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QUANTUM GROUPS, DISCRETE MAGNUS EXPANSION, PRE-LIE & TRIDENDRIFORM ALGEBRAS

ANASTASIA DOIKOU

ABSTRACT. We review the discrete evolution problem and the corresponding solution as a discrete Dyson series in order to rigorously derive a discrete version of Magnus expansion. We also systematically derive the discrete analogue of the pre-Lie Magnus expansion and show that the elements of the discrete Dyson series are expressed in terms of a tridendriform algebra action. Key links between quantum algebras, tridendriform and pre-Lie algebras are then established. This is achieved by examining tensor realizations of quantum groups, such as the Yangian. We show that these realizations can be expressed in terms of tridendriform and pre-Lie algebras actions. The continuous limit as expected provides the corresponding non-local charges of the Yangian as members of the pre-Lie Magnus expansion.

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1. INTRODUCTION

In the present study we identify interesting links between quantum groups [25, 19, 27], and tridendriform [30, 31] and pre-Lie algebras (also studied under the name chronological algebras) [2, 26, 47] (see also [5, 35] for a recent reviews). Specifically, we systematically derive the discrete analogues of Dyson series [20] and Magnus expansion [34] as solutions of a discrete evolution problem. We

then show that the Dyson series members are expressed in terms of a tridendriform algebra action, whereas the discrete Magnus expansion members are derived in relation to a pre-Lie algebra action in analogy to the continuous case (see also relevant findings in [22]). The use of Rota-Baxter operators [6, 39] has been essential in expressing the discrete series in connection with tridendriform and pre-Lie algebra actions. On the other hand, tensor realizations of quantum groups [25, 19], such as the Yangian [19], are also solutions of a discrete evolution problem. Hence, we deduce that the coproducts of the elements of the Yangian can be re-expressed in terms of suitable tridendriform and pre-Lie algebras actions.

Before we describe in detail what is achieved in each section we first recall the general set up and some necessary preliminaries on the Magnus expansion as a solution of a linear evolution problem, whereas in the subsequent section we briefly recall basic notions on Rota-Baxter, pre-Lie and tridendriform algebras. We note that interesting relations between pre-Lie algebras, rooted tree graphs, (tri)dendriform and Rota-Baxter algebras have been reported (see for instance [30, 22, 21, 23]), whereas links between tridendriform, Rota-Baxter and (quasi)shuffle algebras have been also revealed in [32]. The recent findings on the relationships between pre-Lie algebras and braces (nilpotent rings) [41, 46], have generated increased interest on these distinct algebraic structures opening up unexplored research avenues. It is worth pointing out that the notion of infinitesimal Hopf algebras and their connections to pre-Lie and dendriform algebras have been explored in [3]. In this study however we establish links with typical Hopf algebras, such as the Yangian that also appear in quantum integrable systems. Some of these profound emerging relations in the frame of classical and quantum integrability lie in the epicenter of our analysis, while others will be further examined and extended in future works.

We start our discussion by recalling the initial value problem associated to a linear differential equation. Indeed, let A, T be in general some linear operators e.g. ($T, A \in \text{End}(\mathbb{C}^N)$) depending on two parameters, $\xi \in \mathbb{R}, \alpha \in \mathbb{C}$, such that

$$\partial_\xi T(\xi, \alpha) = \alpha A(\xi)T(\xi, \alpha), \quad T(x_0, \alpha) = T_0. \quad (1.1)$$

The formal solution of the evolution equation above can be given as (we consider simple initial conditions $T(x_0, \alpha) = 1$)

$$T(x, \alpha) = \widehat{\text{Pexp}}\left(\alpha \int_{x_0}^x A(\xi) d\xi\right), \quad x > x_0. \quad (1.2)$$

The latter solution is a *path ordered* exponential (monodromy), which is formally expressed in terms of Dyson series [20]

$$\widehat{\text{Pexp}}\left(\alpha \int_{x_0}^x A(\xi) d\xi\right) = \sum_{n=0}^{\infty} \alpha^n \int_{x_0}^x dx_n A(x_n) \int_{x_0}^{x_n} dx_{n-1} A(x_{n-1}) \dots \int_{x_0}^{x_2} dx_1 A(x_1).$$

Magnus [34] suggested that the solution $T(x, \alpha)$ of the linear evolution problem can be expressed as a real exponential, i.e. $T(x, \alpha) = e^{\mathbb{Q}(x, \alpha)}$ such that $e^{\mathbb{Q}(x, \alpha)} := 1 + \sum_{n=1}^{\infty} \frac{\mathbb{Q}^n(x, \alpha)}{n!}$, where the following formal series are considered $T(x, \alpha) = 1 + \sum_{n=1}^{\infty} \alpha^n T^{(n)}(x)$, $\mathbb{Q}(x, \alpha) = \sum_{n=1}^{\infty} \alpha^n \mathbb{Q}^{(n)}(x)$

and

$$T^{(n)}(x) = \int_{x_0}^x dx_n A(x_n) \int_{x_0}^{x_n} dx_{n-1} A(x_{n-1}) \dots \int_{x_0}^{x_2} dx_1 A(x_1). \quad (1.3)$$

Comparing the α series expansion of $T(x, \alpha)$ and $e^{\mathbb{Q}(x, \alpha)}$ we obtain the coefficients $\mathbb{Q}^{(n)}$ as symmetric polynomials of $T^{(n)}$ ($T^{(n)}$, $\mathbb{Q}^{(n)}$ below depend on x):

$$\mathbb{Q}^{(1)} = T^{(1)}, \quad \mathbb{Q}^{(2)} = T^{(2)} - \frac{1}{2}(T^{(1)})^2, \quad (1.4)$$

$$\mathbb{Q}^{(3)} = T^{(3)} - \frac{1}{2}(T^{(1)}T^{(2)} + T^{(2)}T^{(1)}) + \frac{1}{3}(T^{(1)})^3, \quad \dots \quad (1.5)$$

and vice versa all $T^{(m)}$'s can be expressed in terms of $\mathbb{Q}^{(m)}$'s,

$$T^{(1)} = \mathbb{Q}^{(1)}, \quad T^{(2)} = \mathbb{Q}^{(2)} + \frac{1}{2}(\mathbb{Q}^{(1)})^2, \quad (1.6)$$

$$T^{(3)} = \mathbb{Q}^{(3)} + \frac{1}{2}(\mathbb{Q}^{(1)}\mathbb{Q}^{(2)} + \mathbb{Q}^{(2)}\mathbb{Q}^{(1)}) + \frac{1}{3!}(\mathbb{Q}^{(1)})^3, \quad \dots \quad (1.7)$$

Every element $\mathbb{Q}^{(m)}$ of the series expansion can be obtained by means of the generic recursive formula (see for instance [8] and references therein):

$$\mathbb{Q}^{(m)} = T^{(m)} - \sum_{k=2}^m (-1)^m \frac{\Pi_k^{(m)}}{k}, \quad n > 1, \quad (1.8)$$

where $\Pi_k^{(n)} = \sum T^{(j_1)} \dots T^{(j_k)} : j_1 + \dots + j_k = n$ and satisfy the recursion formula:

$$\Pi_k^{(n)} = \sum_{m=1}^{n-k+1} \Pi_1^{(m)} \Pi_{k-1}^{(n-m)} \quad \text{and} \quad \Pi_1^{(n)} = T^{(n)}, \quad \Pi_n^{(n)} = (T^{(1)})^n. \quad (1.9)$$

Hence, we obtain, after recalling (4.3), the explicit expressions for the first three terms of the series (*Magnus expansion*):

$$\mathbb{Q}^{(1)}(x) = \int_{x_0}^x dx_1 A(x_1), \quad (1.10)$$

$$\mathbb{Q}^{(2)}(x) = \frac{1}{2} \int_{x_0}^x dx_2 \int_{x_0}^{x_2} dx_1 [A(x_2), A(x_1)],$$

$$\mathbb{Q}^{(3)}(x) = \frac{1}{6} \int_{x_0}^x dx_3 \int_{x_0}^{x_3} dx_2 \int_{x_0}^{x_2} dx_1 ([A(x_3), [A(x_2), A(x_1)] + [[A(x_3), A(x_2)], A(x_1)]).$$

Remark 1.1. *Magnus (Theorem [34]) obtained the general expression for $\mathbb{Q}(x)$ as an infinite series involving Bernoulli's numbers (see also [8] and references therein). We first introduce some useful notation: Let A, B be linear operators, and recall the binary operation $[\ , \] : (A, B) \mapsto AB - BA$, i.e. the familiar Lie commutator. We also define*

$$ad_A B = [A, B], \quad ad_A^n = [A, ad_A^{n-1} B], \quad ad_A^0 B = B, \quad (1.11)$$

and recall the Bernoulli numbers B_n defined as $\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}$. Then $\mathbb{Q}(x)$ can be expressed in a compact form as

$$\mathbb{Q}(x) = \int_{x_0}^x ds \sum_{n=0}^{\infty} \frac{B_n}{n!} ad_{\mathbb{Q}(x)}^n A(s). \quad (1.12)$$

Expressions (1.10) can be then obtained from (1.12) by iteration. For a detailed discussion on Magnus expansion, convergence issues, expansion generators and applications the interested reader is referred for instance to [8] and references therein.

After the brief review on Magnus expansion we recall in the subsequent section some of the fundamental notions necessary of our analysis in Sections 3 and 4. More precisely, we describe below what is achieved in each section.

- In Subsection 2.1 we recall the definitions of Rota-Baxter, pre-Lie and tridendriform algebras and we then discuss the connections among these algebras. To illustrate these relations we use two simple examples, which will be exploited anyway in this study. In subsection 2.2 in order to further motivate the study of deep interconnections among seemingly distinct algebraic structures we recall the passage from pre-Lie algebras to braces [41, 46].
- In section 3 the rigorous derivation of the discrete analogue of Magnus expansion is exhibited, after having first derived the discrete version of Dyson series. These derivations are realized by means of the discrete analogue of the evolution problem (1.1). Linearization of the discrete evolution problem leads naturally to the continuous equation (1.1). Furthermore, we show that the discrete Dyson series are expressed in terms of a tridendriform algebra actions, whereas the Magnus expansion is expressed in terms of a pre-Lie algebra action. Construction of a brace multiplication from this pre-Lie algebra immediately follows. Explicit expressions of the first few elements of both expansions are provided. Taking the continuum limits of these expressions we recover the continuous Magnus expression. and the pre-Lie Magnus formula. In subsection 3.1 we consider alternative discrete Dyson and Magnus expansions, which again can be expressed using tridendriform and pre-Lie algebra actions. The various discrete expansions are associated to distinct quantum algebras as will be transparent in section 4. In subsection 3.2 we present two basic examples/applications related to the “backward” and “forward” Dyson and Magnus expansions. The first example is related to gauge transformations of matrix valued fields, whereas the second one describes the discrete and continuous evolution problem of certain classes of open boundary systems.
- In section 4 we investigate the strong ties between quantum algebras, specifically the Yangian and tridendriform and pre-Lie algebras. We first recall the derivation of the Yangian via the Faddeev-Reshetikhin-Takhtajan (FRT) construction. We extract an alternative set of generators using the Lie exponential of the solutions of the FRT relation and derive the defining algebraic relations of the alternative set of generators. We then move on to study tensor realizations of the Yangian for both sets of generators and express co-products of the Yangian generators using actions of pre-Lie and tridendriform algebras. The classical Yangian is briefly discussed after we introduce the notions of the classical r -matrix and

Sklyanin's bracket [24]. We conclude that the non-local charges of the classical Yangian are naturally expressed in terms of (tri)dendriform algebra actions.

2. PRELIMINARIES

2.1. Rota Baxter, pre-Lie & tridendriform algebras. We shall now briefly recall the notions of pre-Lie algebras [2, 26] and the pre-Lie Magnus expansion, which will be essential to our analysis here. Before we comment on this interesting connection we first introduce the definitions of Rota-Baxter and pre-Lie algebras [6, 39] (see also [5, 22, 23, 35] and references therein).

