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Numerical analysis of two-dimensional Navier–Stokes equations with additive stochastic forcing

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We propose and study a temporal and a spatio-temporal discretisation of the two-dimensional stochastic Navier–Stokes equations in bounded domains supplemented with no-slip boundary conditions. Considering additive noise, we base its construction on the related nonlinear random partial differential equation, which is solved by a transform of the solution of the stochastic Navier–Stokes equations. We show a *strong* rate (up to) 1 in probability for a corresponding discretisation in space and time (and space-time).

Keywords: stochastic Navier–Stokes equations; error in probability; space-time discretisation; convergence rates.

1 Introduction

We are concerned with the numerical approximation of the two-dimensional stochastic Navier–Stokes equations in a bounded smooth domain $\mathcal{O} \subset \mathbb{R}^2$ with no-slip boundary conditions. They describe the flow of a homogeneous incompressible fluid in terms of the velocity field \mathbf{u} and pressure function p —which are both defined on a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P})$, and read as

$$\begin{cases} \mathbf{d}\mathbf{u} = \mu \Delta \mathbf{u} dt - (\nabla \mathbf{u})\mathbf{u} dt - \nabla p dt + \Phi dW & \text{in } \mathcal{Q}_T, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \mathcal{Q}_T, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \mathcal{O}, \end{cases} \quad (1.1)$$

\mathbb{P} -a.s. in $\mathcal{Q}_T := (0, T) \times \mathcal{O}$, with terminal time $T > 0$, the viscosity $\mu > 0$ and \mathbf{u}_0 a given initial datum. The momentum equation is driven by a cylindrical Wiener process W and the diffusion coefficient Φ takes values in the space of Hilbert–Schmidt operators; see Section 2.1 for details.

The existence, regularity and long-time behaviour of (probabilistically) strong solutions to (1.1) have been studied extensively over the last three decades, and we refer to [Kuksin & Shirikyan \(2012\)](#) for a complete picture. In most of the available numerical works for (1.1) that address *strong* convergence analysis (we refer to e.g., [Carelli & Prohl, 2012](#); [Bessaih & Millet, 2019, 2022](#); [Breit & Dodgson, 2021](#)), periodic boundary conditions are assumed. This enables $\int (\nabla \mathbf{u})\mathbf{u} \cdot \Delta \mathbf{u} dx = 0$ as a key property to validate higher moment bounds for the solution of (1.1) in strong spatial norms; for this and related regularity properties we refer to [Capiński & Cutland \(1991\)](#), [Capiński \(1993\)](#), [Kuksin & Shirikyan \(2012\)](#). The convergence analysis for related discretisation schemes then has to cope with the interplay

of the quadratic nonlinearity with the (possibly multiplicative) noise term in (1.1): without the noise term, a standard Gronwall argument suffices to validate optimal convergence rates for a (finite element-based) space-time discretisation of (1.1); see e.g. Heywood & Rannacher (1982). This argument may not be applied directly any more in the presence of noise, where only *moments* of solutions of (1.1) are available (and evenly so for a stable space-time discretisation), rather than *uniform* bounds with respect to $\omega \in \Omega$: in Carelli & Prohl (2012) the convergence analysis therefore separately estimates error amplification through the nonlinearity in (3.5) on ‘large’ nested local sets $(\Omega_{t_{m-1}})_m \subset \Omega$, and proves ‘smallness’ of errors with the help of discrete stability properties on the complementary ‘small’ sets $(\Omega \setminus \Omega_{t_{m-1}})_m$. This approach has been further refined in Bessaih & Millet (2019, 2020, 2022), Breit & Dodgson (2021).

Bessaih & Millet (2019, 2020, 2022), Breit & Dodgson (2021), Carelli & Prohl (2012) mainly address (1.1) with (Lipschitz) multiplicative noise and use the concept of (*strong*) *orders in probability* for the error analysis of related space-time discretisations; this concept is due to the nonlinear character of the problem, the effect of which is controlled along (families of) local subsets of Ω as mentioned above. We remark that it is weaker compared to *mean-square convergence*, which is usually employed to bound discretisation errors for linear stochastic partial differential equations (PDEs). A parallel numerical program for (1.1) when the noise is *additive* started in Bessaih & Millet (2019, 2020, 2022): these works address the question whether improved convergence results may be obtained for the same schemes, by exploiting additionally available improved stability properties for this data setting (‘exponential moment bounds’). It turns out that this is the case, provided that the strength of noise is small with respect to the viscosity μ : the first result is Bessaih & Millet (2020, Thm. 3.4), which proves a *mean-square convergence* rate, which crucially depends on the ratio of μ , and the strength of noise: for example, the latter is limited to be of order $\mathcal{O}(\mu)$ for $\mu \ll 1$; another result in this direction is asymptotic mean-square convergence order (up to $\frac{1}{2}$) in Bessaih & Millet (2022, Thm. 3.3), provided that the strength of noise is of order $\mathcal{O}(\mu)$.

In this work we provide a new temporal (see (3.5)) and spatio-temporal (see (4.1)–(4.2)) discretisation for (1.1) with additive noise, which is based on the reformulation (3.2) as *random PDE* for the transform $\mathbf{y} = \mathbf{u} - \Phi W$. The motivation is therefore to settle improved convergence behaviour of approximates for \mathbf{y} , which has improved time regularity properties: we will see in Lemma 3.1 that \mathbf{y} is differentiable in time, while \mathbf{u} is only Hölder continuous, with exponent $\alpha < \frac{1}{2}$. In fact, we show strong rate (up to) 1 in probability for (3.5)—which doubles the order in Breit & Dodgson (2021) for the space-time discretisation in Carelli & Prohl (2012) for (1.1) with multiplicative noise; see Theorem 3.6 for semidiscretisation (3.5), and the generalisation of this result in Section 4 for the spatio-temporal discretisation (4.1)–(4.2). To our knowledge, an (up to) first-order temporal discretisation for a nonlinear stochastic partial differential equation (SPDE) with non-Lipschitz nonlinearity was not available in the literature before, and we expect that the conceptual steps for its construction and convergence analysis given in the sections below are applicable to other nonlinear SPDEs with additive noise as well. The error analysis below is performed in the presence of homogeneous Dirichlet boundary data in (1.1), and driving trace-class, solenoidal noise, which vanishes on the boundary. We emphasise that our approach does not require any smallness condition on the noise. It is also worth pointing out that our convergence results improve to a *mean-square convergence* rate 1 for the stochastic Stokes equations with additive noise; see Remark 4.2.

We recall that $\int (\nabla \mathbf{u}) \mathbf{u} \cdot \Delta \mathbf{u} \, dx = 0$ does not hold any more for this boundary value problem for (1.1), and also higher moment bounds for \mathbf{u} in strong norms seem beyond reach, which then affects the efficiency of the strategy of proof in Bessaih & Millet (2019, 2020, 2022), Breit & Dodgson (2021), Carelli & Prohl (2012), which separately studies errors on ‘large’ nested local sets $(\Omega_{t_{m-1}})_m \subset \Omega$,

and small ones. Therefore, a different strategy to verify rates with the help of (discrete) stopping times and truncated nonlinearities was developed in Breit & Prohl (2022) for (a spatial discretisation of) (2.6) and iterates $(\mathbf{u}_m)_m$, which was motivated by the analytical works by Glatt-Holz & Ziane (2009), Mikulevicius (2009). We use related concepts in Section 3 to establish stability and convergence rates for (solenoidal) iterates $(\mathbf{y}_m)_m$ with zero boundary data from the temporal semidiscretisation (3.5), which satisfy $\mathbf{y}_m = \mathbf{u}_m - \Phi W(t_m)$. It is worth pointing out that we work here under more restrictive assumptions on the noise compared to our previous paper (Breit & Prohl, 2022); this allows us to prove higher-order estimates for the temporal approximations in Lemma 2.8, which turns out to be useful for the error analysis of $(\mathbf{y}_m)_m$ from (3.5). For this it is crucial to assume that the diffusion coefficient vanishes on $\partial\mathcal{O}$. Similarly, the analysis in Section 3.1 heavily relies on the solenoidal character of the noise (conceptionally, this assumption can be dropped in the case of periodic boundary conditions at least as far as the purely temporal discretisation is concerned).

Section 4 then proposes the Ladyzhenskaya-Babuska-Brezzi-stable mixed finite element scheme (4.1)–(4.2) for iterates $(\mathbf{Y}_m)_m$, for which we quantify the error $\max_m \|\mathbf{y}_m - \mathbf{Y}_m\|_{L_x^2}$ in particular: its proof rests on essentially considering a ‘finite element version’ $(\mathbf{U}_m)_m$ in (4.4) of $(\mathbf{u}_m)_m$, which solves (4.5) as a spatial discretisation of (2.6). To proceed like this here, i.e., to focus on the sequence $(\mathbf{U}_m)_m$ rather than $(\mathbf{Y}_m)_m$, is justified for the *spatial* error analysis part, where the limited temporal regularity of the driving Wiener process does not spoil the order. The main result is then given in Theorem 4.1, which rests on discrete stability results for (2.6) that are collected in Section 2.4.

REMARK 1.1 The research in this paper is to some extent motivated by the weak error analysis where one expects twice the rate from the strong error. A space-time discretization of a simplified stochastic Navier–Stokes system with a finite-dimensional noise without spatial dependence on the two-dimensional torus is considered in Dörsek (2012), and weak error estimates are obtained by semigroup methods: in a first step, (strong) optimal error estimates are shown for the spectral Galerkin method; then, its time discretisation by Strang’s operator splitting method is studied, where weak error bounds in Dörsek (2012, Cor. 4.3) contain restrictive couplings of spatio-temporal discretization scales.

2 Mathematical framework

2.1 Probability set-up

Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The process W is a cylindrical Wiener process, i.e., $W(t) = \sum_{k \geq 1} \beta_j(t) e_j$ with $(\beta_j)_{j \geq 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathfrak{F}_t)_{t \geq 0}$, and $(e_j)_{j \geq 1}$ a complete orthonormal system in a separable Hilbert space \mathfrak{U} . We assume that the diffusion coefficient Φ belongs to the set of Hilbert–Schmidt operators $L_2(\mathfrak{U}; \mathbb{H})$, where \mathbb{H} can take the role of various Hilbert spaces. Of particular importance are spaces of solenoidal vector fields, such as $L_{\text{div}}^2(\mathcal{O}; \mathbb{R}^2)$ and $W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)$. They are defined as the closure of the solenoidal $C_c^\infty(\mathcal{O}; \mathbb{R}^2)$ -functions¹ with respect to the $L^2(\mathcal{O}; \mathbb{R}^2)$ - or $W_0^{1,2}(\mathcal{O}; \mathbb{R}^2)$ -norm, respectively. Given $\Phi \in L^2(\mathfrak{U}; L_{\text{div}}^2(\mathcal{O}; \mathbb{R}^2))$ the stochastic integral

$$t \mapsto \int_0^t \Phi \, dW$$

¹ We denote this space by $C_{c,\text{div}}^\infty(\mathcal{O}; \mathbb{R}^2)$ in the following.

is a well-defined process taking values in $L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2)$ (see Da Prato & Zabczyk, 1992 for a detailed construction). Moreover, we can multiply by test functions to obtain

$$\left\langle \int_0^t \Phi \, dW, \phi \right\rangle_{L^2_x} = \sum_{j \geq 1} \int_0^t \langle \Phi e_j, \phi \rangle_{L^2_x} \, d\beta_j \quad \forall \phi \in L^2(\mathcal{O}; \mathbb{R}^2).$$

Similarly, we can define stochastic integrals with values in $W^{1,2}_{0,\text{div}}(\mathcal{O}; \mathbb{R}^2)$ and $W^{2,2}(\mathcal{O}; \mathbb{R}^2)$, respectively, if Φ belongs to the corresponding class of Hilbert–Schmidt operators.

2.2 The concept of solutions

In dimension two, pathwise uniqueness for analytically weak solutions of (1.1) is known; we refer the reader for instance to Capiński & Cutland (1991), Capiński (1993). Consequently, we may work with the definition of a weak pathwise solution.

DEFINITION 2.1 Suppose that $\Phi \in L_2(\mathfrak{H}; L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2))$. Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ be a given stochastic basis with a complete right-continuous filtration and an (\mathfrak{F}_t) -cylindrical Wiener process W . Let \mathbf{u}_0 be an \mathfrak{F}_0 -measurable random variable with values in $L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2)$. Then \mathbf{u} is called a *weak pathwise solution* to (1.1) with the initial condition \mathbf{u}_0 provided

1. the velocity field \mathbf{u} is (\mathfrak{F}_t) -adapted and

$$\mathbf{u} \in C([0, T]; L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2)) \cap L^2(0, T; W^{1,2}_{0,\text{div}}(\mathcal{O}; \mathbb{R}^2)) \quad \mathbb{P}\text{-a.s.},$$

2. the momentum equation²

$$\begin{aligned} & \int_{\mathcal{O}} \mathbf{u}(t) \cdot \boldsymbol{\varphi} \, dx - \int_{\mathcal{O}} \mathbf{u}_0 \cdot \boldsymbol{\varphi} \, dx \\ &= \int_0^t \int_{\mathcal{O}} \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \, dx \, dt - \mu \int_0^t \int_{\mathcal{O}} \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \, dx \, ds + \int_{\mathcal{O}} \int_0^t \Phi \cdot \boldsymbol{\varphi} \, dW \, dx \end{aligned}$$

holds \mathbb{P} -a.s. for all $\boldsymbol{\varphi} \in W^{1,2}_{0,\text{div}}(\mathcal{O}; \mathbb{R}^2)$ and all $t \in [0, T]$.

THEOREM 2.2 Suppose that $\Phi \in L_2(\mathfrak{H}; L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2))$. Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ be a given stochastic basis with a complete right-continuous filtration and an (\mathfrak{F}_t) -cylindrical Wiener process W . Let \mathbf{u}_0 be an \mathfrak{F}_0 -measurable random variable such that $\mathbf{u}_0 \in L^r(\Omega; L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2))$ for some $r > 2$. Then there exists a unique weak pathwise solution to (1.1) in the sense of Definition 2.1 with the initial condition \mathbf{u}_0 .

We give the definition of a strong pathwise solution to (1.1), which exists up to a stopping time \mathfrak{t} . The velocity field belongs \mathbb{P} -a.s. to $C([0, \mathfrak{t}]; W^{1,2}_{0,\text{div}}(\mathcal{O}; \mathbb{R}^2))$.

DEFINITION 2.3 Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ be a given stochastic basis with a complete right-continuous filtration and an (\mathfrak{F}_t) -cylindrical Wiener process W . Let \mathbf{u}_0 be an \mathfrak{F}_0 -measurable random variable with

² We denote by $\mathbf{A} : \mathbf{B}$ the inner product between two 2×2 -matrices \mathbf{A}, \mathbf{B} .

values in $W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)$. The tuple $(\mathbf{u}, \mathfrak{t})$ is called a *local strong pathwise solution* to (1.1) with the initial condition \mathbf{u}_0 provided

1. \mathfrak{t} is a \mathbb{P} -a.s. strictly positive (\mathfrak{F}_t) -stopping time;
2. the velocity field \mathbf{u} is (\mathfrak{F}_t) -adapted and

$$\mathbf{u}(\cdot \wedge \mathfrak{t}) \in C_{\text{loc}}([0, \infty); W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)) \cap L_{\text{loc}}^2(0, \infty; W^{2,2}(\mathcal{O}; \mathbb{R}^2)) \quad \mathbb{P}\text{-a.s.},$$

3. the momentum equation

$$\begin{aligned} & \int_{\mathcal{O}} \mathbf{u}(\tau \wedge \mathfrak{t}) \cdot \boldsymbol{\varphi} \, dx - \int_{\mathcal{O}} \mathbf{u}_0 \cdot \boldsymbol{\varphi} \, dx \\ &= - \int_0^{\tau \wedge \mathfrak{t}} \int_{\mathcal{O}} (\nabla \mathbf{u}) \mathbf{u} \cdot \boldsymbol{\varphi} \, dx \, ds + \mu \int_0^{\tau \wedge \mathfrak{t}} \int_{\mathcal{O}} \Delta \mathbf{u} \cdot \boldsymbol{\varphi} \, dx \, ds + \int_{\mathcal{O}} \int_0^{\tau \wedge \mathfrak{t}} \Phi \cdot \boldsymbol{\varphi} \, dW \, dx \end{aligned} \tag{2.1}$$

holds \mathbb{P} -a.s. for all $\boldsymbol{\varphi} \in C_{c,\text{div}}^\infty(\mathcal{O}; \mathbb{R}^2)$ and all $\tau \geq 0$.

