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The combined viscous semi-classical limit for a quantum hydrodynamic system with barrier potential

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Abstract

We investigate the viscous model of quantum hydrodynamics, which describes the charge transport in a certain semiconductor. Quantum mechanical effects lead to third order derivatives, turning the stationary system into an elliptic system of mixed order in the sense of Douglis–Nirenberg. In the case most relevant to applications, the semiconductor device features a piecewise constant barrier potential. In the case of thermodynamic equlibrium, we obtain asymptotic expansions of interfacial layers of the particle density in neighbourhoods of the jump points of this barrier potential, and we present rigorous proofs of uniform estimates of the remainder terms in these asymptotic expansions.

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1. Introduction

The ongoing miniaturisation of electronical devices requires the investigation of mathematical models for the electron transport that include quantum mechanical terms. One of these models is the isentropic viscous quantum hy-

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drodynamic model

$$\begin{cases} \partial_t n - \operatorname{div} J = \nu \Delta n, \\ \partial_t J - \operatorname{div}\left(\frac{J \otimes J}{n}\right) - \nabla p(n) + n\nabla(V + V_B) + \frac{\varepsilon^2}{6} n \nabla \frac{\Delta\sqrt{n}}{\sqrt{n}} = \nu \Delta J - \frac{J}{\tau}, \\ \lambda^2 \Delta V = n - \mathcal{C}, \end{cases} \tag{1}$$

formulated for the unknown functions (n, J, V), and the independent variables are $t \in \mathbb{R}$ as time, and $x \in \mathbb{R}^d$ as space. The unknown functions are the (positive) scalar electron density n, the vectorial electric current density J, and the scalar electric potential V. The item $p(n)$ is a generic pressure term, and a common choice is $p(n) = Tn + \mu n$, with a temperature T given by a relation $T(n) = T_0 n^{\gamma-1}$ for a positive constant T_0 and some $\gamma \geq 1$, and $\mu > 0$. Furthermore, the barrier potential $V_B = V_B(x)$ and the doping profile $\mathcal{C} = \mathcal{C}(x)$ of the semiconductor are given functions that describe certain material properties; these two functions are typically piecewise constant, and they are of crucial importance for the working principle of devices as the resonant tunnel diode. The purpose of this paper is to study analytically the behaviour of the solutions (n, J, V) near the jump points of the barrier potential V_B.

Additionally, we have certain positive physical constants, which have been scaled for ease of notation: The Planck constant ε, a relaxation time τ, the Debye length λ, and a viscosity constant ν.

A model (1) without the viscosity terms on the right hand side was proposed in [11] as a variant of the classical Euler–Poisson system, augmented by a term

$$\frac{\varepsilon^2}{6} n \nabla \frac{\Delta\sqrt{n}}{\sqrt{n}} =: \frac{\varepsilon^2}{6} n \nabla B(n)$$

that involves the Bohm potential $B(n)$ and describes quantum mechanical effects. The expectation is that this term is negligible in those regions where the electron flow can be described in terms of classical physics (i.e., in some regions far away from jump points of V_B).

There are various ways to derive (inviscid) quantum hydrodynamic models; we mention the traditional moment method applied to the collision Wigner equation [11], an approach via WKB wave functions from the Schrödinger Poisson system [14], and the entropy minimization approach [16]. Augmenting the Wigner equation with a Fokker Planck operator that describes the interaction of the electrons with the phonons of the crystal lattice, the dissipation terms $\nu\Delta n$ and $\nu\Delta J$ appear, see [3]. For an overview of this field, we refer to [1] and [15].

The quantum mechanical effects enter the system mainly via the Bohm term $B(n)$, which introduces third order spatial derivatives into the momentum balance equation, which complicates analytical studies of (1) considerably, compare [4], [5], [6], [10] for results on the transient problem without barrier potential. Further analytical difficulties arise from the barrier potential V_B having jumps, and in that situation the second equation of (1) must be understood in the distributional sense. We are not aware of any analytical results concerning the transient system (1), however we mention numerical simulations in [8], [11], [13], [17], [18], and [19].

We focus our attention to a one-dimensional, stationary system,

$$\begin{cases} J' = -\nu n'' \quad \text{in } [0,1], \\ 2\varepsilon^2 n \left(\dfrac{\sqrt{n}''}{\sqrt{n}} \right)' - \nu J'' - (p(n))' + \dfrac{J}{\tau} = \left(\dfrac{J^2}{n} \right)' - n(V + V_B)' \quad \text{in } [0,1], \\ \lambda^2 V'' = n - \mathcal{C} \quad \text{in } [0,1]. \end{cases} \tag{2}$$

For such a stationary system (without barrier potential), the existence of solutions was shown in [12], assuming small applied voltages $V(1) - V(0)$ and small currents J, which corresponds to a subsonic condition for the moving electrons. Although formulated for the isothermal case $p(n) = (T_0 + \mu)n$, the results of [12] seem to generalize to the case of general pressure terms $p(n)$. And we also mention [9], where it was shown (in the isothermal case) that solutions $(n, V, J) \in W^{2,2}(0,1) \times W^{2,2}(0,1) \times W^{1,2}(0,1)$ to (2) for given (possibly large) Dirichlet boundary values for V and periodic boundary values for n do exist.

The purpose of the present paper is to extend the solution theory of [9] towards an asymptotic expansion of the solution, for vanishing values of the quantum mechanical parameters ε and ν, focussing on the equilibrium case. See also [20] for further results. It turns out that we find a similar asymptotic expansion of the particle density n as in [2], [7] for a stationary quantum drift diffusion model.

The solution theory in [9] is based on a reformulation of the system (2) by means of a viscosity-adjusted Fermi level

$$\begin{cases} F = -(V + V_B) + h(n) + \dfrac{\nu}{\tau} \ln(n) - 2(\varepsilon^2 + \nu^2) \dfrac{\sqrt{n}''}{\sqrt{n}}, \\ nF' = -\left(\dfrac{J_0^2}{n} \right)' + 2 J_0 \nu \left(2 \dfrac{\sqrt{n}''}{\sqrt{n}} - \dfrac{(n')^2}{2n^2} \right) + \dfrac{J_0}{\tau}, \end{cases}$$

where the electric current was replaced by the identity $J = -\nu n' + J_0$ for some constant of integration J_0, and where $h\colon (0,\infty) \to \mathbb{R}$ is the enthalpy to p, satisfying $sh'(s) = p'(s)$ $(s > 0)$. This reformulation reveals the characteristic parameter $\kappa = \varepsilon^2 + \nu^2$, which is the coefficient of the derivative of highest order. It is expectable that the solutions $n = n_\kappa$ depend significantly on κ. Even in the one-dimensional setting, it seems to be a delicate matter whether the solutions n_κ converge to a limiting function n_0 as κ tends to zero, and uniform pointwise bounds for the electron densities from above and away from zero are not known to hold (except in the equilibrium case). In this paper, we consider the thermal equilibrium case of the Fermi levels, which refers to the case of $F \equiv \text{const}$ and $J_0 = 0$. We also assume the physically reasonable situation where $V_B + V_\kappa$ vanishes at the endpoints of the interval $[0,1]$. By a straightforward generalisation of the approach of [9], the known results read as follows, formulated in terms of $u := \sqrt{n}$:

Theorem 1.1 ([9],[20]). *Let $\varepsilon, \lambda, \nu, \tau > 0$, suppose $\mathcal{C}, V_B \in L^\infty(0,1)$, and assume that the Fermi level is a constant function $F \in \mathbb{R}$. Let h be the enthalpy to the smooth and strictly monotonically increasing pressure term p, which fulfills $sh'(s) = p'(s)$, $s > 0$, and additionally assume*

(i) $\lim_{s\to 0} h(s) = -\infty$ *and* $\lim_{s\to\infty} h(s) = \infty$.

(ii) $s \mapsto \sqrt{s}h(s)$ *is continuous in* $[0,\infty)$ *and* $s \mapsto \frac{\ln(s)}{h(s)}$ *is continuous in* $(0,\infty)$.

(iii) For any positive $f \in W^{1,2}(0,1)$, *there holds* $h(f), \frac{\ln(f)}{h(f)} \in W^{1,2}(0,1)$ *with the chain rule being valid.*

Then, for any $\kappa := \varepsilon^2 + \nu^2$, there exists a solution $(u_\kappa, V_\kappa) \in W^{2,2}(0,1) \times W^{2,2}(0,1)$ to the system of equations

$$\begin{cases} 2\kappa u_\kappa'' = -\Big(F + V_B + V_\kappa - h(u_\kappa^2) - \dfrac{\nu}{\tau}\ln(u_\kappa^2)\Big)\, u_\kappa, \quad \text{in } [0,1], \\ \lambda^2 V_\kappa'' = u_\kappa^2 - \mathcal{C}, \quad \text{in } [0,1], \\ V_\kappa(0) + V_B(0) = 0, \qquad V_\kappa(1) + V_B(1) = 0, \end{cases} \tag{3}$$

for homogeneous Neumann boundary conditions $u_\kappa'(0) = u_\kappa'(1) = 0$ and periodic boundary conditions $u_\kappa(0) = u_\kappa(1)$, $u_\kappa'(0) = u_\kappa'(1)$, respectively. Moreover, for any total mass $C^* > 0$ there exist solutions $(u_\kappa, V_\kappa) \in W^{2,2}(0,1) \times W^{2,2}(0,1)$, $\beta_\kappa \in \mathbb{R}$, to the system of equations

$$\begin{cases} 2\kappa u_\kappa'' = -\Big(F + \beta_\kappa + V_B + V_\kappa - h(u_\kappa^2) - \dfrac{\nu}{\tau}\ln(u_\kappa^2)\Big)\, u_\kappa, \quad \text{in } [0,1], \\ \lambda^2 V_\kappa'' = u_\kappa^2 - \mathcal{C}, \quad \text{in } [0,1], \\ \displaystyle\int_0^1 u_\kappa^2 = C^*, \\ V_\kappa(0) + V_B(0) = 0, \qquad V_\kappa(1) + V_B(1) = 0, \end{cases} \tag{4}$$

for homogeneous Neumann boundary conditions for u_κ and periodic boundary conditions for u_κ, respectively. The functions $n_\kappa = u_\kappa^2$, V_κ and $J_\kappa = -\nu n_\kappa'$ form a solution to the viscous quantum hydrodynamic system (2) and admit the respective boundary values. There exist constants $C_0, \ldots, C_4 > 0$ such that, for $0 < \kappa < \kappa_0$,

$$C_0 \le u_\kappa \le C_1, \qquad \|V_\kappa\|_{C^1(0,1)} \le C_2, \qquad |\beta_\kappa| \le C_3,$$
$$\|u_\kappa'\|_{L^\infty(0,1)} \le C_4 \kappa^{-1/2}.$$