Definition 2.1. *A Rota-Baxter algebra is a k -vector space \mathcal{A} equipped with a linear map $R : \mathcal{A} \rightarrow \mathcal{A}$, such that $\forall a, b \in \mathcal{A}$*

$$R(a)R(b) = R(R(a)b + aR(b) + \theta ab), \quad (2.1)$$

where $\theta \in k$ is a fixed parameter.

The map R is called a Rota-Baxter operator of weight θ .

Definition 2.2. *A pre-Lie algebra is a k -vector space A with a binary operation $\triangleright : (a, b) \mapsto a \triangleright b$, such that it satisfies the pre-Lie identity $\forall a, b, c \in A$,*

$$(a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) = (b \triangleright a) \triangleright c - b \triangleright (a \triangleright c). \quad (2.2)$$

Analogously, a right pre-Lie algebra with a binary operation \triangleleft can be defined with a pre-Lie identity:

$$(a \triangleleft b) \triangleleft c - a \triangleleft (b \triangleleft c) = (a \triangleleft c) \triangleleft b - a \triangleleft (c \triangleleft b). \quad (2.3)$$

In the following Proposition it is shown that Rota-Baxter operators of weight θ can be used to construct a pre-Lie algebra action (see also for instance [5, 22] and references therein).

Proposition 2.3. *Let $R : \mathcal{A} \rightarrow \mathcal{A}$ be a Rota-Baxter operator of weight θ and let \triangleright be a binary operation: $(a, b) \mapsto a \triangleright b$, $\forall a, b \in \mathcal{A}$ such that*

$$a \triangleright b := [R(a), b] + \theta ab. \quad (2.4)$$

Then the binary operation \triangleright satisfies the pre-Lie identity (2.2).

Proof. Using definition (2.4) and assuming (in accordance to the cases examined here) that $R(a + b) = R(a) + R(b)$ we first show that $R(-x) = R(0) - R(x)$, indeed

$$R(x - x) = R(0) \Rightarrow R(x) + R(-x) = R(0) \Rightarrow R(-x) = R(0) - R(x). \quad (2.5)$$

We then explicitly compute:

$$\begin{aligned} (x \triangleright y) \triangleright z &= R(R(x)y - yR(x) + \theta xy)z - zR(R(x)y - yR(x) + \theta xy) \\ &+ \theta(R(x)yz - yR(x)z + \theta xyz). \end{aligned} \quad (2.6)$$

Similarly, using (2.1):

$$\begin{aligned} x \triangleright (y \triangleright z) &= R(R(x)y + xR(y) + \theta xy)z + zR(R(y)x + yR(x) + \theta yx) \\ &+ \theta(R(x)yz - yzR(x) + xR(y)z - xzR(y) + \theta xyz) - R(x)zR(y) - R(y)zR(x) \end{aligned} \quad (2.7)$$

And from the two last expressions we conclude that (recall $R(a + b) = R(a) + R(b)$ and $R(-x) = R(0) - R(x)$):

$$\begin{aligned} (x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = & - R(R(x)y + xR(y))z - zR(R(x)y + xR(y)) \\ & - \theta z(R(xy) + R(yx)) - \theta(yR(x) + xR(y)) \\ & + \theta(yzR(x) + xzR(y)) + R(x)zR(y) + R(y)zR(x). \end{aligned} \quad (2.8)$$

The latter expression is symmetric in x and y and leads to the pre-Lie algebra identity. \square

We present two simple examples of Rota-Baxter operators, which will be used in our analysis.

Example 2.4. *A simple example of Rota-Baxter operator is given by the ordinary Riemann integral, which is a weight zero Rota-Baxter map (other examples can be found for instance in [22] and references therein). Indeed, let $S(f)_x := \int_0^x f(\zeta)d\zeta$, then*

$$\begin{aligned} S(f)_x S(g)_x &= \int_0^x f(\zeta)d\zeta \int_0^x g(\xi)d\xi = \int_0^x d\zeta \int_0^\zeta d\xi f(\zeta)g(\xi) + \int_0^x d\zeta \int_0^\zeta d\xi f(\xi)g(\zeta) \\ &= S(S(f)_\zeta g_\zeta + f_\zeta S(g)_\zeta)_x \end{aligned} \quad (2.9)$$

i.e. S is a Rota-Baxter operator of zero weight.

Let A, B be linear operators that depend on a continuous parameter x , we define the following action

$$(A \triangleright B)(x) := \left[\int_0^x A(s)ds, B(x) \right], \quad (2.10)$$

which provides a non-commutative binary operation, e.g. A and B can be matrix valued functions of x . It turns out according to Proposition 2.3 that the binary operation defined in (2.10) satisfies the pre-Lie identity (2.2).

The elements of the Magnus expansion can be re-expressed in terms of a pre-Lie algebra action –pre-Lie Magnus expansion– (see also [8, 22] and references therein). It is a matter of some tedious computation and use of the form of expressions (1.10) and (2.10) to show that

$$\begin{aligned} \mathbb{Q}^{(1)}(x) &= \int_{x_0}^x dx_1 A(x_1), \\ \mathbb{Q}^{(2)}(x) &= -\frac{1}{2} \int_{x_0}^x dx_2 (A \triangleright A)(x_2), \\ \mathbb{Q}^{(3)}(x) &= \int_{x_0}^x dx_3 \left(\frac{1}{4} ((A \triangleright A) \triangleright A)(x_3) + \frac{1}{12} (A \triangleright (A \triangleright A))(x_3) \right). \end{aligned} \quad (2.11)$$

Example 2.5. *Let f, g be linear operators and let the map $\Sigma : \Sigma(f)_n = \sum_{m=1}^{n-1} f_m$. Then*

$$\Sigma(f)\Sigma(g) = \Sigma(\Sigma(f)g + f\Sigma(g) + fg), \quad (2.12)$$

i.e. Σ is a Rota-Baxter operator of weight one.

Indeed, we rewrite equation (2.12) with the suitable discrete variables:

$$\begin{aligned}
 \Sigma(f)_n \Sigma(g)_n &= \sum_{m=1}^{n-1} f_m \sum_{k=1}^{n-1} g_k = \sum_{m>k} f_m g_k + \sum_{m<k} f_k g_m + \sum_{m=1}^{n-1} f_n g_n \\
 &= \sum_{m=1}^{n-1} (f_m \Sigma(g)_m + \Sigma(f)_m g_m + f_m g_m) \\
 &= \Sigma(f_m \Sigma(g)_m + \Sigma(f)_m g_m + f_m g_m)_n,
 \end{aligned}$$

and this concludes the proof.

As mentioned in the Introduction interesting relationships between pre-Lie algebras, rooted tree graphs, dendriform and Rota-Baxter algebras already exist (see for instance [30, 31]), whereas links between tridendriform, Rota-Baxter and (quasi)shuffle algebras have been also studied in [32]. We shall briefly recall now the relation between pre-Lie algebras and (tri)-dendriform algebras.

Definition 2.6. A **tridendriform algebra** \mathcal{D} is a k -vector space equipped with three binary operations \prec, \succ, \cdot and the following axioms [30, 31] :

- (1) $(a \prec b) \prec c = a \prec (b \prec c + b \succ c + b \cdot c)$
- (2) $(a \prec b) \succ c = a \succ (b \prec c)$
- (3) $a \succ (b \succ c) = (a \prec b + a \succ b + a \cdot b) \succ c$
- (4) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- (5) $(a \succ b) \cdot c = a \succ (b \cdot c)$
- (6) $(a \prec b) \cdot c = a \cdot (b \succ c)$
- (7) $(a \cdot b) \prec c = a \cdot (b \prec c)$.

A *dendriform algebra* is defined by setting the product \cdot to zero in the above axioms, consequently the rules of a dendriform algebra are given in terms of axioms (1)-(3) without the \cdot term.

Remark 2.7. We note that the product $x * y = x \prec y + x \succ y + x \cdot y$ is associative [31], whereas the product $x \triangleright y = x \succ y - y \prec x + x \cdot y$ ($x \triangleleft y = x \prec y - y \succ x + x \cdot y$) defines a pre-Lie algebra \mathcal{A} , $\forall x, y \in \mathcal{D}$. In the case of a dendriform algebra the term $x \cdot y$ is not present in both products, i.e. $x * y = x \prec y + x \succ y$ and $x \triangleright y = x \succ y - y \prec x$ ($x \triangleleft y = x \prec y - y \succ x$).

The binary actions of the (tri)dendriform algebra can be defined $\forall x, y \in \mathcal{A}$ in terms of Rota-Baxter operators as

$$x \succ y := R(x)y, \quad x \prec y := xR(y), \quad a \cdot b = \theta ab, \quad (2.13)$$

which leads to the findings of Proposition 2.3.

2.2. From pre-Lie algebras to braces. We report in this subsection the recent findings on the relations between pre-Lie algebras and braces [41, 46], and in particular the passage from pre-Lie algebras to braces. Before we discuss this passage we recall the definition of a brace [40, 9] and we note that braces were essentially introduced in order to derive set-theoretic solutions of the Yang-Baxter equation [40]. The already known relationships between braces, the Yang-Baxter equation

and quantum integrability [13, 14], as well as the passage described below are expected to lead to even deeper associations and the study of possibly novel algebraic structures.

Definition 2.8. *A left brace is a set B together with two group operations $+, \circ : B \times B \rightarrow B$, the first is called addition and the second is called multiplication, such that $\forall a, b, c \in B$,*

$$a \circ (b + c) = a \circ b - a + a \circ c. \quad (2.14)$$

The additive identity of a left brace B will be denoted by 0 and the multiplicative identity by 1 , and in every left brace $0 = 1$. Also, let $(N, +, \cdot)$ be an associative ring which is a nilpotent. For $a, b \in N$ define $a \circ b = a \cdot b + a + b$, then $(N, +, \circ)$ is a brace [40].

The group of formal flows constructed from a pre-Lie algebra was introduced in [2] (see also [35]). We summarize below the passage from pre-Lie algebras to [41, 46] after recalling the notion of the group of formal flows [2, 35]. We also assume as in [46] that A is a nilpotent pre-Lie algebra.

- (1) Let $a \in A$, and let $L_a : A \rightarrow A$ denote the left multiplication by a , so $L_a(b) := a \triangleright b$. We define $L_c \triangleright L_b(a) := L_c(L_b(a)) = c \triangleright (b \triangleright a)$, and

$$e^{L_a}(b) = b + a \triangleright b + \frac{1}{2!}a \triangleright (a \triangleright b) + \frac{1}{3!}a \triangleright (a \triangleright (a \triangleright b)) + \dots$$

- (2) We formally consider the element $\mathbf{1}$, such that $\mathbf{1} \triangleright a = a \triangleright \mathbf{1} = a$ in the pre-Lie algebra (as in [35]) and define

$$W(a) := e^{L_a}(\mathbf{1}) - \mathbf{1} = a + \frac{1}{2!}a \triangleright a + \frac{1}{3!}a \triangleright (a \triangleright a) + \dots$$

$W(a) : A \rightarrow A$ is a bijective function, provided that A is a nilpotent pre-Lie algebra.

- (3) Let $\Omega(a) : A \rightarrow A$ be the inverse function to the function $W(a)$, i.e. $\Omega(W(a)) = W(\Omega(a)) = a$. Following [35] the first terms of Ω are

$$\Omega(a) = a - \frac{1}{2}a \triangleright a + \frac{1}{4}(a \triangleright a) \triangleright a + \frac{1}{12}a \triangleright (a \triangleright a) + \dots$$

- (4) We define the multiplication,

$$a \circ b := a + e^{L_{\Omega(a)}}(b).$$

The addition is the same as in the pre-Lie algebra A . It was shown in [2] that (A, \circ) is a group. It is then straightforward to show that $(A, \circ, +)$ is a left brace, indeed

$$a \circ (b + c) + a = a + e^{L_{\Omega(a)}}(b + c) + a = (a + e^{L_{\Omega(a)}}(b)) + (a + e^{L_{\Omega(a)}}(c)) = a \circ b + a \circ c.$$

The above formula can also be written using the Baker-Campbell-Hausdorff (BCH) formula, (see [2, 35]). We first recall that the Lie algebra $L(A)$ is obtained from a pre-Lie algebra A by defining $[a, b] = a \triangleright b - b \triangleright a$ (with the same addition as A). By means of the BCH formula, $e^{L_a}(e^{L_b}(\mathbf{1})) = e^{L_{C(a,b)}}(\mathbf{1})$, the element $C(a, b)$ can be represented in the form of a series as $C(a, b) = a + b + \frac{1}{2}[a, b] + \frac{1}{12}([a, [a, b]] + [b, [b, a]]) + \dots$

Lemma 2.9. *The following formula for the multiplication \circ defined above holds (see e.g. [35]):*

$$W(a) \circ W(b) = W(C(a, b)),$$

where $C(a, b)$ is obtained using the BCH series in the Lie algebra $L(A)$.