We finally define a maximal strong pathwise solution.

DEFINITION 2.4 (Maximal strong pathwise solution). Fix a stochastic basis with a cylindrical Wiener process and an initial condition as in Definition 2.3. A triplet

$$(\mathbf{u}, (\mathfrak{t}_R)_{R \in \mathbb{N}}, \mathfrak{t})$$

is a maximal strong pathwise solution to system (1.1) provided

1. \mathfrak{t} is a \mathbb{P} -a.s. strictly positive (\mathfrak{F}_t) -stopping time;
2. $(\mathfrak{t}_R)_{R \in \mathbb{N}}$ is an increasing sequence of (\mathfrak{F}_t) -stopping times such that $\mathfrak{t}_R < \mathfrak{t}$ on the set $[\mathfrak{t} < \infty]$, $\lim_{R \rightarrow \infty} \mathfrak{t}_R = \mathfrak{t}$ \mathbb{P} -a.s. and

$$\mathfrak{t}_R := \inf \{t \in [0, \infty) : \|\mathbf{u}(t)\|_{W_x^{1,2}} \geq R\} \quad \text{on } [\mathfrak{t} < \infty], \tag{2.2}$$

with the convention that $\mathfrak{t}_R = \infty$ if the set above is empty;

3. each triplet $(\mathbf{u}, \mathfrak{t}_R)$, $R \in \mathbb{N}$ is a local strong pathwise solution in the sense of Definition 2.3.

We talk about a global solution if we have (in the framework of Definition 2.4) $\mathfrak{t} = \infty$ \mathbb{P} -a.s. The following result is shown in Mikulevicius (2009) (see also Glatt-Holz & Ziane, 2009 for a similar statement).

THEOREM 2.5 Suppose that $\Phi \in L_2(\mathcal{U}; W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$ and $\mathbf{u}_0 \in L^2(\Omega, W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$. Then there is a unique global maximal strong pathwise solution to (1.1) in the sense of Definition 2.4.

2.3 Regularity of solutions

The following lemma is a special case of Breit & Prohl (2022, Lemma 3.1), where multiplicative noise is also considered.

LEMMA 2.6

- (a) Assume that $\mathbf{u}_0 \in L^r(\Omega, L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2))$ for some $r \geq 2$ and that $\Phi \in L_2(\mathfrak{U}; L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2))$. Then we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \int_{\mathcal{O}} |\mathbf{u}(t)|^2 dx + \int_0^T \int_{\mathcal{O}} |\nabla \mathbf{u}|^2 dx dt \right]^{\frac{r}{2}} \leq c \mathbb{E} \left[1 + \|\mathbf{u}_0\|_{L^2_x}^2 \right]^{\frac{r}{2}}, \tag{2.3}$$

where \mathbf{u} is the weak pathwise solution to (1.1); cf. Definition 2.1.

- (b) Assume that $\mathbf{u}_0 \in L^r(\Omega, W^{1,2}_{0,\text{div}}(\mathcal{O}; \mathbb{R}^2))$ for some $r \geq 2$ and $\Phi \in L_2(\mathfrak{U}; W^{1,2}_{0,\text{div}}(\mathcal{O}; \mathbb{R}^2))$. Then we have

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} \int_{\mathcal{O}} |\nabla \mathbf{u}(t \wedge t_R)|^2 dx + \int_0^{T \wedge t_R} \int_{\mathcal{O}} |\nabla^2 \mathbf{u}|^2 dx dt \right]^{\frac{r}{2}} \\ \leq cR^r \mathbb{E} \left[1 + \|\mathbf{u}_0\|_{W^{1,2}_x}^2 \right]^{\frac{r}{2}}, \end{aligned} \tag{2.4}$$

where $(\mathbf{u}, (t_R)_{R \in \mathbb{N}}, \mathfrak{t})$ is the maximal strong pathwise solution to (1.1); cf. Definition 2.4.

- (c) Assume that $\mathbf{u}_0 \in L^r(\Omega, W^{2,2}(\mathcal{O}; \mathbb{R}^2)) \cap L^{5r}(\Omega, W^{1,2}_{0,\text{div}}(\mathcal{O}; \mathbb{R}^2))$ for some $r \geq 2$ and $\Phi \in L_2(\mathfrak{U}; W^{1,2}_{0,\text{div}} \cap W^{2,2}(\mathcal{O}; \mathbb{R}^2))$. Then we have

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} \int_{\mathcal{O}} |\nabla^2 \mathbf{u}(t \wedge t_R)|^2 dx + \int_0^{T \wedge t_R} \int_{\mathcal{O}} |\nabla^3 \mathbf{u}|^2 dx dt \right]^{\frac{r}{2}} \\ \leq cR^r \mathbb{E} \left[1 + \|\mathbf{u}_0\|_{W^{2,2}_x}^2 \right]^{\frac{r}{2}}, \end{aligned} \tag{2.5}$$

where $(\mathbf{u}, (t_R)_{R \in \mathbb{N}}, \mathfrak{t})$ is the maximal strong pathwise solution to (1.1); cf. Definition 2.4.

Here $c = c(r, T, \Phi)$ is independent of R .

2.4 Estimates for the time-discrete solution

We now consider a temporal approximation of (1.1) on an equidistant partition of $[0, T]$ with mesh size $\tau = T/M$, and set $t_m = m\tau$. Let \mathbf{u}_0 be an \mathfrak{F}_0 -measurable random variable with values in $L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2)$. For $1 \leq m \leq M$, we aim at constructing iteratively a sequence of \mathfrak{F}_{t_m} -measurable random variables \mathbf{u}_m with values in $W^{1,2}_{0,\text{div}}(\mathcal{O}; \mathbb{R}^2)$ such that for every $\phi \in W^{1,2}_{0,\text{div}}(\mathcal{O}; \mathbb{R}^2)$ it holds true \mathbb{P} -a.s. that

$$\begin{aligned} \int_{\mathcal{O}} \mathbf{u}_m \cdot \phi dx - \tau \int_{\mathcal{O}} \mathbf{u}_m \otimes \mathbf{u}_{m-1} : \nabla \phi dx \\ = -\mu\tau \int_{\mathcal{O}} \nabla \mathbf{u}_m : \nabla \phi dx + \int_{\mathcal{O}} \mathbf{u}_{m-1} \cdot \phi dx + \int_{\mathcal{O}} \Phi \Delta_m W \cdot \phi dx, \end{aligned} \tag{2.6}$$

where $\Delta_m W = W(t_m) - W(t_{m-1})$. For given \mathbf{u}_{m-1} and $\Delta_m W$, to verify the existence of a unique \mathbf{u}_m solving (2.6) is straightforward since it is linear in \mathbf{u}_m . Hence we can rewrite (2.6) as

$$\mathbf{u}_m + \tau \mathcal{P}(\nabla \mathbf{u}_m) \mathbf{u}_{m-1} = \mu \tau \mathcal{A} \mathbf{u}_m + \mathbf{u}_{m-1} + \Phi \Delta_m W \quad (1 \leq m \leq M), \tag{2.7}$$

an identity that holds \mathbb{P} -a.s. in $L^2(\mathcal{O}; \mathbb{R}^2)$. Here $\mathcal{P} : L^2(\mathcal{O}; \mathbb{R}^2) \rightarrow L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2)$ denotes the Helmholtz projection and $\mathcal{A} = \mathcal{P} \Delta$ is the Stokes operator.

LEMMA 2.7 Assume that $\mathbf{u}_0 \in L^{2q}(\Omega, L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2))$ for some $q \in \mathbb{N}$. Suppose that $\Phi \in L_2(\mathfrak{U}; L^2_{\text{div}}(\mathcal{O}; \mathbb{R}^2))$. Then the iterates $(\mathbf{u}_m)_{m=1}^M$ given by (2.6) satisfy the following estimate uniformly in M :

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|\mathbf{u}_m\|_{L^2_x}^{2q} + \tau \sum_{m=1}^M \|\mathbf{u}_m\|_{L^2_x}^{2q-2} \|\nabla \mathbf{u}_m\|_{L^2_x}^2 \right] \leq c, \tag{2.8}$$

where $c = c(q, T, \Phi, \mathbf{u}_0) > 0$.

Proof. The proof of (2.8) is identical to Brzeźniak *et al.* (2013, Lemma 3.1). The latter is for periodic boundary conditions, but in the case of no-slip boundary conditions the same arguments apply. Also note that we consider a semi-implicit algorithm, which again does not impact the proof since the convective term still cancels when testing with \mathbf{u}_m . \square

For $R_1 > 0$, we define the (discrete) (\mathfrak{F}_{t_m}) -stopping time

$$\mathfrak{s}_{R_1}^d := \min_{0 \leq m \leq M} \left\{ t_m : \sum_{n=0}^m \tau \|\mathbf{u}_n\|_{L^2_x}^2 \|\nabla \mathbf{u}_n\|_{L^2_x}^2 \geq R_1^4 \right\}.$$

We set $\mathfrak{s}_{R_1}^d = t_M$ if the set above is empty. Note that $\mathfrak{s}_{R_1}^d \in \{t_m\}_{m=0}^M$, with random index $j_{R_1} \in \mathbb{N}_0 \cap [0, M]$, such that $\mathfrak{s}_{R_1}^d = t_{j_{R_1}}$. Now we use the discrete energy estimate (2.8) (for $q = 2$), and Markov’s inequality to validate (assuming that $\mathbf{u}_0 \in L^4(\Omega, W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$)

$$\begin{aligned} \mathbb{P}[\mathfrak{s}_{R_1}^d < T] &\leq \mathbb{P} \left[\sum_{n=0}^M \tau \|\mathbf{u}_n\|_{L^2_x}^2 \|\nabla \mathbf{u}_n\|_{L^2_x}^2 \geq R_1^4 \right] \\ &\leq \frac{1}{R_1^4} \mathbb{E} \left[\sum_{n=0}^M \tau \|\mathbf{u}_n\|_{L^2_x}^2 \|\nabla \mathbf{u}_n\|_{L^2_x}^2 \right] \leq \frac{c}{R_1^4}, \end{aligned} \tag{2.9}$$

and thus

$$\lim_{R_1 \rightarrow \infty} \mathbb{P}[\mathfrak{s}_{R_1}^d < T] = 0. \tag{2.10}$$

Similarly to Lemma 2.6, higher-order estimates can only be achieved with the help of a stopping time. We now derive some uniform estimates for the solution of the time-discrete problem, which hold up to the discrete stopping time $\mathfrak{s}_{R_1}^d$ with $R_1 > 0$.

LEMMA 2.8 Assume that $\mathbf{u}_0 \in L^{2q}(\Omega, W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$ for some $q \in \mathbb{N}$. Suppose that $\Phi \in L_2(\mathfrak{A}; W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$. Then the iterates $(\mathbf{u}_m)_{m=1}^M$ given by (2.6) satisfy the following estimates uniformly in M :

$$\mathbb{E} \left[\max_{1 \leq m \leq j_{R_1}} \|\mathbf{u}_m\|_{W_x^{1,2}}^{2q} + \sum_{m=1}^{j_{R_1}} \tau \|\mathbf{u}_m\|_{W_x^{1,2}}^{2q-2} \|\nabla^2 \mathbf{u}_m\|_{L_x^2}^2 \right] \leq c e^{cR_1^4}, \tag{2.11}$$

where $c = c(q, T, \Phi, \mathbf{u}_0) > 0$ is independent of R_1 .

Proof. We proceed formally; a rigorous proof can be obtained using a Galerkin approximation. In order to prove (2.11) we multiply (2.6) by $\mathcal{A}\mathbf{u}_m$ and integrate in space yielding

$$\begin{aligned} & \int_{\mathcal{O}} \nabla(\mathbf{u}_m - \mathbf{u}_{m-1}) : \nabla \mathbf{u}_m \, dx + \mu \tau \int_{\mathcal{O}} |\mathcal{A}\mathbf{u}_m|^2 \, dx \\ &= -\tau \int_{\mathcal{O}} (\nabla \mathbf{u}_m) \mathbf{u}_{m-1} \cdot \mathcal{A}\mathbf{u}_m \, dx + \int_{\mathcal{O}} \mathcal{A}\mathbf{u}_m \cdot \Phi \Delta_m W \, dx. \end{aligned}$$

For $\delta > 0$ we estimate

$$\begin{aligned} \int_{\mathcal{O}} (\nabla \mathbf{u}_m) \mathbf{u}_{m-1} \cdot \mathcal{A}\mathbf{u}_m &\leq \|\mathbf{u}_{m-1}\|_{L_x^4} \|\nabla \mathbf{u}_m\|_{L_x^4} \|\mathcal{A}\mathbf{u}_m\|_{L_x^2} \\ &\leq c \|\mathbf{u}_{m-1}\|_{L_x^2}^{\frac{1}{2}} \|\nabla \mathbf{u}_{m-1}\|_{L_x^2}^{\frac{1}{2}} \|\nabla \mathbf{u}_m\|_{L_x^2}^{\frac{1}{2}} \|\mathcal{A}\mathbf{u}_m\|_{L_x^2}^{\frac{3}{2}} \\ &\leq c(\delta) \|\mathbf{u}_{m-1}\|_{L_x^2}^2 \|\nabla \mathbf{u}_{m-1}\|_{L_x^2}^2 \|\nabla \mathbf{u}_m\|_{L_x^2}^2 + \delta \|\mathcal{A}\mathbf{u}_m\|_{L_x^2}^2. \end{aligned} \tag{2.12}$$

Summing then shows

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{O}} |\nabla \mathbf{u}_m|^2 \, dx + \frac{1}{2} \sum_{n=1}^m \int_{\mathcal{O}} |\nabla(\mathbf{u}_n - \mathbf{u}_{n-1})|^2 \, dx + \frac{\mu}{2} \sum_{n=1}^m \tau \int_{\mathcal{O}} |\mathcal{A}\mathbf{u}_n|^2 \, dx \\ &\leq \frac{1}{2} \int_{\mathcal{O}} |\nabla \mathbf{u}_0|^2 \, dx + c \sum_{n=1}^m \tau \|\mathbf{u}_{n-1}\|_{L_x^2}^2 \|\nabla \mathbf{u}_{n-1}\|_{L_x^2}^2 \|\nabla \mathbf{u}_n\|_{L_x^2}^2 \\ &\quad + \mathcal{M}^1(t_m) + \mathcal{M}_m^2, \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}^1(t) &= \int_{\mathcal{O}} \int_0^t \sum_{n=1}^M \mathbf{1}_{[t_{n-1}, t_n)} \Phi \, dW \cdot \mathcal{A}\mathbf{u}_{n-1} \, dx, \\ \mathcal{M}_m^2 &= \sum_{n=1}^m \int_{\mathcal{O}} \int_{t_{n-1}}^{t_n} \Phi \, dW \cdot \mathcal{A}(\mathbf{u}_n - \mathbf{u}_{n-1}) \, dx. \end{aligned}$$

By the discrete Gronwall lemma we have \mathbb{P} -a.s.