2. Statement of the problem and main result

Throughout the paper, we always assume that the situation of Theorem 1.1 is given and that u_κ, V_κ are corresponding solutions. In case that the total mass C^* is prescribed for u_κ^2, let β_κ be the corresponding Lagrange multiplier appearing in (4). We investigate the behavior of solutions as κ tends to zero for the case of piecewise constant barrier potentials V_B, which is the physically most relevant case. Formally letting $\kappa = 0$ in (4), we expect that potentially existing limiting functions u_0 and V_0 fulfill the identities

$$0 = -(F + \beta_0 + V_B + V_0 - h(u_0^2))u_0 \quad \text{and} \quad \lambda^2 V_0'' = u_0^2 - \mathcal{C}. \tag{5}$$

The first limiting equation, however, shows that the expected limit u_0 will jump at the jump points of V_B, and therefore, convergence of $(u_\kappa)_{\kappa\to 0}$ to u_0 is not possible in strong topologies like $L^\infty(0,1)$. However, convergence of the sequence $(V_\kappa)_{\kappa\to 0}$ in $W^{1,2}(0,1)$ will be shown by monotonicity arguments; and as a consequence we also obtain L^p convergence of the sequence $(u_\kappa)_{\kappa\to 0}$, which is locally uniform away from the jump points of V_B. Our first main theorem reads as follows:

Theorem 2.1. *Let the situation of Theorem 1.1 be given and assume that the barrier potential V_B is a piecewise constant function. Then there exist $V_0 \in W^{2,2}(0,1)$, $u_0 \in L^\infty(0,1)$ and (if appropriate) $\beta_0 \in \mathbb{R}$ solving (5) and the Dirichlet boundary conditions for V such that*

$$\|V_0 - V_\kappa\|_{W^{1,2}(0,1)} \le C\kappa^{1/4}, \tag{6}$$

$$|\beta_0 - \beta_\kappa| \le C\kappa^{1/4}, \tag{7}$$

$$\|u_0^2 - u_\kappa^2\|_{L^p(0,1)} \le C\kappa^{1/4p} \quad (0 < \kappa < \kappa_0). \tag{8}$$

Moreover, for any subinterval $[s_0, s_1] \subset [0,1]$ of length $L := s_1 - s_0$, where V_B is constant, it holds

$$\|u_0^2 - u_\kappa^2\|_{L^\infty(s_0 + L\kappa^{1/4}, s_1 - L\kappa^{1/4})} \le C\kappa^{1/4}. \tag{9}$$

Since all solutions u_κ ($\kappa > 0$) are continuously differentiable by the Sobolev embedding theorem, and the sequence $(u_\kappa)_{\kappa\to 0}$ converges uniformly in the interior of subintervals, where V_B is constant, c.f. (9), the functions u_κ are expected to form interface layers near the jump points of V_B. The quantum term $\kappa u_\kappa''$ has a non-small value only in this layer regime, and it is natural to expect a layer width of order $\mathcal{O}(\kappa^{1/2})$. Our second main theorem makes this statement rigorous, by means of an analytically proven remainder estimate:

Theorem 2.2 (Zeroth order asymptotic expansion). *Let the situation of Theorem 2.1 be given. Let $s_1 = 0$, $s_{N+1} = 1$ and $s_2, \dots, s_N$ be the jump points of V_B. There exist $W_\kappa \colon [0,1] \to \mathbb{R}$ and a positive function $c_0 \colon [0,1] \to (0,\infty)$ which fulfills $c_0(x) = u_0(s_i\pm)$, $(i = 1, \dots, N+1)$, in half-sided neighbourhoods of s_i, such that*

$$\left\| u_\kappa - u_0 \frac{W_\kappa}{c_0} \right\|_{L^2(0,1)} + \|V_\kappa - V_0\|_{W^{1,2}(0,1)} \le C\kappa^{1/2} \tag{10}$$

and

$$\left\| u_\kappa - u_0 \frac{W_\kappa}{c_0} \right\|_{L^\infty(0,1)} \le C\kappa^{1/4}. \tag{11}$$

Near any jump point s_i of V_B, the functions W_κ locally admit a representation

$$W_\kappa(x) = w\left(\frac{x - s_i}{\kappa^{1/2}}\right)$$

for a function $w \in C^1(\mathbb{R})$ with $\lim_{y\to\pm\infty} w(y) = u_0(s_i\pm)$ and exponential convergence to both limits.

The proofs of both results rely on the spatial dimension being one — we use Sobolev's embedding theorem and ODE techniques extensively.

The structure of the paper is as follows. In Section 3, we show various bounds on derivatives of u_κ, and the key result is (17), which shows that the Bohm potential term $B(u_\kappa^2)$ is indeed negligible in the exterior region, which is, by convention, "far away" from the jumps of V_B. Then Theorem 2.1 is proved in Section 4 by means of monotonicity principles. This gives us the first term u_0 of the asymptotic expansion of u_κ in the exterior region. Section 5 contains results on the asymptotic expansion of u_κ in a certain interior region (which is "near the jumps of V_B"), and on the matching of both asymptotic expansions, see Lemmas 5.3 and 5.5. Choosing a different set of multipliers, we then improve the remainder estimates in Section 6, concluding the proof of Theorem 2.2.

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3. First estimates to solutions

In the thermal equilibrium case, we have $J_0 = 0$ and a constant Fermi level $F \in \mathbb{R}$, so that $J = -\nu n'$. Using this and $\frac{\sqrt{n}''}{\sqrt{n}} = \frac{n''}{2n} - \frac{n'^2}{n^2}$, the weak formulation of the second equation of (2) reads as

$$-\int_0^1 \left(p'(n) + \frac{\nu}{\tau}\right) n'\varphi - (\varepsilon^2 + \nu^2) \int_0^1 \left(\frac{n'^2}{n}\right)' \varphi - (\varepsilon^2 + \nu^2) \int_0^1 n''\varphi'$$
$$+ \int_0^1 nV'\varphi - \int_0^1 V_B (n\varphi)' = 0 \qquad (\varphi \in C_c^\infty(0,1)). \tag{12}$$

Lemma 3.1 (Basic exterior estimates to solutions).
In the situation of Theorem 1.1, assume that V_B is constant in some non-trivial

interval $[s_0, s_1] \subset (0,1)$. *Then, for* $C^* = \int_0^1 n\,\mathrm{d}x$, *the estimate*

$$K_0 \int\limits_{s_0+\sigma L}^{s_1-\sigma L} u'^2\,\mathrm{d}x + (\varepsilon^2+\nu^2) \int\limits_{s_0+\sigma L}^{s_1-\sigma L} u''^2 + \frac{u'^4}{24u^2}\,\mathrm{d}x \le \frac{(\varepsilon^2+\nu^2)CC^*}{\sigma^4 L^4} + \frac{CC^*}{K_0\lambda^4}, \qquad 0<\sigma<\frac{1}{2}, \tag{13}$$

holds, where K_0 *only depends on* C_0, C_1, p; *and* C *only depends on* $\|\mathcal{C}\|_{L^\infty(0,1)}$. *Additionally,* $L := s_1 - s_0$ *is the length of the interval* $[s_0, s_1]$. *Consequently, for* $I_\kappa := [s_0 + \kappa^{1/4}L, s_1 - \kappa^{1/4}L]$, *there holds*

$$\|u'\|_{L^2(I_\kappa)} \le C, \tag{14}$$

$$\|\kappa u''\|_{L^2(I_\kappa)} \le C\kappa^{1/2}, \tag{15}$$

$$\|u'\|_{L^\infty(I_\kappa)} \le C\kappa^{-1/4}, \tag{16}$$

$$\left\|2\kappa\frac{u''}{u}\right\|_{L^\infty(I_\kappa)} \le C\kappa^{1/4}, \tag{17}$$

with some constant C *which does not depend on* $0<\kappa<\kappa_0$.

