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The exponential Lie series for continuous semimartingales

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We consider stochastic differential systems driven by continuous semimartingales and governed by non-commuting vector fields. We prove that the logarithm of the flowmap is an exponential Lie series. This relies on a natural change of basis to vector fields for the associated quadratic covariation processes, analogous to Stratonovich corrections. The flowmap can then be expanded as a series in compositional powers of vector fields and the logarithm of the flowmap can thus be expanded in the Lie algebra of vector fields. Further, we give a direct explicit proof of the corresponding Chen–Strichartz formula which provides an explicit formula for the Lie series coefficients. Such exponential Lie series are important in the development of strong Lie group integration schemes that ensure approximate solutions themselves lie in any homogeneous manifold on which the solution evolves.

1. Introduction

We are concerned with Itô stochastic differential systems driven by continuous semimartingales and governed by non-commuting vector fields of the following form

$$Y_t = Y_0 + \sum_{i=1}^d \int_0^t V_i(Y_\tau) dX_\tau^i,$$

for time $t \in [0, T]$ for some $T > 0$. Here the solution process Y_t is \mathbb{R}^N -valued for some $N \in \mathbb{N}$. For each $i = 1, \dots, d$, the X_t^i are driving scalar continuous semimartingales, and associated with each are governing vector fields V_i which are sufficiently smooth and in general non-commuting. Our goal herein is to compute the logarithm of the flowmap for such a system, i.e. the exponential series for the flowmap, and establish that it is a Lie series. The exponential series for the flowmap for stochastic differential systems driven by general continuous semimartingales was derived in Ebrahimi-Fard, Malham, Patras and Wiese [11]. What we achieve that is new in this paper is we:

- (i) Establish the abstract algebraic structures that underlie the flowmap and computation of functions of the flowmap in the context of general continuous semimartingales;
- (ii) Show by a suitable change of coordinates, the exponential series is a Lie series, and give a direct explicit proof of the corresponding Chen–Strichartz formula which provides an explicit formula for the Lie series coefficients.

The key idea that underlies establishing the exponential series as a Lie series is to express the flowmap in Fisk–Stratonovich form for which the standard rules of calculus apply; see Protter [33]. A crucial integral ingredient in this step is that the Fisk–Stratonovich formulation of the flowmap can be expanded in a basis of terms involving solely compositions of vector fields—without any second order partial differential operators. We can then compute the logarithm of the Fisk–Stratonovich representation of the flowmap. This can be accomplished in principle via the classical Chen–Strichartz formula using the shuffle relations satisfied by multiple Fisk–Stratonovich integrals. We subsequently convert the multiple Fisk–Stratonovich integrals back into multiple Itô integrals. This procedure thus generates an Itô exponential Lie series. Our proof utilizes that the logarithm is a Lie element and the Dynkin–Friedrichs–Specht–Wever Theorem to expand the logarithm in Lie polynomials of the vector fields, see for example Theorem 1.4 and Lemma 3.8 in Reutenauer [34]. It is otherwise self-contained. That the logarithm of the flowmap is in fact an exponential Lie series is important for example, for the development of strong stochastic Lie group integration methods. See Malham and Wiese [27] for the development of such methods for Stratonovich stochastic differential systems driven by Wiener processes, for example those based on the Castell–Gaines numerical simulation approach, see Castell and Gaines [4,5].

The development of exponential solution series for deterministic systems originates with the work of Magnus [26] and Chen [6] in the 1950's, and more recently with Strichartz [36]. Its development and early application to stochastic systems is represented by the work of Azencott [1], Ben Arous [2], Castell and Gaines [4,5] and Baudoin [3]. Also see Fliess [13] and Lyons [25] for its development in control and in the theory of rough paths, respectively. The shuffle product was cemented in firm foundations by the work of Eilenberg and Mac Lane [12] and Schützenberger [35], also in the 1950's. The quasi-shuffle product is a natural extension of the shuffle product, for example to multiple Itô integrals. For a selective insight into its recent development in this context, see Gaines [16,17], Hoffman [19], Ebrahimi-Fard and Guo [9], Hoffman and Ihara [20] and Curry, Ebrahimi-Fard, Malham and Wiese [7].

Our paper is structured as follows. In Section 2 we derive the Itô chain rule and flowmap for systems driven by continuous semimartingales. Then in Section 3 we establish the abstract algebraic structures that underpin the flowmap and its logarithm. We endeavour to keep the connection to the stochastic differential system of interest and provide illustrative examples. We define the transformation to Stratonovich form we require in Section 4 and prove that the

exponential series is a Lie series. Our direct explicit derivation of the Chen–Strichartz coefficients is provided in Section 5. Lastly, we provide some concluding remarks in Section 6.

2. Itô chain rule and flowmap

Consider the Itô stochastic differential system given in the introduction of the form

$$Y_t = Y_0 + \sum_{i=1}^d \int_0^t V_i(Y_\tau) dX_\tau^i.$$

Here for each $i = 1, \dots, d$, the X_t^i are driving scalar continuous semimartingales on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the *usual conditions* of completeness and right-continuity. We assume without loss of generality that the X^i are chosen such that the quadratic covariations $[X^i, X^j] = 0$ for all $i \neq j$. The V_i are associated governing vector fields which we assume are sufficiently smooth and in general non-commuting. We suppose the solution process Y_t , which is \mathbb{R}^N -valued for some $N \in \mathbb{N}$, exists on some finite or possibly infinite time interval. In coordinates the vector fields V_i for each $i = 1, \dots, d$ act as first order partial differential operators on any function $f: \mathbb{R}^N \rightarrow \mathbb{R}$ as follows

$$V_i: f(Y) \mapsto \sum_{j=1}^N V_i^j(Y) \partial_{Y_j} f(Y).$$

For brevity we will often express this vector field action as $(V_i \cdot \partial)f(Y)$ or $V_i \circ f \circ Y$.

Definition 2.1 (Flowmap). *We define the flowmap φ_t as the map prescribing the transport of the initial data $f \circ Y_0$ to the solution $f \circ Y_t$ at time t for any smooth function f on \mathbb{R}^N , i.e. $\varphi_t: f \circ Y_0 \mapsto f \circ Y_t$.*

The solution $Y_t = \varphi_t \circ \text{id} \circ Y_0$ corresponds to the choice $f = \text{id}$, the identity map. The Itô chain rule is the key to developing the Taylor series expansion for the solution Y_t about the initial data. The Itô chain rule implies that for any function $f: \mathbb{R}^N \rightarrow \mathbb{R}$, the quantity $f(Y_t)$ satisfies

$$f(Y_t) = f(Y_0) + \sum_{i=1}^d \int_0^t (V_i \cdot \partial) f(Y_\tau) dX_\tau^i + \frac{1}{2} \sum_{i=1}^d \int_0^t (V_i \otimes V_i: \partial^2) f(Y_\tau) d[X^i, X^i]_\tau,$$

see for example Protter [33]. In this formula we have used the notation

$$(V_i \otimes V_i: \partial^2) f(Y) := \sum_{j,k=1}^N V_i^j(Y) V_i^k(Y) \partial_{Y_j} \partial_{Y_k} f(Y),$$

while for each $i = 1, \dots, d$ the terms $[X^i, X^i]$ represent the quadratic variation of X^i . At this stage it makes sense to extend, first our set of driving continuous semimartingales to include these quadratic variations, and second, our governing vector fields to include the associated second order partial differential operators shown above. Thus for $i = 1, \dots, d$ we set $D_i := V_i \cdot \partial$ and

$$X^{[i,i]} := [X^i, X^i] \quad \text{and} \quad D_{[i,i]} := \frac{1}{2} V_i \otimes V_i: \partial^2.$$

Then the Itô chain rule takes the form

$$f \circ Y_t = f \circ Y_0 + \sum_{a \in \mathbb{A}} \int_0^t D_a \circ f \circ Y_\tau dX_\tau^a,$$

where \mathbb{A} denotes the alphabet of letters $\{1, \dots, d, [1, 1], \dots, [d, d]\}$. Iterating this chain rule produces the formal Taylor series expansion for the solution and thus flowmap given by

$$\varphi_t = \sum_w I_w(t) D_w.$$

Here the sum is over all words/multi-indices w that can be constructed from the alphabet \mathbb{A} . All the stochastic information is encoded in the multiple stochastic Itô integrals $I_w = I_w(t)$ while the geometric information is encoded through the composition of partial differential operators D_w . For a word $w = a_1 \cdots a_n$ these terms are $D_w := D_{a_1} \circ \cdots \circ D_{a_n}$ and

$$I_w := \int_{0 \leq \tau_1 \leq \cdots \leq \tau_n \leq t} dX_{\tau_1}^{a_1} \cdots dX_{\tau_n}^{a_n}.$$

It is natural to abstract the solution flowmap and view it as an object of the form

$$\sum w \otimes w,$$

which lies in a tensor product of two algebras. The algebra on the left is associated with multiple integrals and the algebra on the right is associated with partial differential operators. The algebra on the left should be endowed with a quasi-shuffle product, to reflect the fact that the real product between two multiple Itô integrals I_u and I_v generates a sum over all multiple Itô integrals generated by the quasi-shuffle of the words u and v ; we define this product precisely, presently. The algebra on the right should be endowed with a concatenation product, to reflect the fact that the composition of two differential operators D_u and D_v generates the differential operator equivalent to that represented by the concatenation of the words u and v . In the next section we define these underlying concatenation and quasi-shuffle algebras, their corresponding Hopf algebras and the algebras associated with endomorphisms on them. These algebras prove useful in the following sections, they keep our proofs direct and succinct.

