *Part 2: Right-continuity.* Fix $t_* \in I$ and small $\varepsilon > 0$. As in Part 1, let \tilde{v}_0 be the extension of $v_0 = v_0 \cdot \chi_{[a, t_*)}$ satisfying (A.13). In particular, from (A.13) and (A.15), we have

$$(A.19) \quad \|\tilde{v}_0\|_{U^p(\mathbb{R})} \leq \|u\|_{U^p([a, t_*])} + \frac{\varepsilon}{4}.$$

Note that $\tilde{v}_0 = 0$ on $(-\infty, a)$.

Let $w = \tilde{u} - \tilde{v}_0$, where $\tilde{u} \in U^p(\mathbb{R})$ is an extension of u from I onto \mathbb{R} such that $\tilde{u} = 0$ on $(-\infty, a)$. Since $w \in U^p(\mathbb{R})$, we can write $w = \sum_{j=1}^{\infty} \lambda_j a_j$ for some $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$ and some sequence $\{a_j\}_{j=1}^{\infty}$ of U^p -atoms. Since $w = \tilde{u} - \tilde{v}_0 = 0$ on $(-\infty, t_*]$, we can assume that $\text{supp}(a_j) \subset (t_*, \infty)$ for all j . Then, we can choose large $J = J(\varepsilon) \in \mathbb{N}$ such that

$$(A.20) \quad \sum_{j=J+1}^{\infty} |\lambda_j| < \frac{\varepsilon}{4}.$$

Noting that $w_J := \sum_{j=1}^J \lambda_j a_j$ is a finite linear combination of characteristic functions, there exists $\delta_0 > 0$ such that w_J is constant on $[t_*, t_* + \delta_0) \subset I$. Define λ_0 , ϕ , and a_0 by

$$\lambda_0 := \|w_J(t_*)\|_H, \quad \phi := \lambda_0^{-1} w_J(t_*), \quad \text{and} \quad a_0 := \chi_{[t_*, \infty)} \phi.$$

Then, define \tilde{w} by $\tilde{w}(t) := w_J(t_*) \chi_{[t_*, \infty)} + \sum_{j=J+1}^{\infty} \lambda_j a_j = \lambda_0 a_0 + \sum_{j=J+1}^{\infty} \lambda_j a_j$. Note that $\tilde{w} = 0$ on $(-\infty, t_*]$. It follows from (A.20) that

$$(A.21) \quad \|\tilde{w}\|_{U^p(\mathbb{R})} \leq |\lambda_0| + \sum_{j=J+1}^{\infty} |\lambda_j| < \frac{\varepsilon}{2},$$

since we have

$$\lambda_0 = \left\| w(t_*) - \sum_{j=J+1}^{\infty} \lambda_j a_j(t_*) \right\|_H = \left\| \sum_{j=J+1}^{\infty} \lambda_j a_j(t_*) \right\|_H \leq \sum_{j=J+1}^{\infty} |\lambda_j| < \frac{\varepsilon}{4}.$$

Here, we used the fact that $\|a_j\|_H \leq 1$ for a U^p -atom a . By construction, we have $\text{supp}(\tilde{w}) \subset [t_*, \infty)$. Then, noting that $u - \tilde{v}_0 = \tilde{w}$ on $(-\infty, t_* + \delta_0) \subset I$, it follows from (A.19) and (A.21) that

$$\begin{aligned} \|u\|_{U^p([a, t_* + \delta])} &\leq \|\tilde{v}_0\|_{U^p([a, t_* + \delta])} + \|\tilde{w}\|_{U^p([a, t_* + \delta])} \leq \|\tilde{v}_0\|_{U^p(\mathbb{R})} + \|\tilde{w}\|_{U^p(\mathbb{R})} \\ &\leq \|u\|_{U^p([a, t_*])} + \frac{3}{4}\varepsilon \end{aligned}$$

for any $\delta \in (0, \delta_0]$. Therefore, the mapping $t \mapsto \|u\|_{U^p([a, t])}$ is right-continuous at $t = t_*$. \square