Proof. Let $a_0 := \frac{1}{\sigma^4 L^4}$ and define $\psi \in W_0^{1,2}(0,1)$ by

$$\psi(x) := \begin{cases} a_0(x-s_0)^4, & s_0 \le x \le s_0+\sigma L, \\ 1, & s_0+\sigma L \le x \le s_1 - \sigma L, \\ a_0(s_1-x)^4, & s_1-\sigma L \le x \le s_1, \\ 0, & x \notin [s_0, s_1]. \end{cases}$$

Using $\varphi := \frac{u'}{u}\psi$ as a test function in (12), we obtain in terms of $u=\sqrt{n}$

$$\begin{aligned} 2\int_0^1 &\left(p'(u^2)+\frac{\nu}{\tau}\right) u'^2\psi\,\mathrm{d}x \\ &+ 2(\varepsilon^2+\nu^2)\int_0^1 2(u'^2)'\frac{u'}{u}\psi + (u''u+u'^2)\left(\frac{u'}{u}\psi\right)'\,\mathrm{d}x \\ = &\int_0^1 uu'V'\psi\,\mathrm{d}x, \end{aligned}$$

since V_B is constant on $\operatorname{supp}\psi$. We abbreviate this identity by

$$I_1 + 2(\varepsilon^2+\nu^2)I_2 = J_1.$$

By assumption, we have $p'(\xi)>0$ for $\xi>0$ and $C_0^2 \le u^2(x) \le C_1^2$ for $x\in[0,1]$ from Theorem 1.1, so that $p'(u^2(x)) \ge K_0 > 0$ for all $x \in [0,1]$ and some K_0. Then,

$$2\left(K_0+\frac{\nu}{\tau}\right)\int_0^1 u'^2\psi\,\mathrm{d}x \le I_1.$$

Re-ordering terms in I_2, we find

$$I_2 = \int_0^1 \left(u''^{\,2} + 4\frac{u''u'^{\,2}}{u} - \frac{u'^{\,4}}{u^2} \right) \psi + \left(u'u'' + \frac{u'^{\,3}}{u} \right) \psi' \,\mathrm{d}x. \tag{18}$$

An integration by parts yields

$$0 = \int_0^1 \left(3\frac{u''u'^{\,2}}{u} - \frac{u'^{\,4}}{u^2} \right) \psi + \frac{u'^{\,3}}{u} \psi' \,\mathrm{d}x. \tag{19}$$

Now we form (18)$-\frac{4}{3}$(19), and the result is

$$\begin{aligned} I_2 &= \int_0^1 \left(u''^{\,2} + \frac{1}{3}\frac{u'^{\,4}}{u^2} \right) \psi + u''u'\psi' - \frac{1}{3}\frac{u'^{\,3}}{u}\psi' \,\mathrm{d}x \\ &=: \int_0^1 \left(u''^{\,2} + \frac{1}{3}\frac{u'^{\,4}}{u^2} \right) \psi \,\mathrm{d}x + I_{2,1} + I_{2,2}. \end{aligned}$$

Because $|\psi'| \leq \frac{4}{\sigma L}\psi^{3/4}$, exploiting Young's inequality with exponents 2, 4, 4 gives

$$\begin{aligned} |I_{2,1}| &\leq \int_0^1 \left| u''\psi^{1/2} \right| \cdot \left| \frac{u'\psi^{1/4}}{\sqrt{2}u^{1/2}} \right| \cdot \frac{4\sqrt{2}u^{1/2}}{\sigma L} \,\mathrm{d}x \\ &\leq \int_0^1 \frac{1}{2} u''^{\,2}\psi \,\mathrm{d}x + \int_0^1 \frac{u'^{\,4}}{16u^2}\psi \,\mathrm{d}x + \int_0^1 \frac{256u^2}{\sigma^4 L^4}\chi_{[s_0,s_1]} \,\mathrm{d}x. \end{aligned}$$

Using Young's inequality with the exponents $\frac{4}{3}$ and 4, we further obtain

$$|I_{2,2}| \leq \frac{1}{3}\int_0^1 \frac{(u'^4\psi)^{3/4}}{u^{3/2}} \cdot \frac{4u^{1/2}}{\sigma L} \,\mathrm{d}x \leq \int_0^1 \frac{u'^4}{4u^2}\psi + \frac{64u^2}{3\sigma^4 L^4}\chi_{[s_0,s_1]} \,\mathrm{d}x.$$

Since $\int_0^1 u^2 \,\mathrm{d}x = C^*$, we infer

$$\int_0^1 \frac{1}{2}u''^2\psi \,\mathrm{d}x + \left(\frac{1}{3} - \frac{1}{16} - \frac{1}{4} \right) \int_0^1 \frac{u'^4}{u^2}\psi \,\mathrm{d}x \leq I_2 + \frac{256C^*}{\sigma^4 L^4} + \frac{64C^*}{3\sigma^4 L^4}.$$

Thus,

$$(\varepsilon^2 + \nu^2) \int_0^1 \left(u''^2 + \frac{1}{24}\frac{u'^{\,4}}{u^2} \right) \psi \,\mathrm{d}x \leq 2(\varepsilon^2 + \nu^2) I_2 + \frac{2(\varepsilon^2 + \nu^2)CC^*}{\sigma^4 L^4}.$$

Concerning the right hand side J_1, we estimate

$$|J_1| \leq K_0 \int_0^1 u'^{\,2}\psi \,\mathrm{d}x + \frac{C\|V'\|^2_{L^\infty(0,1)}C^*}{K_0} \leq K_0 \int_0^1 u'^{\,2}\psi \,\mathrm{d}x + \frac{CC^*}{\lambda^4 K_0}.$$

Combining all estimates, inequality (13) follows. For $\sigma = \kappa^{1/4}$, this immediately yields inequalities (14) and (15); and by interpolation we also obtain (16). As

$\frac{u''}{u}$ is smooth in I_κ, we find

$$2\kappa\left(\frac{u''}{u}\right)' = -\Big(F + V_B + V + \beta - h(u^2) - \frac{\nu}{\tau}\ln(u^2)\Big)'$$
$$= -V' + 2\left(p'(u^2) + \frac{\nu}{\tau}\right)\frac{u'}{u}$$

as an equality in I_κ. The right hand side is uniformly bounded in $L^2(I_\kappa)$ by inequality (14), the pointwise upper and lower bounds to u and the uniform boundedness of $\|V\|_{W^{2,2}(0,1)}$. Joining this bound with inequality (15), we obtain

$$\left\|2\kappa\frac{u''}{u}\right\|_{W^{1,2}(I_\kappa)} \le C.$$

Interpolating this inequality with estimate (15), inequality (17) follows. □

4. Exterior convergence results

Estimate (17) already shows that the quantum mechanical Bohm term $\kappa\frac{u''_\kappa}{u_\kappa}$ decays in the interior of subintervals, where V_B is constant. We are now in the position to prove the convergence of $(V_\kappa)_{\kappa\to 0}$ and consequently, convergence of $(u_\kappa)_{\kappa\to 0}$ also follows.

Proof of Theorem 2.1. Let $I^1,\dots,I^N$ be the maximal intervals in which V_B is constant and denote by $I^1_\kappa,\dots,I^N_\kappa$ the corresponding subintervals introduced in Lemma 3.1. By assumption, $h^{-1}\colon \mathbb{R}\to(0,\infty)$ exists and u_κ^2 can be expressed by

$$u_\kappa^2 = h^{-1}(F + V_B + V_\kappa + \beta_\kappa) + r_\kappa,$$

where

$$r_\kappa := h^{-1}\left(F + V_B + V_\kappa + \beta_\kappa + 2\kappa\frac{u''_\kappa}{u_\kappa} - \frac{\nu}{\tau}\ln(u_\kappa^2)\right) - h^{-1}(F + V_B + V_\kappa + \beta_\kappa).$$

By Lipschitz continuity of h^{-1},

$$|r_\kappa(x)| \le C\left(\left|2\kappa\frac{u''_\kappa(x)}{u_\kappa(x)}\right| + \kappa^{1/2}\right) \quad (x\in(0,1)),$$

which implies together with inequalities (15) and (17)

$$\|r_\kappa\|_{L^2(I_\kappa)} \le C\kappa^{1/2}, \tag{20}$$
$$\|r_\kappa\|_{L^\infty(I_\kappa)} \le C\kappa^{1/4}, \tag{21}$$

for $0<\kappa<\kappa_0$. Let $0<\kappa_2\le\kappa_1<\kappa_0$, abbreviate $u_i := u_{\kappa_i}$, $V_i := V_{\kappa_i}$, $\beta_i := \beta_{\kappa_i}$, $r_i := r_{\kappa_i}$ and define $\widetilde{V}_i := V_i + \beta_i$ for $i = 1,2$. In the following, we may formally consider $\beta_i = 0$ for $i = 1, 2$ if the additional constraint $\int u_\kappa^2 = C^*$ is not demanded for the solutions. An integration by parts yields

$$\begin{aligned}\lambda^2 \int_0^1 (V_1' - V_2')^2 \,\mathrm{d}x &= -\int_0^1 \left(u_1^2 - u_2^2\right)\left(\widetilde{V}_1 - \widetilde{V}_2\right) \mathrm{d}x \\ &= -\sum_{i=1}^N \int_{I^i \setminus I^i_{\kappa_1}} \left(u_1^2 - u_2^2\right)\left(\widetilde{V}_1 - \widetilde{V}_2\right) \mathrm{d}x \\ &\qquad - \sum_{i=1}^N \int_{I^i_{\kappa_1}} \left(u_1^2 - u_2^2\right)\left(\widetilde{V}_1 - \widetilde{V}_2\right) \mathrm{d}x \\ &=: S_1 + S_2.\end{aligned}$$