Proof. The proof is immediate from the definition of the brace multiplication:

$$\begin{aligned} W(a) \circ W(b) &= W(a) + e^{L_{\Omega(W(a))}}(W(b)) = e^{L_a}(\mathbf{1}) - \mathbf{1} + e^{L_a}(e^{L_b}(\mathbf{1}) - \mathbf{1}) = \\ &= e^{L_a}(e^{L_b}(\mathbf{1})) - \mathbf{1} = e^{L_{C(a,b)}}(\mathbf{1}) - \mathbf{1} = W(C(a, b)). \quad \square \end{aligned}$$

Example 2.10. *We recall the expression from Magnus expansion $\mathbb{Q}(x, \alpha) = \sum_{m>0} \mathbb{Q}^{(m)}(x) \alpha^m$ (2.11). Let $\mathbb{Q}(x, \alpha) = \int_{x_0}^x \Omega(\xi, \alpha; A) d\xi$, then from (2.11) we read off:*

$$\Omega(\xi, \alpha; A) = \alpha A(\xi) - \frac{1}{2} \alpha^2 (A \triangleright A)(\xi) + \frac{1}{4} \alpha^3 ((A \triangleright A) \triangleright A)(\xi) + \frac{1}{12} \alpha^3 (A \triangleright (A \triangleright A))(\xi) + \dots,$$

where for any linear operators $A, B : (L_A(B))(x) := (A \triangleright B)(x) = [\int_{x_0}^x A(s) ds, B(x)]$, and

$$(e^{L_A}(B))(x) = B(x) + (A \triangleright B)(x) + \frac{1}{2!} (A \triangleright (A \triangleright B))(x) + \frac{1}{3!} (A \triangleright (A \triangleright (A \triangleright B)))(x) + \dots$$

And in this case the brace multiplication is defined as: $(A \circ B)(x) := A(x) + (e^{L_{\Omega(\alpha, A)}}(B))(x)$ (see also [2]).

3. DISCRETE MAGNUS EXPANSION & PRE-LIE ALGEBRAS

In Section 1 we described the solution of the linear evolution problem expressed as Dyson and Magnus series and we recalled the pre-Lie Magnus expansion. We are now focusing on the discrete evolution problem and the derivation of the discrete analogue of the Magnus expansion. Our starting point will be the discrete evolution problem; indeed let \mathbb{L}, T be some linear operators depending on some discrete index $n \in \mathbb{Z}^+$ and an extra parameter $\alpha \in \mathbb{C}$ (e.g. $\mathbb{L}, T \in \text{End}(\mathbb{C}^{\mathcal{N}})$):

$$T_{n+1}(\alpha) = \mathbb{L}_n(\alpha) T_n(\alpha). \quad (3.1)$$

The solution of the difference equation above is found by iteration and is given by $T_{N+1}(\alpha) = \mathbb{L}_N(\alpha) \dots \mathbb{L}_1(\alpha)$ (let us choose for simplicity $T_1(\alpha) = 1$ as initial condition), and as in the continuous case we express the monodromy matrix as $T_{N+1}(\alpha) = 1 + \sum_{n>1} \alpha^n T^{(n)}(N+1)$.

We consider the general scenario, where the \mathbb{L} -operator is formally expressed as $\mathbb{L}(\alpha) = 1 + \sum_{k>1} \alpha^k L^{(k)}$, where in general $L^{(m)}$ are linear operators (or algebraic objects), for instance for our purposes here we will be considering $L^{(m)} \in \text{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathfrak{A}$, where \mathfrak{A} is some quantum algebra to be identified in the subsequent sections, or $L^{(m)} \in \text{End}(\mathbb{C}^{\mathcal{N}})$. Notice that \mathbb{L} may be re-expressed as $\mathbb{L}(\alpha) = e^{\mathbb{Q}(\alpha)}$ (see next section for a detailed exposition and relation to the Yangian algebra). Considering the generic form of \mathbb{L} we obtain the form of the coefficients of the monodromy:

$$T^{(m)}(N+1) = \sum_{n=1}^N \sum_{k=1}^m \left(L_n^{(m_k)} \sum_{n_{k-1}=1}^{n-1} L_{n_{k-1}}^{(m_{k-1})} \dots \sum_{n_1=1}^{n_2-1} L_{n_1}^{(m_1)} \right)_{\sum_{j=1}^k m_j = m}. \quad (3.2)$$

In the following Proposition we express the elements of the discrete analogue of Dyson's series (3.2) in terms of the tridendriform algebra.

Proposition 3.1. *The elements of the discrete analogue of Dyson series (3.2) are expressed in terms of the tridendriform action \prec as*

$$T^{(m)}(N+1) = \sum_{n=1}^N \sum_{k=1}^m \left(L^{(m_k)} \prec (L^{(m_{k-1})} \prec (\dots \prec (L^{(m_2)} \prec L^{(m_1)})) \dots) \right)_n \Big|_{\sum_{j=1}^k m_j = m}. \quad (3.3)$$

Proof. We first recall (2.13) and the fact that in this case the Rota-Baxter operator of weight one is the summation, i.e. $R(x) := \Sigma(x)_n = \sum_{m=1}^{n-1} x_m$, then $(x \prec y)_n = x_n \Sigma(y)_n$, where x, y are in general some linear operators. Using the definitions of Σ and the explicit expressions of $T^{(m)}$, $m \in \{1, 2, \dots, N\}$ (3.2), we write

$$T^{(m)}(N+1) = \sum_{n=1}^N L_n^{(m_k)} \Sigma(L^{(m_{k-1})} \Sigma(L^{(m_{k-2})} \Sigma(\dots \Sigma(L^{(m_1)})) \dots))_n \Big|_{\sum_{j=1}^k m_j = m}. \quad (3.4)$$

Via $(x \prec y)_n = x_n \Sigma(y)_n$, expression (3.4) leads to (3.3).

For instance, for $n = 1$, $T^{(1)}(N+1) = \sum_{n=1}^N L_n^{(1)}$, for $n = 2$

$$\begin{aligned} T^{(2)}(N+1) &= \sum_{n=1}^N \left(L_n^{(1)} \sum_{m=1}^{n-1} L_m^{(1)} + L_n^{(2)} \right) = \sum_{n=1}^N \left(L_n^{(1)} \Sigma(L^{(1)})_n + L_n^{(2)} \right) \\ &= \sum_{n=1}^N \left((L^{(1)} \prec L^{(1)})_n + L_n^{(2)} \right) \end{aligned}$$

for $n = 3$,

$$\begin{aligned} T^{(3)}(N+1) &= \sum_{n=1}^N \left(L_n^{(1)} \sum_{m=1}^{n-1} L_m^{(1)} \sum_{k=1}^{m-1} L_k^{(1)} + L_n^{(1)} \sum_{m=1}^{n-1} L_m^{(2)} + L_n^{(2)} \sum_{m=1}^{n-1} L_m^{(1)} + L_n^{(3)} \right) \\ &= \sum_{n=1}^N \left(L_n^{(1)} \Sigma(L^{(1)} \Sigma(L^{(1)}))_n + L_n^{(1)} \Sigma(L^{(2)})_n + L_n^{(2)} \Sigma(L^{(1)})_n + L_n^{(3)} \right) \\ &= \sum_{n=1}^N \left((L^{(1)} \prec (L^{(1)} \prec L^{(1)}))_n + (L^{(1)} \prec L^{(2)})_n + (L^{(2)} \prec L^{(1)})_n + L_n^{(3)} \right). \quad \square \end{aligned}$$

We come now to the precise derivation of the discrete Magnus expansion. We consider the Lie exponential $T_{N+1}(\alpha) = e^{\mathbb{Q}_{N+1}(\alpha)}$, $\mathbb{Q}_{N+1}(\alpha) = \sum_{m=1}^{\infty} \alpha^m \mathbb{Q}^{(m)}(N+1)$, which will lead to the discrete analogue of Magnus expansion; expressions (1.4), (1.5), (1.8) and (1.9) hold.

Lemma 3.2. *The quantities $\mathbb{Q}^{(k)}(N+1) := \sum_{n=1}^N \Omega_n^{(k)}$, where $\Omega_n^{(k)} = \mathbb{Q}^{(k)}(n+1) - \mathbb{Q}^{(k)}(n)$, are expressed explicitly as:*

$$\begin{aligned} \mathbb{Q}^{(1)}(N+1) &= \sum_{n=1}^N L_n^{(1)}, \\ \mathbb{Q}^{(2)}(N+1) &= \frac{1}{2} \sum_{n > n_1=1}^N [L_n^{(1)}, L_{n_1}^{(1)}] - \frac{1}{2} \sum_{n=1}^N (L_n^{(1)})^2 + \sum_{n=1}^N L_n^{(2)}, \end{aligned}$$

$$\begin{aligned}
 \mathbb{Q}^{(3)}(N+1) &= \sum_{n=1}^N \left(\frac{1}{6} \sum_{n_2 > n_1 = 1}^{n-1} ([L_n^{(1)}, [L_{n_2}^{(1)}, L_{n_1}^{(1)}]] + [[L_n^{(1)}, L_{n_2}^{(1)}], L_{n_1}^{(1)}) \right. \\
 &+ \frac{1}{6} \sum_{n_1=1}^{n-1} (L_{n_1}^{(1)} [L_{n_1}^{(1)}, L_n^{(1)}] + [L_{n_1}^{(1)}, L_n^{(1)}] L_n^{(1)}) \\
 &+ \frac{1}{6} \sum_{n_1=1}^{n-1} ([L_{n_1}^{(1)}, (L_n^{(1)})^2] + [(L_{n_1}^{(1)})^2, L_n^{(1)}]) - \frac{1}{2} \sum_{n=1}^N (L_n^{(1)} L_n^{(2)} + L_n^{(2)} L_n^{(1)}) \\
 &+ \left. \frac{1}{3} (L_n^{(1)})^3 - \frac{1}{2} \sum_{m=1}^{n-1} ([L_m^{(1)}, L_n^{(2)}] + [L_m^{(2)}, L_n^{(1)}]) + L_n^{(3)} \right), \dots \tag{3.5}
 \end{aligned}$$

Proof. Recursive expressions (1.8), (1.9) apparently still hold, but now $T^{(m)}$ are given by (3.2). Notice that in the discrete case both $T_{N+1}(\lambda)$ and $T^{(m)}(N+1)$ depend on a discrete parameter N , which replaces the continuum parameter x of the continuous analogue discussed in Section 1. The quantities $\Omega^{(k)}$ can be immediately read from (3.5), and are the discrete analogues of the derivatives $\dot{\mathbb{Q}}^{(k)}(x)$, of Magnus expansion in Section 1. \square

Corollary 3.3. *Let A, B be linear operators (or generic algebraic objects) that depend on a discrete parameter $n \in \mathbb{Z}^+$, and define the binary operation $\triangleright: (A, B) \mapsto A \triangleright B$, such that*

$$(A \triangleright B)_n := \left[\sum_{m=1}^{n-1} A_m, B_n \right] + A_n B_n. \tag{3.6}$$

Then the pre-Lie identity is satisfied, i.e.

$$((A \triangleright B) \triangleright C)_n - (A \triangleright (B \triangleright C))_n = ((B \triangleright A) \triangleright C)_n - (B \triangleright (A \triangleright C))_n. \tag{3.7}$$

Proof. The proof is immediate by means of Proposition 2.3 and Example 2.5. \square

Proposition 3.4. *The elements of the discrete Magnus expansion (3.5) can be re-expressed in terms of the pre-Lie action (3.6).*