$$\begin{aligned} & \frac{1}{2} \max_{1 \leq m \leq j_{R_1}} \int_{\mathcal{O}} |\nabla \mathbf{u}_m|^2 \, dx + \frac{1}{2} \sum_{n=1}^{j_{R_1}} \int_{\mathcal{O}} |\nabla(\mathbf{u}_n - \mathbf{u}_{n-1})|^2 \, dx + \frac{\mu}{2} \sum_{n=1}^{j_{R_1}} \tau \int_{\mathcal{O}} |\mathcal{A}\mathbf{u}_n|^2 \, dx \\ & \leq ce^{cR_1^2} \left(\int_{\mathcal{O}} |\nabla \mathbf{u}_0|^2 \, dx + \max_{1 \leq m \leq j_{R_1}} |\mathcal{M}^1(t_m)| + \max_{1 \leq m \leq j_{R_1}} |\mathcal{M}_m^2| \right) \end{aligned}$$

using that

$$\sum_{n=1}^{j_{R_1}} \tau \|\mathbf{u}_{n-1}\|_{L_x^2}^2 \|\nabla \mathbf{u}_{n-1}\|_{L_x^2}^2 \leq R_1^4$$

by the definition of j_{R_1} . Since \mathbf{u}_{m-1} is $\mathfrak{F}_{t_{m-1}}$ -measurable and $s_{R_1}^d$ is an (\mathfrak{F}_{t_m}) -stopping time we know that $\mathcal{M}^1(t \wedge s_{R_1}^d)$ is an (\mathfrak{F}_t) -martingale. Consequently, by the Burkholder–Davis–Gundy inequality, $\Phi \in L_2(\mathfrak{U}; W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$ and Young’s inequality, and for $\kappa > 0$,

$$\begin{aligned} \mathbb{E} \left[\max_{1 \leq m \leq j_{R_1}} |\mathcal{M}^1(t_m)| \right] & \leq \mathbb{E} \left[\max_{s \in [0, T]} |\mathcal{M}^1(s \wedge s_{R_1}^d)| \right] \\ & \leq c \mathbb{E} \left[\left(\int_0^{T \wedge s_{R_1}^d} \sum_{n=1}^M \mathbf{1}_{[t_{n-1}, t_n)} \|\Phi\|_{L_2(\mathfrak{U}, L_x^2)}^2 \|\mathcal{A}\mathbf{u}_{n-1}\|_{L_x^2}^2 \, d\sigma \right)^{\frac{1}{2}} \right] \\ & \leq c \mathbb{E} \left[\left(\sum_{n=1}^{j_{R_1}-1} \|\mathcal{A}\mathbf{u}_n\|_{L_x^2}^2 \right)^{\frac{1}{2}} \right] \\ & \leq c(\kappa) + \kappa \mathbb{E} \left[\sum_{n=1}^{j_{R_1}-1} \tau \|\mathcal{A}\mathbf{u}_n\|_{L_x^2}^2 \right]. \end{aligned} \tag{2.13}$$

Furthermore, we have

$$\begin{aligned} \mathbb{E} \left[\max_{1 \leq m \leq j_{R_1}} |\mathcal{M}_m^2| \right] & \leq \kappa \mathbb{E} \left[\sum_{n=1}^{j_{R_1}} \|\nabla(\mathbf{u}_n - \mathbf{u}_{n-1})\|_{L_x^2}^2 \right] + c_\kappa \mathbb{E} \left[\sum_{n=1}^{j_{R_1}} \left\| \int_{t_{n-1}}^{t_n} \nabla \Phi \, dW \right\|_{L_x^2}^2 \right] \\ & \leq \kappa \mathbb{E} \left[\sum_{n=1}^{j_{R_1}} \|\nabla(\mathbf{u}_n - \mathbf{u}_{n-1})\|_{L_x^2}^2 \right] + c_\kappa \mathbb{E} \left[\tau \sum_{n=1}^M \|\Phi\|_{L_2(\mathfrak{U}; W_x^{1,2})}^2 \right] \\ & \leq \kappa \mathbb{E} \left[\sum_{n=1}^{j_{R_1}} \|\nabla(\mathbf{u}_n - \mathbf{u}_{n-1})\|_{L_x^2}^2 \right] + c_\kappa \end{aligned} \tag{2.14}$$

due to Young’s inequality, Itô isometry and $\Phi \in L_2(\mathcal{U}; W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$. Absorbing the κ -terms we conclude (b) for $q = 1$. The case $q \geq 2$ follows similarly by multiplying with $\|\mathbf{u}_m\|_{W_x^{1,2}}^{2q-2}$ and iterating (see Brzeźniak *et al.*, 2013, Lemma 3.1 for details). \square

For $R_2 > 0$, we define the (discrete) (\mathfrak{F}_{t_m}) -stopping time

$$t_{R_2}^d := \min_{0 \leq m \leq M} \left\{ t_m : \|\nabla \mathbf{u}_m\|_{L_x^2}^2 + \sum_{n=1}^m \tau \|\nabla^2 \mathbf{u}_n\|_{L_x^2}^2 \geq R_2^2 \right\},$$

and set $t_{R_2}^d = t_M$ if the set above is empty. Note that $t_{R_2}^d \in \{t_m\}_{m=0}^M$, with random index $i_{R_2} \in \mathbb{N}_0 \cap [0, M]$, such that $t_{R_2}^d = t_{i_{R_2}}$. The crucial observation is now that

$$\lim_{R_1 \rightarrow \infty} \mathbb{P}[t_{R_2(R_1)}^d < T] = 0, \tag{2.15}$$

provided we choose $R_2 = R_2(R_1)$ such that $e^{cR_1^4} = o(R_2(R_1)^2)$. We argue as in Glatt-Holz & Ziane (2009, Sec. 4.3) to show (2.15) and estimate

$$\mathbb{P}[t_{R_2}^d < T] \leq \mathbb{P}[\{t_{R_2}^d < T\} \cap \{s_{R_1}^d \geq T\}] + \mathbb{P}[s_{R_1}^d < T]. \tag{2.16}$$

By the notation introduced before this lemma, the first term on the right-hand side of (2.16) can thus be bounded by

$$\begin{aligned} & \mathbb{P} \left[\left\{ \|\nabla \mathbf{u}_{i_{R_2}}\|_{L_x^2}^2 + \sum_{n=1}^{i_{R_2}} \tau \|\nabla^2 \mathbf{u}_n\|_{L_x^2}^2 \geq R_2^2 \right\} \cap \{s_{R_1}^d \geq T\} \right] \\ & \leq \mathbb{P} \left[\left\{ \|\nabla \mathbf{u}_{i_{R_2} \wedge j_{R_1}}\|_{L_x^2}^2 + \sum_{n=1}^{i_{R_2} \wedge j_{R_1}} \tau \|\nabla^2 \mathbf{u}_n\|_{L_x^2}^2 \geq R_2^2 \right\} \right]. \end{aligned} \tag{2.17}$$

By Markov’s inequality, the right-hand side is bounded by

$$\frac{1}{R_2^2} \mathbb{E} \left[\|\nabla \mathbf{u}_{i_{R_2} \wedge j_{R_1}}\|_{L_x^2}^2 + \sum_{n=1}^{i_{R_2} \wedge j_{R_1}} \tau \|\nabla^2 \mathbf{u}_n\|_{L_x^2}^2 \right].$$

Using Lemma 2.8 and choosing R_2 such that $e^{cR_1^4} = o(R_2^2)$, we may deduce

$$\lim_{R_2 \rightarrow \infty} \frac{1}{R_2^2} \mathbb{E} \left[\|\nabla \mathbf{u}_{i_{R_2} \wedge j_{R_1}}\|_{L_x^2}^2 + \sum_{n=1}^{i_{R_2} \wedge j_{R_1}} \tau \|\nabla^2 \mathbf{u}_n\|_{L_x^2}^2 \right] = 0, \tag{2.18}$$

and then (2.15) follows from (2.16).

Equation (2.6) has been stated for solenoidal test functions $\phi \in W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)$; for $\phi \in W_0^{1,2}(\mathcal{O}; \mathbb{R}^2)$, the additional term

$$-\tau \int_{\mathcal{O}} p_m \operatorname{div} \phi \, dx$$

would appear on the left-hand side of (2.6). The \mathbb{P} -a.s. unique solvability for $(\mathbf{u}_m, p_m) \in W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2) \times L^2(\mathcal{O})/\mathbb{R}$ follows from standard arguments. Moreover, exploiting that $\Phi \in L_2(\mathfrak{U}; L_{\text{div}}^2(\mathcal{O}; \mathbb{R}^2))$, we easily deduce

$$\mathbb{E} \left[\sum_{m=1}^{j_{R_1}} \tau \|\nabla p_m\|_{L_x^2}^2 \right] \leq c e^{cR_1^4}, \tag{2.19}$$

from (2.11), where $c = c(q, T, \Phi, \mathbf{u}_0) > 0$ is independent of R_1 : therefore, we start from (2.6) \mathbb{P} -a.s. in strong form,

$$\mathbf{u}_m - \mu \tau \Delta \mathbf{u}_m + \tau \operatorname{div}(\mathbf{u}_m \otimes \mathbf{u}_{m-1}) + \tau \nabla p_m = \mathbf{u}_{m-1} + \Phi \Delta_m W, \tag{2.20}$$

which we multiply with ∇p_m and integrate in space; a standard calculation then leads to

$$\tau \|\nabla p_m\|^2 \leq C \tau (\|\Delta \mathbf{u}_m\|^2 + \|\mathbf{u}_{m-1}\|_{L_x^2} \|\nabla \mathbf{u}_{m-1}\|_{L_x^2} \|\nabla \mathbf{u}_m\|_{L_x^2} \|\Delta \mathbf{u}_m\|_{L_x^2}) + \frac{\tau}{2} \|\nabla p_m\|^2.$$

Hence (2.11) settles (2.19).

REMARK 2.9

- (a) It is possible to prove Lemma 2.8 for nonlinear multiplicative noise also, provided suitable growth conditions for the derivatives of the diffusion coefficient with respect to the velocity field are assumed.
- (b) The assumption that the noise vanishes at the boundary is crucial for the estimates in Lemma 2.8. Otherwise we are unable to integrate by parts in \mathcal{M}_m^2 below (2.12), and it is unclear how to proceed. Interestingly, this limitation does not occur in the time-continuous counterpart of Lemma 2.8; see Breit & Prohl, (2022, Lemma 3.1). However, in Lemma 2.8 the noise does not need to be solenoidal; this assumption is only needed for estimate (2.19) and will become crucial in the next section.
- (c) Without the vanishing trace condition for Φ we are currently unable to control the error between the temporal discretisation and the spatio-temporal discretisation. On account of this in Breit & Prohl (2022) we studied directly the error between the exact solution and its space-time approximation.

3 Asymptotic strong first-order error bounds for the time discretisation

If \mathbf{u} is the weak pathwise solution defined on $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ with an (\mathfrak{F}_t) -cylindrical Wiener process W (recall Definition 2.1 and Theorem 2.2), we may consider the transform

$$\mathbf{y}(t) = \mathbf{u}(t) - \int_0^t \Phi \, dW(s) = \mathbf{u}(t) - \Phi W(t) \quad \forall t \geq 0. \tag{3.1}$$

We denote

$$\begin{aligned} \mathcal{L}_1^W(\mathbf{y}) &:= (\nabla[\Phi W])\mathbf{y}, & \mathcal{L}_2^W(\mathbf{y}) &:= (\nabla\mathbf{y})[\Phi W], \\ \mathcal{L}_3^W(t) &:= (\nabla[\Phi W])[\Phi W], & \mathcal{L}^W &:= \mathcal{L}_1^W + \mathcal{L}_2^W + \mathcal{L}_3^W. \end{aligned}$$

Then $\mathbf{y} : [0, T] \times \mathcal{O} \times \Omega \rightarrow \mathbb{R}^2$ solves the random PDE

$$\begin{cases} \partial_t \mathbf{y} = \mu \mathcal{A} \mathbf{y} - \mathcal{P}[(\nabla \mathbf{y}) \mathbf{y}] + \mu \mathcal{A}[\Phi W] - \mathcal{P}[\mathcal{L}^W(\mathbf{y})] & \text{in } \mathcal{Q}_T, \\ \operatorname{div} \mathbf{y} = 0 & \text{in } \mathcal{Q}_T, \\ \mathbf{y}(0) = \mathbf{u}_0 & \text{in } \mathcal{O}. \end{cases} \tag{3.2}$$

Note that for \mathbb{P} -a.a. $\omega \in \Omega$ the function $\mathbf{y}(\omega, \cdot)$ is a solution to the Navier–Stokes equations with right-hand side

$$\mathbf{f} := \mu \mathcal{A}[\Phi W] - \mathcal{P}[\mathcal{L}^W(\mathbf{y})].$$

Standard regularity results apply provided Φ is sufficiently regular. In particular, $\partial_t \mathbf{y} \in L^2(0, T; L^2_{\operatorname{div}}(\mathcal{O}; \mathbb{R}^2))$ holds \mathbb{P} -a.s.

LEMMA 3.1 Suppose $\mathbf{u}_0 \in L^r(\Omega; L^2_{\operatorname{div}}(\mathcal{O}; \mathbb{R}^2))$ for some $r > 2$ and $\Phi \in L_2(\mathfrak{U}; W^{1,2}_{0,\operatorname{div}}(\mathcal{O}; \mathbb{R}^2))$. Let \mathbf{u} be the unique weak pathwise solution to (1.1).

- (a) Assume additionally $\mathbf{u}_0 \in W^{1,2}_{0,\operatorname{div}}(\mathcal{O}; \mathbb{R}^2)$ \mathbb{P} -a.s. and $\Phi \in L_2(\mathfrak{U}; W^{2,2}(\mathcal{O}; \mathbb{R}^2))$. Then $\partial_t \mathbf{y} \in L^2(0, T; L^2_{\operatorname{div}}(\mathcal{O}; \mathbb{R}^2))$ \mathbb{P} -a.s. and for a.a. $t \in (0, T)$,

$$\int_{\mathcal{O}} |\partial_t \mathbf{y}|^2 \, dx \leq c \left[\|\Phi W\|^2_{W^{2,2}_x} + \|\mathbf{u}\|^2_{W^{2,2}_x} + \|\Phi W\|^4_{W^{1,2}_x} + \|\mathbf{u}\|^4_{W^{1,2}_x} \right]. \tag{3.3}$$

- (b) Assume additionally $\mathbf{u}_0 \in W^{2,2} \cap W^{1,2}_{0,\operatorname{div}}(\mathcal{O}; \mathbb{R}^2)$ \mathbb{P} -a.s. and $\Phi \in L_2(\mathfrak{U}; W^{3,2}(\mathcal{O}; \mathbb{R}^2))$. Then $\partial_t \mathbf{y} \in L^2(0, T; W^{1,2}_{\operatorname{div}}(\mathcal{O}; \mathbb{R}^2))$ \mathbb{P} -a.s. and

$$\int_{\mathcal{Q}_T} |\partial_t \nabla \mathbf{y}|^2 \, dx \, dt \leq c \int_0^T \left[\|\Phi W\|^2_{W^{3,2}_x} + \|\mathbf{u}\|^2_{W^{3,2}_x} + \|\Phi W\|^4_{W^{2,2}_x} + \|\mathbf{u}\|^4_{W^{2,2}_x} \right] dt. \tag{3.4}$$

Proof. The proof follows directly from (3.2)₁ by estimating the right-hand side in L^2_x and $W^{1,2}_x$ respectively. This only uses Ladyshenskaya’s inequality to estimate the quadratic terms. Note that our assumptions imply sufficient regularity of \mathbf{u} by Lemma 2.6. □

Combining Lemmas 3.1 and 2.6 we obtain the following.