Using the uniform pointwise boundedness of u_κ^2, Young's inequality, the Sobolev embedding $W^{1,2}(0,1) \hookrightarrow L^\infty(0,1)$ and Poincaré's inequality, we find

$$\begin{aligned}S_1 &\le C \sum_{i=1}^N \|\widetilde{V}_1 - \widetilde{V}_2\|_{L^\infty(I^i \setminus I^i_{\kappa_1})} \kappa_1^{1/4} \\ &\le C\gamma^2 \|\widetilde{V}_1 - \widetilde{V}_2\|^2_{W^{1,2}(0,1)} + \gamma^{-2}\kappa_1^{1/2} \\ &\le K_1 \gamma^2 \left(\|V_1' - V_2'\|^2_{L^2(0,1)} + |\beta_1 - \beta_2|^2\right) + \gamma^{-2}\kappa_1^{1/2},\end{aligned}$$

where $K_1 > 0$ is a constant und $\gamma > 0$ is a free parameter which will be chosen later on. Concerning S_2, we calculate

$$\begin{aligned}&-\sum_{i=1}^N \int_{I^i_{\kappa_1}} \left(u_1^2 - u_2^2\right)\left(\widetilde{V}_1 - \widetilde{V}_2\right) \mathrm{d}x \\ &\quad = -\sum_{i=1}^N \int_{I^i_{\kappa_1}} \left(h^{-1}(F + V_B + \widetilde{V}_1) - h^{-1}(F + V_B + \widetilde{V}_2)\right)\left(\widetilde{V}_1 - \widetilde{V}_2\right) \mathrm{d}x \\ &\qquad - \sum_{i=1}^N \int_{I^i_{\kappa_1}} (r_1 - r_2)\left(\widetilde{V}_1 - \widetilde{V}_2\right) \mathrm{d}x.\end{aligned}$$

Now we certainly find a positive number K_2 with

$$\frac{1}{K_2} \le h'(s^2) \le K_2, \qquad C_0 \le s \le C_1. \tag{22}$$

Then $(h^{-1})'$ enjoys the same bounds, and we get

$$\begin{aligned}&-\sum_{i=1}^N \int_{I^i_{\kappa_1}} \left(h^{-1}(F + V_B + \widetilde{V}_1) - h^{-1}(F + V_B + \widetilde{V}_2)\right)\left(\widetilde{V}_1 - \widetilde{V}_2\right) \mathrm{d}x \\ &\quad \le -K_2^{-1} \sum_{i=1}^N \|\widetilde{V}_1 - \widetilde{V}_2\|^2_{L^2(I^i_{\kappa_1})}.\end{aligned}$$

The Cauchy-Schwarz inequality in combination with both the uniform boundedness of $(\widetilde{V}_\kappa)_{0<\kappa<\kappa_0}$ in $L^2(0,1)$ and estimate (20) implies

$$-\sum_{i=1}^N \int_{I^i_{\kappa_1}} (r_1 - r_2)\left(\widetilde{V}_1 - \widetilde{V}_2\right) \mathrm{d}x \le \sum_{i=1}^N \|r_1 - r_2\|_{L^2(I^i_{\kappa_1})} \|\widetilde{V}_1 - \widetilde{V}_2\|_{L^2(I^i_{\kappa_1})} \le K_3 \kappa_1^{1/2},$$

and we conclude that

$$S_2 \le -K_2^{-1} \sum_{i=1}^N \|\widetilde{V}_1 - \widetilde{V}_2\|^2_{L^2(I^i_{\kappa_1})} + K_3 \kappa_1^{1/2}.$$

Combining all estimates, we obtain

$$\left(\lambda^2 - K_1\gamma^2\right) \|V_1' - V_2'\|^2_{L^2(0,1)} \le K_1 \gamma^2 |\beta_1 - \beta_2|^2 - K_2^{-1} \sum_{i=1}^N \|\widetilde{V}_1 - \widetilde{V}_2\|^2_{L^2(I^i_{\kappa_1})} + \gamma^{-2}\kappa_1^{1/2} + K_3 \kappa_1^{1/2}. \tag{23}$$

Let $\delta > 0$ be a parameter to be determined later on. For small κ_1, we certainly have $\sum_{i=1}^N \mathrm{meas}(I^i_{\kappa_1}) \ge \frac{1}{2}$, and then we may estimate

$$\frac{1}{2}\delta |\beta_1 - \beta_2|^2 \le \delta \sum_{i=1}^N \|\beta_1 - \beta_2\|^2_{L^2(I^i_{\kappa_1})} \le 2\delta \sum_{i=1}^N \|\widetilde{V}_1 - \widetilde{V}_2\|^2_{L^2(I^i_{\kappa_1})} + 2\delta K_4 \|V_1' - V_2'\|^2_{L^2(0,1)} \tag{24}$$

with some constant $K_4 > 0$ arising from the Poincaré inequality. Adding estimates (23) and (24), the inequality

$$\left(\lambda^2 - K_1\gamma^2 - 2\delta K_4\right) \|V_1' - V_2'\|^2_{L^2(0,1)} + \left(\frac{\delta}{2} - K_1\gamma^2\right) |\beta_1 - \beta_2|^2 \le \left(2\delta - K_2^{-1}\right) \sum_{i=1}^N \|\widetilde{V}_1 - \widetilde{V}_2\|^2_{L^2(I^i_{\kappa_1})} + \gamma^{-2}\kappa_1^{1/2} + K_3\kappa_1^{1/2}$$

follows. Choosing δ and γ sufficiently small, one obtains

$$\|V_1' - V_2'\|^2_{L^2(0,1)} + |\beta_1 - \beta_2|^2 \le C\kappa_1^{1/2},$$

which yields inequalities (6) and (7). To prove the convergence results on u_κ, observe that

$$u_0^2 := h^{-1}(F + V_B + V_0 + \beta_0) \in L^\infty(0,1)$$

is a positive function on $[0,1]$. For the function

$$k_\nu := h + \tfrac{\nu}{\tau} \ln$$

it is easily seen that $\|h^{-1} - k_\nu^{-1}\|_{L^\infty(k_{\nu_0}(I))} \le C\nu$ for small $0 < \nu < \nu_0$ and an interval $I = [a,b] \subset (0,\infty)$. Since V_κ, β_κ and $2\kappa\frac{u_\kappa''}{u_\kappa}$ are uniformly bounded in $L^\infty(I_\kappa^i)$ for $0 < \kappa < \kappa_0$ and $i = 1,\ldots,N$, the local Lipschitz continuity of k_ν^{-1}, the convergence results for V_κ and β_κ and inequality (17) readily yield

$$\|u_0^2 - u_\kappa^2\|_{L^\infty(I_\kappa^i)} \le C\kappa^{1/4}.$$

From this, the L^p-estimates for $u_\kappa - u_0$ are easily obtained by a trivial estimate of the appearing integrals over the regimes outside the subintervals $I_\kappa^1,\ldots,I_\kappa^N$, because the functions u_κ are uniformly bounded. □

5. Derivation of the zeroth order asymptotic expansion

We now derive differential equations describing the functions W_κ from Theorem 2.2 locally at any jump point s_0 of V_B. As a preliminary step, we need to show that the derivatives $u_\kappa'(s_0)$ are not too small — they are of order $\kappa^{-1/2}$.

Lemma 5.1. *In the situation of Theorem 1.1, assume that V_B is piecewise constant, and let s_0 be a jump point of V_B. Then there exists a constant $C_5 > 0$ such that*

$$C_4\kappa^{-1/2} \ge \|u_\kappa'\|_{L^\infty(0,1)} \ge |u_\kappa'(s_0)| \ge C_5\kappa^{-1/2}, \tag{25}$$

for $0 < \kappa < \kappa_0$. We also have, for such κ,

$$u_\kappa^2(s_0) = \frac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{V_B(s_0+) - V_B(s_0-)} + \mathcal{O}(\kappa^{1/4}). \tag{26}$$

Remark 5.2. *We remark that $u_\kappa^2(s_0)$ is (up to an error of size $\kappa^{1/4}$) between the left and right limits $u_0^2(s_0-)$ and $u_0^2(s_0+)$, because the extended mean value theorem gives us*

$$\frac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{V_B(s_0+) - V_B(s_0-)} = \frac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{h(u_0^2(s_0+)) - h(u_0^2(s_0-))} = \frac{p'(\xi)}{h'(\xi)} = \xi,$$

for some ξ between $u_0^2(s_0+)$ and $u_0^2(s_0-)$.

Proof of Lemma 5.1. We choose, for $z > 0$,

$$K(z) = zk(z) - p(z) - \frac{\nu}{\tau}z, \tag{27}$$

$$H(z) = zh(z) - p(z) \tag{28}$$

as primitive functions of k and h, respectively. We may unite the differential equations for u_κ from (3) and (4) into the equation

$$2\kappa u_\kappa'' = -(F + \beta_\kappa + V_B + V_\kappa - k(u_\kappa^2))u_\kappa, \tag{29}$$

tacitly making the convention $\beta_\kappa = 0$ in the case without mass balance. Then (6) and (7) imply

$$\begin{aligned} -2\kappa u_\kappa'' &= (F + \beta_0 + V_B + V_0 - k(u_\kappa^2))u_\kappa + (\beta_\kappa - \beta_0 + V_\kappa - V_0)u_\kappa \\ &= (h(u_0^2) - k(u_\kappa^2))u_\kappa + \mathcal{O}(\kappa^{1/4}), \end{aligned}$$

with $\mathcal{O}(\kappa^{1/4})$ meant in $L^\infty(0,1)$. At the jump points of V_B, this equation is to be understood in the sense of one-sided limits. We also obtain

$$-2\kappa u_\kappa'' = (h(u_0^2) - h(u_\kappa^2))u_\kappa + \mathcal{O}(\kappa^{1/4}). \tag{30}$$