Proof. We recall the definition of the pre-Lie binary operation (3.6), then

$$(x \triangleright x)_n = \left[\sum_{m=1}^{n-1} x_m, x_n \right] + x_n^2, \tag{3.8}$$

also, from expressions (2.6) and (2.7), for $R \rightarrow \Sigma$, and by setting $x = y = z$ we obtain:

$$\begin{aligned}
 \frac{1}{4}((x \triangleright x) \triangleright x)_n + \frac{1}{12}(x \triangleright (x \triangleright x))_n &= \frac{1}{6} \sum_{n_2 > n_1 = 1}^{n-1} \left([x_n, [x_{n_2}, x_{n_1}]] + [[x_n, x_{n_2}], x_{n_1}] \right) \\
 &+ \frac{1}{6} \sum_{n_1=1}^{n-1} \left(x_{n_1} [x_{n_1}, x_n] + [x_{n_1}, x_n] x_n \right) \\
 &+ \frac{1}{6} \sum_{n_1=1}^{n-1} \left([x_{n_1}, x_n^2] + [x_{n_1}^2, x_n] \right) + \frac{1}{3} x_n^3. \tag{3.9}
 \end{aligned}$$

Comparing expressions (3.5) with (3.8) and (3.9) we conclude

$$\begin{aligned}
\mathbb{Q}^{(1)}(N+1) &= \sum_{n=1}^N L_n^{(1)}, \\
\mathbb{Q}^{(2)}(N+1) &= -\frac{1}{2} \sum_{n=1}^N (L^{(1)} \triangleright L^{(1)})_n + \sum_{n=1}^N L_n^{(2)}, \\
\mathbb{Q}^{(3)}(N+1) &= \sum_{n=1}^N \left(\frac{1}{4} ((L^{(1)} \triangleright L^{(1)}) \triangleright L^{(1)})_n + \frac{1}{12} (L^{(1)} \triangleright (L^{(1)} \triangleright L^{(1)}))_n \right) \\
&\quad - \frac{1}{2} \sum_{n=1}^N \left((L^{(2)} \triangleright L^{(1)})_n + (L^{(1)} \triangleright L^{(2)})_n \right) + \sum_{n=1}^N L_n^{(3)}. \tag{3.10}
\end{aligned}$$

The above expressions provide elegant expressions of the discrete analogues of the pre-Lie Magnus expansion. Higher order terms are computed by iteration via (1.8), (1.9). \square

Remark 3.5. *It is worth focusing on the simple linear case, where $\mathbb{L}(\alpha) = 1 + \alpha\mathbb{P}$, i.e. $L^{(1)} = \mathbb{P}$ and $L^{(m)} = 0 \forall m > 1$. Then we obtain the following explicit expressions, which provide a simpler discrete analogue of Dyson's series:*

$$T_{N+1}(\alpha) = 1 + \sum_{m=1}^N \alpha^m T^{(m)}(N+1), \quad T^{(m)}(N+1) = \sum_{n_m > \dots > n_1 = 1}^N \mathbb{P}_{n_m} \dots \mathbb{P}_{n_1}. \tag{3.11}$$

According to Proposition 3.1 the elements of the discrete Dyson series (3.11) are expressed in terms of the tridendriform action \prec as

$$T^{(m)}(N+1) = \sum_{n=1}^N \left(\mathbb{P} \prec (\mathbb{P} \prec (\mathbb{P} \prec (\dots \prec (\mathbb{P} \prec \mathbb{P}))) \dots) \right)_n, \tag{3.12}$$

with m \mathbb{P} -terms. We also recall that $T := e^{\mathbb{Q}}$, then via (1.4) and (1.5) we immediately obtain the formulas for $\mathbb{Q}^{(m)}(N+1)$, given the findings of the generic case, and the pre-Lie discrete Magnus expansion in this case takes the simple form resembling structurally the continuum case,

$$\begin{aligned}
\mathbb{Q}^{(1)}(N+1) &= \sum_{n=1}^N \mathbb{P}_n, \\
\mathbb{Q}^{(2)}(N+1) &= -\frac{1}{2} \sum_{n=1}^N (\mathbb{P} \triangleright \mathbb{P})_n \\
\mathbb{Q}^{(3)}(N+1) &= \sum_{n=1}^N \left(\frac{1}{4} ((\mathbb{P} \triangleright \mathbb{P}) \triangleright \mathbb{P})_n + \frac{1}{12} (\mathbb{P} \triangleright (\mathbb{P} \triangleright \mathbb{P}))_n \right). \tag{3.13}
\end{aligned}$$

Higher order terms are obtained via (1.8), (1.9). Expressions (3.13) are much more concise compared to the general ones (3.10), and apparently similar to the corresponding continuous formulas, given that certain ‘‘boundary terms’’ are missing in the linear scenario.

Remark 3.6. *We briefly discuss now the associated brace structure emerging from the discrete pre-Lie Magnus expansion, which provides one more example of brace construction in accordance*

to Subsection 2.2. We recall expressions $\mathbb{Q}_{N+1}(\alpha) = \sum_{m>0} \mathbb{Q}^{(m)}(N+1)\alpha^m$ (2.11). Let $\mathbb{Q}_{N+1}(\alpha) = \sum_{n=1}^N \Omega_n(\alpha; \mathbb{P})$, then from (3.13) we read off:

$$\Omega_n(\alpha; \mathbb{P}) = \alpha \mathbb{P}_n - \frac{1}{2} \alpha^2 (\mathbb{P} \triangleright \mathbb{P})_n + \frac{1}{4} \alpha^3 ((\mathbb{P} \triangleright \mathbb{P}) \triangleright \mathbb{P})_n + \frac{1}{12} \alpha^3 (\mathbb{P} \triangleright (\mathbb{P} \triangleright \mathbb{P}))_n + \dots, \quad (3.14)$$

where we recall that for \mathbb{P}, B being linear operators $(L_{\mathbb{P}}(B))_n := (\mathbb{P} \triangleright B)_n = [\sum_{m=1}^{n-1} \mathbb{P}_m, B_n] + \mathbb{P}_n B_n$, and

$$(e^{L_{\mathbb{P}}(B)})_n = B_n + (\mathbb{P} \triangleright B)_n + \frac{1}{2!} (\mathbb{P} \triangleright (\mathbb{P} \triangleright B))_n + \frac{1}{3!} (\mathbb{P} \triangleright (\mathbb{P} \triangleright (\mathbb{P} \triangleright B)))_n + \dots$$

And in this case the brace multiplication is defined as: $(\mathbb{P} \circ B)_n = \mathbb{P}_n + (e^{L_{\Omega(\alpha, \mathbb{P})}}(B))_n$.

Remark 3.7. (The continuum limit). Recall the discrete evolution problem (3.1), then rescale $\alpha \rightarrow \delta \alpha$ ($\delta \ll 1$) and consider the general form $\mathbb{L}(\lambda) = 1 + \sum_{m>1} \alpha^m \delta^n L^{(m)}$. Expression (3.1) can be rewritten as (keeping only linear terms in δ)

$$T_{n+1}(\alpha) = (1 + \alpha \delta L_{n+1}^{(1)}) T_n(\alpha). \quad (3.15)$$

By considering the following ‘‘dictionary’’ as $\delta \rightarrow 0$: $L_{n+1}^{(1)} \rightarrow A(x)$, and $\frac{\Psi_{n+1} - \Psi_n}{\delta} \rightarrow \partial_{\xi} \Psi(\xi)$ we arrive at the continuum limit of (3.1), which is the linear evolution problem

$$\partial_{\xi} T(\xi, \alpha) = \alpha A(\xi) T(\xi, \alpha). \quad (3.16)$$

Detailed proof on the continuum limit of the discrete monodromy that lead to continuum monodromy, based on a ‘‘power counting rule’’ is given in [4]. The counting rule relies on the fact $\delta \sum_{n=1}^N f_n \rightarrow \int_0^x f(\xi) d\xi$, and terms of the form:

$$\delta \sum_{j=1}^m n_j \sum_{k_1, k_2, \dots, k_l} L_{k_1}^{(n_1)} \dots L_{k_m}^{(n_m)} \rightarrow 0 \text{ in the continuum limit for } \sum_{j=1}^m n_j > m.$$

In the continuum limit the monodromy matrix T is expressed as a Dyson series with terms that are written, via (3.11), (3.12) in terms of a dendriform action $(A \prec B)(x) := A(x) \int_0^x B(\xi) d\xi$, as

$$T^{(m)}(x) = \int_0^x d\zeta \left(A \prec (A \prec (A \prec (\dots \prec (A \prec A) \dots))) \right) (\zeta). \quad (3.17)$$

3.1. An alternative discrete expansion. Let us now consider a slightly different scenario, where the \mathbb{L} -operator in (3.1) is of the form $\mathbb{L}(\alpha) = M + \alpha L$, $M \neq 1$ and M, L are some linear operators. Before we state the main findings in the next Proposition it is useful to introduce some notation. We set:

- (1) $\mathbb{M}_{N+1, n-1} := M_N \dots M_n M_{n-1}$
- (2) $\hat{M} := M^{-1}$ and $\hat{\mathbb{M}}_{n-1, N+1} = \hat{M}_{n-1} \hat{M}_n \dots \hat{M}_N$
- (3) $\mathbb{M}_{N+1, 1} := \mathbb{M}_{N+1}$ and $\hat{\mathbb{M}}_{1, N+1} := \hat{\mathbb{M}}_{N+1}$.

Proposition 3.8. Let \mathbb{L} in (3.1) be of the form $\mathbb{L}(\alpha) = M + \alpha L$, and the monodromy be $T_{N+1}(\alpha) = \sum_{n=0}^N \alpha^n T^{(n)}(N+1)$, where in general $M \neq 1$ and L are linear operators. Then the monodromy matrix $T_{N+1}(\alpha)$ can be expressed as

$$T_{N+1}(\alpha) = e^{\mathbb{Q}_{N+1}(\alpha)} \mathbb{M}_{N+1}, \quad (3.18)$$

where $\mathbb{Q}_{N+1}(\alpha) = \sum_{m=1}^{N+1} \alpha^m \mathbb{Q}^{(m)}(N+1)$ is the discrete Magnus expansion as in Remark 3.5 and the elements $\mathbb{Q}^{(m)}(N+1)$ are given in (3.13) as the pre-Lie Magnus expansion.

Proof. We start with the standard ordered expansion of the monodromy matrix, given the choice $\mathbb{L}(\alpha) = M + \alpha L$ we obtain

$$\begin{aligned} T_{N+1}(\alpha) &= \mathbb{M}_{N+1} + \alpha \sum_{n=1}^N \mathbb{M}_{N+1} \dots \mathbb{M}_{n+1} \mathbb{L}_n \mathbb{M}_{n-1} \dots \mathbb{M}_1 \\ &+ \alpha^2 \sum_{n>m=1}^N \mathbb{M}_N \dots \mathbb{M}_{n+1} \mathbb{L}_n \mathbb{M}_{n-1} \dots \mathbb{M}_{m+1} \mathbb{L}_m \mathbb{M}_{m-1} \dots \mathbb{M}_1 + \dots \end{aligned} \quad (3.19)$$

after some lengthy, but straightforward computation and after defining:

$$\mathbf{P}_n := \mathbb{M}_{N,n-1} \mathbb{L}_n \hat{\mathbb{M}}_n \hat{\mathbb{M}}_{n-1,N}, \quad (3.20)$$

we arrive at

$$T_{N+1}(\alpha) = \left(1 + \sum_{m=1}^N \alpha^m \sum_{n_m > \dots > n_1 = 1}^N \mathbf{P}_{n_m} \mathbf{P}_{n_{m-1}} \dots \mathbf{P}_{n_1} \right) \mathbb{M}_{N+1}. \quad (3.21)$$

The bracket in the expression above is just the discrete ordered expansion (3.12), which as shown in the previous subsection can be expressed as the discrete analogue of Magnus expansion, and also in terms of a suitable pre-Lie action. Notice however that contrary to the previous analysis of Section 3, the objects \mathbf{P}_n (3.20) are not local anymore, but are “dressed” quantities via a gauge transformation $\mathbb{M}_{N,n-1}$. \square

3.2. Gauge fields & open boundary systems. We consider in this subsection two fundamental problems/applications very much associated to the derivations of the previous subsection. The first problem is the transformation of the field \mathbb{L} (A in the continuous case) via a gauge transformation, and the second one is the construction of the evolution problem and solution for certain classes of systems with open boundary conditions.