3. Quasi-shuffle Hopf algebras and endomorphisms

Our exposition here is based on Reutenauer [34], Hoffman [19] and Hoffman and Ihara [20]. Let \mathbb{A} denote a countable alphabet and $\mathbb{K}\mathbb{A}$ the vector space with \mathbb{A} as basis and \mathbb{K} a field of characteristic zero. Suppose there is a commutative and associative product $[\cdot, \cdot]$ on $\mathbb{K}\mathbb{A}$. We use $\mathbb{K}\langle\mathbb{A}\rangle$ to denote the non-commutative polynomial algebra over \mathbb{K} generated by monomials (or words) we can construct from the alphabet \mathbb{A} . We denote by \mathbb{A}^* the free monoid of words on \mathbb{A} .

Definition 3.1 (Bilinear form). *We define the bilinear form $\langle \cdot, \cdot \rangle: \mathbb{K}\langle\mathbb{A}\rangle \otimes \mathbb{K}\langle\mathbb{A}\rangle \rightarrow \mathbb{K}$ for any words $u, v \in \mathbb{A}^*$ to be*

$$\langle u, v \rangle := \begin{cases} 1, & \text{if } u = v, \\ 0, & \text{if } u \neq v. \end{cases}$$

This is equivalent to the scalar product given in Reutenauer [34, p. 17] and Hoffman [19, p. 57]. For this scalar product, the free monoid \mathbb{A}^* forms an orthonormal basis. We will always assume that \mathbb{A} equipped with $[\cdot, \cdot]$ satisfies the following finiteness condition: for all letters $c \in \mathbb{A}$ the cardinality of the set $\{a, b \in \mathbb{A}: \langle [a, b], c \rangle \neq 0\}$ is finite. It is satisfied, though not restricted to, when \mathbb{A} is finite. The following example illustrates the case of a possibly infinite alphabet.

Example 3.2. Consider a minimal family of general semimartingales $\{X^1, \dots, X^d\}$ in the sense outlined in Curry, Ebrahimi-Fard, Malham and Wiese [7]. We do not restrict ourselves here to continuous semimartingales. However, a collection of independent continuous semimartingales, or a collection of independent Lévy processes, is a minimal family. We can construct a countable alphabet \mathbb{A} as outlined therein as follows. With each semimartingale we associate a letter $1, \dots, d$. In addition, inductively for $n \geq 2$, we assign a distinct new letter for each nested quadratic covariation process $[X^{k_1}, [X^{k_2}, [\dots [X^{k_{n-1}}, X^{k_n}] \dots]]$ with $k_i \in \{1, \dots, d\}$ for $i = 1, \dots, n$, “provided it is not in the linear span of $\{X^1, \dots, X^d\}$ and previously constructed ones”. Due to commutativity the order of the letters in the n -fold nested bracket is irrelevant and associativity means that we can render all n -fold nested brackets to the canonical form of left to right bracketing shown, or more conveniently $[X^{k_1}, \dots, X^{k_n}]$. We denote the new distinct letters by $[k_1, \dots, k_n]$. Hence our underlying countable alphabet \mathbb{A} consists of the letters $1, \dots, d$ and all

possible nested brackets $[\cdot, \cdot]$ generated in this manner. For convenience we set $[X^i] \equiv X^i$ and thus also $[i] \equiv i$ on $\mathbb{K}\langle\mathbb{A}\rangle$.

We use $\mathbb{K}\langle\mathbb{A}\rangle$ to also denote the concatenation algebra of words with concatenation as product. If u and v are words in $\mathbb{K}\langle\mathbb{A}\rangle$, then their concatenation is $uv \in \mathbb{K}\langle\mathbb{A}\rangle$.

Definition 3.3 (Quasi-Shuffle product). *For words u, v and letters a, b the quasi-shuffle product $*$ on $\mathbb{K}\langle\mathbb{A}\rangle$ is generated recursively by the formulae: $u * 1 = 1 * u = u$, where '1' represents the empty word, and*

$$ua * vb = (u * vb)a + (ua * v)b + (u * v)[a, b].$$

Endowed with this product $\mathbb{K}\langle\mathbb{A}\rangle$ is a commutative and associative algebra called the quasi-shuffle algebra which we denote by $\mathbb{K}\langle\mathbb{A}\rangle_*$; see Hoffman [19]. In the special case when the generator $[\cdot, \cdot]$ is identically zero on $\mathbb{K}\mathbb{A}$, it reverts to the shuffle algebra $\mathbb{K}\langle\mathbb{A}\rangle_{\sqcup}$ of words with shuffle \sqcup as product, where $ua \sqcup vb = (u \sqcup vb)a + (ua \sqcup v)b$.

Example 3.4. The quasi-shuffle of the words 12 and 34 is given by $12 * 34 = 1234 + 3412 + 1342 + 3142 + 1324 + 3124 + 1[2, 3]4 + [1, 3]42 + 3[1, 4]2 + [1, 3]24 + 13[2, 4] + 31[2, 4] + [1, 3][2, 4]$.

Example 3.5. A minimal family of semimartingales generates a quasi-shuffle algebra. This is proved in Curry *et al.* [7].

Definition 3.6 (Deconcatenation and de-quasi-shuffle coproducts). *We define the deconcatenation coproduct $\Delta: \mathbb{K}\langle\mathbb{A}\rangle \rightarrow \mathbb{K}\langle\mathbb{A}\rangle \otimes \mathbb{K}\langle\mathbb{A}\rangle$ for any word $w \in \mathbb{K}\langle\mathbb{A}\rangle$ by*

$$\Delta(w) := \sum_{u,v} \langle uv, w \rangle u \otimes v.$$

We also define the de-quasi-shuffle coproduct $\Delta': \mathbb{K}\langle\mathbb{A}\rangle \rightarrow \mathbb{K}\langle\mathbb{A}\rangle \otimes \mathbb{K}\langle\mathbb{A}\rangle$ for any word $w \in \mathbb{K}\langle\mathbb{A}\rangle$ by

$$\Delta'(w) := \sum_{u,v} \langle u * v, w \rangle u \otimes v.$$

The finiteness condition on \mathbb{A} ensures that Δ' is well defined. Endowed with the concatenation product and de-quasi-shuffle coproduct $\mathbb{K}\langle\mathbb{A}\rangle$ is a Hopf algebra which we also denote by $\mathbb{K}\langle\mathbb{A}\rangle$. No confusion should arise from the context. In addition, when endowed with the quasi-shuffle product and deconcatenation coproduct $\mathbb{K}\langle\mathbb{A}\rangle$ is another Hopf algebra which we denote by $\mathbb{K}\langle\mathbb{A}\rangle_*$. The antipodes in both cases are given in Hoffman [19]. We denote by $\text{End}(\mathbb{K}\langle\mathbb{A}\rangle_*)$ the \mathbb{K} -module of linear endomorphisms of $\mathbb{K}\langle\mathbb{A}\rangle_*$.

Definition 3.7 (Convolution products). *Suppose X and Y are two linear endomorphisms on the Hopf quasi-shuffle algebra $\mathbb{K}\langle\mathbb{A}\rangle_*$. We define their quasi-shuffle convolution product $X * Y$ by the formula $X * Y := \text{quas} \circ (X \otimes Y) \circ \Delta$, where 'quas' denotes the quasi-shuffle product on $\mathbb{K}\langle\mathbb{A}\rangle_*$.*

Remark 3.8. We use the same notation for the quasi-shuffle convolution product as for the underlying product, no confusion should arise from the context. There is also a concatenation convolution product $\text{conc} \circ (X \otimes Y) \circ \Delta'$ on $\text{End}(\mathbb{K}\langle\mathbb{A}\rangle)$, where 'conc' denotes the concatenation.

In other words, since deconcatenation Δ splits any word w into the sum of all two-partitions $u \otimes v$ with $u, v \in \mathbb{A}^*$, including when u or v are the empty word 1, we see that

$$(X * Y)(w) = \sum_{uv=w} X(u) * Y(v).$$

Now consider the following algebra which plays an essential role hereafter,

$$\mathbb{K}\langle\mathbb{A}\rangle_* \bar{\otimes} \mathbb{K}\langle\mathbb{A}\rangle.$$

This is the complete tensor product of the Hopf quasi-shuffle algebra on the left and the Hopf concatenation algebra on the right; see Reutenauer [34, p. 18, 29]. It is itself an associative Hopf algebra. The product of any two elements in this tensored Hopf algebra, which extends linearly, is naturally given by

$$(u \otimes v)(u' \otimes v') = (u * u') \otimes (vv').$$

Remark 3.9. In our context, the significance of this tensor algebra is that it is the natural abstract setting for the flowmap.