Now we have on the one hand, in the sense of distributions,

$$\begin{aligned} &\big((F + \beta_\kappa + V_B + V_\kappa)u_\kappa^2 - K(u_\kappa^2)\big)' \\ &\quad = 2(F + \beta_\kappa + V_B + V_\kappa - k(u_\kappa^2))u_\kappa u_\kappa' + (V_B + V_\kappa)' u_\kappa^2 \\ &\quad = -2\kappa((u_\kappa')^2)' + (V_B + V_\kappa)' u_\kappa^2, \end{aligned}$$

and on the other hand, we have

$$\begin{aligned} &\big((F + \beta_\kappa + V_B + V_\kappa)u_\kappa^2 - K(u_\kappa^2)\big)' \\ &\quad = \Big((F + \beta_\kappa + V_B + V_\kappa - k(u_\kappa^2))u_\kappa^2 + p(u_\kappa^2) + \frac{\nu}{\tau}u_\kappa^2\Big)' \\ &\quad = \Big(-2\kappa u_\kappa'' u_\kappa + p(u_\kappa^2) + \frac{\nu}{\tau}u_\kappa^2\Big)'. \end{aligned}$$

Now let $[s_0, s_1]$ be a maximal interval of length $L := s_1 - s_0$ where V_B is constant, and consider x_0, x_1 with $s_0 < x_0 < x_1 < s_1$. Then we have

$$-2\kappa((u_\kappa')^2)' + V_\kappa' u_\kappa^2 = \Big(-2\kappa u_\kappa'' u_\kappa + p(u_\kappa^2) + \frac{\nu}{\tau}u_\kappa^2\Big)', \qquad \text{on } [x_0, x_1].$$

We integrate over $[x_0, x_1]$:

$$\begin{aligned} &- 2\kappa(u_\kappa')^2\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} V_\kappa' u_\kappa^2 \,\mathrm{d}x \\ &\quad = -2\kappa u_\kappa''(x_1)u_\kappa(x_1) + 2\kappa u_\kappa''(x_0)u_\kappa(x_0) + \Big(p(u_\kappa^2) + \frac{\nu}{\tau}u_\kappa^2\Big)\Big|_{x_0}^{x_1}. \end{aligned}$$

Now we utilise (30) for $u_\kappa''(x_0)$ and send x_0 to s_0:

$$\begin{aligned} &2\kappa(u_\kappa'(s_0))^2 - 2\kappa(u_\kappa'(x_1))^2 + p(u_\kappa^2(s_0)) + \int_{s_0}^{x_1} V_\kappa' u_\kappa^2 \,\mathrm{d}x \\ &\quad = -2\kappa u_\kappa''(x_1)u_\kappa(x_1) - (h(u_0^2(s_0+)) - h(u_\kappa^2(s_0)))u_\kappa^2(s_0) + p(u_\kappa^2(x_1)) + \mathcal{O}(\kappa^{1/4}), \end{aligned}$$

having recalled that u_0^2 jumps at s_0. Let us re-arrange this identity into

$$\begin{aligned} &2\kappa(u_\kappa'(s_0))^2 + (h(u_0^2(s_0+)) - h(u_\kappa^2(s_0)))u_\kappa^2(s_0) + p(u_\kappa^2(s_0)) - p(u_0^2(s_0+)) \\ &\quad = 2\kappa(u_\kappa'(x_1))^2 - 2\kappa u_\kappa''(x_1)u_\kappa(x_1) - \int_{s_0}^{x_1} V_\kappa' u_\kappa^2 \,\mathrm{d}x \\ &\qquad + \big(p(u_\kappa^2(x_1)) - p(u_0^2(x_1))\big) + \big(p(u_0^2(x_1)) - p(u_0^2(s_0+))\big) + \mathcal{O}(\kappa^{1/4}). \end{aligned} \tag{31}$$

We now consider x_1 as a variable in the interval $J_\kappa := [s_0 + \kappa^{1/4}L, s_0 + 2\kappa^{1/4}L]$, and we evaluate the $L^2(J_\kappa)$ norms of both sides of this identity. To this end, we define

$$C_\kappa^+ := -\frac{1}{2}\Big((h(u_0^2(s_0+)) - h(u_\kappa^2(s_0)))u_\kappa^2(s_0) + p(u_\kappa^2(s_0)) - p(u_0^2(s_0+))\Big). \quad (32)$$

Then we have

$$\begin{aligned}
&\left|\kappa(u_\kappa'(s_0))^2 - C_\kappa^+\right| \cdot \kappa^{1/8} \\
&\le \kappa \left\|(u_\kappa')^2\right\|_{L^2(J_\kappa)} + \kappa \left\|u_\kappa''\right\|_{L^2(J_\kappa)} \cdot \left\|u_\kappa\right\|_{L^\infty(J_\kappa)} \\
&\quad + \frac{1}{2}\left\|\int_{s_0}^{(\cdot)} V_\kappa'(s)u_\kappa^2(s)\,\mathrm{d}s\right\|_{L^2(J_\kappa)} + \frac{1}{2}\left\|p(u_\kappa^2(\cdot)) - p(u_0^2(\cdot))\right\|_{L^2(J_\kappa)} \\
&\quad + \frac{1}{2}\left\|p(u_0^2(\cdot)) - p(u_0^2(s_0+))\right\|_{L^2(J_\kappa)} + \mathcal{O}(\kappa^{3/8}).
\end{aligned}$$

Now we estimate the terms of the right hand side one after the other. From the inequalities (14) and (16) it follows that

$$\kappa \left\|(u_\kappa')^2\right\|_{L^2(J_\kappa)} \le \kappa \left\|u_\kappa'\right\|_{L^2(J_\kappa)} \left\|u_\kappa'\right\|_{L^\infty(J_\kappa)} \le C\kappa^{3/4}.$$

Next, inequality (15) and the uniform estimates for u_κ show

$$\kappa \left\|u_\kappa''\right\|_{L^2(J_\kappa)} \left\|u_\kappa\right\|_{L^\infty(J_\kappa)} \le C\kappa^{1/2}.$$

Since $\|V_\kappa'\|_{L^\infty(0,1)}$ and $\|u_\kappa\|_{L^\infty(0,1)}$ are uniformly bounded, it holds

$$\left\|\int_{s_0}^{(\cdot)} V_\kappa'(s)u_\kappa^2(s)\,\mathrm{d}s\right\|_{L^2(J_\kappa)} \le \left(\int_{\kappa^{1/4}L}^{2\kappa^{1/4}L} (Ct)^2\,\mathrm{d}t\right)^{1/2} \le C\kappa^{3/8}.$$

By Lipschitz continuity of p on compact subsets of $(0,\infty)$, (9) implies

$$\left\|p(u_\kappa^2(\cdot)) - p(u_0^2(\cdot))\right\|_{L^2(J_\kappa)} \le C\kappa^{3/8}.$$

Because $u_0^2 = h^{-1}(F + V_B(s_0+) + V_0 + \beta_0)$ on $[s_0, s_0 + L]$, we also see

$$\left\|p(u_0^2(\cdot)) - p(u_0^2(s_0+))\right\|_{L^2(J_\kappa)} \le C\left\|V_0(\cdot) - V_0(s_0)\right\|_{L^2(J_\kappa)} \le C\kappa^{3/8}.$$

The result then is

$$|\kappa u_\kappa'(s_0)^2 - C_\kappa^+| \le C\kappa^{1/4}. \quad (33)$$

This does not yet prove the desired lower bound on $|u_\kappa'(s_0)|$, because C_κ^+ might be very close to zero. However, we can repeat the above reasoning with another interval $[s_{-1}, s_0]$ on which V_B is constant, resulting in $|\kappa u_\kappa'(s_0)^2 - C_\kappa^-| \le C\kappa^{1/4}$ for a constant

$$C_\kappa^- := -\frac{1}{2}\Big((h(u_0^2(s_0-)) - h(u_\kappa^2(s_0)))u_\kappa^2(s_0) + p(u_\kappa^2(s_0)) - p(u_0^2(s_0-))\Big). \quad (34)$$

It follows

$$C_\kappa^+ + C_\kappa^- = g_+(u_\kappa^2(s_0)) + g_-(u_\kappa^2(s_0)),$$

where we have introduced functions $g_\pm\colon [0,\infty) \to \mathbb{R}$ defined by

$$g_\pm(z) := \frac{1}{2}\Big((h(z) - h(u_0^2(s_0\pm)))z + p(u_0^2(s_0\pm)) - p(z)\Big). \tag{35}$$

There is a Taylor expansion hidden in $g_\pm$:

$$\begin{aligned} g_\pm(z) &= \frac{1}{2}\Big(H(z) - H(u_0^2(s_0\pm)) - H'(u_0^2(s_0\pm)) \cdot (z - u_0^2(s_0\pm))\Big) \\ &= \frac{1}{4}H''(\xi) \cdot \left(z - u_0^2(s_0\pm)\right)^2, \end{aligned}$$

with some ξ between z and $u_0^2(s_0\pm)$. Then (22) implies

$$\frac{1}{4K_2}\left(z - u_0^2(s_0\pm)\right)^2 \le g_\pm(z) \le \frac{K_2}{4}\left(z - u_0^2(s_0\pm)\right)^2, \tag{36}$$

which brings us to

$$C_\kappa^+ + C_\kappa^- \ge \frac{1}{8K_2}\left(u_0^2(s_0+) - u_0^2(s_0-)\right)^2.$$

We clearly have

$$|2\kappa u_\kappa'(s_0)^2 - (C_\kappa^+ + C_\kappa^-)| \le C\kappa^{1/4},$$

which finally yields $\kappa u_\kappa'(s_0)^2 \ge C_5^2 > 0$ for all $0 < \kappa < \kappa_0$ and a certain C_5.