COROLLARY 3.2 Assume that $\mathbf{u}_0 \in L^r(\Omega, W^{2,2}_{\operatorname{div}}(\mathcal{O}; \mathbb{R}^2)) \cap L^{5r}(\Omega, W^{1,2}_{0,\operatorname{div}}(\mathcal{O}; \mathbb{R}^2))$ for some $r \geq 2$ and that $\Phi \in L_2(\mathfrak{U}; W^{3,2}(\mathcal{O}; \mathbb{R}^2)) \cap L_2(\mathfrak{U}; W^{1,2}_{0,\operatorname{div}}(\mathcal{O}; \mathbb{R}^2))$. Let \mathbf{u} be the unique weak pathwise solution to (1.1). Then we have for any $R > 0$,

$$\mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \int_{\mathcal{O}} |\partial_t \mathbf{y}(t \wedge \tau_R)|^2 \, dx \right)^{\frac{r}{2}} \right] \leq c(T, \Phi, \mathbf{u}_0) R^{2r},$$

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^{t_R} \int_{\mathcal{O}} |\partial_t \nabla \mathbf{y}|^2 \, dx \, dt \right)^{\frac{r}{2}} \right] &\leq c(T, \Phi, \mathbf{u}_0) R^{2r}, \\ \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \int_{\mathcal{O}} |\nabla \mathbf{y}(t \wedge t_R)|^2 \, dx \right)^{\frac{r}{2}} \right] &\leq c(T, \Phi, \mathbf{u}_0) R^r. \end{aligned}$$

3.1 A stable time-discretisation

Due to the time-regularity of \mathbf{y} stated in Lemma 3.1 above we expect a strong order (up to) 1 in probability for the semiimplicit temporal discretisation of (1.1). We consider an equidistant partition of $[0, T]$ with mesh size $\tau = T/M$ and set $t_m = m\tau$. Let \mathbf{u}_0 be an \mathfrak{F}_0 -measurable random variable with values in $W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)$. Furthermore, we assume that $\Phi \in L_2(\mathfrak{U}; W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$. Given $\mathbf{y}_0 = \mathbf{u}_0$ we seek an \mathfrak{F}_{t_m} -measurable, $W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)$ -valued random variable \mathbf{y}_m ($1 \leq m \leq M$) such that

$$\begin{aligned} \frac{\mathbf{y}_m - \mathbf{y}_{m-1}}{\tau} - \mu \mathcal{A} \mathbf{y}_m &= \mu \mathcal{A}[\Phi W(t_m)] - \mathcal{P}[(\nabla \mathbf{y}_m) \mathbf{y}_{m-1}] + \mathcal{P}[\mathcal{L}^m(\mathbf{y}_{m-1}, \mathbf{y}_m)], \\ \text{where } \mathcal{L}^m(\mathbf{y}_{m-1}, \mathbf{y}_m) &= \mathcal{L}_1^{W(t_m)}(\mathbf{y}_{m-1}) + \mathcal{L}_2^{W(t_{m-1})}(\mathbf{y}_m) + \mathcal{L}_1^{W(t_m)}(\Phi W(t_{m-1})). \end{aligned} \tag{3.5}$$

This system can be written (for \mathbb{P} -a.a. $\omega \in \Omega$) as a steady Navier–Stokes problem with forcing $\mu \mathcal{A}[\Phi W(t_m)]$ perturbed by the linear term $\mathcal{P}[\mathcal{L}^m(\mathbf{y}_{m-1}, \mathbf{y}_m)]$. Clearly, we have a continuous dependence on \mathbf{y}_{m-1} , $W(t_{m-1})$ and $W(t_m)$, which yields the correct measurability. If also $\Phi \in L_2(\mathfrak{U}; W^{2,2}(\mathcal{O}; \mathbb{R}^2))$, we have more regularity and it holds that $\mathbf{y}_m \in W^{2,2}(\mathcal{O}; \mathbb{R}^2)$ \mathbb{P} -a.s. (and similarly for $W^{3,2}(\mathcal{O}; \mathbb{R}^2)$ instead of $W^{2,2}(\mathcal{O}; \mathbb{R}^2)$). Setting

$$\mathbf{u}_m := \mathbf{y}_m + \Phi W(t_m) \quad (1 \leq m \leq M), \tag{3.6}$$

accordingly gives with $\Delta_m W := W(t_m) - W(t_{m-1})$,

$$(\mathbf{u}_m - \mathbf{u}_{m-1}) - \mu \tau \mathcal{A} \mathbf{u}_m + \tau \mathcal{P}(\nabla \mathbf{u}_m) \mathbf{u}_{m-1} = \Phi \Delta_m W \quad (1 \leq m \leq M) \tag{3.7}$$

and $\mathbf{u}_0 = \mathbf{y}_0$, which was considered in Section 2.4. Inequality (3.10) of the following lemma is based on corresponding stability bounds for $(\mathbf{u}_m)_{m=1}^M$ from Lemma 2.8. We define

$$\tilde{t}_{R_1}^d := \min_{0 \leq m \leq M} \left\{ t_m \leq t_{R_1} : \sup_{t \in [0, t_m]} \|\Phi(W_t)\|_{W_x^{2,2}} \geq R_2(R_1) \right\} \wedge \mathfrak{s}_{R_1}^d \wedge t_{R_2(R_1)}^d, \tag{3.8}$$

where the minimum of the empty set is defined as t_M and $R_2(R_1)$ is chosen in accordance with (3.9) below. Finally, m_{R_1} denotes the unique index in $\{1, \dots, M\}$ with $t_{m_{R_1}} = \tilde{t}_{R_1}^d$.

Since $\Phi \in L_2(\mathfrak{U}; W^{3,2}(\mathcal{O}; \mathbb{R}^2))$ we clearly have

$$\mathbb{E} \left[\sup_{0 \leq t \leq t_M} \|\Phi W(t)\|_{W_x^{2,2}} \right] \leq c.$$

Hence we can control the size of $\{\tilde{t}_{R_1}^d < T\}$ by

$$\begin{aligned} \mathbb{P}[\{\tilde{t}_{R_1}^d < T\}] &\leq \mathbb{P}[\{t_{R_2(R_1)}^d < T\}] \\ &\quad + \mathbb{P}[\{s_{R_1}^d < T\}] + \mathbb{P}\left[\sup_{0 \leq t \leq t_M} \|\Phi W(t)\|_{W_x^{2,2}} \geq R_2(R_1)\right] \\ &\rightarrow 0, \end{aligned} \tag{3.9}$$

provided we choose R_2 such that $e^{cR_1^4} = o(R_2(R_1)^2)$ (recall (2.15)). Note that we have \mathbb{P} -a.s.

$$\begin{aligned} \sup_{0 \leq t \leq t_{m_{R_1}}} &\left(\|\Phi W(t)\|_{W_x^{2,2}}^2 + \|\mathbf{u}(t)\|_{W_x^{1,2}}^2 + \|\mathbf{y}(t)\|_{W_x^{1,2}}^2\right) \\ &\quad + \max_{1 \leq n \leq m_{R_1}-1} \|\nabla \mathbf{u}_n\|_{L_x^2}^2 + \sum_{n=0}^{m_{R_1}-1} \tau \|\nabla^2 \mathbf{u}_n\|_{L_x^2}^2 \\ &\quad + \max_{1 \leq n \leq m_{R_1}-1} \|\nabla \mathbf{y}_n\|_{L_x^2}^2 + \sum_{n=0}^{m_{R_1}-1} \tau \|\nabla^2 \mathbf{y}_n\|_{L_x^2}^2 \leq R_2(R_1)^2. \end{aligned}$$

LEMMA 3.3 Assume that $\mathbf{u}_0 \in L^{2q}(\Omega, W^{3,2}(\mathcal{O}; \mathbb{R}^2)) \cap L^{2q+2}(\Omega, W_{0,\text{div}}^{1,2}(\Omega; \mathbb{R}^2))$ for some $q \in \mathbb{N}$ and that $\Phi \in L_2(\mathfrak{U}; W^{3,2}(\mathcal{O}; \mathbb{R}^2)) \cap L_2(\mathfrak{U}; W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$. Then the iterates $(\mathbf{y}_m)_{m=1}^M$ given by (3.5) satisfy the following estimates uniformly in M :

$$\mathbb{E}\left[\max_{1 \leq m \leq m_{R_1}} \|\mathbf{y}_m\|_{W_x^{1,2}}^{2q} + \sum_{m=1}^{m_{R_1}} \tau \|\nabla \mathbf{y}_m\|_{L_x^2}^{2q-2} \|\mathcal{A}\mathbf{y}_m\|_{L_x^2}^2\right] \leq c e^{cR_2(R_1)^4}, \tag{3.10}$$

$$\mathbb{E}\left[\max_{1 \leq m \leq m_{R_1}} \|\mathbf{y}_m\|_{W_x^{2,2}}^{2q} + \sum_{m=1}^{m_{R_1}} \tau \|\mathcal{A}\mathbf{y}_m\|_{L_x^2}^{2q-2} \|\nabla \mathcal{A}\mathbf{y}_m\|_{L_x^2}^2\right] \leq c e^{cR_2(R_1)^4}, \tag{3.11}$$

$$\mathbb{E}\left[\sum_{m=1}^{m_{R_1}} \tau \|\mathbf{y}_m\|_{W_x^{2,2}}^{2q-4} \left\|\frac{\mathbf{y}_m - \mathbf{y}_{m-1}}{\tau}\right\|_{W_x^{1,2}}^2\right] \leq c e^{cR_2(R_1)^4} \quad (\text{for } q \geq 2), \tag{3.12}$$

where $c = c(q, T, \mathbf{u}_0) > 0$ is independent of R_1 .

Proof. **1.** Proof of (3.10). The proof is similar to the corresponding estimate (2.11) given in Lemma 2.8. Equation (3.5) can be rewritten as (see also equation (3.2))

$$\mathbf{y}_m - \mathbf{y}_{m-1} - \mu\tau \mathcal{A}\mathbf{y}_m = \mu\tau \mathcal{A}[\Phi W(t_m)] - \tau \mathcal{P}[(\nabla \mathbf{y}_m)\mathbf{y}_{m-1}] + \tau \mathcal{P}[\mathcal{L}^m(\mathbf{y}_{m-1}, \mathbf{y}_m)].$$

Multiplying by $\mathcal{A}\mathbf{y}_m$ and integrating in space yields

$$\begin{aligned} & \int_{\mathcal{O}} \nabla(\mathbf{y}_m - \mathbf{y}_{m-1}) : \nabla \mathbf{y}_m \, dx + \mu \tau \int_{\mathcal{O}} |\mathcal{A}\mathbf{y}_m|^2 \, dx \\ &= \tau \int_{\mathcal{O}} (\nabla \mathbf{y}_m) \mathbf{y}_{m-1} \cdot \mathcal{A}\mathbf{y}_m \, dx - \mu \tau \int_{\mathcal{O}} \mathcal{A}[\Phi W(t_m)] \cdot \mathcal{A}\mathbf{y} \, dx - \tau \int_{\mathcal{O}} \mathcal{L}^m(\mathbf{y}_{m-1}, \mathbf{y}_m) \cdot \mathcal{A}\mathbf{y}_m \, dx. \end{aligned}$$

As in (2.12) we have for $\delta > 0$,

$$\int_{\mathcal{O}} (\nabla \mathbf{y}_m) \mathbf{y}_{m-1} \cdot \mathcal{A}\mathbf{y}_m \, dx \leq c(\delta) \|\mathbf{y}_{m-1}\|_{L_x^2}^2 \|\nabla \mathbf{y}_{m-1}\|_{L_x^2}^2 \|\nabla \mathbf{y}_m\|_{L_x^2}^2 + \delta \|\mathcal{A}\mathbf{y}_m\|_{L_x^2}^2$$

such that for δ sufficiently small,

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{O}} |\nabla \mathbf{y}_m|^2 \, dx + \frac{1}{2} \int_{\mathcal{O}} |\nabla(\mathbf{y}_m - \mathbf{y}_{m-1})|^2 \, dx + \frac{\mu}{2} \tau \int_{\mathcal{O}} |\mathcal{A}\mathbf{y}_m|^2 \, dx \\ & \leq \frac{1}{2} \int_{\mathcal{O}} |\nabla \mathbf{y}_{m-1}|^2 \, dx + c \sum_{n=1}^m \tau \|\mathbf{y}_{n-1}\|_{L_x^2}^2 \|\nabla \mathbf{y}_{n-1}\|_{L_x^2}^2 \|\nabla \mathbf{y}_n\|_{L_x^2}^2 \\ & \quad + c \tau \int_{\mathcal{O}} |\mathcal{A}[\Phi W(t_m)]|^2 \, dx + c \tau \int_{\mathcal{O}} |\mathcal{L}^m(\mathbf{y}_{m-1}, \mathbf{y}_m)|^2 \, dx. \end{aligned} \tag{3.13}$$

Since $\Phi \in L_2(\mathcal{U}; W^{2,2}(\mathcal{O}))$ we have

$$\mathbb{E} \left[\tau \sum_{m=1}^M \int_{\mathcal{O}} |\mathcal{A}[\Phi W(t_m)]|^2 \, dx \right] \leq c,$$

and choosing $m = m_{R_1}$ yields

$$\begin{aligned} & \mathbb{E} \left[\sum_{m=1}^{m_{R_1}} \tau \int_{\mathcal{O}} |\mathcal{L}^m(\mathbf{y}_{m-1}, \mathbf{y}_m)|^2 \, dx \right] \\ & \leq \mathbb{E} \left[\sum_{m=1}^{m_{R_1}} \tau (\|\Phi W(t_{m-1})\|_{W_x^{1,2}}^4 + \|\Phi W(t_m)\|_{W_x^{1,2}}^4 + \|\mathbf{u}_{m-1}\|_{W_x^{1,2}}^4 + \|\mathbf{u}_m\|_{W_x^{1,2}}^4) \right] \leq cR_2(R_1)^4 \end{aligned}$$

using (2.11) also. We conclude by the discrete Gronwall lemma,

$$\begin{aligned} & \mathbb{E} \left[\max_{1 \leq n \leq m_{R_1}} \int_{\mathcal{O}} |\nabla \mathbf{y}_n|^2 \, dx \right] + \mathbb{E} \left[\sum_{m=1}^m \int_{\mathcal{O}} |\nabla(\mathbf{y}_m - \mathbf{y}_{m-1})|^2 \, dx \right] \\ & \quad + \frac{\mu}{2} \mathbb{E} \left[\sum_{m=1}^{m_{R_1}} \tau \int_{\mathcal{O}} |\mathcal{A}\mathbf{y}_m|^2 \, dx \right] \\ & \leq ce^{cR_2(R_1)^4} \mathbb{E} \left[\int_{\mathcal{O}} |\nabla \mathbf{u}_0|^2 \, dx + 1 \right] \end{aligned}$$

using that

$$\sum_{n=1}^{m_{R_1}} \tau_n \|\mathbf{y}_{n-1}\|_{L_x^2}^2 \|\nabla \mathbf{y}_{n-1}\|_{L_x^2}^2 \leq R_2(R_1)^4$$

by the definition of m_{R_1} . This proves (3.10) for $q = 1$. To obtain the estimate in (3.10) for $q = 2$ we multiply (3.13) by $\int_{\mathcal{O}} |\nabla \mathbf{y}_m|^2 dx$ and obtain

$$\begin{aligned} & \frac{1}{4} \|\nabla \mathbf{y}_m\|_{L_x^2}^4 + \frac{1}{4} (\|\nabla \mathbf{y}_m\|_{L_x^2}^2 - \|\nabla \mathbf{y}_{m-1}\|_{L_x^2}^2)^2 \\ & + \frac{1}{2} \|\nabla(\mathbf{y}_m - \mathbf{y}_{m-1})\|_{L_x^2}^2 \|\nabla \mathbf{y}_m\|_{L_x^2}^2 + \frac{\mu}{2} \tau \|\nabla \mathbf{y}_m\|_{L_x^2}^2 \|\mathcal{A} \mathbf{y}_m\|_{L_x^2}^2 \\ & \leq \frac{1}{4} \|\nabla \mathbf{y}_{m-1}\|_{L_x^2}^4 + c \tau \|\nabla \mathbf{y}_m\|_{L_x^2}^2 (\|\Phi W(t_{m-1})\|_{W_x^{1,2}}^4 + \|\mathbf{u}_{m-1}\|_{W_x^{1,2}}^4) \\ & + c \tau \|\nabla \mathbf{y}_m\|_{L_x^2}^2 (\|\Phi W(t_m)\|_{W_x^{1,2}}^4 + \|\mathbf{u}_m\|_{W_x^{1,2}}^4) + c \tau \|\mathbf{y}_{m-1}\|_{L_x^2}^2 \|\nabla \mathbf{y}_{m-1}\|_{L_x^2}^2 \|\nabla \mathbf{y}_m\|_{L_x^2}^4 \\ & \leq \frac{1}{4} \|\nabla \mathbf{y}_{m-1}\|_{L_x^2}^4 + c \tau (\|\Phi W(t_{m-1})\|_{W_x^{1,2}}^6 + \|\mathbf{u}_{m-1}\|_{W_x^{1,2}}^6 + \|\Phi W(t_m)\|_{W_x^{1,2}}^6 + \|\mathbf{u}_m\|_{W_x^{1,2}}^6) \\ & + c \tau \|\mathbf{y}_{m-1}\|_{L_x^2}^2 \|\nabla \mathbf{y}_{m-1}\|_{L_x^2}^2 \|\nabla \mathbf{y}_m\|_{L_x^2}^4. \end{aligned}$$

Again we can control the second last term by means of (2.11) and the last one with the help of the discrete Gronwall lemma (and the definition of m_{R_1}). We obtain (3.10) for $q = 2$. We can prove similarly the claim for $q \in \mathbb{N}$ by iteration following the strategy of Brzeźniak *et al.* (2013, Lemma 3.1).