Gauge fields. Let T_n be a solution of the difference equation (3.1) and \hat{T}_n a solution of

$$\hat{T}_{n+1}(\alpha) = \hat{\mathbb{L}}_n(\alpha) \hat{T}_n(\alpha). \quad (3.22)$$

Let also $G(\alpha)$ be a linear operator, such that $\hat{T}_n(\alpha) = G_n(\alpha) T_n(\alpha)$, then via (3.1), (3.22), we conclude for the transformed field, $\hat{\mathbb{L}}_n(\alpha) = G_{n+1}(\alpha) \mathbb{L}_n(\alpha) G_n^{-1}(\alpha)$. Assuming that the operators $\mathbb{L}_n, \hat{\mathbb{L}}_n$ are given we solve the following difference equation to identify the gauge transformation

$$G_{n+1}(\alpha) = \hat{\mathbb{L}}_n(\alpha) G_n(\alpha) \mathbb{L}_n^{-1}(\alpha). \quad (3.23)$$

Lemma 3.9. *We recall that $\hat{T}_{N+1}(\alpha) := \hat{\mathbb{L}}_N(\alpha) \dots \hat{\mathbb{L}}_1(\alpha)$ and $T_{N+1}^{-1}(\alpha) := \mathbb{L}_1^{-1}(\alpha) \dots \mathbb{L}_N^{-1}(\alpha)$. The solution of the difference equation (3.23) is given by $G_{N+1}(\alpha) = \hat{T}_{N+1}(\alpha) G_1(\alpha) T_{N+1}^{-1}(\alpha)$, $G_1(\alpha)$ is some generic initial value.*

Proof. The solution of (3.23) is obtained directly by iteration. \square

Similarly, in the continuous case (see also Remark 3.7) we can either consider the continuous limit of the above expressions or directly apply the gauge transformation: $\hat{T}(x, \alpha) = G(x, \alpha) T(x, \alpha)$, where T satisfies (1.1) and \hat{T} satisfies $\partial_\xi \hat{T}(\xi, \alpha) = \alpha \hat{A}(\xi) \hat{T}(\xi, \alpha)$. Hence, we conclude that the

transformed field is given as $\hat{A}(\xi) = G(\xi, \alpha)AG^{-1}(\xi, \alpha) + \alpha^{-1}\partial_\xi G(\xi, \alpha)G^{-1}(\xi, \alpha)$, which is the familiar transformation of a gauge field A . Suppose that A, \hat{A} are given, then $G(\xi, \alpha)$ satisfies the evolution problem

$$\partial_\xi G(\xi, \alpha) = \alpha\hat{A}(\xi)G(\xi, \alpha) - \alpha G(\xi, \alpha)A(\xi). \quad (3.24)$$

The solution of the latter equation is given by $G(x, \alpha) = \hat{T}(x, \alpha)G_0(\alpha)T^{-1}(x, \alpha)$, where we recall that, $T(x, \alpha) = \widehat{\text{Pexp}}\left(\alpha \int_0^x A(\xi)d\xi\right)$ (similarly for $\hat{T}(x, \alpha)$) and $G_0(\alpha)$ is some initial value at $x = 0$.

This type of problems systematically appear in the context of integrable systems, where a Lax pair and strong compatibility conditions (zero curvature condition, i.e. equations of motion) exist [29, 1, 24]. In this frame dressing schemes [49, 17, 36] are used in order to obtain solutions of the associated integrable non-linear ODEs and PDEs that emerge from the zero curvature condition.

Typically, in integrable systems due to the existence of a Lax pair or the existence of a classical or quantum R -matrix [24, 25] (see also [11] and references therein) it turns out that $\text{tr}T(\alpha)$ provides a hierarchy of conserved quantities; for instance the Hamiltonian of the system with periodic boundary conditions is a member of this hierarchy (a more detailed discussion will follow later in the text). We are regarding in the next section both quantum and classical integrable systems, and we establish fundamental connections with the findings of the present section. A relevant construction, which leads to systems with open boundary conditions is presented below.

Open boundary conditions. We now focus on the evolution problem associated to integrable systems with open boundary conditions, although we are not going to discuss here about the notion of integrability for such systems (the interested reader is referred to [45] for a detailed exposition). In order to describe discrete systems with open boundary conditions within integrability [45], we also need to consider the following difference equation

$$\hat{T}_{n+1}(\alpha) = \hat{T}_n(\alpha)\hat{\mathbb{L}}_n(\alpha). \quad (3.25)$$

The solution of equation (3.25) is found by iteration and is given by $\hat{T}_{N+1}(\alpha) = \hat{\mathbb{L}}_1(\alpha)\dots\hat{\mathbb{L}}_N(\alpha)$ (we choose for simplicity $\hat{T}_1(\alpha) = 1$ as initial condition). We express the monodromy as $\hat{T}_{N+1}(\alpha) = 1 + \sum_{n>1} \alpha^n \hat{T}^{(n)}(N+1)$ and recall the exponential map $\hat{T}_{N+1}(\alpha) = e^{\hat{\mathbb{Q}}_{N+1}(\alpha)}$; expressions analogous to (4.5), (1.5), (1.8) and (1.9) naturally hold.

We consider the general scenario, where $\hat{\mathbb{L}}(\alpha) = 1 + \sum_{k>1} \alpha^k \hat{L}^{(k)}$, and in general $\hat{L}^{(m)}$ are linear operators, then the form of the coefficients of the monodromy \hat{T} are given as

$$\hat{T}^{(m)}(N+1) = \sum_{n_1=1}^N \sum_{k=1}^m \left(\sum_{n_k=1}^{n_{k-1}-1} \hat{L}_{n_k}^{(m_k)} \sum_{n_{k-1}=1}^{n_{k-2}-1} \hat{L}_{n_{k-1}}^{(m_{k-1})} \dots \sum_{n_2=1}^{n_1-1} \hat{L}_{n_1}^{(m_2)} \hat{L}_{n_1}^{(m_1)} \right)_{\sum_{j=1}^k m_j=m}. \quad (3.26)$$

We may than express the elements of the discrete analogue of Dyson's series (3.26) in terms of the tridendriform algebra in analogy to Proposition 3.1.

Lemma 3.10. *The elements of the generalized discrete Dyson series (3.26) are expressed in terms of the tridendriform action \succ as*

$$\hat{T}^{(m)}(N+1) = \sum_{n=1}^N \sum_{k=1}^m \left((\dots (\hat{L}^{(m_k)} \succ \hat{L}^{(m_{k-1})}) \succ \dots) \succ \hat{L}^{(m_1)} \right)_n \Big|_{\sum_{j=1}^k m_j = m}. \quad (3.27)$$

For instance, for $n = 1$, $\hat{T}^{(1)}(N+1) = \sum_{n=1}^N \hat{L}_n^{(1)}$, for $n = 2$ and $n = 3$

$$\begin{aligned} \hat{T}^{(2)}(N+1) &= \sum_{n=1}^N \left((\hat{L}^{(1)} \succ \hat{L}^{(1)})_n + \hat{L}_n^{(2)} \right) \\ \hat{T}^{(3)}(N+1) &= \sum_{n=1}^N \left(((\hat{L}^{(1)} \succ \hat{L}^{(1)}) \succ \hat{L}^{(1)})_n + (\hat{L}^{(1)} \succ L^{(2)})_n + (\hat{L}^{(2)} \succ \hat{L}^{(1)})_n + \hat{L}_n^{(3)} \right). \end{aligned}$$

Proof. The proof goes along the same lines as in Proposition 3.1. \square

We recall the exponential map $\hat{T}(\alpha) = e^{\hat{Q}(\alpha)}$, $\hat{Q}(\alpha) = \sum_{m=1}^{\infty} \alpha^m \hat{Q}^{(m)}$, which will lead to the discrete analogue of Magnus expansion.

Lemma 3.11. *The quantities $\hat{Q}^{(k)}(N+1) := \sum_{n=1}^N \hat{\Omega}_n^{(k)}$, where $\hat{\Omega}_n^{(k)} = \hat{Q}^{(k)}(n+1) - \hat{Q}^{(k)}(n)$, are expressed explicitly as:*

$$\begin{aligned} \hat{Q}^{(1)}(N+1) &= \sum_{n=1}^N \hat{L}_n^{(1)}, \\ \hat{Q}^{(2)}(N+1) &= \frac{1}{2} \sum_{n>n_1=1}^N [\hat{L}_{n_1}^{(1)}, L_n^{(1)}] - \frac{1}{2} \sum_{n=1}^N (\hat{L}_n^{(1)})^2 + \sum_{n=1}^N \hat{L}_n^{(2)}, \\ \hat{Q}^{(3)}(N+1) &= \sum_{n=1}^N \left(\frac{1}{6} \sum_{m>k=1}^{n-1} ([\hat{L}_k^{(1)}, [\hat{L}_m^{(1)}, \hat{L}_n^{(1)}]] + [[\hat{L}_k^{(1)}, \hat{L}_m^{(1)}], \hat{L}_n^{(1)}) \right) \\ &\quad + \frac{1}{6} \sum_{m=1}^{n-1} (\hat{L}_n^{(1)} [\hat{L}_n^{(1)}, \hat{L}_m^{(1)}] + [\hat{L}_n^{(1)}, \hat{L}_m^{(1)}] \hat{L}_m^{(1)}) \\ &\quad + \frac{1}{6} \sum_{m=1}^{n-1} ([\hat{L}_n^{(1)}, (\hat{L}_m^{(1)})^2] + [(\hat{L}_n^{(1)})^2, \hat{L}_m^{(1)}]) - \frac{1}{2} \sum_{n=1}^N (\hat{L}_n^{(1)} \hat{L}_n^{(2)} + \hat{L}_n^{(2)} \hat{L}_n^{(1)}) \\ &\quad + \frac{1}{3} (\hat{L}_n^{(1)})^3 - \frac{1}{2} \sum_{m=1}^{n-1} ([\hat{L}_n^{(1)}, \hat{L}_m^{(2)}] + [\hat{L}_n^{(2)} \hat{L}_m^{(1)}]) + \hat{L}_n^{(3)}, \dots \end{aligned} \quad (3.28)$$

Proof. The proof is along the lines of Lemma 3.2. \square

We are now in the position to express the discrete Magnus expansion elements in terms of the Pre-Lie action \triangleleft . Let A, B be linear operators that depend on a discrete parameter $n \in \mathbb{N}$ and define the binary operation $\triangleleft : (A, B) \mapsto A \triangleleft B$, such that

$$(A \triangleleft B)_n := [A_n, \sum_{m=1}^{n-1} B_m] + A_n B_n. \quad (3.29)$$

Then the pre-Lie identity is satisfied, i.e.

$$((A \triangleleft B) \triangleleft B)_n - (A \triangleleft (B \triangleleft C))_n = ((A \triangleleft C) \triangleleft B)_n - (A \triangleleft (C \triangleleft B))_n. \quad (3.30)$$

Proposition 3.12. *The elements of the discrete Magnus expansion (3.28) can be re-expressed in terms of the pre-Lie action (3.29).*

Proof. Recalling that $(a \triangleleft b)_n = [a_n, \sum_{m=1}^{n-1} b_m] + a_n b_n$, we conclude

$$\begin{aligned} \hat{\mathbb{Q}}^{(1)}(N+1) &= \sum_{n=1}^N \hat{L}_n^{(1)}, \\ \hat{\mathbb{Q}}^{(2)}(N+1) &= -\frac{1}{2} \sum_{n=1}^N (\hat{L}^{(1)} \triangleleft \hat{L}^{(1)})_n + \sum_{n=1}^N \hat{L}_n^{(2)}, \\ \hat{\mathbb{Q}}^{(3)}(N+1) &= \sum_{n=1}^N \left(\frac{1}{12} ((\hat{L}^{(1)} \triangleleft \hat{L}^{(1)}) \triangleleft \hat{L}^{(1)})_n + \frac{1}{4} (\hat{L}^{(1)} \triangleleft (\hat{L}^{(1)} \triangleleft \hat{L}^{(1)}))_n \right) \\ &\quad - \frac{1}{2} \sum_{n=1}^N \left((\hat{L}^{(2)} \triangleleft \hat{L}^{(1)})_n + (\hat{L}^{(1)} \triangleleft \hat{L}^{(2)})_n \right) + \sum_{n=1}^N \hat{L}_n^{(3)}. \quad \square \end{aligned}$$

A remark similar to Remark 3.5 applies here too by setting $\hat{L}^{(1)} = \hat{\mathbb{P}}$, $\hat{L}^{(j)} = 0$, $\forall j > 1$.