To prove (26), we remark that $|C_\kappa^+ - C_\kappa^-| \le C\kappa^{1/4}$, hence

$$\begin{aligned} \mathcal{O}(\kappa^{1/4}) &\ge \frac{|C_\kappa^+ - C_\kappa^-|}{|h(u_0^2(s_0+)) - h(u_0^2(s_0-))|} \\ &= \frac{1}{2}\left|-u_\kappa^2(s_0) + \frac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{h(u_0^2(s_0+)) - h(u_0^2(s_0-))}\right|, \end{aligned}$$

where we have exploited (32) and (34). □

Lemma 5.3 (**Derivation of the zeroth order asymptotic expansion**). *In the situation of Theorem 1.1, assume that V_B is piecewise constant and let s_0 be a jump point of V_B. Assume that $[s_0, s_0 + L]$ is a maximal interval where V_B is constant. Then there exist $C_6 > 0$, $w\colon \mathbb{R} \to \mathbb{R}$ such that*

$$\left\| u_\kappa(\cdot) - w\left(\frac{\cdot - s_0}{\kappa^{1/2}}\right)\right\|_{L^\infty([s_0, s_0 + C_6\kappa^{1/2}])} \le C\kappa^{1/4}. \tag{37}$$

Moreover, w converges exponentially fast to $u_0(s_0+)$ for $y \to \infty$,

$$|w(y) - u_0(s_0+)| + |w'(y)| + |w''(y))| \le C\exp(-C_7 y), \quad (y \ge 0), \tag{38}$$

so that

$$\left\| w\left(\frac{\cdot}{\kappa^{1/2}}\right) - u_0(s_0+)\right\|_{L^1(0,\infty)} \le \frac{C}{C_7}\kappa^{1/2}. \tag{39}$$

Proof. We rewrite (31): Rename x_1 to $x \in [s_0, s_0 + 2\kappa^{1/4}L]$, recall that the left hand side as well as the integral are items of size $\mathcal{O}(\kappa^{1/4})$, and apply (30) for $u_\kappa''(x_1)$. Then we get

$$2\kappa(u_\kappa'(x))^2 = (h(u_\kappa^2(x)) - h(u_0^2(x)))u_\kappa^2(x) + p(u_0^2(s_0+)) - p(u_\kappa^2(x)) + \mathcal{O}(\kappa^{1/4}).$$

Observe that the Lipschitz continuities of h and u_0 imply

$$h(u_0^2(s_0+)) - h(u_0^2(x)) = \mathcal{O}(\kappa^{1/4}),$$

hence we find

$$\kappa(u_\kappa'(x))^2 = g_+(u_\kappa^2(x)) + \mathcal{O}(\kappa^{1/4}), \qquad s_0 < x \leq s_0 + 2\kappa^{1/4}L. \tag{40}$$

We wish to extract the root here, and therefore we think about the sign of $u_\kappa'(s_0)$. From (40) and (36) we learn that $u_\kappa'(x)$ can change its sign (for the mentioned x) only if $u_\kappa^2(x)$ is near $u_0^2(s_0+)$. On the other hand, (9) tells us that $u_\kappa^2(x) - u_0^2(s_0+) = \mathcal{O}(\kappa^{1/4})$ is a small number for $x = s_0 + 2\kappa^{1/4}L$. Hence it is possible to conclude: if $u_0^2(s_0-) < u_0^2(s_0+)$, then $u_\kappa'(s_0) > 0$, and *vice versa*. Without loss of generality, we assume this case. From Remark 5.2, (25) and (36) we then also find that $u_\kappa^2(s_0) \leq u_0^2(s_0+) - c$ holds for some positive c. Next we get

$$\kappa^{1/2}u_\kappa'(x) = \sqrt{g_+(u_\kappa^2(x))} + \mathcal{O}(\kappa^{1/4}), \qquad s_0 < x \leq s_0 + C_6\kappa^{1/2},$$

and C_6 will be chosen later in such a way that $g_+(u_\kappa^2(x)) \geq c > 0$ is ensured for the mentioned x and some small constant c, making the manipulation on the right hand side valid.

Introducing the variable transformation $y = \frac{1}{\kappa^{1/2}}(x - s_0)$, we consider the initial value problem for a function w,

$$\begin{cases} w'(y) = +\sqrt{g_+(w^2(y))}, & 0 < y < \infty, \\ w(0) = \dfrac{p(u_0^2(s_0+)) - p(u_0^2(s_0-))}{V_B(s_0+) - V_B(s_0-)}, \end{cases} \tag{41}$$

compare (26). The life span of w is *a priori* not known, but the constant function $\widehat{w}(y) :\equiv u_0(s_0+)$ solves the same differential equation and has an initial value $\widehat{w}(0) > w(0)$, hence the uniqueness principle gives $w(y) < \widehat{w}(y)$, making a blowup of w impossible. The classical theory of upper and lower solutions (c.f. [21, II §9 IV]) can be applied: Let $\underline{w}$ and $\overline{w}$ be functions solving

$$\underline{w}'(y) = \frac{1}{2\sqrt{K_2}}\left|\underline{w}^2(y) - u_0^2(s_0+)\right|, \qquad \underline{w}(0) = w(0),$$
$$\overline{w}'(y) = \frac{\sqrt{K_2}}{2}\left|\overline{w}^2(y) - u_0^2(s_0+)\right|, \qquad \overline{w}(0) = w(0),$$

compare (36). Then $\underline{w}(y) \leq w(y) \leq \overline{w}(y) < \widehat{w}(y)$, for $0 \leq y < \infty$, and in particular, (38) follows (using (41) for estimating w' and w'').

Now we pick a small number $c > 0$, determine C_6 by $g_+(\overline{w}^2(y)) > 2c$ on $[0, C_6]$, and classical perturbation arguments then show (37). $\square$

Remark 5.4. *Let us summarize what we have obtained so far:* (9) *provides us the starting term of an asymptotic expansion of* $u_\kappa(x)$*, valid for points* x *whose distance to the nearest jump point of* V_B *is at least* $\mathcal{O}(\kappa^{1/4})$*. On the other hand,* (37) *takes care of those* x *whose distance to the nearest jump point is at most* $\mathcal{O}(\kappa^{1/2})$*. Hence a gap remains between both regions, and this gap is handled in the next lemma. The bound* $\kappa^{1/8}$ *in* (42) *will be improved in Section 6 to* $\kappa^{1/4}$*.*

Lemma 5.5 (**Preliminary estimates for the zeroth order asymptotic expansion**)**.** *Let* w *be the function constructed in Lemma 5.3 for a jump point* s_0 *of* V_B*. Let* $[s_0, s_0 + L]$ *be a maximal interval where* V_B *is constant. Then*

$$\left\| u_\kappa(\cdot) - \frac{u_0(\cdot)}{u_0(s_0+)} w\left(\frac{\cdot - s_0}{\kappa^{1/2}}\right) \right\|_{L^\infty(s_0, s_0+L/2)} \leq C\kappa^{1/8}, \qquad 0 < \kappa < \kappa_0. \tag{42}$$

A similar estimate is valid in a left neighbourhood of the jump point s_0*.*

Then there are open disjoint neighbourhoods Ω_i *of the jump points* s_i *of* V_B*, there is a function* $c_0\colon [0,1] \to (0,\infty)$ *and a family of functions* W_κ *such that:*

$$W_\kappa \in C^1([0,1];\mathbb{R}) \quad \text{and} \quad W_\kappa \text{ has piecewise } C^2 \text{ regularity},$$
$$c_0 \quad \text{has piecewise } C^2 \text{ regularity},$$
$$c_0(x) = u_0(s_i\pm) \text{ in } \Omega_i,$$
$$W_\kappa(x) = w\left(\frac{x - s_i}{\kappa^{1/2}}\right) \text{ in } \Omega_i,$$
$$2\kappa W_\kappa''(x) = W_\kappa(x)\left(h(W_\kappa^2(x)) - h(u_0^2(s_i\pm))\right) \text{ in } \Omega_i, \tag{43}$$
$$|W_\kappa''(x)| \leq C \text{ outside } \cup_i \Omega_i,$$
$$\left\| \frac{W_\kappa}{c_0} - 1 \right\|_{L^1(0,1)} \leq C\kappa^{1/2}, \tag{44}$$
$$\left\| u_\kappa - u_0 \frac{W_\kappa}{c_0} \right\|_{L^\infty(0,1)} \leq C\kappa^{1/8}.$$

Proof. Without loss of generality, we assume $u_0^2(s_0-) < u_0^2(s_0+)$, which corresponds to $u_\kappa'(s_0) > 0$, $u_\kappa^2(s_0) < u_0^2(s_0+)$ and $w'(y) > 0$ everywhere. By construction of w, it is easily seen that estimate (42) even holds with the better rate $\kappa^{1/4}$ in the regimes $[s_0, s_0 + C_6\kappa^{1/2}]$ and $[s_0 + \kappa^{1/4}L, s_0 + L/2]$. Now we treat the remaining part $(s_0 + C_6\kappa^{1/2}, s_0 + \kappa^{1/4}L)$, and our first step is to think about how large can $u_\kappa^2(x)$ be on the interval $J_\kappa := [s_0, s_0 + \kappa^{1/4}L]$. The maximum can not be attained on the left endpoint s_0, by assumption. If the maximum is attained at the right endpoint, then (9) and the Lipschitz continuity of u_0 imply $\max_{J_\kappa} u_\kappa^2(x) \leq u_0^2(s_0+) + C\kappa^{1/4}$. And if the maximum is attained inside of J_κ, then (40) and (36) yield

$$\max_{J_\kappa} u_\kappa^2(x) \leq u_0^2(s_0+) + C\kappa^{1/8}. \tag{45}$$

After this preparation, we now consider the local extrema of the function

$$d(x) := u_\kappa(x) - w\left(\frac{x - s_0}{\kappa^{1/2}}\right), \quad x \in [s_0 + C_6\kappa^{1/2}, s_0 + \kappa^{1/4}L].$$

We know that $|d| = \mathcal{O}(\kappa^{1/4})$ at the endpoints of this interval, and if x_* is an interior local extremum of d, then $u_* := u_\kappa(x_*)$ and $w_* := w((x_* - s_0)/\kappa^{1/2})$ satisfy

$$g_+(u_*^2) + \mathcal{O}(\kappa^{1/4}) = g_+(w_*^2), \tag{46}$$

by (40) and (41). Put $c_0 := u_0(s_0+)$. Now we distinguish the cases $u_* \le c_0$ and $u_* > c_0$, and we remark that the inequality $w_* < c_0$ holds in both of them.