2. Proof of (3.11). We apply ∇ to (3.5) and multiply eventually by $\nabla \mathcal{A} \mathbf{y}_m$, which yields

$$\begin{aligned} & \frac{1}{2} \left(\|\mathcal{A} \mathbf{y}_m\|_{L_x^2}^2 - \|\mathcal{A} \mathbf{y}_{m-1}\|_{L_x^2}^2 + \|\mathcal{A}(\mathbf{y}_m - \mathbf{y}_{m-1})\|_{L_x^2}^2 \right) + \frac{\tau}{\mu} 2 \|\nabla \mathcal{A} \mathbf{y}_m\|_{L_x^2}^2 \\ & \leq c \tau \left(\|\nabla \mathcal{A}[\Phi W(t_m)]\|_{L_x^2}^2 + \tau \left\langle \nabla[(\nabla \mathbf{y}_m) \mathbf{y}_m], \nabla \mathcal{A} \mathbf{y}_m \right\rangle_{L_x^2} \right. \\ & \quad \left. + \tau \left\langle \nabla[\mathcal{P} \mathcal{L}^m(\mathbf{y}_{m-1}, \mathbf{y}_m)], \nabla \mathcal{A} \mathbf{y}_m \right\rangle_{L_x^2} \right). \end{aligned} \tag{3.14}$$

We deal independently with the terms on the right-hand side. First of all, we have for $\delta > 0$,

$$\begin{aligned} & \left\langle \nabla[(\nabla \mathbf{y}_m) \mathbf{y}_{m-1}], \nabla \mathcal{A} \mathbf{y}_m \right\rangle_{L_x^2} \\ & \leq c(\delta) \|\nabla[(\nabla \mathbf{y}_m) \mathbf{y}_{m-1}]\|_{L_x^2}^2 + \delta \|\nabla \mathcal{A} \mathbf{y}_m\|_{L_x^2}^2 \\ & \leq c(\delta) \|\nabla^2 \mathbf{y}_m\|_{L_x^2}^2 \|\mathbf{y}_{m-1}\|_{L_x^\infty}^2 + \|\nabla \mathbf{y}_m\|_{L_x^4}^2 \|\nabla \mathbf{y}_{m-1}\|_{L_x^4}^2 + \delta \|\nabla \mathcal{A} \mathbf{y}_m\|_{L_x^2}^2 \\ & \leq c(\delta) \|\mathbf{y}_m\|_{W_x^{2,2}}^2 \|\mathbf{y}_{m-1}\|_{W_x^{2,2}}^2 + \delta \|\nabla \mathcal{A} \mathbf{y}_m\|_{L_x^2}^2. \end{aligned}$$

For the remaining term we use Hölder’s inequality, Young’s inequality and the interpolation $\|\mathcal{A}\mathbf{y}_m\|_{L_x^2}^3 \leq \|\nabla\mathbf{y}_m\|_{L_x^2}^{\frac{3}{2}}\|\nabla\mathcal{A}\mathbf{y}_m\|_{L_x^2}^{\frac{3}{2}}$ to obtain

$$\begin{aligned} & \tau \left\langle \nabla [(\nabla\mathbf{y}_m)[\Phi W(t_{m-1})]], \nabla\mathcal{A}\mathbf{y}_m \right\rangle_{L_x^2} \\ & \leq \frac{\tau\mu}{8} \|\nabla\mathcal{A}\mathbf{y}_m\|_{L_x^2}^2 + c\tau \left(\|\nabla\mathbf{y}_m\|_{L_x^2}^4 + \|\nabla[\Phi W(t_m)]\|_{L_x^\infty}^4 \right) \\ & \quad + c\tau \left(\|\mathcal{A}\mathbf{y}_m\|_{L_x^2}^3 + \|\Phi W(t_{m-1})\|_{L_x^\infty}^6 \right) \\ & \leq \frac{\tau\mu}{4} \|\nabla\mathcal{A}\mathbf{y}_m\|_{L_x^2}^2 + c\tau \left(1 + \|\nabla\mathbf{y}_m\|_{L_x^2}^6 + \|\Phi W(t_{m-1})\|_{W_x^{3,2}}^6 \right). \end{aligned}$$

The last two terms can be controlled using (3.10) and $\Phi \in L_2(\mathcal{U}; W^{3,2}(\mathcal{O}; \mathbb{R}^2))$. Similarly, we have

$$\begin{aligned} & \tau \left\langle \nabla [(\nabla[\Phi W(t_m)])\mathbf{y}_{m-1}], \nabla\mathcal{A}\mathbf{y}_m \right\rangle_{L_x^2} \\ & \leq \frac{\tau\mu}{4} \|\nabla\mathcal{A}\mathbf{y}_m\|_{L_x^2}^2 + c\tau \left(\|\nabla\mathbf{y}_{m-1}\|_{W_x^{1,2}}^4 + \|\Phi W(t_m)\|_{W_x^{3,2}}^4 \right) \end{aligned}$$

and

$$\begin{aligned} & \tau \left\langle \nabla [(\nabla[\Phi W(t_m)])(\Phi W(t_{m-1}))], \nabla\mathcal{A}\mathbf{y}_m \right\rangle_{L_x^2} \\ & \leq \frac{\tau\mu}{4} \|\nabla\mathcal{A}\mathbf{y}_m\|_{L_x^2}^2 + c\tau \|\Phi W(t_{m-1})\|_{W_x^{3,2}}^4 + c\tau \|\Phi W(t_m)\|_{W_x^{3,2}}^4, \end{aligned}$$

which can be controlled accordingly. Absorbing the $\|\nabla\mathcal{A}\mathbf{y}_m\|_{L_x^2}^2$ -terms and iterating and applying the discrete Gronwall lemma we conclude

$$\mathbb{E} \left[\max_{1 \leq m \leq m_{R_1}} \|\mathbf{y}_m\|_{W_x^{2,2}}^2 \right] + \mu \mathbb{E} \left[\sum_{m=1}^{m_{R_1}} \tau \|\nabla\mathcal{A}\mathbf{y}_m\|_{L_x^2}^2 \right] \leq ce^{cR_2(R_1)^4} \mathbb{E} \left[\|\mathbf{u}_0\|_{W_x^{3,2}}^2 + 1 \right],$$

using that

$$\sum_{n=1}^{m_{R_1}} \tau \|\mathbf{y}_{n-1}\|_{W_x^{2,2}}^2 \leq R_2(R_1)^2$$

by the definition of m_{R_1} . This proves (3.11) for $q = 1$. The proof for $q \geq 2$ follows by multiplying (3.14) with $\|\mathcal{A}\mathbf{y}_m\|_{L_x^2}^{2q-2}$ ($q \geq 1$) and proceeding recursively (using also (3.10)).

3. Proof of (3.12). We apply ∇ and $\|\cdot\|_{L_x^2}^2$ to (3.5) such that

$$\begin{aligned} \left\| \nabla \frac{\mathbf{y}_m - \mathbf{y}_{m-1}}{\tau} \right\|_{L_x^2}^2 &\leq c \|\nabla \mathcal{A} \mathbf{y}_m\|_{L_x^2}^2 + c \|\nabla \mathcal{A}[\Phi W(t_m)]\|_{L_x^2}^2 + c \|\nabla \mathcal{P}[(\nabla \mathbf{y}_m) \mathbf{y}_{m-1}]\|_{L_x^2}^2 \\ &\quad + c \|\nabla \mathcal{P}[\mathcal{L}^m(\mathbf{y}_{m-1}, \mathbf{y}_m)]\|_{L_x^2}^2 \\ &\leq c \|\mathbf{y}_m\|_{W_x^{3,2}}^2 + c \|\Phi W(t_m)\|_{W_x^{3,2}}^2 + c \|\mathbf{y}_{m-1}\|_{W_x^{2,2}}^4 + c \|\mathbf{y}_m\|_{W_x^{2,2}}^4 \\ &\quad + c \|\Phi W(t_{m-1})\|_{W_x^{2,2}}^4 + c \|\Phi W(t_m)\|_{W_x^{2,2}}^4, \end{aligned}$$

where we used Ladyshenskaya’s inequality. Multiplying by $\tau \|\mathbf{y}_m\|_{W_x^{2,2}}^{2q-4}$ and summing with respect to m yields (iii)=(3.12) due to (ii)=(3.11) and $\Phi \in L_2(\mathcal{U}; W^{3,2}(\mathcal{O}; \mathbb{R}^2))$. We can again iterate the argument for $q \geq 3$. □

REMARK 3.4 It is crucial that the diffusion coefficient in (3.2) is solenoidal in order to perform the integration by parts at the beginning of the proof of Lemma 3.3. In the general case we must define \mathbf{y} by $\mathbf{y}(t) = \mathbf{u}(t) - \mathcal{P}\Phi W(t)$ with the Helmholtz projection \mathcal{P} . Even if Φ vanishes on $\partial\mathcal{O}$, we do not necessarily have the same for $\mathcal{P}\Phi W(t)$ (and \mathbf{y}).

3.2 Temporal error analysis

For the purpose of the error analysis we define for $R > 0$ and $m \in \{1, \dots, M\}$,

$$\tilde{t}_m^R := t_m \wedge \tilde{t}_R^d, \quad \tilde{\tau}_n^R = \tilde{t}_n^R - \tilde{t}_{n-1}^R. \tag{3.15}$$

We consider the stopped process

$$\tilde{\mathbf{y}}_m^R := \begin{cases} \mathbf{y}_m & \text{in } \{t_m = \tilde{t}_m^R\}, \\ \mathbf{y}_{\mathfrak{m}_R} & \text{in } \{t_m > \tilde{t}_m^R\}, \end{cases} \tag{3.16}$$

where \mathfrak{m}_R is defined before (3.8). The main effort of this section is to prove the following error estimate.

THEOREM 3.5 Assume that $\mathbf{u}_0 \in L^r(\Omega, W^{3,2}(\mathcal{O}; \mathbb{R}^2)) \cap L^{5r}(\Omega, W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$ for some $r \geq 8$ and that $\Phi \in L_2(\mathcal{U}; W_{0,\text{div}}^{1,2} \cap W^{3,2}(\mathcal{O}; \mathbb{R}^2))$. Let \mathbf{y} be the solution to (3.2), and $(\mathbf{y}_m)_{m=1}^M$ be the solution to (3.5). Then we have the error estimate

$$\max_{1 \leq m \leq M} \mathbb{E} \left[\|\mathbf{y}(\tilde{t}_m^R) - \tilde{\mathbf{y}}_m^R\|_{L_x^2}^2 + \sum_{n=1}^m \tilde{\tau}_n^R \|\nabla \mathbf{y}(\tilde{t}_n^R) - \nabla \tilde{\mathbf{y}}_n^R\|_{L_x^2}^2 \right] \leq c \tau^2 e^{cR_2(R)^4}. \tag{3.17}$$

The convergence in probability of the original scheme (3.5) is now a direct consequence of Theorem 3.5: defining R by $R_2(R) = c^{-1/4} \sqrt[4]{-\varepsilon \log \tau}$, where $\varepsilon > 0$ is arbitrary, we have for any

$\xi > 0, \alpha < 1,$

$$\begin{aligned} & \max_{1 \leq m \leq M} \mathbb{P} \left[\|\mathbf{y}(t_m) - \mathbf{y}_m\|_{L_x^2}^2 + \sum_{n=1}^m \tau \|\nabla \mathbf{y}(t_n) - \nabla \mathbf{y}_n\|_{L_x^2}^2 > \xi \tau^{2\alpha-2\varepsilon} \right] \\ & \leq \max_{1 \leq m \leq M} \mathbb{P} \left[\|\mathbf{y}(\tilde{t}_m^R) - \tilde{\mathbf{y}}_m^R\|_{L_x^2}^2 + \sum_{n=1}^m \tilde{\tau}_n^R \|\nabla \mathbf{y}(\tilde{t}_n^R) - \nabla \tilde{\mathbf{y}}_n^R\|_{L_x^2}^2 > \xi \tau^{2\alpha-2\varepsilon} \right] \\ & \quad + \mathbb{P}[\{\tilde{t}_M^R < T\}] \rightarrow 0 \end{aligned}$$

as $\tau \rightarrow 0$ by (3.9) (recall that $t_R \rightarrow \infty$ \mathbb{P} -a.s. by Theorem 2.2). Relabeling α we have proved the following result.

THEOREM 3.6 Assume that $\mathbf{u}_0 \in L^r(\Omega, W^{3,2}(\mathcal{O}; \mathbb{R}^2)) \cap L^{5r}(\Omega, W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$ for some $r \geq 8$ and that $\Phi \in L_2(\mathcal{U}; W_{0,\text{div}}^{1,2} \cap W^{3,2}(\mathcal{O}; \mathbb{R}^2))$. Let

$$(\mathbf{u}, (t_R)_{R \in \mathbb{N}}, \mathbf{t})$$

be the unique global maximal strong solution to (1.1) from Theorem 2.5. Then we have for any $\xi > 0, \alpha < 1,$

$$\max_{1 \leq m \leq M} \mathbb{P} \left[\|\mathbf{y}(t_m) - \mathbf{y}_m\|_{L_x^2}^2 + \sum_{n=1}^m \tau \|\nabla \mathbf{y}(t_n) - \nabla \mathbf{y}_n\|_{L_x^2}^2 > \xi \tau^{2\alpha} \right] \rightarrow 0$$

as $\tau \rightarrow 0$, where \mathbf{y} is the solution to (3.2) and $(\mathbf{y}_m)_{m=1}^M$ is the solution to (3.5).

Proof of Theorem 3.5. We integrate (3.2) over $(\tilde{t}_{m-1}^R, \tilde{t}_m^R)$, and subtract this equation from (3.5) (multiplied by $\tilde{\tau}_m^R$). Then $\mathbf{e}_\ell := \mathbf{y}(\tilde{t}_\ell^R) - \tilde{\mathbf{y}}_\ell^R$ (for $\ell \in \{m-1, m\}$) solves

$$\begin{aligned} \langle \mathbf{e}_m - \mathbf{e}_{m-1} - \mu \tilde{\tau}_m^R \mathcal{A} \mathbf{e}_m, \boldsymbol{\phi} \rangle_{L_x^2} &= \left\langle \mu \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \int_s^{\tilde{t}_m^R} \partial_t \nabla \mathbf{y}(\xi) \, d\xi \, ds, \nabla \boldsymbol{\phi} \right\rangle_{L_x^2} \\ &\quad + \left\langle \mu \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{A}[\Phi(W_s - W_{\tilde{t}_m^R}^R)] \, ds, \boldsymbol{\phi} \right\rangle + \langle \text{NLT}_m, \boldsymbol{\phi} \rangle_{L_x^2} \end{aligned} \tag{3.18}$$

for all $\boldsymbol{\phi} \in W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)$, where

$$\begin{aligned} \text{NLT}_m &= \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \left(\mathcal{P}[(\nabla \mathbf{y})\mathbf{y} + \mathcal{L}^W(\mathbf{y})] - \mathcal{P}[(\nabla \tilde{\mathbf{y}}_m^R)\tilde{\mathbf{y}}_{m-1}^R + \mathcal{L}^m(\tilde{\mathbf{y}}_{m-1}^R, \tilde{\mathbf{y}}_m^R)] \right) ds \\ &= \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{P}[(\nabla \mathbf{y})\mathbf{y} - (\nabla \tilde{\mathbf{y}}_m^R)\tilde{\mathbf{y}}_{m-1}^R] \, ds + \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{P}[\nabla[\Phi W_s]\mathbf{y}(s) - \nabla[\Phi W_{\tilde{t}_m^R}^R]\tilde{\mathbf{y}}_{m-1}^R] \, ds \end{aligned}$$

$$\begin{aligned}
 &+ \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{P} \left[(\nabla \mathbf{y}(s)) [\Phi W_s] - \nabla \tilde{\mathbf{y}}_m^R [\Phi W_{\tilde{t}_{m-1}^R}] \right] ds \\
 &+ \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{P} \left[\nabla [\Phi W_s] [\Phi W_s] - \nabla [\Phi W_{\tilde{t}_m^R}] [\Phi W_{\tilde{t}_{m-1}^R}] \right] ds \\
 &=: \text{NLT}_m^1 + \text{NLT}_m^2 + \text{NLT}_m^3 + \text{NLT}_m^4.
 \end{aligned}$$