Any endomorphism $X \in \text{End}(\mathbb{K}\langle \mathbb{A} \rangle_*)$ can be completely described by the image in $\mathbb{K}\langle \mathbb{A} \rangle_* \overline{\otimes} \mathbb{K}\langle \mathbb{A} \rangle$ of the map

$$X \mapsto \sum_{w \in \mathbb{A}^*} X(w) \otimes w,$$

see Reutenauer [34, p. 29]. Note that the identity endomorphism ‘id’ on $\mathbb{K}\langle \mathbb{A} \rangle_*$ maps onto $\sum w \otimes w$. Indeed the embedding $\text{End}(\mathbb{K}\langle \mathbb{A} \rangle_*) \rightarrow \mathbb{K}\langle \mathbb{A} \rangle_* \overline{\otimes} \mathbb{K}\langle \mathbb{A} \rangle$ defined by this map is an algebra homomorphism for the quasi-shuffle convolution product. The unit endomorphism ν on the algebra $\text{End}(\mathbb{K}\langle \mathbb{A} \rangle_*)$ sends non-empty words to zero and the empty word to itself. This embedding provides a mechanism for representing functions of $\sum w \otimes w$ in $\text{End}(\mathbb{K}\langle \mathbb{A} \rangle_*)$. Before we demonstrate this, we need the following.

Definition 3.10 (Augmented ideal projector). *We use \mathfrak{J} to denote the augmented ideal projector. This is the linear endomorphism on $\mathbb{K}\langle \mathbb{A} \rangle_*$ or $\mathbb{K}\langle \mathbb{A} \rangle$ that sends every non-empty word to itself and the empty word to zero. From the definition of the unit endomorphism ν given above we see that $\mathfrak{J} \equiv \text{id} - \nu$.*

We observe we can apply a power series function such as the logarithm function to the element $\sum_w w \otimes w$ in $\mathbb{K}\langle \mathbb{A} \rangle_* \overline{\otimes} \mathbb{K}\langle \mathbb{A} \rangle$ as follows. If ‘1’ represents the empty word, $|w|$ represents the length of the word w and $c_k := (-1)^{k-1} \frac{1}{k}$ for all $k \in \mathbb{N}$, then by direct computation we find

$$\begin{aligned} \log \left(\sum_{w \in \mathbb{A}^*} w \otimes w \right) &= \sum_{k \geq 1} c_k \left(\sum_{w \in \mathbb{A}^*} w \otimes w - 1 \otimes 1 \right)^k \\ &= \sum_{k \geq 1} c_k \left(\sum_{w \in \mathbb{A}^* \setminus \{1\}} w \otimes w \right)^k \\ &= \sum_{k \geq 1} c_k \sum_{u_1, \dots, u_k \in \mathbb{A}^* \setminus \{1\}} (u_1 * \dots * u_k) \otimes (u_1 \dots u_k) \\ &= \sum_{w \in \mathbb{A}^*} \left(\sum_{k=1}^{|w|} c_k \sum_{u_1 \dots u_k = w} u_1 * \dots * u_k \right) \otimes w \\ &= \sum_{w \in \mathbb{A}^*} \left(\sum_{k \geq 1} c_k \mathfrak{J}^{*k} \right) \circ w \otimes w, \end{aligned}$$

where \mathfrak{J}^{*k} denotes the k th quasi-shuffle convolution power of the augmented ideal projector \mathfrak{J} . We emphasize the elements in the partition $u_1 \dots u_k = w$ in the sum in the penultimate line are all non-empty. Note that words cannot be deconcatenated further than all the letters it contains, and thus $\mathfrak{J}^{*k}(w)$ is zero if w has length less than k . We conclude the action of the logarithm function power series on $\sum_w w \otimes w$ can be represented by a corresponding power series endomorphism in $\text{End}(\mathbb{K}\langle \mathbb{A} \rangle_*)$. This will prove useful in Sections 4 and 5 so we summarize the result as follows.

Lemma 3.11 (Logarithm convolution power series). *The logarithm of $\sum_w w \otimes w$ is given by*

$$\log \left(\sum_{w \in \mathbb{A}^*} w \otimes w \right) = \sum_{w \in \mathbb{A}^*} \log^*(\text{id}) \circ w \otimes w,$$

where

$$\log^*(\text{id}) := \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \mathfrak{J}^{*k}.$$

We will often abbreviate $\log^*(\text{id}) \circ w$ to $\log^*(w)$.

We also note that equivalently, the embedding $\text{End}(\mathbb{K}\langle \mathbb{A} \rangle) \rightarrow \mathbb{K}\langle \mathbb{A} \rangle_* \overline{\otimes} \mathbb{K}\langle \mathbb{A} \rangle$ given by

$$Y \mapsto \sum_{w \in \mathbb{A}^*} w \otimes Y(w),$$

is an algebra homomorphism for the concatenation convolution product.

Definition 3.12 (Adjoint endomorphisms). *Two endomorphisms X and Y are adjoints if the images of*

$$X \mapsto \sum_w w \otimes X(w) \quad \text{and} \quad Y \mapsto \sum_w Y(w) \otimes w,$$

are equal. Hereafter we use X^\dagger to denote the adjoint of X .

This coincides with X and X^\dagger being adjoints in the sense $\langle X^\dagger(u), v \rangle = \langle u, X(v) \rangle$ for all $u, v \in \mathbb{A}^*$; see Reutenauer [34, Section 1.5].

There is a natural isomorphism between the Hopf shuffle and quasi-shuffle algebras discovered by Hoffman [19] which will play an important role here; also see Foissy, Patras and Thibon [14] for a theoretical perspective. To describe the isomorphism succinctly we need to introduce the notion of composition action on words. For any natural number n , we use $\mathbb{C}(n)$ to denote the set of compositions of n , i.e. the set of all tuples of natural numbers whose sum is n . A given composition λ in $\mathbb{C}(n)$ will have say $\ell \leq n$ components so $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $\lambda_1 + \dots + \lambda_\ell = n$. For such a λ we define the following simple multi-index functions, $|\lambda| := \ell$ as well as

$$\Sigma(\lambda) := \lambda_1 + \dots + \lambda_\ell, \quad \Pi(\lambda) := \lambda_1 \cdots \lambda_\ell \quad \text{and} \quad \Gamma(\lambda) := \lambda_1! \cdots \lambda_\ell!.$$

We now define the composition action on words and the exponential map from Hoffman [19].

Definition 3.13 (Composition action). *For a given word $w = a_1 \cdots a_n$ and composition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ in $\mathbb{C}(n)$ we define the action $\lambda \circ w$ to be*

$$\lambda \circ w := [a_1 \cdots a_{\lambda_1}] [a_{\lambda_1+1} \cdots a_{\lambda_1+\lambda_2}] \cdots [a_{\lambda_1+\dots+\lambda_{\ell-1}+1} \cdots a_n],$$

where the brackets are concatenated. Here for any word $w = a_1 \cdots a_n$ the notation $[w]$ denotes the n -fold nested bracket described above.

Definition 3.14 (Hoffman exponential). *We define the map $\text{exp}_H: \mathbb{K}\langle \mathbb{A} \rangle_{\sqcup} \rightarrow \mathbb{K}\langle \mathbb{A} \rangle_*$ as that which leaves the empty word unchanged and for any non-empty word w is*

$$\text{exp}_H(w) := \sum_{\lambda \in \mathbb{C}(|w|)} \frac{1}{\Gamma(\lambda)} \lambda \circ w.$$

Its inverse $\text{log}_H: \mathbb{K}\langle \mathbb{A} \rangle_* \rightarrow \mathbb{K}\langle \mathbb{A} \rangle_{\sqcup}$ for any word w is given by

$$\text{log}_H(w) := \sum_{\lambda \in \mathbb{C}(|w|)} \frac{(-1)^{\Sigma(\lambda)-|\lambda|}}{\Pi(\lambda)} \lambda \circ w.$$

Hoffman [19] proved the exponential map is an isomorphism from the Hopf shuffle algebra $\mathbb{K}\langle\mathbb{A}\rangle_{\sqcup}$ to the Hopf quasi-shuffle algebra $\mathbb{K}\langle\mathbb{A}\rangle_*$.

Example 3.15. In the case of the word $w = a_1 a_2 a_3$ the Hoffman exponential is given by $\exp_{\mathbb{H}}(a_1 a_2 a_3) = a_1 a_2 a_3 + \frac{1}{2}[a_1, a_2]a_3 + \frac{1}{2}a_1[a_2, a_3] + \frac{1}{6}[a_1, a_2, a_3]$.