Now that we have the discrete Magnus expansions for both the “forward” and “backward” evolution problems we may identify the solution of the open boundary problem [45].

Lemma 3.13. *Let $T_{N+1}(\alpha) = \mathbb{L}_N(\alpha) \dots \mathbb{L}_1(\alpha)$ and $\hat{T}_{N+1}(\alpha) = \hat{\mathbb{L}}_1(\alpha) \dots \hat{\mathbb{L}}_N(\alpha)$ be solutions of (3.1) and (3.25) respectively. Let also $K(\alpha)$ be a linear operator, then the quantity $\mathbb{T}_{N+1}(\alpha) = T_{N+1}(\alpha)K(\alpha)\hat{T}_{N+1}(\alpha)$ [45], is a solution of the difference equation*

$$\mathbb{T}_{n+1}(\alpha) = \mathbb{L}_n(\alpha)\mathbb{T}_n(\alpha)\hat{\mathbb{L}}_n(\alpha). \quad (3.31)$$

Proof. The proof is straightforward by means of (3.1) and (3.25). \square

Remark 3.14. *We consider $\hat{\mathbb{L}}(\alpha) = \mathbb{L}^{-1}(-\alpha)$ and consequently $\hat{T}_{N+1}(\alpha) = T_{N+1}^{-1}(-\alpha)$. This case is associated to the so-called reflection algebra in integrable systems [10, 45]. In the continuum limit, after recalling $\mathbb{T}_{n+1} \rightarrow \mathbb{T}(x+\delta)$, $\mathbb{L}_n \rightarrow 1 + \delta\alpha A(x)$ and keeping linear terms we obtain from (3.31)*

$$\partial_\xi \mathbb{T}(\xi; \alpha) = \alpha A(\xi)\mathbb{T}(\xi; \alpha) + \alpha \mathbb{T}(\xi; \alpha)A(\xi), \quad (3.32)$$

which is the evolution equation for a system with open boundary conditions. The solution of (3.32) is given by $\mathbb{T}(\xi, \alpha) = T(\xi, \alpha)K(\alpha)\hat{T}(\xi, \alpha)$, where $T(\xi, \alpha) = e^{\mathbb{Q}(\xi, \alpha)}$ is given in (1.2), and $\hat{T}(\xi, \alpha) = T^{-1}(\xi, -\alpha)$. Notice that although (3.24) and (3.32) are mathematically similar, they have distinct physical interpretations as already pointed out.

4. QUANTUM GROUPS AS TRIDENDRIFORM & PRE-LIE ALGEBRAS

After the derivation of the discrete analogue of the Magnus expansion and its relation to pre-Lie algebras we are ready to study some key relationships between quantum groups, tridendriform and pre-Lie algebras. We employ the Yangian $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$ [19, 37] as our key paradigm and we show that the N -coproducts of the Yangian elements can be re-expressed in terms of suitable tridendriform and pre-Lie algebra actions.

We first recall the derivation of quantum groups associated to solutions $R : \mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}} \rightarrow \mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}}$ of the Yang-Baxter equation (YBE) [7, 48]

$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2), \quad (4.1)$$

where $u_1, u_2 \in \mathbb{C}$. Let $R = \sum_x a_x \otimes b_x$ then in the “index notation”: $R_{12} = \sum_x a_x \otimes b_x \otimes 1_V$, $R_{23} = 1_V \otimes \sum_x a_x \otimes b_x$, and $R_{13} = \sum_x a_x \otimes 1_V \otimes b_x$. For the derivation of a quantum algebra associated to an R -matrix we employ the FRT construction. Given a solution of the YBE the associated quantum algebra is a quotient of the free algebra, generated by $\{L_{x,y}^{(m)} | x, y \in \{1, \dots, \mathcal{N}\}\}$ and relations:

$$R_{12}(u_1, u_2) \mathbb{L}_1(u_1) \mathbb{L}_2(u_2) = \mathbb{L}_2(u_2) \mathbb{L}_1(u_1) R_{12}(u_1, u_2), \quad (4.2)$$

where $\mathbb{L}(\lambda) \in \text{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathfrak{A}$ and \mathfrak{A}^1 is the quantum algebra defined by (4.2). Let $\mathbb{L}(u) = \sum_{m=0}^{\infty} u^m L^{(m)}$, and $e_{x,y}$ be $\mathcal{N} \times \mathcal{N}$ matrices with entries $(e_{x,y})_{z,w} = \delta_{x,z} \delta_{y,w}$. Then in the index notation, $R_{12} = R \otimes 1_{\mathfrak{A}}$ and

$$L_1^{(m)} = \sum_{x,y \in 1}^{\mathcal{N}} e_{x,y} \otimes 1_V \otimes L_{x,y}^{(m)}, \quad L_2^{(m)} = \sum_{x,y \in 1}^{\mathcal{N}} 1_V \otimes e_{x,y} \otimes L_{x,y}^{(m)}.$$

4.1. The Yangian $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$. We focus now on the Yangian $\mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$; in this case $\alpha^{-1} = u := \lambda$ (additive parameter) and $R(u_1, u_2) = R(\lambda_1 - \lambda_2)$. Specifically, $R(\lambda_1, \lambda_2) = (\lambda_1 - \lambda_2)1_{V \otimes V} + \mathcal{P}$, where $\mathcal{P} = \sum_{i,j=1}^{\mathcal{N}} e_{i,j} \otimes e_{j,i}$ is the permutation operator. Via the fundamental relation (4.2) we obtain the algebraic relations among the generators [37]:

$$\left[L_{i,j}^{(n+1)}, L_{k,l}^{(m)} \right] - \left[L_{i,j}^{(n)}, L_{k,l}^{(m+1)} \right] = L_{k,j}^{(m)} L_{i,l}^{(n)} - L_{k,j}^{(n)} L_{i,l}^{(m)}, \quad i, j, k, l \in \{1, \dots, \mathcal{N}\}. \quad (4.3)$$

We recall the Lie exponential is written as $e^{\mathbb{Q}} := 1 + \sum_{n=1}^{\infty} \frac{\mathbb{Q}^n}{n!}$. We focus here on the case were $\mathbb{L}, \mathbb{Q} \in \text{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathfrak{Y}(\mathfrak{gl}_{\mathcal{N}})$, and \mathbb{L} satisfies relation (4.2). We express the generic solution of (4.2) as $\mathbb{L}(\lambda) = e^{\mathbb{Q}(\lambda)}$ and consider the formal λ series expansions:

$$\mathbb{L}(\lambda) = 1_{V \otimes \mathfrak{Y}} + \sum_{m=1}^{\infty} \lambda^{-m} L^{(m)}, \quad \mathbb{Q}(\lambda) = \sum_{m=1}^{\infty} \lambda^{-m} \mathbb{Q}^{(m)}. \quad (4.4)$$

¹Notice that in \mathbb{L} in addition to the indices 1 and 2 in (4.2) there is also an implicit “quantum index” n associated to \mathfrak{A} , which for now is omitted, i.e. one writes $\mathbb{L}_{1n}, \mathbb{L}_{2n}$.

Then comparing the series expansion $\mathbb{L}(\lambda)$ and $e^{\mathbb{Q}(\lambda)}$, using also (4.4), we obtain expressions of $\mathbb{Q}^{(m)}$ in terms of symmetric polynomials of $L^{(m)}$ (see (1.4), (1.5), $\mathbb{Q}^{(n)} \rightarrow \mathbb{Q}^{(n)}$, $T^{(n)} \rightarrow L^{(n)}$).

$$\mathbb{Q}^{(1)} = L^{(1)}, \quad \mathbb{Q}^{(2)} = L^{(2)} - \frac{1}{2}(L^{(1)})^2 \quad (4.5)$$

$$\mathbb{Q}^{(3)} = L^{(3)} - \frac{1}{2} \left(L^{(1)}L^{(2)} + L^{(2)}L^{(1)} \right) + \frac{1}{3}(L^{(1)})^3, \quad \dots \quad (4.6)$$

and vice versa as in (1.6), (1.7), i.e. similarly, the formal logarithm can be defined such that $\ln(\mathbb{L}(\lambda)) = \mathbb{Q}(\lambda)$.

Our aim now is to derive an alternative set of generators of the Yangian based on expressions (4.5), (4.6). Indeed, let us focus in the first few explicit exchange relations from (4.3)

$$(1) \quad n = 0, m = 1 \quad (L_{i,j}^{(0)} = \delta_{i,j}):$$

$$[L_{i,j}^{(1)}, L_{k,l}^{(1)}] = \delta_{i,l}L_{k,j}^{(1)} - \delta_{k,j}L_{i,l}^{(1)}$$

the latter are the familiar $\mathfrak{gl}_{\mathcal{N}}$ exchange relations.

$$(2) \quad n = 2, m = 0:$$

$$[L_{i,j}^{(2)}, L_{k,l}^{(1)}] = \delta_{i,l}L_{k,j}^{(2)} - \delta_{k,j}L_{i,l}^{(2)}$$

$$(3) \quad n = 2, m = 1:$$

$$[L_{i,j}^{(3)}, L_{k,l}^{(1)}] - [L_{i,j}^{(2)}, L_{k,l}^{(2)}] = L_{k,j}^{(1)}L_{i,l}^{(2)} - L_{k,j}^{(2)}L_{i,l}^{(1)}$$

$$(4) \quad n = 3, m = 0$$

$$[L_{i,j}^{(3)}, L_{k,l}^{(1)}] = \delta_{i,l}L_{k,j}^{(3)} - \delta_{k,j}L_{i,l}^{(3)}$$

Lemma 4.1. *The algebraic relations for the alternative set of generators of the Yangian, $\mathbb{Q}_{i,j}^{(m)}$, $i, j \in \{1, \dots, \mathcal{N}\}$, $m \in \{1, 2, \dots\}$ are given:*

$$\begin{aligned} [\mathbb{Q}_{i,j}^{(1)}, \mathbb{Q}_{k,l}^{(1)}] &= \delta_{i,l}\mathbb{Q}_{k,j}^{(1)} - \delta_{k,j}\mathbb{Q}_{i,l}^{(1)} \\ [\mathbb{Q}_{i,j}^{(1)}, \mathbb{Q}_{k,l}^{(2)}] &= \delta_{i,l}\mathbb{Q}_{k,j}^{(2)} - \delta_{k,j}\mathbb{Q}_{i,l}^{(2)} \\ [\mathbb{Q}_{i,j}^{(2)}, \mathbb{Q}_{k,l}^{(2)}] &= \delta_{i,l}\mathbb{Q}_{k,j}^{(3)} - \delta_{k,j}\mathbb{Q}_{i,l}^{(3)} - \frac{1}{4}\mathbb{Q}_{k,j}^{(1)} \sum_{x=1}^{\mathcal{N}} \mathbb{Q}_{i,x}^{(1)}\mathbb{Q}_{x,l}^{(1)} + \frac{1}{4} \sum_{x=1}^{\mathcal{N}} \mathbb{Q}_{k,x}^{(1)}\mathbb{Q}_{x,j}^{(1)}\mathbb{Q}_{i,l}^{(1)} \\ &+ \frac{1}{12} \left(\delta_{k,j} \sum_{x,y=1}^{\mathcal{N}} \mathbb{Q}_{i,x}^{(1)}\mathbb{Q}_{x,y}^{(1)}\mathbb{Q}_{y,l}^{(1)} - \delta_{i,l} \sum_{x,y=1}^{\mathcal{N}} \mathbb{Q}_{k,x}^{(1)}\mathbb{Q}_{x,y}^{(1)}\mathbb{Q}_{y,j}^{(1)} \right), \dots \end{aligned} \quad (4.7)$$