Assume $u_* \le c_0$. Without loss of generality, we also suppose $w_* \le u_*$. Noticing that g_+ is monotonically decreasing on $[w_*^2, c_0^2]$ and convex there, a Taylor expansion, $g_+''(z) = h'(z)/2$, and (22) then tell us

$$\begin{aligned} C\kappa^{1/4} \ge g_+(w_*^2) - g_+(u_*^2) &= g_+'(u_*^2) \cdot (w_*^2 - u_*^2) + \frac{1}{2} g_+''(\xi) \cdot (w_*^2 - u_*^2)^2 \\ &\ge 0 + \frac{1}{4K_2}(w_*^2 - u_*^2)^2, \end{aligned}$$

which settles the first case $u_* \le c_0$.

We come to the slightly harder case $u_* > c_0$, in which we clearly have

$$\|d\|_{L^\infty(s_0 + C_6\kappa^{1/2}, s_0 + \kappa^{1/4}L)} = |u_* - c_0| + |c_0 - w_*|,$$

and the first item on the right is bounded by $C\kappa^{1/8}$, from (45). From this first term estimate, (46), and twice (36) we then deduce that

$$\begin{aligned} C\kappa^{1/4} \ge \frac{K_2}{4}\left(u_*^2 - c_0^2\right)^2 &\ge g_+(u_*^2) \ge g_+(w_*^2) - \mathcal{O}(\kappa^{1/4}) \\ &\ge \frac{1}{4K_2}\left(w_*^2 - c_0^2\right)^2 - \mathcal{O}(\kappa^{1/4}), \end{aligned}$$

which brings us to $|c_0 - w_*| \le C\kappa^{1/8}$. This concludes the second case, and (42) is readily seen.

Now we match the various asymptotic expansions of u_κ. Let $\{s_2, s_3, \dots, s_N\} \subset (0, 1)$ be the jump points of V_B, increasingly ordered, and define $s_1 := 0$, $s_{N+1} := 1$. Let $\Omega_1, \dots, \Omega_{N+1}$ be disjoint open neighbourhoods of $s_1, \dots, s_{N+1}$. For $i = 2, \dots, N$, let $w_{+,i}$ and $w_{-,i}$ be the layer profiles near s_i, defined similarly to (41). Observe that their derivatives match, $w_{-,i}'(0) = w_{+,i}'(0)$, by the very choice of $w_{\pm,i}(0)$. Then we construct a function $\widetilde{w}_\kappa^i \in C^1(\mathbb{R})$ that makes the transition between $u_0(s_i-)$ and $u_0(s_i+)$ in Ω_i (having only an exponentially small error at the endpoints of Ω_i):

$$\widetilde{w}_\kappa^i(x) := \begin{cases} w_{-,i}\left(\frac{x - s_i}{\kappa^{1/2}}\right) & : x \le s_i, \\ w_{+,i}\left(\frac{x - s_i}{\kappa^{1/2}}\right) & : x \ge s_i. \end{cases}$$

Outside the jump point set, $\widetilde{w}_\kappa^i$ has C^2 regularity. We also define $\widetilde{w}_\kappa^1 \equiv u_0(0)$ and $\widetilde{w}_\kappa^{N+1} \equiv u_0(1)$ as constant functions. Next we choose a partition of unity: Define $s_0 := -1$, $s_{N+2} := 2$, and select functions $(\varphi_i)_{i=1,\dots,N+1} \subset C^\infty(\mathbb{R},[0,1])$ such that $\operatorname{supp}\varphi_i \Subset (s_{i-1}, s_{i+1})$, $\varphi_i \equiv 1$ in a Ω_i, where $i = 1,\dots,N+1$; and $\sum_{j=1}^{N+1} \varphi_j(x) = 1$ on $[0,1]$. Then the function W_κ is given as

$$W_\kappa(x) := \sum_{i=1}^{N+1} \widetilde{w}_\kappa^i(x)\varphi_i(x).$$

Moreover, define the piecewise constant functions $\widetilde{c}_0^1 \equiv u_0(0)$, $\widetilde{c}_0^{N+1} \equiv u_0(1)$ and $\widetilde{c}_0^i = \chi_{(-\infty,s_i]}u_0(s_i-) + \chi_{(s_i,\infty)}u_0(s_i+)$ for $i = 2,\dots,N$. Finally, we put

$$\widetilde{W}_\kappa(x) := \sum_{i=1}^{N+1} \frac{\widetilde{w}_\kappa^i(x)}{\widetilde{c}_0^i(x)}\varphi_i(x), \qquad c_0(x) := \frac{W_\kappa(x)}{\widetilde{W}_\kappa(x)}.$$

The inequality (44) follows from (39), and (43) is easily checked. We remark

$$\left|\widetilde{W}_\kappa(x) - 1\right| \le \sum_{i=1}^{N+1} \frac{|\widetilde{w}_\kappa^i(x) - \widetilde{c}_0^i(x)|}{\widetilde{c}_0^i(x)}\varphi_i(x) \le C\exp\left(-C\kappa^{-1/2}\right), \tag{47}$$

valid for $x \notin \cup_i\Omega_i$. □

6. Refined remainder estimates

We complete the proof of Theorem 2.2 by considering differential equations to the remainder terms

$$R_{u_\kappa} := u_\kappa - \frac{u_0}{c_0}W_\kappa, \quad R_{V_\kappa} := V_\kappa - V_0, \quad R_{\beta_\kappa} := \beta_\kappa - \beta_0, \quad R_{V_\kappa,\beta_\kappa} := R_{V_\kappa} + R_{\beta_\kappa}.$$

Then R_{V_κ} and R_{u_κ} solve the differential equations

$$\lambda^2 R''_{V_\kappa} = 2u_\kappa R^2_{u_\kappa} - R^2_{u_\kappa} + u_0^2\left(\frac{W_\kappa^2}{c_0^2} - 1\right), \tag{48}$$

$$\frac{2\kappa}{W_\kappa}\left(W_\kappa^2\left(\frac{R_{u_\kappa}}{W_\kappa}\right)'\right)' \tag{49}$$

$$= -R_{V_\kappa,\beta_\kappa}u_\kappa - \left(h(u_0^2) - k(u_\kappa^2)\right)u_\kappa - 2\kappa W''_\kappa\frac{u_\kappa}{W_\kappa} - \frac{2\kappa}{W_\kappa}\left(W_\kappa^2\left(\frac{u_0}{c_0}\right)'\right)',$$

where we have recalled (5) and (29).

Proof of Theorem 2.2. We will show

$$\kappa\left\|W_\kappa\left(\frac{R_{u_\kappa}}{W_\kappa}\right)'\right\|^2_{L^2(0,1)} + \|R'_{V_\kappa}\|^2_{L^2(0,1)} + \|R_{u_\kappa}\|^2_{L^2(0,1)} \le C\kappa, \tag{50}$$

from which we obtain estimate (10) by Poincaré's inequality. Estimate (11) follows by interpolation of the inequalities $\left\|\left(\frac{R_{u_\kappa}}{W_\kappa}\right)'\right\|_{L^2(0,1)} \leq C$ and $\left\|\frac{R_{u_\kappa}}{W_\kappa}\right\|_{L^2(0,1)} \leq C\kappa^{1/2}$, as W_κ is uniformly bounded from above and away from zero.

Multiplying equation (49) by R_{u_κ} and integrating by parts we obtain

$$
\begin{aligned}
&2\kappa \int_0^1 W_\kappa^2 \left(\frac{R_{u_\kappa}}{W_\kappa}\right)'^2 \mathrm{d}x \qquad (51)\\
&\quad = \int_0^1 R_{V_\kappa,\beta_\kappa} R_{u_\kappa} u_\kappa \,\mathrm{d}x + \int_0^1 \left(h(u_0^2) - k(u_\kappa^2)\right) u_\kappa R_{u_\kappa} \,\mathrm{d}x \\
&\qquad + 2\kappa \int_0^1 W_\kappa'' \frac{u_\kappa}{W_\kappa} R_{u_\kappa} \,\mathrm{d}x + 2\kappa \int_0^1 W_\kappa \left(\frac{R_{u_\kappa}}{W_\kappa}\right)' \cdot W_\kappa \left(\frac{u_0}{c_0}\right)' \mathrm{d}x.
\end{aligned}
$$

By Young's inequality it follows that

$$
2\kappa \int_0^1 W_\kappa \left(\frac{R_{u_\kappa}}{W_\kappa}\right)' \cdot W_\kappa \left(\frac{u_0}{c_0}\right)' \mathrm{d}x \leq \kappa \left\| W_\kappa \left(\frac{R_{u_\kappa}}{W_\kappa}\right)' \right\|_{L^2(0,1)}^2 + C\kappa. \qquad (52)
$$

Concerning R_{β_κ}, observe that (5) and (29) give us the representation

$$
R_{\beta_\kappa} = -R_{V_\kappa} - \left(h(u_0^2) - k(u_\kappa^2)\right) - 2\kappa \frac{u_\kappa''}{u_\kappa}.
$$