We test (3.18) with \mathbf{e}_m and apply expectations. The left-hand side then is

$$\frac{1}{2} \mathbb{E} \left[(\|\mathbf{e}_m\|_{L_x^2}^2 - \|\mathbf{e}_{m-1}\|_{L_x^2}^2 + \|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2) \right] + \mu \mathbb{E} \left[\tilde{\tau}_m^R \|\nabla \mathbf{e}_m\|_{L_x^2}^2 \right]. \tag{3.19}$$

The first term on the right-hand side of (3.18) may be bounded by

$$\begin{aligned}
 &\delta \mathbb{E} \left[\tilde{\tau}_m^R \|\nabla \mathbf{e}_m\|_{L_x^2}^2 \right] + c(\delta) \mathbb{E} \left[\tilde{\tau}_m^R \left(\int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \|\nabla \partial_t \mathbf{y}(\xi)\|_{L_x^2} d\xi \right)^2 \right] \\
 &\leq \delta \mathbb{E} \left[\tilde{\tau}_m^R \|\nabla \mathbf{e}_m\|_{L_x^2}^2 \right] + c(\delta) \tau^2 \mathbb{E} \left[\int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \|\nabla \partial_t \mathbf{y}\|_{L_x^2}^2 d\xi \right], \tag{3.20}
 \end{aligned}$$

where $\delta > 0$ is arbitrary. For the second term on the right-hand side of (3.18), we use independence of increments of the Wiener process to conclude that it equals

$$\begin{aligned}
 &\mathbb{E} \left[\mu \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{A}[\Phi(W_s - W_{\tilde{t}_m^R})] ds, \mathbf{e}_m - \mathbf{e}_{m-1} \right\rangle_{L_x^2} \right] \\
 &\leq \frac{1}{2} \mathbb{E} \left[\|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2 \right] + c \mathbb{E} \left[\left\| \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{A}[\Phi(W_s - W_{\tilde{t}_m^R})] ds \right\|_{L_x^2}^2 \right], \tag{3.21}
 \end{aligned}$$

where the last term is bounded by

$$\tau^2 \mathbb{E} \left[\sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\Phi(W_s - W_{\tilde{t}_m^R})\|_{W_x^{2,2}}^2 \right] \leq \tau^2 \mathbb{E} \left[\sup_{s \in [t_{m-1}, t_m]} \|\Phi(W_s - W_{t_m})\|_{W_x^{2,2}}^2 \right] \leq c\tau^3$$

due to Doob’s maximum inequality and $\Phi \in L_2(\mathcal{U}; W^{2,2}(\mathcal{O}; \mathbb{R}^2))$. Now we consider the errors due to nonlinear effects that are contained in NLT_m : we start with the ‘quadratic terms’:

$$\begin{aligned}
 \text{NLT}_m^1 &= \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{P} \left[(\nabla \mathbf{y}(s)) \mathbf{y}(s) - (\nabla \mathbf{y}(\tilde{t}_m^R)) \mathbf{y}(\tilde{t}_m^R) + (\nabla \mathbf{y}(\tilde{t}_m^R)) \mathbf{y}(\tilde{t}_m^R) - (\nabla \tilde{\mathbf{y}}_m^R) \tilde{\mathbf{y}}_{m-1}^R \right] ds \\
 &= \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{P} \left[(\nabla [\mathbf{y}(s) - \mathbf{y}(\tilde{t}_m^R)]) \mathbf{y}(s) + (\nabla \mathbf{e}_m) \mathbf{y}(\tilde{t}_m^R) + (\nabla \tilde{\mathbf{y}}_m^R) \mathbf{e}_m \right] ds
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \mathcal{P} \left[(\nabla \tilde{\mathbf{y}}_m^R) (\tilde{\mathbf{y}}_m^R - \tilde{\mathbf{y}}_{m-1}^R) + \nabla \mathbf{y}(\tilde{t}_m^R) (\mathbf{y}(s) - \mathbf{y}(\tilde{t}_m^R)) \right] ds \\
 & =: \sum_{i=1}^5 \text{NLT}_m^{1,i}.
 \end{aligned}$$

Note that $\langle \text{NLT}_m^{1,2}, \mathbf{e}_m \rangle_{L_x^2} = 0$ such that we only have to consider $\text{NLT}_m^{1,1}$, $\text{NLT}_m^{1,3}$, $\text{NLT}_m^{1,4}$ and $\text{NLT}_m^{1,5}$. Thanks to $\mathbf{y}(s) - \mathbf{y}(\tilde{t}_m^R) = \int_{\tilde{t}_m^R}^s \partial_t \mathbf{y}(\xi) d\xi$, we rewrite $\langle \text{NLT}_m^{1,1}, \mathbf{e}_m \rangle_{L_x^2}$ as

$$\begin{aligned}
 & \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \left(\int_{\tilde{t}_m^R}^s \nabla \partial_t \mathbf{y}(\xi) d\xi \right) \mathbf{y}(s) ds, \mathbf{e}_m \right\rangle_{L_x^2} \\
 & \leq \tilde{\tau}_m^R \sup_{\tilde{t}_{m-1}^R \leq s \leq \tilde{t}_m^R} \left(\left\| \int_s^{\tilde{t}_m^R} \nabla \partial_t \mathbf{y}(\xi) d\xi \right\|_{L_x^2} \|\mathbf{y}(s)\|_{L_x^4} \right) \|\mathbf{e}_m\|_{L_x^4} \\
 & \leq \delta \tilde{\tau}_m^R \|\nabla \mathbf{e}_m\|_{L_x^2}^2 + c(\delta) \tau^2 \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \|\nabla \partial_t \mathbf{y}(s)\|_{L_x^2}^2 ds \sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\nabla \mathbf{y}(s)\|_{L_x^2}^2 \\
 & \leq \delta \tilde{\tau}_m^R \|\nabla \mathbf{e}_m\|_{L_x^2}^2 + c(\delta) \tau^2 R_2(R)^2 \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \|\nabla \partial_t \mathbf{y}(s)\|_{L_x^2}^2 ds
 \end{aligned}$$

for $\delta > 0$ using Corollary 3.2 and recalling that $\tilde{t}_m^R \leq t_R$. Moreover, we have

$$\begin{aligned}
 \langle \text{NLT}_m^{1,3}, \mathbf{e}_m \rangle_{L_x^2} & \leq \tilde{\tau}_m^R \|\mathbf{e}_m\|_{L_x^4}^2 \|\nabla \tilde{\mathbf{y}}_m^R\|_{L_x^2} \leq c \tilde{\tau}_m^R \|\mathbf{e}_m\|_{L_x^2} \|\nabla \mathbf{e}_m\|_{L_x^2} \|\nabla \tilde{\mathbf{y}}_m^R\|_{L_x^2} \\
 & \leq \delta \tilde{\tau}_m^R \|\nabla \mathbf{e}_m\|_{L_x^2}^2 + c(\delta) \tilde{\tau}_m^R R_2(R)^2 \|\mathbf{e}_m\|_{L_x^2}^2
 \end{aligned}$$

by definition of \tilde{t}_m^R . Arguing similarly as for the estimates for $\text{NLT}_m^{1,1}$ above and using Sobolev’s embedding $W_0^{1,2}(\mathcal{O}) \hookrightarrow L^4(\mathcal{O})$ we have

$$\begin{aligned}
 \langle \text{NLT}_m^{1,5}, \mathbf{e}_m \rangle_{L_x^2} & = \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \nabla \mathbf{y}(\tilde{t}_m^R) \left(\int_{\tilde{t}_m^R}^s \partial_t \mathbf{y}(\xi) d\xi \right) ds, \mathbf{e}_m \right\rangle_{L_x^2} \\
 & \leq \tilde{\tau}_m^R \sup_{\tilde{t}_{m-1}^R \leq s \leq \tilde{t}_m^R} \left(\left\| \int_s^{\tilde{t}_m^R} \partial_t \mathbf{y}(\xi) d\xi \right\|_{L_x^4} \|\nabla \mathbf{y}(s)\|_{L_x^2} \right) \|\mathbf{e}_m\|_{L_x^4} \\
 & \leq \delta \tilde{\tau}_m^R \|\nabla \mathbf{e}_m\|_{L_x^2}^2 + c(\delta) \tau^2 \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \|\nabla \partial_t \mathbf{y}(s)\|_{L_x^2}^2 ds \sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\nabla \mathbf{y}(s)\|_{L_x^2}^2
 \end{aligned}$$

such that we obtain the same estimate for its expectation. For $\text{NLT}_m^{1,4}$ we employ a discrete version of this argument to obtain

$$\begin{aligned} \langle \text{NLT}_m^{1,4}, \mathbf{e}_m \rangle_{L_x^2} &\leq \tilde{\tau}_m^R \|\tilde{\mathbf{y}}_m^R - \tilde{\mathbf{y}}_{m-1}^R\|_{L_x^4} \|\nabla \tilde{\mathbf{y}}_m^R\|_{L_x^2} \|\mathbf{e}_m\|_{L_x^4} \\ &\leq \delta \tilde{\tau}_m^R \|\nabla \mathbf{e}_m\|_{L_x^2}^2 + c(\delta)\tau^3 \left\| \frac{\nabla(\tilde{\mathbf{y}}_m^R - \tilde{\mathbf{y}}_{m-1}^R)}{\tau} \right\|_{L_x^2}^2 \|\nabla \tilde{\mathbf{y}}_m^R\|_{L_x^2}^2. \end{aligned}$$

Note that the last term can be controlled by means of equation (3.12) from Lemma 3.3 (iii) giving the bound $c(\delta)\tau^2 e^{cR_2(R)^4}$ for its expectation (in summed form).

For the second term in NLT we write

$$\begin{aligned} \langle \text{NLT}_m^2, \mathbf{e}_m \rangle_{L_x^2} &= \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} (\nabla[\Phi W_s] \mathbf{y}(s) - \nabla[\Phi W_{\tilde{t}_m^R}] \tilde{\mathbf{y}}_{m-1}^R) ds, \mathbf{e}_m \right\rangle_{L_x^2} \\ &= \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \nabla[\Phi(W_s - W_{\tilde{t}_m^R})] \mathbf{y}(s) ds, \mathbf{e}_m \right\rangle_{L_x^2} \\ &\quad + \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \nabla[\Phi W_{\tilde{t}_{m-1}^R}] \mathbf{e}_{m-1} ds, \mathbf{e}_m \right\rangle_{L_x^2} \\ &\quad + \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \nabla[\Phi(W_{\tilde{t}_m^R} - W_{\tilde{t}_{m-1}^R})] \mathbf{e}_{m-1} ds, \mathbf{e}_m \right\rangle_{L_x^2} \\ &\quad + \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \nabla[\Phi W_{\tilde{t}_m^R}] (\mathbf{y}(s) - \mathbf{y}(\tilde{t}_{m-1}^R)) ds, \mathbf{e}_m \right\rangle_{L_x^2} \\ &=: E_m^{2,1} + E_m^{2,2} + E_m^{2,3} + E_m^{2,4}. \end{aligned}$$

For the first term we further split

$$\begin{aligned} E_m^{2,1} &= \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \nabla[\Phi(W_s - W_{\tilde{t}_m^R})] \mathbf{y}(s) ds, \mathbf{e}_m - \mathbf{e}_{m-1} \right\rangle_{L_x^2} \\ &\quad + \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \nabla[\Phi(W_s - W_{\tilde{t}_m^R})] (\mathbf{y}(s) - \mathbf{y}(\tilde{t}_{m-1}^R)) ds, \mathbf{e}_{m-1} \right\rangle_{L_x^2} \\ &\quad + \left\langle \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \nabla[\Phi(W_s - W_{\tilde{t}_m^R})] \mathbf{y}(\tilde{t}_{m-1}^R) ds, \mathbf{e}_{m-1} \right\rangle_{L_x^2} \\ &=: E_m^{2,1,1} + E_m^{2,1,2} + E_m^{2,1,3}. \end{aligned}$$

We notice that $\mathbb{E}[\sum_{n=1}^m E_n^{2,1,3}] = 0$. Indeed, $\sum_{n=1}^m E_n^{2,1,3}$ can be written as the decomposition of the (\mathfrak{F}_{t_m}) -martingale

$$\sum_{n=1}^m \left\langle \int_{t_{n-1}}^{t_n} \nabla[\Phi(W_s - W_{t_n})] \mathbf{y}(t_{m-1}) \, ds, \mathbf{e}_{m-1} \right\rangle_{L_x^2}$$

with the (\mathfrak{F}_{t_m}) -stopping time m_R defined after (3.16). Hence it is an (\mathfrak{F}_{t_m}) -martingale in its own right. We estimate

$$\begin{aligned} & \mathbb{E}[E_m^{2,1,1}] \\ & \leq \mathbb{E} \left[\tilde{\tau}_m^R \sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\nabla[\Phi(W_s - W_{\tilde{t}_m^R})]\|_{L_x^4} \sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\mathbf{y}(s)\|_{L_x^4} \|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2} \right] \\ & \leq c(\delta) \tau^2 \mathbb{E} \left[\sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\nabla^2[\Phi(W_s - W_{\tilde{t}_m^R})]\|_{L_x^2}^2 \sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\nabla \mathbf{y}(s)\|_{L_x^2}^2 \right] \\ & \quad + \delta \mathbb{E}[\|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2] \\ & \leq c(\delta) \tau^2 R_2(R)^2 \mathbb{E} \left[\sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\nabla^2[\Phi(W_s - W_{\tilde{t}_m^R})]\|_{L_x^2}^2 \right] \\ & \quad + \delta \mathbb{E}[\|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2] \\ & \leq c(\delta) \tau^3 R_2(R)^2 + \delta \mathbb{E}[\|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2], \end{aligned}$$

using Doob’s maximal inequality, $\Phi \in L_2(\mathfrak{U}; W^{2,2}(\mathcal{O}; \mathbb{R}^2))$ and Corollary 3.2 (recalling that $\tilde{t}_m^R \leq t_R$). Similarly, it holds that

$$\begin{aligned} & \mathbb{E}[E_m^{2,1,2}] \\ & \leq c(\delta) \tilde{\tau}_m^R \mathbb{E} \left[\sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\nabla[\Phi(W_s - W_{\tilde{t}_m^R})]\|_{L_x^4}^2 \sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\mathbf{y}(s) - \mathbf{y}(\tilde{t}_{m-1}^R)\|_{L_x^2}^2 \right] \\ & \quad + \delta \tilde{\tau}_m^R \mathbb{E}[\|\mathbf{e}_{m-1}\|_{L_x^4}^2] \\ & \leq c(\delta) \tau^2 \left(\mathbb{E} \left[\sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\nabla^2[\Phi(W_s - W_{\tilde{t}_m^R})]\|_{L_x^2}^4 \right] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \|\partial_t \mathbf{y}(\xi)\|_{L_x^2}^4 \, d\xi \right] \right)^{\frac{1}{2}} \\ & \quad + \delta \tilde{\tau}_m^R \mathbb{E}[\|\nabla \mathbf{e}_{m-1}\|_{L_x^2}^2] \\ & \leq c(\delta) \tau^3 R^4 + \delta \tilde{\tau}_m^R \mathbb{E}[\|\nabla \mathbf{e}_{m-1}\|_{L_x^2}^2], \end{aligned}$$

using Corollary 3.2. Furthermore, we obtain

$$\mathbb{E}[E_m^{2,2}] \leq \tilde{\tau}_m^R \mathbb{E} \left[\|\nabla \Phi(W_{\tilde{t}_{m-1}^R})\|_{L_x^4} \|\mathbf{e}_{m-1}\|_{L_x^2} \|\mathbf{e}_m\|_{L_x^4} \right]$$

$$\begin{aligned} &\leq c\tilde{\tau}_m^R R_2(R)\mathbb{E}[\|\mathbf{e}_m\|_{L_x^2}\|\mathbf{e}_m\|_{L_x^2}^{\frac{1}{2}}\|\nabla\mathbf{e}_m\|_{L_x^2}^{\frac{1}{2}}] \\ &\leq c(\delta)\tilde{\tau}_m^R R_2(R)^2\mathbb{E}[\|\mathbf{e}_{m-1}\|_{L_x^2}^2 + \|\mathbf{e}_m\|_{L_x^2}^2] + \delta\tilde{\tau}_m^R\mathbb{E}[\|\nabla\mathbf{e}_m\|_{L_x^2}^2] \end{aligned}$$

by definition of $\tilde{\tau}_m^R$. The term $E_m^{2,3}$ has to be split in the same fashion as

$$\begin{aligned} E_m^{2,3} &= \left\langle \int_{\tilde{\tau}_{m-1}^R}^{\tilde{\tau}_m^R} \nabla[\Phi(W_{\tilde{\tau}_m^R} - W_{\tilde{\tau}_{m-1}^R})]\mathbf{e}_m \, ds, \mathbf{e}_m - \mathbf{e}_{m-1} \right\rangle_{L_x^2} \\ &\quad + \left\langle \int_{\tilde{\tau}_{m-1}^R}^{\tilde{\tau}_m^R} \nabla[\Phi(W_{\tilde{\tau}_m^R} - W_{\tilde{\tau}_{m-1}^R})]\mathbf{e}_{m-1} \, ds, \mathbf{e}_{m-1} \right\rangle_{L_x^2} \\ &=: E_m^{2,3,1} + E_m^{2,3,2}, \end{aligned}$$

where again $\mathbb{E}[\sum_{n=1}^m E_n^{2,3,2}] = 0$.³ Proceeding similarly to the estimates of $E_m^{2,1}$ we have

$$\begin{aligned} \mathbb{E}[E_m^{2,3,1}] &\leq c(\delta)\tau^2\left(\mathbb{E}\left[\sup_{s\in[\tilde{\tau}_{m-1}^R, \tilde{\tau}_m^R]} \|\nabla^2[\Phi(W_s - W_{\tilde{\tau}_m^R})]\|_{L_x^2}^4\right]\right)^{\frac{1}{2}}(\mathbb{E}\|\nabla\mathbf{e}_m\|_{L_x^2}^4)^{\frac{1}{2}} \\ &\quad + \delta[\mathbb{E}\|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2] \\ &\leq c(\delta)\tau^3 R_2(R)^2 + \delta\mathbb{E}[\|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2]. \end{aligned}$$