The adjoint of the Hoffman exponential $\exp_{\mathbb{H}}^{\dagger}: \mathbb{K}\langle\mathbb{A}\rangle_{\text{conc}, \Delta'} \rightarrow \mathbb{K}\langle\mathbb{A}\rangle_{\text{conc}, \delta}$ is an isomorphism and defined explicitly as follows. Here we distinguish between two Hopf algebras with concatenation as product, we have $\mathbb{K}\langle\mathbb{A}\rangle_{\text{conc}, \Delta'}$ with de-quasi-shuffle Δ' as coproduct and $\mathbb{K}\langle\mathbb{A}\rangle_{\text{conc}, \delta}$ with deshuffle δ as coproduct. Then for any letter a from the alphabet \mathbb{A} we have

$$\exp_{\mathbb{H}}^{\dagger}(a) := \sum_{n \geq 1} \sum_{[a_1, \dots, a_n] = a} \frac{1}{n!} a_1 \dots a_n.$$

See Hoffman [19] for more details. Note $\exp_{\mathbb{H}}^{\dagger}$ is a homomorphism for the concatenation product. Its inverse $\log_{\mathbb{H}}^{\dagger}: \mathbb{K}\langle\mathbb{A}\rangle_{\text{conc}, \delta} \rightarrow \mathbb{K}\langle\mathbb{A}\rangle_{\text{conc}, \Delta'}$ for any letter a from the alphabet as above is

$$\log_{\mathbb{H}}^{\dagger}(a) := \sum_{n \geq 1} \sum_{[a_1, \dots, a_n] = a} \frac{(-1)^{n-1}}{n} a_1 \dots a_n.$$

We note $\mathbb{K}\langle\mathbb{A}\rangle_{\sqcup, \Delta}$ and $\mathbb{K}\langle\mathbb{A}\rangle_{\text{conc}, \delta}$ are dual Hopf algebras as well as $\mathbb{K}\langle\mathbb{A}\rangle_{*, \Delta}$ and $\mathbb{K}\langle\mathbb{A}\rangle_{\text{conc}, \Delta'}$.

4. Exponential Lie series for continuous semimartingales

We show the logarithm of the flowmap for a system of stochastic differential equations, driven by a set of d continuous semimartingales X_t^i and governed by an associated set of non-commuting vector fields V_i , for $i = 1, \dots, d$, can be expressed as a Lie series. We begin by emphasizing that for a given set of orthogonal continuous semimartingales, the only non-zero quadratic variations are those of the form $[X^i, X^i]$ for $i = 1, \dots, d$. In particular all third order $[X^i, X^i, X^i]$ and thus higher order variations are zero. Hence the generator $[\cdot, \cdot]$, for example in the underlying quasi-shuffle algebra, is nilpotent of degree 3.

Remark 4.1. Examples of continuous (local) martingales are Brownian motion, time-changed Brownian motion and stochastic integrals of Brownian motion. The representation results of Doob, of Dambis and of Dubins and Schwarz, and of Knight (see Theorems 3.4.2, 3.4.6 and 3.4.13 in Karatzas and Shreve [22]) show that these examples are fundamental. Hence important examples for continuous semimartingales are those just mentioned to which a continuous finite variation process, i.e. the difference of two real-valued continuous increasing processes, is added.

We saw in Section 2 that the Itô flowmap has the form $\sum_w I_w D_w$ where the sum is over all words constructed from the alphabet \mathbb{A} which contains the letters $1, \dots, d$ as well as the letters $[1, 1], \dots, [d, d]$. We proposed there to represent the Itô flowmap by the abstract expression $\sum_w w \otimes w$ in $\mathbb{K}\langle\mathbb{A}\rangle_* \otimes \mathbb{K}\langle\mathbb{A}\rangle$, the tensor algebra of the Hopf quasi-shuffle and concatenation algebras. We now make this more precise. Let \mathbb{I} denote the algebra generated by multiple Itô integrals $I_w = I_w(t)$ with respect to the semimartingales $\{X^1, \dots, X^d\}$ or any nested quadratic variation processes generated from them, and the constant random variable 1.

Definition 4.2 (Itô word-to-integral map). We denote by $\mu: \mathbb{K}\langle\mathbb{A}\rangle_* \rightarrow \mathbb{I}$ the word-to-integral map $\mu: w \mapsto I_w$ assigning each word $w \in \mathbb{A}^*$ to the corresponding multiple Itô integral I_w .

The Itô word-to-integral map is a quasi-shuffle algebra homomorphism, i.e. we have $\mu(u * v) = \mu(u)\mu(v)$ for any $u, v \in \mathbb{A}^*$; see Curry, Ebrahimi-Fard, Malham and Wiese [7]. Let \mathbb{D} denote the algebra of scalar linear partial differential operators that can be constructed by composition from the partial differential operators D_i .

Definition 4.3 (Itô word-to-partial differential operator map). We denote by $\bar{\mu}: \mathbb{K}\langle\mathbb{A}\rangle \rightarrow \mathbb{D}$ the letter-to-partial differential operator map $\bar{\mu}: i \mapsto D_i$ assigning each letter $i \in \mathbb{A}$ to the corresponding operator D_i . Recall the operators D_i are given, for $i = 1, \dots, d$, by $D_i := V_i \cdot \partial$ and $D_{[i,i]} := \frac{1}{2}V_i \otimes V_i: \partial^2$.

The map $\bar{\mu}$ is a concatenation algebra homomorphism, i.e. we have $\bar{\mu}(uv) = \bar{\mu}(u)\bar{\mu}(v)$ for any $u, v \in \mathbb{A}^*$. Naturally $\mu \otimes \bar{\mu}: \mathbb{K}\langle\mathbb{A}\rangle_* \otimes \mathbb{K}\langle\mathbb{A}\rangle \rightarrow \mathbb{I} \otimes \mathbb{D}$ is also an algebra homomorphism. With this homomorphism construction in place, we observe

$$\sum_w I_w \otimes D_w = (\mu \otimes \bar{\mu}) \circ \left(\sum_w w \otimes w \right).$$

In principle we can compute the logarithm of $\sum_w w \otimes w$ in $\mathbb{K}\langle\mathbb{A}\rangle_* \otimes \mathbb{K}\langle\mathbb{A}\rangle$ in the form of the quasi-shuffle logarithm as outlined in Section 3. However the basis in which we expand the flowmap and its logarithm corresponds, on the right hand side, to the terms D_w in \mathbb{D} which are compositions of the vector fields $V_i \cdot \partial$ and second order partial differential operators $\frac{1}{2}V_i \otimes V_i: \partial^2$ for $i = 1, \dots, d$. The question is, how can we express the logarithm of the flowmap in Lie polynomials or in particular, in Lie brackets of vector fields? The natural resolution is to use the Fisk–Stratonovich representation of the flowmap. From the stochastic analysis perspective the procedure is as follows.

Definition 4.4 (Fisk–Stratonovich integral). For continuous semimartingales H and Z , the Fisk–Stratonovich integral is defined as

$$\int_0^t H_\tau \circ dZ_\tau := \int_0^t H_\tau dZ_\tau + \frac{1}{2}[H, Z]_t,$$

where the \circ indicates Fisk–Stratonovich integration; see Protter [33, p. 216].

Lemma 4.5 (Itô to Fisk–Stratonovich conversion). For $i = 1, \dots, d$ and any function $f \in C^2(\mathbb{R}^N; \mathbb{R})$, for the integral term in the Itô chain rule we have

$$\int_0^t (V_i \cdot \partial) f(Y_\tau) dX_\tau^i = \int_0^t (V_i \cdot \partial) f(Y_\tau) \circ dX_\tau^i - \frac{1}{2}[(V_i \cdot \partial) f(Y), X^i]_t,$$

and

$$[(V_i \cdot \partial) f(Y), X^i]_t = \int_0^t (V_i \cdot \partial)(V_i \cdot \partial) f(Y_\tau) d[X^i, X^i]_\tau.$$

Proof. To establish the first result we set $H = (V_i \cdot \partial) f(Y)$ and $Z = X^i$ in the definition for Fisk–Stratonovich integrals above. For the second result we use the Itô chain rule in Section 2 to substitute for $(V_i \cdot \partial) f(Y_i)$ into the quadratic covariation bracket on the left. Then using that the bracket is nilpotent of degree 3 for continuous semimartingales, and zero if any argument is constant, establishes the result. \square

Substituting the product rule

$$(V_i \cdot \partial)(V_i \cdot \partial) = (V_i \otimes V_i): \partial^2 + (V_i \cdot \partial V_i) \cdot \partial$$

into the second result and that itself into the first result in Lemma 4.5, generates the Fisk–Stratonovich chain rule.