Proof. The proof is based on (4.6) and the exchange relations (1)-(4). \square

Remark 4.2. *The Yangian is a quasi-triangular Hopf algebra on \mathbb{C} [19] equipped with:*

$$(1) \quad \text{A co-product } \Delta : \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}}) \rightarrow \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}}) \otimes \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}}) \text{ such that}$$

$$(id \otimes \Delta)\mathbb{L}(\lambda) = \mathbb{L}_{13}(\lambda)\mathbb{L}_{12}(\lambda), \quad (\Delta \otimes id)\mathbb{L}(\lambda) = \mathbb{L}_{13}(\lambda)\mathbb{L}_{23}(\lambda). \quad (4.8)$$

We define the N -coproduct as $(id \otimes \Delta^{(N)})\mathbb{L}(\lambda) = \mathbb{L}_{0N} \dots \mathbb{L}_{01}$.

$$(2) \quad \text{A counit } \epsilon : \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}}) \rightarrow \mathbb{C}, \text{ such that } (\epsilon \otimes id)\mathbb{L}(\lambda) = 1_{\mathfrak{A}}, \quad (id \otimes \epsilon)\mathbb{L}(\lambda) = 1_V.$$

$$(3) \quad \text{An antipode } S : \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}}) \rightarrow \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}}) : (S \otimes id)\mathbb{L}(\lambda) = \mathbb{L}^{-1}(\lambda), \quad (id \otimes S)\mathbb{L}^{-1}(\lambda) = \mathbb{L}(\lambda).$$

Specifically, for the generators of the Yangian algebra (Hopf algebra):

$$\begin{aligned}\Delta(Q_{ab}^{(1)}) &= Q_{ab}^{(1)} \otimes 1 + 1 \otimes Q_{ab}^{(1)} \\ \Delta(Q_{ab}^{(2)}) &= Q_{ab}^{(2)} \otimes 1 + 1 \otimes Q_{ab}^{(2)} + \frac{1}{2} \sum_{d=1}^n (Q_{ad}^{(1)} \otimes Q_{db}^{(1)} - Q_{db}^{(1)} \otimes Q_{ad}^{(1)}).\end{aligned}\quad (4.9)$$

Also, $\epsilon(Q_{ab}^{(1)}) = \epsilon(Q_{ab}^{(2)}) = 0$, and $S(Q_{ab}^{(1)}) = -Q_{ab}^{(1)}$, $S(Q_{ab}^{(2)}) = -Q_{ab}^{(2)} + \frac{1}{2}Q_{ab}^{(1)}$.

The Yangian as a Hopf algebra is associative, and the n -coproducts can be derived by iteration via $\Delta^{(n+1)} = (id \otimes \Delta^{(n)})\Delta = (\Delta^{(n)} \otimes id)\Delta$.

In the next subsection we examine tensor realizations of the Yangian and express the N -coproducts of the algebra generators as elements of the discrete Magnus expansion. We also express coproducts of the algebra in terms of suitable tridendriform and pre-Lie algebra actions.

4.2. Tensor realizations of the Yangian & Pre-Lie algebras. In order to demonstrate the links between the Yangian, tridendriform and pre-Lie algebras we consider tensor realizations of the Yangian. Let the solution of relation (4.2) be of the general form $\mathbb{L}(\lambda) = 1_{V \otimes \mathcal{Y}} + \sum_{m>1} \frac{L^{(m)}}{\lambda^m} \in \text{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})$, with the R -matrix being the Yangian $R(\lambda) = 1_{V \otimes V} + \lambda^{-1}\mathcal{P}$ and $L^{(m)} = \sum_{x,y=1}^{\mathcal{N}} e_{x,y} \otimes L_{x,y}^{(m)}$, where $L_{x,y}^{(m)}$ are the generators of the Yangian $\mathfrak{gl}_{\mathcal{N}}$. We introduce the monodromy matrix $T \in \text{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}})^{\otimes N}$:

$$T_{0,12\dots N}(\lambda) := (id \otimes \Delta^{(N)})\mathbb{L}(\lambda) = \mathbb{L}_{0N}(\lambda) \dots \mathbb{L}_{01}(\lambda), \quad (4.10)$$

which also satisfies the algebraic relation (4.2) and is a solution of the discrete evolution problem (3.1). Historically, the index 0 is called ‘‘auxiliary’’, whereas the indices $1, 2, \dots, N$ are called ‘‘quantum’’, and they are usually suppressed for simplicity, i.e. we simply write $T_{0,N+1}$ (or T_{N+1}). We also note that a standard simple solution of the fundamental relation (4.2) for the Yangian is $\mathbb{L}(\lambda) = 1 + \lambda^{-1}\mathbb{P}$, where $\mathbb{P} = \sum_{i,j=1}^{\mathcal{N}} e_{i,j} \otimes \mathbb{P}_{i,j}$ and $\mathbb{P}_{i,j} \in \mathfrak{gl}_{\mathcal{N}} : [\mathbb{P}_{i,j}, \mathbb{P}_{k,l}] = \delta_{i,l}\mathbb{P}_{k,j} - \delta_{k,j}\mathbb{P}_{i,l}$.

Remark 4.3. We define the transfer matrix $\mathfrak{t}_{N+1}(\lambda) = \text{tr}_0(T_{0,N+1}(\lambda)) \in \text{End}((\mathbb{C}^{\mathcal{N}})^{\otimes N})$. The monodromy matrix T satisfies (4.2) (it is a tensor realization of the algebra defined by (4.2)), and hence it is shown that the transfer matrix provides a family of mutually commuting quantities [25]

($\mathfrak{t}_{N+1}(\lambda) = \lambda^N \sum_{k=1}^N \frac{\mathfrak{t}_{N+1}^{(k)}}{\lambda^k}$):

$$\left[\mathfrak{t}_{N+1}(\lambda), \mathfrak{t}_{N+1}(\mu) \right] = 0 \Rightarrow \left[\mathfrak{t}_{N+1}^{(k)}, \mathfrak{t}_{N+1}^{(l)} \right] = 0. \quad (4.11)$$

These commutation relations guarantee the ‘‘quantum integrability’’ of a spin-chain like system with periodic boundary conditions. For instance the Hamiltonian and momentum of the system belong to the family of the mutual commuting quantities.

Proposition 4.4. Let $L_{a,b}^{(m)}$, $a, b \in \{1, 2, \dots, \mathcal{N}\}$, $m = \{1, 2, \dots\}$ be the generators of the Yangian $\mathfrak{gl}_{\mathcal{N}}$ (4.3). The coproducts of the algebra generators are expressed in terms of tridendriform algebras actions \prec .

Proof. Before we carry on with the proof it is useful to recall the tensor index notation: let A, B be elements of some algebra \mathfrak{A} , then $A_n B_m = B_m A_n = B_m \otimes A_n$ $n > m$, where the indices n, m denote the position of the objects A, B on an N -tensor product, i.e.

$$A_n := 1 \otimes \dots \otimes \underbrace{A}_{n^{\text{th}} \text{ position}} \otimes 1 \dots \otimes 1. \quad (4.12)$$

We recall that the quantum monodromy is expressed as $T_{N+1}(\alpha) = 1 + \sum_{m>1} \alpha^m T^{(m)}(N+1)$ ($\alpha = \frac{1}{\lambda}$), with coefficients given in (3.2) and in terms of a tridendriform action in (3.3). We also recall that $T_{N+1}(\lambda) = (\text{id} \otimes \Delta^{(N)})\mathbb{L}(\lambda)$, hence $T_{a,b}^{(m)}(N+1) = \Delta^{(N)}(L_{a,b}^{(m)})$ (Δ is an algebra homomorphism), which leads to

$$\begin{aligned} \Delta^{(N)}(L_{a,b}^{(m)}) &= \sum_{n>\dots>n_1=1}^N \sum_{k=1}^m \left((L_{a,b_k}^{(m_k)})_{n_k} (L_{b_k,b_{k-1}}^{(m_{k-1})})_{n_{k-1}} \dots (L_{b_1,b}^{(m_1)})_{n_1} \right) \Big|_{\sum_{j=1}^k m_j=m} \\ &= \sum_{n=1}^N \sum_{k=1}^m \left(L_{a,b_k}^{(m_k)} \prec (L_{b_k,b_{k-1}}^{(m_{k-1})} \prec (\dots \prec (L_{b_2,b_1}^{(m_2)} \prec L_{b_1,b}^{(m_1)}) \dots)) \right) \Big|_{\sum_{j=1}^k m_j=m}. \end{aligned}$$

We read off the coproducts for the first three orders $m = 1, 2, 3$:

$$\begin{aligned} (1) \quad \Delta^{(N)}(L_{a,b}^{(1)}) &= \sum_{n=1}^N (L_{a,b}^{(1)})_n, \\ (2) \quad \Delta^{(N)}(L_{a,b}^{(2)}) &= \sum_{n=1}^N (L_{a,b}^{(2)})_n + \sum_{n=1}^N (L_{a,b_1}^{(1)} \prec L_{b_1,b}^{(1)})_n. \\ (3) \quad \Delta^{(N)}(L_{a,b}^{(3)}) &= \sum_{n=1}^N (L_{a,b}^{(3)})_n + \sum_{n=1}^N \left((L_{a,b_1}^{(2)} \prec L_{b_1,b}^{(1)})_n + (L_{a,b_1}^{(1)} \prec L_{b_1,b}^{(2)})_n \right) \\ &\quad + \sum_{n=1}^N (L_{a,b_2}^{(1)} \prec (L_{b_2,b_1}^{(1)} \prec L_{b_1,b}^{(1)}))_n. \quad \square \end{aligned}$$

We note that a presentation of the Yangian is given via the evaluation homomorphism, $\text{ev} : \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}}) \rightarrow \mathfrak{gl}_{\mathcal{N}}$, such that $L_{a,b}^{(m)} \mapsto \theta^m \mathbb{P}_{a,b}$, $\theta \in \mathbb{C}$, and $\mathbb{P}_{a,b}$ are the $\mathfrak{gl}_{\mathcal{N}}$ generators. We also recall that $\mathbb{L}(\lambda) = \sum_{m>0} \lambda^{-m} \mathbb{P}^m$ is a solution of (4.2), hence the following map also exists, $\sigma : \mathcal{Y}(\mathfrak{gl}_{\mathcal{N}}) \rightarrow \mathfrak{gl}_{\mathcal{N}}$, such that $L_{a,b}^{(m)} \mapsto (\mathbb{P}^m)_{a,b}$. Before we come to the second key link between the coproducts of the alternative Yangian generators $Q_{a,b}$ and pre-Lie algebras, we first introduce a useful Lemma and some handy notation.

Lemma 4.5. *Let x, y be general linear operators and $R(x), R(y)$ be Rota-Baxter operators, such that $[R(x), y] = [R(y), x] = 0$. Assume also that the actions of a tridendriform algebra are defined in (2.13), then $x \prec y = y \succ x$.*

Proof. From the definitions in (2.13) we immediately conclude that $x \prec y = y \succ x$. □

Remark 4.6. *We introduce some notation that will be used in the following Proposition. Let $\mathcal{O} \in \text{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathfrak{A}$, then $\mathcal{O}_n = \sum_{a,b} e_{a,b} \otimes (\mathcal{O}_{a,b})_n$, where $(\mathcal{O}_{a,b})_n$ is defined as in (4.12). We*

assume that $A, B \in \text{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathfrak{A}$, and we define

$$\begin{aligned} ((A \triangleright B)_n)_{a,b} &:= \left(\left[\sum_{m=1}^n A_m, B_n \right] + A_n B_n \right)_{a,b} \\ &= \sum_{c=1}^{\mathcal{N}} \left(\sum_{m=1}^{n-1} (A_{a,c})_m (B_{c,b})_n - (B_{a,c})_n \sum_{m=1}^{n-1} (A_{c,b})_m + (A_{a,c})_n (B_{c,b})_n \right). \end{aligned} \quad (4.13)$$

Recalling the definitions of the tridendriform actions (2.13), we conclude for the above expression

$$((A \triangleright B)_n)_{a,b} = \sum_{c=1}^{\mathcal{N}} \left((A_{a,c} \succ B_{c,b})_n - (B_{a,c} \prec A_{c,b})_n + (A_{a,c} B_{c,b})_n \right). \quad (4.14)$$

Due to the fact that $[(A_{a,c})_m, (B_{c,b})_n] = 0$, $n \neq m$, Lemma 4.5 applies.