Finally, we have

$$\begin{aligned} \mathbb{E}[E_m^{2,4}] &\leq \tilde{\tau}_m^R\left(\mathbb{E}\left[\|\nabla[\Phi(W_{\tilde{\tau}_m^R})]\|_{L_x^4}^4\right]\right)^{\frac{1}{4}}\left(\mathbb{E}\|\mathbf{y}(s) - \mathbf{y}(\tilde{\tau}_{m-1}^R)\|_{L_x^2}^4\right)^{\frac{1}{4}}(\mathbb{E}[\|\nabla\mathbf{e}_m\|_{L_x^2}^2])^{\frac{1}{2}} \\ &\leq c\tau^2\left(\mathbb{E}\int_{\tilde{\tau}_{m-1}^R}^s \|\partial_t\mathbf{y}(\xi)\|_{L_x^4}^4 \, d\xi\right)^{\frac{1}{2}}(\mathbb{E}[\|\nabla\mathbf{e}_m\|_{L_x^2}^2])^{\frac{1}{2}} \leq c(\delta)\tau^4 R^4 + \delta\mathbb{E}[\tilde{\tau}_m^R\|\nabla\mathbf{e}_m\|_{L_x^2}^2] \end{aligned}$$

using Corollary 3.2 and $\Phi \in L_2(\mathcal{U}; W^{2,2}(\mathcal{O}; \mathbb{R}^2))$. The estimates for NLT_m^3 are analogues and lead to the same result. For the final term we split

$$\begin{aligned} \langle NLT_m^4, \mathbf{e}_m \rangle_{L_x^2} &= \left\langle \int_{\tilde{\tau}_{m-1}^R}^{\tilde{\tau}_m^R} (\nabla[\Phi W_s]\Phi W_s - \nabla[\Phi W_{\tilde{\tau}_m^R}]\Phi W_{\tilde{\tau}_{m-1}^R}) \, ds, \mathbf{e}_m - \mathbf{e}_{m-1} \right\rangle_{L_x^2} \\ &\quad + \left\langle \int_{\tilde{\tau}_{m-1}^R}^{\tilde{\tau}_m^R} (\nabla[\Phi W_s]\Phi W_s - \nabla[\Phi W_{\tilde{\tau}_m^R}]\Phi W_{\tilde{\tau}_{m-1}^R}) \, ds, \mathbf{e}_{m-1} \right\rangle_{L_x^2} \\ &=: E_m^{4,1} + E_m^{4,2}. \end{aligned}$$

³ Similarly to $E_n^{2,1,3}$ we can write $\sum_{n=1}^m E_n^{2,3,2}$ as the decomposition of a discrete martingale with m_R .

By Young’s inequality we have for $\delta > 0$ arbitrary,

$$\begin{aligned} & \mathbb{E}[\mathbb{E}_m^{4,1}] \\ & \leq \delta \mathbb{E}[\|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2] + c(\delta) \mathbb{E} \left\| \int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} (\nabla[\Phi W_s] \Phi W_s - \nabla[\Phi W_{\tilde{t}_m^R}] \Phi W_{\tilde{t}_{m-1}^R}) \, ds \right\|_{L_x^2}^2 \\ & \leq \delta \mathbb{E}[\|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2] + c(\delta) \tau^2 \mathbb{E} \left[\sup_{s \in [\tilde{t}_{m-1}^R, \tilde{t}_m^R]} \|\nabla[\Phi W_s] \Phi W_s - \nabla[\Phi W_{\tilde{t}_m^R}] \Phi W_{\tilde{t}_{m-1}^R}\|_{L_x^2}^2 \right] \\ & \leq \delta \mathbb{E}[\|\mathbf{e}_m - \mathbf{e}_{m-1}\|_{L_x^2}^2] + c(\delta) \tau^3 R_2(R)^2, \end{aligned}$$

also using Doob’s maximal inequality, the definition of \tilde{t}_m^R and $\Phi \in L_2(\mathcal{U}; W^{3,2}(\mathcal{O}; \mathbb{R}^2))$ (together with the Sobolev embedding $W^{3,2}(\mathcal{O}) \hookrightarrow W^{1,\infty}(\mathcal{O})$). In order to estimate $\mathbb{E}^{4,2}$ we write

$$\begin{aligned} & \mathbb{E}[\mathbb{E}^{4,2}] \\ & = \mathbb{E} \left[\left\langle \mathbb{E} \left[\int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \nabla[\Phi W_s] \Phi W_s - \nabla[\Phi W_{\tilde{t}_m^R}] \Phi W_{\tilde{t}_{m-1}^R} \Big| \mathfrak{F}_{t_{m-1}} \right] ds, \mathbf{e}_{m-1} \right\rangle_{L_x^2} \right] \\ & \leq \mathbb{E} \left[\left\langle \mathbb{E} \left[\int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \|\nabla[\Phi W_s] \Phi W_s - \nabla[\Phi W_{\tilde{t}_m^R}] \Phi W_{\tilde{t}_{m-1}^R}\|_{L_x^2} \Big| \mathfrak{F}_{t_{m-1}} \right] ds, \mathbf{e}_{m-1} \right\rangle_{L_x^2} \right] \\ & \leq \delta \mathbb{E}[\tilde{\tau}_m^R \|\mathbf{e}_{m-1}\|_{L_x^2}^2] + c(\delta) \tau^2 \mathbb{E} \left[\mathbb{E} \left[\sup_{t,s,\tau \in [t_{m-1}, t_m]} \|\nabla[\Phi W_s] \Phi W_s - \nabla[\Phi W_t] \Phi W_\tau\|_{L_x^2}^2 \Big| \mathfrak{F}_{t_{m-1}} \right] \right] \\ & = \delta \mathbb{E}[\|\mathbf{e}_{m-1}\|_{L_x^2}^2] + c(\delta) \tau^2 \mathbb{E} \left[\sup_{t,s,\tau \in [t_{m-1}, t_m]} \|\nabla[\Phi W_s] \Phi W_s - \nabla[\Phi W_t] \Phi W_\tau\|_{L_x^2}^2 \right]. \end{aligned}$$

Using Doob’s maximal inequality and the assumption $\Phi \in L_2(\mathcal{U}; W^{3,2}(\mathcal{O}; \mathbb{R}^2))$ (as in the estimates for $\mathbb{E}_m^{4,1}$) we conclude that

$$\mathbb{E}[\mathbb{E}^{4,1}] \leq \delta \mathbb{E}[\|\mathbf{e}_{m-1}\|_{L_x^2}^2] + c(\delta) R_2(R)^2 \tau^3.$$

Combining everything and choosing δ conveniently small we obtain

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{\mathcal{O}} |\mathbf{e}_m|^2 \, dx + \frac{1}{2} \int_{\mathcal{O}} |\mathbf{e}_m - \mathbf{e}_{m-1}|^2 \, dx + \tilde{\tau}_m^R \frac{\mu}{2} \int_{\mathcal{O}} |\nabla \mathbf{e}_m|^2 \, dx \right) \right] \\ & \leq \mathbb{E} \left[\int_{\mathcal{O}} |\mathbf{e}_{m-1}|^2 \, dx \right] + c R_2(R)^2 \mathbb{E} \left[\tilde{\tau}_m^R \int_{\mathcal{O}} |\mathbf{e}_{m-1}|^2 \, dx + \tilde{\tau}_m^R \int_{\mathcal{O}} |\mathbf{e}_m|^2 \, dx \right] \\ & \quad + c \tau^2 R_2(R)^2 \mathbb{E} \left[\int_{\tilde{t}_{m-1}^R}^{\tilde{t}_m^R} \|\nabla \partial_t \mathbf{y}(s)\|_{L_x^2}^2 \, ds \right] + c \tau^3 e^{c R_2(R)^4}. \end{aligned}$$

Iterating this inequality and noticing that

$$\sum_{n=1}^M \mathbb{E} \left[\int_{\tilde{t}_{n-1}^R}^{\tilde{t}_n^R} \|\nabla \partial_t \mathbf{y}(s)\|_{L_x^2}^2 \, ds \right] \leq \mathbb{E} \left[\int_0^{t_R} \|\nabla \partial_t \mathbf{y}(s)\|_{L_x^2}^2 \, ds \right] \leq c R^4,$$

by Corollary 3.2 as well as $\mathbf{e}_0 = 0$, yields

$$\begin{aligned} & \max_{1 \leq n \leq M} \mathbb{E} \left[\int_{\mathcal{O}} |\mathbf{e}_n|^2 \, dx \right] + \mathbb{E} \left[\sum_{n=1}^M \tilde{\tau}_n^R \int_{\mathcal{O}} |\nabla \mathbf{e}_n|^2 \, dx \right] \\ & \leq cK(R)^2 \mathbb{E} \left[\sum_{n=1}^M \tilde{\tau}_n^R \int_{\mathcal{O}} |\mathbf{e}_n|^2 \, dx \right] + c\tau^2 e^{cR_2(R)^4}. \end{aligned}$$

Applying Gronwall’s lemma shows

$$\begin{aligned} & \max_{1 \leq n \leq M} \mathbb{E} \left[\int_{\mathcal{O}} |\mathbf{e}_n|^2 \, dx \right] + \mathbb{E} \left[\sum_{n=1}^M \tilde{\tau}_n^R \int_{\mathcal{O}} |\nabla \mathbf{e}_n|^2 \, dx \right] \\ & \leq c\tau^2 e^{cK(R)^4}, \end{aligned}$$

which gives the claim. □

4 Optimal weak order in probability for LBB-stable space-time discretisations of (1.1)

Section 3 validates the (strong) order (up to) 1 in probability for semidiscretisation (3.5) for $W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)$ -valued iterates $(\mathbf{y}_m)_m$; the scheme was set up to confine to approximating only the *timely regular part* \mathbf{y} of the strong solution $\mathbf{u} = \mathbf{y} + \Phi W$ of (1.1). We recall that $\mathbf{u}_m := \mathbf{y}_m + \Phi W(t_m)$ solves (2.7). We now propose and analyse an implementable spatial discretisation of (3.5) based on the finite element method (FEM).

The study of temporal semidiscretisations in Section 3 involves exactly divergence-free iterates, and also the used test functions are from the same space $W_{0,\text{div}}^{1,2}(\mathcal{O}, \mathbb{R}^2)$, which allowed us to eliminate the pressure from the problem. This is not the case any more in the setting of a spatio-temporal discretisation—e.g. via well-known LBB-stable mixed finite element pairings $(V^h(\mathcal{O}; \mathbb{R}^2), P^h(\mathcal{O}))$ on quasi-uniform meshes \mathcal{T}_h of size $0 < h \ll 1$ covering \mathcal{O} , which, in particular, involve the space $P^h(\mathcal{O})$ for discrete pressures.

We work with a standard finite element set-up for incompressible fluid mechanics; see e.g. Girault & Raviart (1986). We denote by \mathcal{T}_h a quasi-uniform subdivision of \mathcal{O} into triangles of maximal diameter $h > 0$. For $K \subset \mathcal{O}$ and $\ell \in \mathbb{N}_0$ we denote by $\mathcal{P}_\ell(K)$ the polynomials on K of degree less than or equal to ℓ . Let us characterize the finite element spaces $V^h(\mathcal{O}; \mathbb{R}^2)$ and $P^h(\mathcal{O})$ as

$$\begin{aligned} V^h(\mathcal{O}; \mathbb{R}^2) & := \{ \mathbf{V}_h \in W_0^{1,2}(\mathcal{O}; \mathbb{R}^2) : \mathbf{V}_h|_K \in \mathcal{P}_i(K; \mathbb{R}^2) \, \forall K \in \mathcal{T}_h \}, \\ P^h(\mathcal{O}) & := \{ R_h \in L^2(\mathcal{O})/\mathbb{R} : R_h|_K \in \mathcal{P}_j(K) \, \forall K \in \mathcal{T}_h \}, \end{aligned}$$

for some $i, j \in \mathbb{N}_0$. In order to guarantee stability of our approximations we relate $V^h(\mathcal{O}; \mathbb{R}^2)$ and $P^h(\mathcal{O})$ by the discrete LBB-condition, that is we assume that

$$\sup_{\mathbf{V}_h \in V^h(\mathcal{O}; \mathbb{R}^2)} \frac{\int_{\mathcal{O}} \text{div} \mathbf{V}_h R_h \, dx}{\| \nabla \mathbf{V}_h \|_{L_x^2}} \geq C \| R_h \|_{L_x^2} \quad \forall R_h \in P^h(\mathcal{O}),$$

where $C > 0$ does not depend on h . This gives a relation between i and j (for instance the choice $(i, j) = (1, 0)$ is excluded, whereas $(i, j) = (2, 0)$ is allowed). Finally, we define the space of discretely solenoidal finite element functions by

$$V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2) := \{\mathbf{V}_h \in V^h(\mathcal{O}; \mathbb{R}^2) : \int_{\mathcal{O}} \text{div} \mathbf{V}_h R_h \, dx = 0 \quad \forall R_h \in P^h(\mathcal{O})\}.$$

Let $\Pi_h : L^2(\mathcal{O}; \mathbb{R}^2) \rightarrow V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2)$ be the $L^2(\mathcal{O}; \mathbb{R}^2)$ -orthogonal projection onto $V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2)$; see Heywood & Rannacher (1982), for instance. We are now ready to define an LBB-stable mixed FEM for (3.5) based on mixed finite elements: Let $\mathbf{Y}_0 \in V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2)$. For any $m \geq 1$, find $V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2) \times P^h(\mathcal{O})$ -valued random variables (\mathbf{Y}_m, P_m) such that \mathbb{P} -a.s.