Corollary 4.6 (Fisk–Stratonovich chain rule). For any function $f \in C^1(\mathbb{R}^N; \mathbb{R})$ we have

$$f(Y) = f(Y_0) + \sum_{i=1}^d \int_0^t (V_i \cdot \partial) f(Y_\tau) \circ dX_\tau^i - \frac{1}{2} \sum_{i=1}^d \int_0^t ((V_i \cdot \partial V_i) \cdot \partial) f(Y_\tau) d[X^i, X^i]_\tau.$$

We emphasize, the differential operator in the second term on the right is a vector field. In addition, the usual rules of calculus apply to Fisk–Stratonovich integrals. As in Section 2 for the Itô case, we extend the driving continuous semimartingales and governing vector fields as follows. For $i = 1, \dots, d$ we set $X^{[i,i]} := [X^i, X^i]$ as before, however we now set $V_{[i,i]} := -\frac{1}{2}(V_i \cdot \partial V_i) \cdot \partial$. Then the chain rule above generates the Fisk–Stratonovich representation for the flowmap

$$\sum_w J_w V_w,$$

where the sum is over all words w constructed from the alphabet $\mathbb{A} := \{1, \dots, d, [1, 1], \dots, [d, d]\}$. The multiple stochastic Fisk–Stratonovich integrals $J_w = J_w(t)$ are defined over the same simplex as for the multiple Itô integrals but with each nested integration interpreted in the Fisk–Stratonovich sense. The basis terms V_w are compositions of vector fields from the alphabet \mathbb{A} with the assignment of each letter to each vector field as outlined above. We define the Fisk–Stratonovich word-to-integral map $\nu: \mathbb{K}\langle\mathbb{A}\rangle_{\sqcup} \rightarrow \mathbb{J}$ by $\nu: w \mapsto J_w$, in a similar manner to that for the Itô word-to-integral map. Here \mathbb{J} denotes the algebra generated by multiple Fisk–Stratonovich integrals with respect to the semimartingales $\{X^1, \dots, X^d\}$ or any quadratic variation processes $[X^i, X^i]$ generated from them, and the constant random variable 1. Note the alphabet \mathbb{A} contains all the letters $1, \dots, d$ and $[1, 1], \dots, [d, d]$. Let \mathbb{V} denote the set of partial differential operators constructed by composition from the vector fields V_i and $V_{[i,i]}$ for $i = 1, \dots, d$.

Definition 4.7 (Word-to-vector field map). *We denote by $\bar{\nu}: \mathbb{K}\langle\mathbb{A}\rangle \rightarrow \mathbb{V}$ the letter-to-vector field map $\bar{\nu}: i \mapsto V_i$ assigning each letter $i \in \mathbb{A}$ to the corresponding vector field V_i and each letter $[i, i] \in \mathbb{A}$ to the corresponding vector field $V_{[i,i]}$, for $i = 1, \dots, d$.*

The Fisk–Stratonovich word-to-integral map ν is a shuffle algebra homomorphism while the word-to-vector field map $\bar{\nu}$ is a concatenation algebra homomorphism. Hence the natural abstract setting for the Fisk–Stratonovich representation of the flowmap is the complete tensor algebra $\mathbb{K}\langle\mathbb{A}\rangle_{\sqcup} \otimes \mathbb{K}\langle\mathbb{A}\rangle$. The map $\nu \otimes \bar{\nu}: \mathbb{K}\langle\mathbb{A}\rangle_{\sqcup} \otimes \mathbb{K}\langle\mathbb{A}\rangle \rightarrow \mathbb{J} \otimes \mathbb{V}$ is an algebra homomorphism.

The Fisk–Stratonovich and Itô representations for the flowmap must coincide and so must their logarithms. The Fisk–Stratonovich integrals J_w satisfy the usual rules of calculus, see Protter [33], while the basis terms V_w are compositions of vector fields. Hence the Chen–Strichartz formula applies to the Fisk–Stratonovich representation for the flowmap, see Strichartz [36]. Recall from Lemma 3.11 the quasi-shuffle convolution logarithm of the identity $\log^*(\text{id})$ can be expressed as a power series in the augmented ideal projector \mathfrak{J} . When the quasi-shuffle convolution product reduces to the shuffle convolution product, also denoted \sqcup , the corresponding shuffle convolution power series $\log^{\sqcup}(\text{id})$ is given by $\mathfrak{J} - \frac{1}{2}\mathfrak{J}^{\sqcup 2} + \frac{1}{3}\mathfrak{J}^{\sqcup 3} + \dots$.

Theorem 4.8 (Chen–Strichartz Lie series). *The logarithm of the flowmap has the series representation*

$$\log\left(\sum_w J_w V_w\right) = \sum_w \frac{1}{|w|} J_{\log^{\sqcup}(w)} V_{[w]_{\mathbb{L}}}$$

where $J_{\log^{\sqcup}(w)} = \nu \circ \log^{\sqcup}(w)$ and

$$\log^{\sqcup}(w) = \sum_{\sigma \in \mathbb{S}_{|w|}} c_{\sigma} \sigma^{-1}(w) \quad \text{with} \quad c_{\sigma} := \frac{(-1)^{d(\sigma)}}{|\sigma|} \binom{|\sigma| - 1}{d(\sigma)}^{-1}.$$

Here for any word $w = a_1 \cdots a_n \in \mathbb{A}^*$ the basis terms $V_{[w]_{\mathbb{L}}} := [V_{a_1}, [V_{a_2}, \dots, [V_{a_{n-1}}, V_{a_n}]_{\mathbb{L}} \cdots]_{\mathbb{L}}$, where $[\cdot, \cdot]_{\mathbb{L}}$ is the Lie bracket, are Lie polynomials. The non-negative integers $d(\sigma)$ denote the number of descents in the permutation σ and $|\sigma|$ denotes its length, i.e. if $\sigma \in \mathbb{S}_{|w|}$ then $|\sigma| = |w|$.

Remark 4.9. After resummation, the formula in Theorem 4.8 is that derived by Strichartz [36].

Proof. First, since the Fisk–Stratonovich multiple integrals satisfy the usual rules of calculus, the underlying product is the shuffle product \sqcup ; this is a special case of the quasi-shuffle product

for which the quadratic variation generator $[\cdot, \cdot]$ is identically zero. Hence we can emulate the derivation of the quasi-shuffle convolution given in Section 3 to show that

$$\begin{aligned} \log\left(\sum J_w V_w\right) &= \log \circ (\nu \otimes \bar{\nu}) \circ \left(\sum w \otimes w\right) \\ &= (\nu \otimes \bar{\nu}) \circ \log \circ \left(\sum w \otimes w\right) \\ &= (\nu \otimes \bar{\nu}) \circ \left(\sum \log^{\sqcup}(\text{id}) \circ w \otimes w\right), \end{aligned}$$

where the sums are over all words $w \in \mathbb{A}^*$. The expression $\log^{\sqcup}(\text{id})$ is the shuffle convolution power series for the logarithm on the identity described above. That this power series has the equivalent expansion in terms of inverse permutations with the form for the coefficients c_σ shown, is proved in Section 5 for $\log^*(\text{id})$. See in particular Corollary 5.11. Note that the coefficients $J_{\log^{\sqcup}(w)}$ then refer to the linear combination of multiple Fisk–Stratonovich integrals enumerated by the words generated by the set of permutations shown.

Second, we express the logarithm of the Fisk–Stratonovich flowmap in terms of Lie polynomials as shown. The crucial observation here is that the adjoint of the shuffle convolution logarithm $\log^{\sqcup}(\text{id})$ is the concatenation convolution logarithm on the identity (computed with the deshuffle coproduct δ) which we denote by $\log(\text{id})$. This follows as for any integer $k \geq 1$, the adjoint of $\mathfrak{J}^{\sqcup k}$ is the k th concatenation convolution power of \mathfrak{J} . The concatenation logarithm of the identity is a Lie idempotent known as the Eulerian or Solomon idempotent. Another Lie idempotent is the Dynkin idempotent which has several characterizations, here it suffices to define it as follows. Let $[1 \cdots p]_{\text{L}}$ denote left to right Lie bracketing of the word $1 \cdots p$ so that $[1 \cdots p]_{\text{L}} := [1, [2, [\dots [p-1, p] \dots]]]_{\text{L}}$. The element $\frac{1}{p}[1 \cdots p]_{\text{L}}$ of $\mathbb{K}[\mathbb{S}_p]$ is known as the Dynkin idempotent, where $\mathbb{K}[\mathbb{S}_p]$ is the group algebra over \mathbb{K} for the symmetric group \mathbb{S}_p of order p . We denote the Dynkin idempotent as