Proposition 4.7. *Let $Q_{a,b}^{(m)}$, $a, b \in \{1, 2, \dots, \mathcal{N}\}$ and $m = \{1, 2, \dots\}$ be the alternative generators of the Yangian $\mathfrak{gl}_{\mathcal{N}}$ (Lemma 4.1). The coproducts of these generators are emerging from a pre-Lie algebra and are explicitly expressed in terms of tridendriform algebras actions.*

Proof. First we use the fact that $(\text{id} \otimes \Delta^{(N)})Q^{(m)} = \mathbb{Q}^{(m)}(N+1)$ (4.10), we also recall the expressions $\mathbb{Q}^{(m)}(N+1)$ (3.10), then via (4.5), (4.6) we conclude

$$\begin{aligned} (\text{id} \otimes \Delta^{(N)})Q^{(1)} &= \sum_{n=1}^N Q_n^{(1)}, \\ (\text{id} \otimes \Delta^{(N)})Q^{(2)} &= -\frac{1}{2} \sum_{n=1}^N (Q^{(1)} \triangleright Q^{(1)})_n + \sum_{n=1}^N (Q_n^{(2)} + \frac{1}{2}(Q_n^{(1)})^2), \\ (\text{id} \otimes \Delta^{(N)})Q^{(3)} &= \sum_{n=1}^N \left(\frac{1}{4} ((Q^{(1)} \triangleright Q^{(1)}) \triangleright Q^{(1)})_n + \frac{1}{12} (Q^{(1)} \triangleright (Q^{(1)} \triangleright Q^{(1)}))_n \right) \\ &\quad - \frac{1}{2} \sum_{n=1}^N \left((Q^{(2)} \triangleright Q^{(1)})_n + (Q^{(1)} \triangleright Q^{(2)})_n \right) \\ &\quad - \frac{1}{4} \sum_{n=1}^N \left(((Q^{(1)})^2 \triangleright Q^{(1)})_n + (Q^{(1)} \triangleright (Q^{(1)})^2)_n \right) \\ &\quad + \sum_{n=1}^N (Q_n^{(3)} + \frac{1}{2}(Q_n^{(2)} Q_n^{(1)} + Q_n^{(1)} Q_n^{(2)}) + \frac{1}{6}(Q_n^{(1)})^3), \dots \end{aligned} \quad (4.15)$$

The N -coproducts of the alternative generators $Q_{a,b}^{(m)}$ of the Yangian are extracted from expressions (4.15), by recalling $(\text{id} \otimes \Delta^{(N)})Q^{(n)} = \sum_{a,b} e_{a,b} \otimes \Delta^{(N)}(Q_{a,b}^{(n)})$ and $\mathbb{Q}_{a,b}^{(p)}(N) = \Delta^{(N)}(Q_{a,b}^{(p)})$, $a, b \in \{1, \dots, \mathcal{N}\}$. Specifically, for the first couple of terms, $n = 1, 2$ we obtain from (4.15), Remark 4.6,

$$\begin{aligned} (1) \quad \Delta^{(N)}(Q_{a,b}^{(1)}) &= \sum_{n=1}^N (Q_{a,b}^{(1)})_n, \\ (2) \quad \Delta^{(N)}(Q_{a,b}^{(2)}) &= \sum_{n=1}^N \left((Q_{a,b}^{(2)})_n - \frac{1}{2} \sum_{c=1}^{\mathcal{N}} \left((Q_{a,c}^{(1)} \succ Q_{c,b}^{(1)})_n - (Q_{a,c}^{(1)} \prec Q_{c,b}^{(1)})_n \right) \right). \end{aligned}$$

The notation and the requirements introduced in Remark 4.6 apply in this case. \square

Remark 4.8. *In the special case where $\mathbb{L}(\lambda) = 1 + \frac{1}{\lambda}\mathbb{P}$, i.e. $L^{(1)} = \mathbb{P}$ and $L^{(m)} = 0 \forall m > 0$, and given relation (4.5), (4.6), we conclude that expressions (4.15) reduce to the much more concise formulas (3.13), and Remark 3.6 on the construction of braces from pre-Lie algebras holds.*

We present below two distinct examples/applications associated to the alternative discrete expansion described in Subsection 3.1. More details on these two examples will be presented in future works as they are both of particular interest.

Example 4.9. *Let $R : \mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}} \rightarrow \mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}}$ be a solution of the YBE, of the form $R(\lambda) = r + \lambda^{-1}\mathcal{P}$, where r is also a solution of the YBE and \mathcal{P} is the permutation operator (see e.g.[13, 14, 15]). Such solutions can be obtained for instance from involutive set-theoretic solutions of the YBE [18, 13]. Recall now the expression of the monodromy matrix $T(\lambda) = \mathbb{L}_{\mathcal{N}}(\lambda) \dots \mathbb{L}_1(\lambda)$, where in this example $\mathbb{L}(\lambda) \rightarrow R(\lambda)$. Given that R satisfies the YBE one shows that the monodromy matrix naturally satisfies the (4.2). In this case expressions (3.21) and (3.20), (1)-(3) in Subsection 3.1 hold: $M \rightarrow r$, $L \rightarrow \mathcal{P}$ and $\alpha = \lambda^{-1}$.*

Example 4.10. *Another typical example in this class is the algebra $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$ [27, 28]. In this case $\mathbb{L}(\lambda) = L^+ - e^{-2\lambda}L^-$, where $L^+ = \sum_{a \leq b} e_{a,b} \otimes L_{a,b}^+$ and $L^- = \sum_{a \geq b} e_{a,b} \otimes L_{a,b}^-$. The elements $L_{a,b}^{\pm} \in B^{\pm}$, where B^{\pm} are subalgebras of $\mathfrak{U}_q(\mathfrak{gl}_{\mathcal{N}})$, (upper/lower Borel decomposition; for more details the interested reader is referred to [27, 28, 25]). The associated R -matrix is an $\mathcal{N}^2 \times \mathcal{N}^2$ matrix and is of the form $R(\lambda) = R^+ - e^{-2\lambda}R^-$, where $R^+ = \sum_{a \leq b} e_{a,b} \otimes R_{a,b}^+$ and $R^- = \sum_{a \geq b} e_{a,b} \otimes R_{a,b}^-$, (see detailed expressions in [27, 28]). In this case too expressions (3.21) and (3.20), (1)-(3) in Subsection 3.1 hold: $M \rightarrow L^+$, $L \rightarrow L^-$ and $\alpha = -e^{-2\lambda}$.*

Classical Integrability. Connections between (tri)dendriform, pre-Lie algebras and classical integrable systems and the associated classical deformed algebras are naturally identified given the findings of the previous subsection. The key point in the description and construction of classical integrable systems from the Hamiltonian point of view is the existence of a classical matrix $r \in \text{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}})$ that satisfies the classical YBE [24, 42],

$$[r_{12}(\lambda_1 - \lambda_2), r_{13}(\lambda_1 - \lambda_3)] + [r_{12}(\lambda_1 - \lambda_2) + r_{13}(\lambda_1 - \lambda_3), r_{23}(\lambda_2 - \lambda_3)] = 0. \quad (4.16)$$

The classical YBE can be seen as a linearization of the quantum YBE, i.e. we set $R = 1 + \delta r + \mathcal{O}(\delta^2)$. Also, the classical Lax operator $\mathbb{L}(\lambda) \in \text{End}(\mathbb{C}^{\mathcal{N}})$ associated to a discrete space integrable system satisfies the so called Sklyanin's bracket [24, 43, 44] (i.e. the classical analogue of the fundamental relation (4.2)):

$$\{\mathbb{L}_n(\lambda_1) \otimes \mathbb{L}_m(\lambda_2)\} = [r(\lambda_1 - \lambda_2), \mathbb{L}_n(\lambda_1) \otimes \mathbb{L}_m(\lambda_2)] \delta_{n,m}. \quad (4.17)$$

As in the quantum case algebraic quantities in involution exist, due the existence of an r -matrix. Indeed, the monodromy matrix $T(\lambda) = \mathbb{L}_{\mathcal{N}}(\lambda) \dots \mathbb{L}_1(\lambda) \in \text{End}(\mathbb{C}^{\mathcal{N}})$ also satisfies Sklyanin's bracket and hence it is shown that $\mathfrak{t}(\lambda) = \text{tr}T(\lambda)$ satisfies $\{\mathfrak{t}(\lambda), \mathfrak{t}(\mu)\} = 0$, which, given that $\mathfrak{t}(\lambda) = \sum_{m=1}^{\mathcal{N}} \lambda^{-m} \mathfrak{t}^{(m)}$, leads to a family of quantities in involution $\{\mathfrak{t}^{(m)}, \mathfrak{t}^{(k)}\} = 0$, i.e. classical integrability à la Liouville is shown [24].

In the case of the classical $\mathfrak{gl}_{\mathcal{N}}$ Yangian, the r -matrix is given as $r(\lambda) = \frac{\mathcal{P}}{\lambda}$, where we recall that \mathcal{P} is the $\mathcal{N}^2 \times \mathcal{N}^2$ permutation operator. Also we recall, as in the quantum case, $\mathbb{L}(\lambda) =$

$1 + \sum_{m>1} \lambda^{-m} L^{(m)}$ and $L^{(m)} = \sum_{a,b=1}^{\mathcal{N}} L_{a,b} e_{a,b}$. By substituting \mathbb{L} in (4.17) the classical Yangian relations are recovered, i.e. the classical analogues of (4.3). Everything holds as in the quantum case, but $[\cdot, \cdot] \mapsto -\{\cdot, \cdot\}$, i.e. $L_{a,b}^{(m)}$ are commutative objects, and are the generators of the classical Yangian. $T_{a,b}^{(m)}$ (non-local charges) are the classical analogues of the coproducts $\Delta^{(N)}(L_{a,b}^{(m)})$.

In the continuum case the Lax operator $\mathbb{A}(x, \lambda) \in \text{End}(\mathbb{C}^{\mathcal{N}})$ satisfies a linear Sklyanin's bracket (ultra-local algebra):

$$\{\mathbb{A}(x, \lambda_1) \otimes \mathbb{A}(y, \lambda_2)\} = [r(\lambda_1 - \lambda_2), \mathbb{A}(x, \lambda_1) \otimes 1 + 1 \otimes \mathbb{A}(y, \lambda_2)] \delta(x - y). \quad (4.18)$$

In the case of the Yangian specifically $r(\lambda) = \frac{\mathcal{P}}{\lambda}$ and $\mathbb{A}(x, \lambda) = \frac{1}{\lambda} A(x)$. The monodromy matrix in this case is defined as $T(x, \alpha) = \widehat{\text{Pexp}}\left(\int_0^x \mathbb{A}(\xi, \alpha) d\xi\right)$ (see also (1.2) and Remark 3.6) and satisfies the quadratic relation (4.17), [24, 33]. Also, the expressions for the continuous Magnus expansion (1.10), (1.12) and (2.11) hold ($\alpha = \lambda^{-1}$).

With this we conclude our analysis on the links between quantum and classical algebras, arising in the context of integrable systems, and pre-Lie and (tri)dendriform algebras. Further study on the quantum algebras associated to the Examples 4.9 and 4.10 will follow in future investigations. Example 4.9 in particular is of special interest given recent findings on the characterization of the quantum algebra associated to involutive set-theoretic solutions of the YBE as quasi-bialgebras [15, 16]. These quasi-bialgebras naturally emerge after suitably twisting the Yangian [15]. We note that set-theoretic solutions do not have a classical analogue, a fact that makes Example 4.9 even more intriguing. Moreover, it is known that all involutive set-theoretic solution of the YBE come from braces [9, 40]. This together with the fact that braces are obtained from pre-Lie algebras, while pre-lie algebras in turn are naturally connected to quantum algebras via the FRT construction, give us a strong motivation to further investigate these algebraic structures at the level of solutions, but also at the level of the emerging quantum algebras.

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