$$\begin{aligned} & \int_{\mathcal{O}} (\mathbf{Y}_m - \mathbf{Y}_{m-1}) \cdot \boldsymbol{\varphi} \, dx + \mu \tau \int_{\mathcal{O}} \nabla \mathbf{Y}_m : \nabla \boldsymbol{\varphi} \, dx - \tau \int_{\mathcal{O}} P_m \cdot \text{div} \boldsymbol{\varphi} \, dx \\ &= \tau \int_{\mathcal{O}} \left(\mu \nabla[\Phi W(t_m)] : \nabla \boldsymbol{\varphi} - ((\nabla \mathbf{Y}_m) \mathbf{Y}_{m-1} + \frac{1}{2} [\text{div} \mathbf{Y}_{m-1}] \mathbf{Y}_m) \cdot \boldsymbol{\varphi} \right) dx \\ & \quad + \int_{\mathcal{O}} \widetilde{\mathcal{L}}^m(\mathbf{Y}_{m-1}, \mathbf{Y}_m) \cdot \boldsymbol{\varphi} \, dx, \end{aligned} \tag{4.1}$$

$$\int_{\mathcal{O}} \text{div} \mathbf{Y}_m R \, dx = 0, \tag{4.2}$$

for all $(\boldsymbol{\varphi}, R) \in V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2) \times P^h(\mathcal{O})$, where the following map approximates \mathcal{L}^m from (3.5):

$$\begin{aligned} \widetilde{\mathcal{L}}^m(\mathbf{Y}_{m-1}, \mathbf{Y}_m) &= \nabla \Pi_h[\Phi W(t_m)] \mathbf{Y}_{m-1} + \nabla \mathbf{Y}_m \Pi_h(\Phi W(t_{m-1})) \\ & \quad - \nabla \Pi_h[\Phi W(t_m)] \Pi_h \Phi W(t_{m-1}) + \frac{1}{2} [\text{div} \mathbf{Y}_{m-1}] \Pi_h[\Phi W(t_m)] \\ & \quad + \frac{1}{2} [\text{div} \Pi_h[\Phi W(t_{m-1})]] \Pi_h[\Phi W(t_m)] + \frac{1}{2} [\text{div} \Pi_h[\Phi W(t_{m-1})]] \mathbf{Y}_m. \end{aligned} \tag{4.3}$$

The additional term $\frac{1}{2} \int_{\mathcal{O}} [\text{div} \mathbf{Y}_{m-1}] \mathbf{Y}_m \cdot \boldsymbol{\varphi} \, dx$ in (4.1) is used to control nonlinear effects in the presence of *discretely* divergence-free velocity iterates, as do the last three terms in (4.3). The reason why the scheme is set up in this form will become clear if we consider the $V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2)$ -valued random variable

$$\mathbf{U}_m := \mathbf{Y}_m + \Pi_h[\Phi W(t_m)], \tag{4.4}$$

which solves the equation

$$\begin{aligned} & \int_{\mathcal{O}} \mathbf{U}_m \cdot \boldsymbol{\varphi} \, dx + \tau \int_{\mathcal{O}} ((\nabla \mathbf{U}_m) \mathbf{U}_{m-1} + (\text{div} \mathbf{U}_{m-1}) \mathbf{U}_m) \cdot \boldsymbol{\varphi} \, dx \\ & \quad + \mu \tau \int_{\mathcal{O}} \nabla \mathbf{U}_m : \nabla \boldsymbol{\varphi} \, dx - \tau \int_{\mathcal{O}} P_m \text{div} \boldsymbol{\varphi} \, dx \\ &= \int_{\mathcal{O}} \mathbf{U}_{m-1} \cdot \boldsymbol{\varphi} \, dx + \int_{\mathcal{O}} \Phi \Delta_m W \cdot \boldsymbol{\varphi} \, dx \end{aligned} \tag{4.5}$$

for all $\boldsymbol{\varphi} \in V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2)$. This combined spatio-temporal discretisation with $V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2)$ -valued $(\mathbf{U}_m)_m$ was investigated in Breit & Prohl (2022) in the more general context of multiplicative noise in (1.1); the key observation now is that for the spatial error analysis for (4.5) it suffices to bound $\max_m \|\mathbf{Y}_m - \mathbf{y}_m\|_{L_x^2}$ thanks to (4.4), which only involves spatially regular noise. Therefore, we isolate from the convergence proof for (4.1)–(4.2) those arguments from Breit & Prohl (2022) for (4.5), which address the *spatial discretisation* only. We have the following result.

THEOREM 4.1 Assume that $\mathbf{u}_0 \in L^r(\Omega, W_{\text{div}}^{2,2}(\mathcal{O}; \mathbb{R}^2)) \cap L^{5r}(\Omega, W_{\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2))$ for some $r \geq 8$ and that $\Phi \in L_2(\mathfrak{U}; W_{0,\text{div}}^{1,2} \cap W^{3,2}(\mathcal{O}; \mathbb{R}^2))$. Let

$$(\mathbf{u}, (\mathfrak{t}_R)_{R \in \mathbb{N}}, \mathfrak{t})$$

be the unique global maximal strong solution to (1.1) from Theorem 2.2. Then we have for any $\xi > 0$, $\alpha < 1$,

$$\max_{1 \leq m \leq M} \mathbb{P} \left[\|\mathbf{y}(t_m) - \mathbf{Y}_m\|_{L_x^2}^2 + \sum_{n=1}^m \tau \|\nabla \mathbf{y}(t_n) - \nabla \mathbf{Y}_n\|_{L_x^2}^2 > \xi (\tau^{2\alpha} + h^{2\alpha}) \right] \rightarrow 0$$

as $\tau, h \rightarrow 0$, where \mathbf{y} is the solution to (3.2) and $(\mathbf{Y}_m)_{m=1}^M$ is the solution to (4.1)–(4.2).

Let $\tilde{\mathbf{u}}_0^R := \mathbf{u}_0$. To prepare for the proof of Theorem 4.1 we introduce the $W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)$ -iterates $(\tilde{\mathbf{u}}_m^R)_{m=1}^M$ solving for $m \geq 1$,

$$\begin{aligned} & \int_{\mathcal{O}} \tilde{\mathbf{u}}_m^R \cdot \boldsymbol{\varphi} \, dx + \tilde{\tau}_m^R \int_{\mathcal{O}} (\nabla \tilde{\mathbf{u}}_m^R) \mathbf{u}_{m-1}^R : \nabla \boldsymbol{\varphi} \, dx \\ & + \mu \tilde{\tau}_m^R \int_{\mathcal{O}} \nabla \tilde{\mathbf{u}}_m^R : \nabla \boldsymbol{\varphi} \, dx = \int_{\mathcal{O}} \tilde{\mathbf{u}}_{m-1}^R \cdot \boldsymbol{\varphi} \, dx + \frac{\tilde{\tau}_m^R}{\tau} \int_{\mathcal{O}} \Phi \Delta_m W \cdot \boldsymbol{\varphi} \, dx, \end{aligned} \tag{4.6}$$

for every $\boldsymbol{\varphi} \in W_{0,\text{div}}^{1,2}(\mathcal{O}; \mathbb{R}^2)$, where $\tilde{\mathbf{u}}_m^R = \mathbf{u}_m$ in $\{t_m \leq \tilde{\tau}_m^R\}$ with $\tilde{\tau}_m^R$ and $\tilde{\tau}_m^R$ introduced in (3.15). Accordingly, we define $V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2)$ -valued iterates $(\tilde{\mathbf{U}}_m^R)_m$ solving

$$\begin{aligned} & \int_{\mathcal{O}} \tilde{\mathbf{U}}_m^R \cdot \boldsymbol{\varphi} \, dx + \tilde{\tau}_m^R \int_{\mathcal{O}} ((\nabla \tilde{\mathbf{U}}_m^R) \tilde{\mathbf{U}}_{m-1}^R + (\text{div} \tilde{\mathbf{U}}_{m-1}^R) \tilde{\mathbf{U}}_m^R) \cdot \boldsymbol{\varphi} \, dx \\ & + \mu \tilde{\tau}_m^R \int_{\mathcal{O}} \nabla \tilde{\mathbf{U}}_m^R : \nabla \boldsymbol{\varphi} \, dx = \int_{\mathcal{O}} \tilde{\mathbf{U}}_{m-1}^R \cdot \boldsymbol{\varphi} \, dx + \frac{\tilde{\tau}_m^R}{\tau} \int_{\mathcal{O}} \Phi \Delta_m W \cdot \boldsymbol{\varphi} \, dx \end{aligned} \tag{4.7}$$

for every $\boldsymbol{\varphi} \in V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2)$. Again, $\tilde{\mathbf{U}}_m^R = \mathbf{U}_m$ in $[t_m \leq \tilde{\tau}_m^R]$.

Next, by setting $R = c^{-1/4} \sqrt[4]{-\varepsilon \log h}$ where $\varepsilon > 0$ is arbitrary, we have for any $\xi > 0$,

$$\begin{aligned} & \mathbb{P} \left[\max_{1 \leq m \leq M} \|\mathbf{u}_m - \mathbf{U}_m\|_{L_x^2}^2 + \sum_{m=1}^M \tau \|\nabla \mathbf{u}_m - \nabla \mathbf{U}_m\|_{L_x^2}^2 > \xi h^{2-2\varepsilon} \right] \\ & \leq \mathbb{P} \left[\max_{1 \leq m \leq M} \|\tilde{\mathbf{u}}_m^R - \tilde{\mathbf{U}}_m^R\|_{L_x^2}^2 + \sum_{m=1}^M \tilde{\tau}_m^R \|\nabla \tilde{\mathbf{u}}_m^R - \nabla \tilde{\mathbf{U}}_m^R\|_{L_x^2}^2 > \xi h^{2-2\varepsilon} \right] \\ & \quad + \mathbb{P}[\{\tilde{\tau}_R^d < T\}]. \end{aligned}$$

The last term vanishes \mathbb{P} -a.s. by (3.9) as $R \rightarrow \infty$; for the first term to converge to zero for $h \rightarrow 0$, we use Markov’s inequality, and the bound

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|\tilde{\mathbf{u}}_m^R - \tilde{\mathbf{U}}_m^R\|_{L_x^2}^2 + \sum_{m=1}^M \tilde{\tau}_m^R \|\nabla \tilde{\mathbf{u}}_m^R - \nabla \tilde{\mathbf{U}}_m^R\|_{L_x^2}^2 \right] \leq ce^{cR^4} h^2, \tag{4.8}$$

whose proof will be sketched. Since the error between $\Pi_h \Phi W$ (from (4.4)) and ΦW (from (3.6)) can be controlled by the assumed regularity of Φ and the approximation properties of Π_h , we conclude that

$$\mathbb{P} \left[\max_{1 \leq m \leq M} \|\mathbf{y}_m - \mathbf{Y}_m\|_{L_x^2}^2 + \sum_{m=1}^M \tau \|\nabla \mathbf{y}_m - \nabla \mathbf{Y}_m\|_{L_x^2}^2 > \xi h^{2-2\varepsilon} \right] \rightarrow 0,$$

which yields the claim of Theorem 4.1 due to Theorem 3.5.

We are left with the proof of (4.8). We define the error $\mathbf{E}_m^R = \tilde{\mathbf{u}}_m^R - \tilde{\mathbf{U}}_m^R$. Subtracting (4.7) from (4.6) and using the additive character of the noise gives, for every $\phi \in V_{\text{div}}^h(\mathcal{O}; \mathbb{R}^2)$,

$$\begin{aligned} & \int_{\mathcal{O}} \tilde{\mathbf{E}}_m^R \cdot \phi \, dx + \mu \tilde{\tau}_m^R \int_{\mathcal{O}} \nabla \tilde{\mathbf{E}}_m^R : \nabla \phi \, dx \\ &= \int_{\mathcal{O}} \mathbf{E}_{m-1}^R \cdot \phi \, dx + \tilde{\tau}_m^R \int_{\mathcal{O}} \tilde{p}_m^R \operatorname{div} \phi \, dx \\ & \quad - \tilde{\tau}_m^R \int_{\mathcal{O}} \left((\nabla \tilde{\mathbf{u}}_m^R) \tilde{\mathbf{u}}_{m-1}^R - ((\nabla \tilde{\mathbf{U}}_m^R) \tilde{\mathbf{U}}_{m-1}^R + (\operatorname{div} \tilde{\mathbf{U}}_{m-1}^R) \tilde{\mathbf{U}}_m^R) \right) \cdot \phi \, dx, \end{aligned} \tag{4.9}$$

with

$$\tilde{p}_m^R := \begin{cases} p_m & \text{in } [t_m = \tilde{\tau}_m^R], \\ p_{m_R} & \text{in } [t_m > \tilde{\tau}_m^R], \end{cases}$$

where m_R is defined below (3.8). Here p_m from (2.20) enters due to only *discretely* solenoidal test functions. Setting $\phi = \Pi_h \tilde{\mathbf{E}}_m^R$, standard manipulations then lead to

$$\begin{aligned} & \int_{\mathcal{O}} \frac{1}{2} (|\Pi_h \tilde{\mathbf{E}}_m^R|^2 - |\Pi_h \tilde{\mathbf{E}}_{m-1}^R|^2 + |\Pi_h \tilde{\mathbf{E}}_m^R - \Pi_h \tilde{\mathbf{E}}_{m-1}^R|^2) \, dx + \mu \tilde{\tau}_m^R \int_{\mathcal{O}} |\nabla \tilde{\mathbf{E}}_m^R|^2 \, dx \\ &= -\mu \tilde{\tau}_m^R \int_{\mathcal{O}} \nabla \tilde{\mathbf{E}}_m^R : \nabla (\tilde{\mathbf{u}}_m^R - \Pi_h \tilde{\mathbf{u}}_m^R) \, dx + \tilde{\tau}_m^R \int_{\mathcal{O}} \tilde{p}_m^R \operatorname{div} \Pi_h \tilde{\mathbf{E}}_m^R \, dx \\ & \quad - \tilde{\tau}_m^R \int_{\mathcal{O}} \left((\nabla \tilde{\mathbf{u}}_m^R) \tilde{\mathbf{u}}_{m-1}^R - ((\nabla \tilde{\mathbf{U}}_m^R) \tilde{\mathbf{U}}_{m-1}^R + (\operatorname{div} \tilde{\mathbf{U}}_{m-1}^R) \tilde{\mathbf{U}}_m^R) \right) \cdot \Pi_h \tilde{\mathbf{E}}_m^R \, dx \\ &=: I_1(m) + \dots + I_3(m). \end{aligned}$$

We use well-known approximation properties of Π_h to bound

$$I_1(m) \leq \frac{\mu}{2} \tilde{\tau}_m^R \int_{\mathcal{O}} |\nabla \tilde{\mathbf{E}}_m^R|^2 \, dx + c \tilde{\tau}_m^R h^2 \int_{\mathcal{O}} |\nabla^2 \tilde{\mathbf{u}}_m^R|^2 \, dx,$$

and (2.11) may now be applied to optimally bound the last term. A corresponding longer but elementary estimation, which rests on the same ingredients, bounds $I_3(m)$; see also Breit & Prohl, (2022, bounds for term $I_4(m)$ in the proof of Thm. 4.2). For $I_2(m)$, we use well-known approximation properties for the L^2 -projection $Q_h : L^2(\mathcal{O}) \rightarrow P^h(\mathcal{O})$, and (4.2) and (2.19) to conclude

$$\begin{aligned} I_2(m) &= \tilde{\tau}_m^R \int_{\mathcal{O}} (\tilde{p}^m - Q_h \tilde{p}^m) \operatorname{div} \Pi_h \tilde{\mathbf{E}}_m^R \, dx \\ &\leq \kappa \tilde{\tau}_m^R \int_{\mathcal{O}} |\nabla \tilde{\mathbf{E}}_m^R|^2 \, dx + c(\kappa) \tilde{\tau}_m^R h^2 \int_{\mathcal{O}} |\nabla \tilde{p}_m^R|^2 \, dx; \end{aligned} \quad (4.10)$$

see again Breit & Prohl, (2022, bound for term $I_7(m)$ in the proof of Thm. 4.2). The expectation of the sum of the leading term is bounded by ce^{cR^4} , due to (2.19). The proof of (4.8) (and hence that of Theorem 4.1) is thus complete.

REMARK 4.2 For the stochastic Stokes equation instead of (1.1), and a corresponding simplification of scheme (4.1)–(4.2), we have for all $\alpha < 1$,

$$\max_{1 \leq m \leq M} \mathbb{E} \left[\|\mathbf{y}(t_m) - \mathbf{Y}_m\|_{L^2}^2 + \sum_{n=1}^m \tau \|\nabla \mathbf{y}(t_n) - \nabla \mathbf{Y}_n\|_{L_x^2}^2 \right] \leq c(\tau^2 + h^2). \quad (4.11)$$

The first part of (4.11), which addresses semidiscretisation-in-time results, starts from error identity (3.18) in simplified form—without the term NLT_m . Furthermore, no (discrete) stopping times are needed, the stopping times in the counterparts of Lemmas 3.1, 3.3 and Corollary 3.2 are all equal to T . The second part of (4.11) on semidiscretisation in space starts from error identity (4.9) in simplified form, without the last term that addresses the role of the nonlinearity. Again, involved stopping times are all T , which settles (4.11).

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