$$\theta := \frac{1}{p}[1 \cdots p]_{\text{L}}.$$

That the Eulerian and Dynkin operators are Lie idempotents is proved in Reutenauer [34, p. 195] for example. The Dynkin–Friedrichs–Specht–Wever Theorem is now key. It states any polynomial P in the concatenation Hopf algebra $\mathbb{K}\langle \mathbb{A} \rangle$ lies in the corresponding free Lie algebra if and only if $\theta P = P$; see for example Reutenauer [34, Theorem 1.4]. The image of the Eulerian idempotent is contained in the free Lie algebra associated with $\mathbb{K}\langle \mathbb{A} \rangle$; see Reutenauer [34, Lemma 3.8]. Thus the element $\log(w)$ for any word $w \in \mathbb{K}\langle \mathbb{A} \rangle$ is a Lie element. The Dynkin–Friedrichs–Specht–Wever Theorem thus implies

$$\theta(\log(w)) \equiv \log(w).$$

Now by direct calculation,

$$\begin{aligned} \log\left(\sum w \otimes w\right) &= \sum \log^{\sqcup}(w) \otimes w \\ &= \sum w \otimes \log(w) \\ &= \sum w \otimes \theta(\log(w)) \\ &= \sum \log^{\sqcup}(w) \otimes \theta(w) \\ &= \sum \frac{1}{|w|} \log^{\sqcup}(w) \otimes [w]_{\text{L}}, \end{aligned}$$

where the sums are over all words $w \in \mathbb{A}^*$. Here we used that if X and Y are adjoint endomorphisms, and Z is an endomorphism on the concatenation Hopf algebra, then we have

$$\sum X(w) \otimes Z(w) = \sum w \otimes Z(Y(w)).$$

We applied this identity with $X = \log^{\sqcup}(\text{id})$, $Y = \log(\text{id})$ and $Z = \theta$. □

The Hoffman exponential map \exp_{H} naturally relates Itô and Fisk–Stratonovich multiple integrals as follows; see Kloeden and Platen [23, Remark 5.2.8] for the Wiener process case.

Proposition 4.10 (Itô to Fisk–Stratonovich: Hoffman exponential). *For continuous semimartingales, for any word w in \mathbb{A}^* , we have $J_w = I_{\exp_{\text{H}}(w)}$ where explicitly we have*

$$I_{\exp_{\text{H}}(w)} = \sum_{\lambda \in \mathbb{C}(|w|)} \frac{1}{2^{\Sigma(\lambda) - |\lambda|}} I_{\lambda \circ w} = I_w + \sum_{u \in [[w]]} \frac{1}{2^{|w| - |u|}} I_u.$$

The nilpotency of the generator $[\cdot, \cdot]$ implies the compositions $\lambda \in \mathbb{C}(|w|)$ with a nonzero contribution only contain letters 1 and 2 and so $\Gamma(\lambda) = 2^{\Sigma(\lambda) - |\lambda|}$. Thus the set $[[w]]$ consists of the words we can construct from w by successively replacing any neighbouring pairs ii in w by $[i, i]$.

Proof. Using the definition of the Fisk–Stratonovich integral, for any word $w = a_1 \cdots a_n$ we have

$$J_{a_1 \cdots a_n} = \int J_{a_1 \cdots a_{n-1}} dI_{a_n} + \frac{1}{2} \int J_{a_1 \cdots a_{n-2}} d[I_{a_{n-1}}, I_{a_n}].$$

Recursively applying this formula for $J_{a_1 \cdots a_{n-1}}$ and so forth, generates the result. \square

With this in hand, we deduce the following main result of this section.

Corollary 4.11 (Itô Lie series). *Let c_σ denote the coefficient of $\sigma^{-1}(w)$ in the expression for $\log^{\text{H}}(w)$ above. We can express the Chen–Strichartz Lie series in terms of multiple Itô integrals as follows*

$$\log \left(\sum_w J_w V_w \right) = \sum_w \sum_{\sigma \in \mathbb{S}_{|w|}} \frac{c_\sigma}{|\sigma|} I_{\exp_{\text{H}}(\sigma^{-1}(w))} V_{[\sigma]_{\text{L}}},$$

or equivalently, by resummation of the series,

$$\log \left(\sum_w J_w V_w \right) = \sum_w I_w \left(\sum_{\sigma \in \mathbb{S}_{|w|}} \frac{c_\sigma}{|\sigma|} V_{[\sigma(w)]_{\text{L}}} + \sum_{u \in [[w]]} \frac{1}{2^{|u| - |w|}} \sum_{\sigma \in \mathbb{S}_{|u|}} \frac{c_\sigma}{|\sigma|} V_{[\sigma(u)]_{\text{L}}} \right).$$

Here for any word $w \in \mathbb{A}^*$ the set $[[w]]$ consists of w and all words we construct from w by successively replacing any letter $[i, i]$ with ii , for $i = 1, \dots, d$.

Since the Fisk–Stratonovich and Itô representations for the flowmap must coincide, the expression above must coincide with the logarithm of the Itô representation for the flowmap. The key fact distinguishing the Itô from the Fisk–Stratonovich representation for the flowmap is that the word-to-vector field map $\bar{\nu}: \mathbb{K}\langle \mathbb{A} \rangle_{\text{conc}, \delta} \rightarrow \mathbb{V}$ and word-to-partial differential operator map $\bar{\mu}: \mathbb{K}\langle \mathbb{A} \rangle_{\text{conc}, \Delta'} \rightarrow \mathbb{D}$, which are both concatenation homomorphisms, assign

$$\bar{\mu}: \begin{cases} i & \mapsto V_i \cdot \partial, \\ [i, i] & \mapsto \frac{1}{2} V_i \otimes V_i : \partial^2, \end{cases} \quad \text{and} \quad \bar{\nu}: \begin{cases} i & \mapsto V_i \cdot \partial, \\ [i, i] & \mapsto -\frac{1}{2} (V_i \cdot \partial V_i) \cdot \partial. \end{cases}$$

The Itô representation is constructed by composing the operators on the left shown above, while that for the Fisk–Stratonovich representation is constructed by composing the operators on the right. The operators $\frac{1}{2} V_i \otimes V_i : \partial^2$ and $\frac{1}{2} (V_i \cdot \partial V_i) \cdot \partial$ are both associated with the quadratic variation process $[X^i, X^i]$. The Itô word-to-integral map $\mu: \mathbb{K}\langle \mathbb{A} \rangle_{*, \Delta} \rightarrow \mathbb{I}$ and Fisk–Stratonovich word-to-integral map $\nu: \mathbb{K}\langle \mathbb{A} \rangle_{\text{sh}, \Delta} \rightarrow \mathbb{J}$ both assign $i \mapsto X^i$ and $[i, i] \mapsto [X^i, X^i]$. The former is a quasi-shuffle homomorphism and the latter a shuffle homomorphism and consequently $J_w = I_{\exp_{\text{H}}(w)}$. This relation, together with the calculus product rule given by $(V_i \cdot \partial)(V_i \cdot \partial) = (V_i \otimes V_i) : \partial^2 + (V_i \cdot \partial V_i) \cdot \partial$ underlie the following result. Recall the definition of the Hoffman exponential and logarithm map adjoints in Section 3.

Theorem 4.12 (Itô and Fisk–Stratonovich map relations). *The two word-to-integral maps μ and ν and the word-to-vector field and word-to-partial differential operator maps $\bar{\mu}$ and $\bar{\nu}$ are related as follows*

$$\nu \equiv \mu \circ \exp_{\mathbb{H}} \quad \text{and} \quad \bar{\nu} \equiv \bar{\mu} \circ \log_{\mathbb{H}}^{\dagger}.$$

Proof. The first relation follows directly from $J_w = I_{\exp_{\mathbb{H}}(w)} \Leftrightarrow \nu \circ w = \mu \circ \exp_{\mathbb{H}} \circ w$. The second relation follows from the calculus product rule above which can be expressed in the form

$$\frac{1}{2}\bar{\mu} \circ ii = \bar{\mu} \circ [i, i] - \bar{\nu} \circ [i, i] \Leftrightarrow \bar{\nu} \circ [i, i] = \bar{\mu} \circ ([i, i] - \frac{1}{2}ii) \Leftrightarrow \bar{\nu} \circ [i, i] = \bar{\mu} \circ \log_{\mathbb{H}}^{\dagger} \circ [i, i].$$

Note ii denotes the concatenation of i with i and $\bar{\mu}(ii) = \bar{\nu}(ii)$. The final expression in this sequence follows using the nilpotency of the bracket $[\cdot, \cdot]$. Using that $\log_{\mathbb{H}}^{\dagger} \circ i = i$ for the letters $i = 1, \dots, d$ and $\log_{\mathbb{H}}^{\dagger}$ is a concatenation homomorphism, establishes the second result. \square

Some immediate consequences of this result are as follows. First, algebraically, we observe

$$\begin{aligned} \sum I_w D_w &= \sum \mu \circ w \otimes \bar{\mu} \circ w \\ &= \sum \mu \circ w \otimes \bar{\nu} \circ \exp_{\mathbb{H}}^{\dagger} \circ w \\ &= (\mu \otimes \bar{\nu}) \circ \left(\sum w \otimes \exp_{\mathbb{H}}^{\dagger} \circ w \right) \\ &= (\mu \otimes \bar{\nu}) \circ \left(\sum \exp_{\mathbb{H}} \circ w \otimes w \right) \\ &= \sum \mu \circ \exp_{\mathbb{H}} \circ w \otimes \bar{\nu} \circ w \\ &= \sum \nu \circ w \otimes \bar{\nu} \circ w \\ &= \sum J_w V_w. \end{aligned}$$