Lemma A.8. *Let $s \in \mathbb{R}$ and $I = [a, b) \subset \mathbb{R}$. Given $u \in X^s(I) \cap C(I; H^s(\mathbb{R}^d))$, the mapping $t \in I \mapsto \|u\|_{X^s([a, t])}$ is continuous.*

Proof. First, we claim that the infimum in the definition of the local-in-time X^s -norm on an interval $[a, t)$ is achieved by $u \cdot \chi_{[a, t)}$ for any $t \leq b$. Namely, we have

$$(A.22) \quad \|u\|_{X^s([a, t])} = \|u \cdot \chi_{[a, t)}\|_{X^s(\mathbb{R})}.$$

On the one hand, given any extension v on \mathbb{R} of u restricted to $[a, t)$, i.e. $v|_{[a, t)} = u$, we have

$$(A.23) \quad \|u\|_{X^s([a, t])} \leq \|v\|_{X^s(\mathbb{R})}.$$

On the other hand, by Lemma A.1, we have

$$(A.24) \quad \|u \cdot \chi_{[a, t)}\|_{X^s(\mathbb{R})} = \|v \cdot \chi_{[a, t)}\|_{X^s(\mathbb{R})} \leq \|v\|_{X^s(\mathbb{R})}.$$

Hence, (A.22) follows, since (A.23) and (A.24) hold for any extension v . Moreover, we have

$$(A.25) \quad \begin{aligned} \|u\|_{X^s([a, t])} &= \|u \cdot \chi_{[a, t)}\|_{X^s(\mathbb{R})} = \left(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{2s} \|\mathbf{P}_N u \cdot \chi_{[a, t)}\|_{U_{\Delta}^2(\mathbb{R}; L^2)}^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{2s} \|\mathbf{P}_N u\|_{U_{\Delta}^2([a, t); L^2)}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where the last equality follows from Remark A.2.

Let v be an extension of u onto \mathbb{R} such that $\|v\|_{X^s(\mathbb{R})} < \infty$. Given $\varepsilon > 0$, we can choose $J \in \mathbb{N}$ such that

$$\left(\sum_{j=J}^{\infty} 2^{2js} \|\mathbf{P}_{2^j} v\|_{U_{\Delta}^2(\mathbb{R}; L^2)}^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{4}.$$

Then, we have

$$(A.26) \quad \left(\sum_{j=J}^{\infty} 2^{2js} \|\mathbf{P}_{2^j} u\|_{U_{\Delta}^2([a, t); L^2)}^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{4}$$

for any $t \in I$. Fix $t_* \in I$. By Lemma A.6, for each $j = 0, 1, \dots, J-1$, there exists $\delta_j > 0$ such that

$$(A.27) \quad 2^{js} \left| \|\mathbf{P}_{2^j} u\|_{U_{\Delta}^2([a, t_*]; L^2)} - \|\mathbf{P}_{2^j} u\|_{U_{\Delta}^2([a, t_* + \delta]; L^2)} \right| < \frac{\varepsilon}{2J^{\frac{1}{2}}}$$

for $|\delta| < \delta_j$. Then, by Minkowski's inequality with (A.25), (A.26), and (A.27), we have

$$\begin{aligned} & \left| \|u\|_{X^s([a, t_*])} - \|u\|_{X^s([a, t_* + \delta])} \right| \\ & \leq \left(\sum_{j=0}^{\infty} 2^{2js} \left(\|\mathbf{P}_{2^j} u\|_{U_{\Delta}^2([a, t_*]; L^2)} - \|\mathbf{P}_{2^j} u\|_{U_{\Delta}^2([a, t_* + \delta]; L^2)} \right)^2 \right)^{\frac{1}{2}} \\ & \leq J^{\frac{1}{2}} \max_{j=0, \dots, J-1} 2^{js} \left| \|\mathbf{P}_{2^j} u\|_{U_{\Delta}^2([a, t_*]; L^2)} - \|\mathbf{P}_{2^j} u\|_{U_{\Delta}^2([a, t_* + \delta]; L^2)} \right| + \frac{\varepsilon}{2} \\ & < \varepsilon \end{aligned}$$

for $0 < |\delta| < \min(\delta_0, \dots, \delta_{J-1})$. This proves the lemma. \square

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