Let $I = [s_1, s_2]$ be an interval of length L, on which V_B is constant, and $I_\kappa := [s_1 + \kappa^{1/4} L, s_2 - \kappa^{1/4} L]$. Then, due to (15),

$$
\begin{aligned}
|R_{\beta_\kappa}| &\leq \frac{2}{L} \|R_{\beta_\kappa}\|_{L^1(I_\kappa)} \\
&\leq C \left(\|R_{V_\kappa}\|_{L^1(I_\kappa)} + \left\|h(u_\kappa^2) - h(u_0^2)\right\|_{L^1(I_\kappa)} + \kappa \left\|\ln(u_\kappa^2)\right\|_{L^1(I_\kappa)} + \kappa \left\|\frac{u_\kappa''}{u_\kappa}\right\|_{L^1(I_\kappa)} \right) \\
&\leq C \|R_{V_\kappa}\|_{L^2(0,1)} + C \|R_{u_\kappa}\|_{L^2(0,1)} + C\kappa^{1/2}.
\end{aligned}
$$

Employing (48) one finds

$$
\begin{aligned}
&\frac{\lambda^2}{2} \int_0^1 R_{V_\kappa,\beta_\kappa}'^2 \,\mathrm{d}x \qquad (53)\\
&= -\int_0^1 R_{V_\kappa,\beta_\kappa} R_{u_\kappa} u_\kappa \,\mathrm{d}x + \frac{1}{2} \int_0^1 R_{u_\kappa}^2 R_{V_\kappa,\beta_\kappa} \,\mathrm{d}x \\
&\qquad - \frac{1}{2} \int_0^1 u_0^2 \left(\frac{W_\kappa^2}{c_0^2} - 1\right) R_{V_\kappa,\beta_\kappa} \,\mathrm{d}x \\
&\leq -\int_0^1 R_{V_\kappa,\beta_\kappa} R_{u_\kappa} u_\kappa \,\mathrm{d}x + \frac{1}{2} \|R_{V_\kappa,\beta_\kappa}\|_{L^\infty(0,1)} \cdot \|R_{u_\kappa}\|_{L^2(0,1)}^2 \\
&\qquad + C \|R_{V_\kappa,\beta_\kappa}\|_{L^\infty(0,1)} \left\|\frac{W_\kappa}{c_0} - 1\right\|_{L^1(0,1)}
\end{aligned}
$$

$$\leq -\int_0^1 R_{V_\kappa,\beta_\kappa} R_{u_\kappa} u_\kappa \,\mathrm{d}x + C\kappa^{1/4} \|R_{u_\kappa}\|^2_{L^2(0,1)} + C\kappa^{1/2} \|R_{V_\kappa,\beta_\kappa}\|_{L^\infty(0,1)},$$

using (6), (7), (44). The Sobolev embedding $H^1(0,1) \hookrightarrow L^\infty(0,1)$ and Poincaré's inequality give us

$$\begin{aligned} C\kappa^{1/2} \|R_{V_\kappa,\beta_\kappa}\|_{L^\infty(0,1)} &\leq C\kappa^{1/2} \|R'_{V_\kappa}\|_{L^2(0,1)} + C\kappa^{1/2} \|R_{u_\kappa}\|_{L^2(0,1)} + C\kappa \\ &\leq \frac{\lambda^2}{4} \|R'_{V_\kappa}\|^2_{L^2(0,1)} + C\kappa^{1/2} \|R_{u_\kappa}\|_{L^2(0,1)} + C\kappa \end{aligned} \tag{54}$$

Now we add the inequalities (51), (52), and (53), and we bring (54) into play:

$$\begin{aligned} &\kappa \left\| W_\kappa \left(\frac{R_{u_\kappa}}{W_\kappa} \right)' \right\|^2_{L^2(0,1)} + \frac{\lambda^2}{4} \|R'_{V_\kappa}\|^2_{L^2(0,1)} \\ &\leq \int_0^1 \left(\left(h(u_0^2) - h(u_\kappa^2) \right) + \frac{2\kappa W_\kappa''}{W_\kappa} \right) u_\kappa R_{u_\kappa} \,\mathrm{d}x \\ &\quad + C\kappa + C\kappa^{1/4} \|R_{u_\kappa}\|^2_{L^2(0,1)} + C\kappa^{1/2} \|R_{u_\kappa}\|_{L^2(0,1)}. \end{aligned}$$

To discuss the integral on the right hand side, we distinguish the cases $x \in \Omega_i$ and $x \notin \cup_i \Omega_i$.

Suppose x to be in the right part of Ω_i. Then we conclude from (43) that

$$\begin{aligned} &\left(h(u_0^2) - h(u_\kappa^2) \right)(x) + \frac{2\kappa W_\kappa''(x)}{W_\kappa(x)} \\ =& h(u_0^2(x)) - h(u_\kappa^2(x)) + h(W_\kappa^2(x)) - h(u_0^2(s_i+)) \\ =& \left(h\left(W_\kappa^2(x) \frac{u_0^2(x)}{c_0^2(x)} \right) - h(u_\kappa^2(x)) \right) \\ &\quad + \left(h(W_\kappa^2(x)) - h(u_0^2(s_i+)) - h\left(\frac{W_\kappa^2(x)}{u_0^2(s_i+)} u_0^2(x) \right) + h(u_0^2(x)) \right) \\ =:& T_1(x) + T_2(x). \end{aligned}$$

Here we have used $c_0(x) = u_0(s_i+)$ for these x. By monotonicity of h, we have

$$\begin{aligned} T_1(x) u_\kappa(x) R_{u_\kappa}(x) &= \frac{u_\kappa(x)}{u_\kappa(x) + \frac{u_0(x)}{c_0(x)} W_\kappa(x)} T_1(x) \left(u_\kappa^2(x) - \frac{u_0^2(x)}{c_0^2(x)} W_\kappa^2(x) \right) \\ &\leq -\frac{1}{K_2} \frac{u_\kappa(x)}{u_\kappa(x) + \frac{u_0(x)}{c_0(x)} W_\kappa(x)} \left(u_\kappa^2(x) - \frac{u_0^2(x)}{c_0^2(x)} W_\kappa^2(x) \right)^2 \\ &\leq -C_8 R_{u_\kappa}^2(x), \end{aligned}$$

for a certain positive C_8, and this estimate is even valid for all $x \in [0, 1]$.

The term $T_2(x)$ can be handled using Lemma Appendix A.1:

$$\begin{aligned}|T_2(x)| &\le C\left|\ln\frac{u_0^2(x)}{u_0^2(s_i+)}\right|\cdot\left|\ln\frac{W_\kappa^2(x)}{u_0^2(s_i+)}\right| \\ &\le C|u_0(x)-u_0(s_i+)|\cdot|W_\kappa(x)-u_0(s_i+)| \\ &\le C\kappa^{1/2}\frac{x-s_i}{\kappa^{1/2}}\cdot\exp\left(-C_7\frac{x-s_i}{\kappa^{1/2}}\right),\end{aligned}$$

which implies $\sup_{\Omega_i}|T_2(x)|\le C\kappa^{1/2}$.

Next we handle the case $x\notin\cup_i\Omega_i$. Then we calculate as follows:

$$\begin{aligned}\left|\frac{2\kappa W_\kappa''(x)}{W_\kappa(x)}\right| &\le C\kappa, \\ h(u_0^2(x))-h(u_\kappa^2(x)) &= T_1(x)+h(u_0^2(x))-h\left(W_\kappa^2(x)\frac{u_0^2(x)}{c_0^2(x)}\right) \\ &= T_1(x)+h(u_0^2(x))-h\left(\widetilde{W}_\kappa^2(x)u_0^2(x)\right) \\ &= T_1(x)+\mathcal{O}(\kappa),\end{aligned}$$

by (47). Summing up what we have obtained so far, we get

$$\begin{aligned}&\kappa\left\|W_\kappa\left(\frac{R_{u_\kappa}}{W_\kappa}\right)'\right\|_{L^2(0,1)}^2+\frac{\lambda^2}{4}\|R_{V_\kappa}'\|_{L^2(0,1)}^2 \\ &\qquad\le -C_8\|R_{u_\kappa}\|_{L^2(0,1)}^2+C\kappa+C\kappa^{1/4}\|R_{u_\kappa}\|_{L^2(0,1)}^2+C\kappa^{1/2}\|R_{u_\kappa}\|_{L^2(0,1)},\end{aligned}$$

and now (50) is deduced by Young's inequality. □

Appendix A. Appendix

We conclude this paper with a useful tiny technical lemma.

Lemma Appendix A.1. *Let $I\subset(0,\infty)$ be a compact interval, and $h\colon I\to\mathbb{R}$ be twice continuously differentiable, and define $C_h:=\max_I|h''(t)t^2+h'(t)t|$. If p, q, λ are positive numbers with p, q, λp, $\lambda q\in I$, then*

$$|h(\lambda p)-h(p)-h(\lambda q)+h(q)|\le C_h\left|\ln\frac{q}{p}\right|\cdot|\ln\lambda|.$$

Proof. We define a function g by

$$h(t)=:g(\ln t),\qquad t\in I,$$

and we observe that $g''(\tau)=h''(t)t^2+h'(t)t$ for $t=e^\tau\in I$. Now it suffices to calculate

$$\begin{aligned}h(\lambda p) - h(p) - h(\lambda q) + h(q) &= g(\ln\lambda + \ln p) - g(\ln p) - g(\ln\lambda + \ln q) + g(\ln q) \\ &= \int_{\tau=0}^{\ln\lambda} g'(\tau + \ln p) - g'(\tau + \ln q)\,\mathrm{d}\tau \\ &= \int_{\tau=0}^{\ln\lambda} \int_{\sigma=\ln q}^{\ln p} g''(\tau+\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau.\end{aligned}$$

□

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