In other words we have verified that the Itô and Fisk–Stratonovich flowmaps coincide. Note the transfer of $\exp_{\mathbb{H}}^{\dagger}$ from the right of the tensor product to $\exp_{\mathbb{H}}$ on the left, relies solely on the vector space properties of $\mathbb{K}\langle \mathbb{A} \rangle$. Second, using this result and some of the results stated in the proof of Theorem 4.8 such as that the Eulerian idempotent is a Lie element and the Dynkin–Friedrichs–Specht–Wever Theorem, we observe

$$\begin{aligned} \log \left(\sum I_w D_w \right) &= \log \circ (\mu \otimes \bar{\mu}) \circ \left(\sum w \otimes w \right) \\ &= \log \circ (\nu \otimes \bar{\nu}) \circ \left(\sum w \otimes w \right) \\ &= (\nu \otimes \bar{\nu}) \circ \log \circ \left(\sum w \otimes w \right) \\ &= (\nu \otimes \bar{\nu}) \circ \left(\sum \log^{\sqcup} \circ w \otimes w \right) \\ &= (\nu \otimes \bar{\nu}) \circ \left(\sum \frac{1}{|w|} \log^{\sqcup} \circ w \otimes [w]_{\mathbb{L}} \right) \\ &= (\mu \otimes \bar{\nu}) \circ \left(\sum \frac{1}{|w|} \exp_{\mathbb{H}} \circ \log^{\sqcup} \circ w \otimes [w]_{\mathbb{L}} \right) \\ &= \sum \frac{1}{|w|} I_{\exp_{\mathbb{H}}(\log^{\sqcup}(w))} V_{[w]_{\mathbb{L}}}, \end{aligned}$$

where the sums are over all words $w \in \mathbb{A}^*$. Similarly, we can also perform a resummation of the series as indicated in Corollary 4.11 as follows,

$$\begin{aligned} \log\left(\sum J_w V_w\right) &= (\nu \otimes \bar{\nu}) \circ \left(\sum \log^{\sqcup} \circ w \otimes \theta \circ w\right) \\ &= (\mu \otimes \bar{\nu}) \circ \left(\sum \exp_{\mathbb{H}} \circ \log^{\sqcup} \circ w \otimes \theta \circ w\right) \\ &= (\mu \otimes \bar{\nu}) \circ \left(\sum w \otimes \theta \circ \log \circ \exp_{\mathbb{H}}^{\dagger} \circ w\right), \end{aligned}$$

where

$$\begin{aligned} \log \circ \exp_{\mathbb{H}}^{\dagger} \circ w &= \log \circ \left(w + \sum_{u \in]w[[} \frac{1}{2^{|u|-|w|}} u\right) \\ &= \sum_{\sigma \in \mathbb{S}_{|w|}} c_{\sigma} \sigma(w) + \sum_{u \in]w[[} \frac{1}{2^{|u|-|w|}} \sum_{\sigma \in \mathbb{S}_{|u|}} c_{\sigma} \sigma(u). \end{aligned}$$

These last two results are thus a restatement of the Itô Lie series results in Corollary 4.11.

Remark 4.13. From the algebraic combinatorial computations above, we observe: (1) In the first computation above the transformation from the Itô to Fisk–Stratonovich flowmaps was instigated by the transformation of coordinates $\mu = \nu \circ \exp_{\mathbb{H}}^{\dagger}$. In other words this transformation, which is a direct result of the product rule, encodes all the information required for Itô to Fisk–Stratonovich conversion; (2) Quadratic variations are a natural component in the Fisk–Stratonovich formulation; (3) The encoding which retains the letters $[1, 1], \dots, [d, d]$ as well as $1, \dots, d$ in the alphabet appears to be natural, especially in the context of using the quasi-shuffle machinery provided by Hoffman [19]. Indeed this is also the case for stochastic differential equations driven by Wiener processes for which it is usual to replace the quadratic variation terms by the corresponding drift term; (4) The flowmap satisfies the linear equation $\varphi = \text{id} + \int \varphi dS$ with $S := \sum_i D_i X^i$, see Ebrahimi-Fard, Malham, Patras and Wiese [11]. When the coefficients D_i are constant, the solution is the well-known Doléans-Dade exponential; a representation of it in terms of iterated integrals was derived in Jamshidian [21].

5. Quasi-Shuffle Chen–Strichartz formula

In this section we do not make any nilpotency assumptions on n -fold nested brackets as in Section 4. We derive an explicit formula for the coefficients of the quasi-shuffle convolution logarithm of the identity endomorphism on $\mathbb{K}\langle \mathbb{A} \rangle_*$, i.e. we explicitly enumerate

$$\log^*(\text{id}).$$

This represents the quasi-shuffle logarithm equivalent of the Chen–Strichartz shuffle logarithm formula and was derived in Novelli, Patras and Thibon [30] and generalized to linear matrix valued systems in Ebrahimi-Fard, Malham, Patras and Wiese [11]. Using the notion of surjections instead of permutations and quasi-descents, we can closely follow the development given in Reutenauer [34]. We begin by outlining the theory of surjections and quadratic covariation permutations, which we hereafter call “quasi-permutations”, as well as their action on words. We denote the symmetric group of order p by \mathbb{S}_p and the corresponding group algebra over the field \mathbb{K} by $\mathbb{K}[\mathbb{S}_p]$. The crucial fact about any permutation $\sigma \in \mathbb{K}[\mathbb{S}_p]$, which underlies the classical shuffle Chen–Strichartz formula, is that the inverse σ^{-1} records the following information: “The letter i is at position $\sigma^{-1}(i)$ in σ ”. We exploit the corresponding result for surjections herein.

We shall denote the set of surjective maps from the set of natural numbers $\{1, \dots, p\}$ to the set of natural numbers $\{1, \dots, q\}$ with $q \leq p$ by $\mathbb{S}'_{p,q}$. We set

$$\mathbb{S}'_p := \bigcup_{q \leq p} \mathbb{S}'_{p,q}.$$

Naturally we have $\mathbb{S}_p \subseteq \mathbb{S}'_p$. Associated with each surjection in \mathbb{S}'_p is a quasi-permutation.

Definition 5.1 (Quasi-permutations). *We denote by $\mathbb{C}(\mathbb{S}_p)$ the set of all quasi-permutations, these are all the permutations in \mathbb{S}_p together with all unique words formed by applying all possible composition actions to these permutations, taking care to unify equivalent terms. In other words,*

$$\mathbb{C}(\mathbb{S}_p) := \{\lambda \circ \sigma : \sigma \in \mathbb{S}_p, \lambda \in \mathbb{C}(p)\},$$

where we identify all terms that are equal due to the symmetry and associative properties of the nested bracket operation.

Henceforth we record quasi-permutations simply as $\sigma \in \mathbb{C}(\mathbb{S}_p)$. However, it is always possible to decompose (non-uniquely) any given quasi-permutation into its composition and permutation components, say as $\lambda \circ \rho$ or as a pair (λ, ρ) where λ is a composition and ρ a permutation. Given any quasi-permutation in $\mathbb{C}(\mathbb{S}_p)$, there is a unique surjection in \mathbb{S}'_p that records the position of letters and nested brackets of letters.

Example 5.2. Consider the set of all quasi-permutations $\mathbb{C}(\mathbb{S}_3)$, these are given by the set of \mathbb{S}_3 permutations 123, 132, 213, 231, 312, 321 and $[1, 2]3$, $1[2, 3]$, $[1, 3]2$, $2[1, 3]$, $[2, 3]1$, $3[1, 2]$, $[1, 2, 3]$. The set of all surjections in \mathbb{S}'_3 consists of 123, 132, 213, 312, 231, 321 and 112, 122, 121, 212, 211, 221, 111. Term by term in the order given, we see that the surjections record the corresponding positions of the letters in the quasi-permutations.

Hence, by analogy with permutations, quasi-permutations play the role of generalized permutations, while the corresponding surjections play the role of the inverse permutations by recording the positions of the letters in the corresponding quasi-permutations. Hence we have the corresponding statement to that above and crucial fact about surjections: each surjection ζ corresponding to a given quasi-permutation σ records the information:

The letter i is at position $\zeta(i)$ in σ .

Example 5.3. The surjection 3221 from \mathbb{S}'_4 , tells us that the letter that was in position 1 in the quasi-permutation was sent to position position 3, the letters 2 and 3 were sent to position 2, and the letter 4 was sent to position 1. Hence the corresponding quasi-permutation is $4[2, 3]1$. For another example, if 2312 is a given surjection in \mathbb{S}'_4 mapping $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$ and $4 \mapsto 2$, then the corresponding quasi-permutation is $3[1, 4]2$ which is equal to $3[4, 1]2$.

With each surjection we can associate a quasi-descent set.

Definition 5.4 (Quasi-descent sets). *Given any surjection $\zeta \in \mathbb{S}'_p$ we define its quasi-descent set $\text{Des}(\zeta)$ to be the list of the indices $k \in \{1, \dots, p-1\}$ for which $\zeta(k+1) \leq \zeta(k)$.*

For the particular subset $\mathbb{S}_p \subset \mathbb{S}'_p$ these inequalities would be strict and the indices k would correspond to the classical descent indices. Just as there is an intimate relation between shuffles and descents, there is also one between quasi-shuffles and quasi-descents. The following first key result underlies the whole of this section.

Lemma 5.5 (Quasi-descents and quasi-shuffles). *The set of surjections $\zeta \in \mathbb{S}'_p$ satisfying $\text{Des}(\zeta) \subseteq \{q\}$ for $q < p$, is identical to the set of surjections satisfying $\zeta(1) < \dots < \zeta(q)$ and $\zeta(q+1) < \dots < \zeta(p)$.*

Proof. We observe that for any surjection $\zeta \in \mathbb{S}'_p$ for which $\text{Des}(\zeta) \subseteq \{q\}$ for some natural number $q < p$, then discounting the case when the quasi-descent set is empty, by definition we must have $\zeta(k) \geq \zeta(k+1) \implies k = q \iff k \neq q \implies \zeta(k) < \zeta(k+1)$. The latter condition is equivalent to that in the statement of the lemma. \square

The second key result we establish in this section is a natural consequence.

Corollary 5.6 (Quasi-descents and quasi-shuffles). *Let $q < p$ be natural numbers. If we factorize the word $1 \cdots p = u_1 u_2$ with $|u_1| = q$ and $|u_2| = p - q$, then the quasi-shuffle product of $u_1 * u_2$ is given by*

$$u_1 * u_2 \equiv \sum_{\substack{\zeta(1) < \cdots < \zeta(q) \\ \zeta(q+1) < \cdots < \zeta(p)}} \sigma(\zeta) \equiv \sum_{\text{Des}(\zeta) \subseteq \{q\}} \sigma(\zeta),$$

where $\sigma(\zeta)$ denotes the unique quasi-permutation associated with a given surjection ζ . The first sum is over all $\zeta \in \mathbb{S}'_p$ satisfying the inequalities shown, and the second sum is over all $\zeta \in \mathbb{S}'_p$ such that $\text{Des}(\zeta) \subseteq \{q\}$.

Proof. Recall the definition of the quasi-shuffle product and its generation through the formula

$$\begin{aligned} (1 \cdots q) * (q + 1 \cdots p) &= ((1 \cdots q - 1) * (q + 1 \cdots p)) q \\ &\quad + ((1 \cdots q) * (q + 1 \cdots p - 1)) p \\ &\quad + ((1 \cdots q - 1) * (q + 1 \cdots p - 1)) [q, p]. \end{aligned}$$

We observe that if we recursively apply this formula to obtain on the right hand side the complete sum over all quasi-permutations, then the quasi-shuffle product of u_1 and u_2 is equivalent to the prescription that it is the sum over all quasi-permutations whose corresponding surjections satisfy the set of inequalities $\zeta(1) < \cdots < \zeta(q)$ and $\zeta(q + 1) < \cdots < \zeta(p)$. This establishes the first result. With this in hand, the result of the quasi-descent and quasi-shuffle conditions Lemma 5.5 above implies the equivalence to the second result. \square

The following generalization is then immediate and represents the quasi-shuffle analog of Lemma 3.13 in Reutenauer [34, p. 65].

Corollary 5.7 (Multiple quasi-shuffles and quasi-descents). *Let p_1, \dots, p_k be positive integers of sum p and $S = \{p_1, p_1 + p_2, \dots, p_1 + \cdots + p_{k-1}\}$ be a subset of $\{1, \dots, p - 1\}$. If we factorize the word $1 \cdots p = u_1 \cdots u_k$ with $|u_i| = p_i$ for $i = 1, \dots, k$, then we have*

$$u_1 * \cdots * u_k = \sum_{\text{Des}(\zeta) \subseteq S} \sigma(\zeta).$$

Much like the symmetric group action on words there is an analogous quasi-permutation action on words. Recall that we can decompose any quasi-permutation in $\mathbb{K}[\mathbb{C}(\mathbb{S}_p)]$ as $\sigma = \lambda \circ \rho$, into its composition $\lambda \in \mathbb{C}$ and permutation $\rho \in \mathbb{S}_p$ components.

Definition 5.8 (Quasi-permutation action). *We define the action of $\mathbb{K}[\mathbb{C}(\mathbb{S}_p)]$ on $\mathbb{K}\langle \mathbb{A} \rangle$ for any $\sigma \in \mathbb{K}[\mathbb{C}(\mathbb{S}_p)]$ decomposed as $\sigma = \lambda \circ \rho$ and word $w = a_1 \cdots a_p$ by $\sigma w := \lambda \circ (a_{\rho(1)} \cdots a_{\rho(p)})$.*

We can now construct the quasi-shuffle logarithm of the identity. We start with the quasi-shuffle convolution powers of the augmented ideal projector \mathfrak{J} .

Corollary 5.9 (Convolution powers and descents). *For any $k \geq 1$ and word w we have*

$$\mathfrak{J}^{*k}(w) = \left(\sum_{|S|=k-1} \sum_{\text{Des}(\zeta) \subseteq S} \sigma(\zeta) \right) \circ w.$$

Proof. For any word w the quantity $\mathfrak{J}^{*k}(w)$ is the sum over all possible k -partitions of w , say $v_1 \cdots v_k$, quasi-shuffled together. Hence we have

$$\begin{aligned} \mathfrak{J}^{*k}(w) &= \sum_{v_1 \cdots v_k = w} v_1 * \dots * v_k \\ &= \sum_{u_1 \cdots u_k = 1 \cdots p} (u_1 * \dots * u_k) \circ w \\ &= \sum_{|S|=k-1} \left(\sum_{\text{Des}(\zeta) \subseteq S} \sigma(\zeta) \right) \circ w \\ &= \left(\sum_{|S|=k-1} \sum_{\text{Des}(\zeta) \subseteq S} \sigma(\zeta) \right) \circ w, \end{aligned}$$

where we used Corollary 5.7 in the third step. \square

Corollary 5.10 (Quasi-Shuffle convolution logarithm on words). *The action of the quasi-shuffle convolution logarithm on any word w is as follows*

$$\log^*(w) = \left(\sum_{S \subseteq \{1, \dots, |w|-1\}} \frac{(-1)^{|S|}}{|S|+1} \sum_{\text{Des}(\zeta) \subseteq S} \sigma(\zeta) \right) \circ w.$$

Proof. By direct computation using Corollary 5.9 we find

$$\begin{aligned} \log^*(w) &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \mathfrak{J}^{*k}(w) \\ &= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \left(\sum_{|S|=k-1} \sum_{\text{Des}(\zeta) \subseteq S} \sigma(\zeta) \right) \circ w \\ &= \left(\sum_{|S| \geq 0} \frac{(-1)^{|S|}}{|S|+1} \sum_{\text{Des}(\zeta) \subseteq S} \sigma(\zeta) \right) \circ w \\ &= \left(\sum_{S \subseteq \{1, \dots, |w|-1\}} \frac{(-1)^{|S|}}{|S|+1} \sum_{\text{Des}(\zeta) \subseteq S} \sigma(\zeta) \right) \circ w. \end{aligned}$$

\square

The following characterization of the quasi-shuffle convolution logarithm is the generalization of the standard shuffle convolution logarithm. We shall need the following integral identity for non-negative integers d and r which is proved for example in Reutenauer [34, p. 69]:

$$\int_{-1}^0 x^d (1+x)^r dx = \frac{(-1)^d d! r!}{(d+r+1)!}.$$

Corollary 5.11 (Quasi-Shuffle convolution logarithm endomorphism). *The quasi-shuffle convolution logarithm $\log^*(\text{id})$ acts on $1 \cdots p$ as follows*

$$\log^*(1 \cdots p) = \sum_{\zeta \in \mathbb{S}'_p} \frac{(-1)^{d(\zeta)}}{p} \binom{p-1}{d(\zeta)}^{-1} \sigma(\zeta),$$

where $d(\zeta)$ denotes the number of quasi-descents in ζ .

Proof. We observe from Corollary 5.10 that $\log^*(1 \cdots p)$ consists of a linear combination of quasi-permutations $\sigma(\zeta)$. Hence we directly compute the coefficient of an arbitrary quasi-permutation

$\sigma(\zeta)$ in $\log^*(1 \cdots p)$ which, using the result of Corollary 5.10, is given by

$$\sum_{S \subseteq \{1, \dots, p-1\} : \text{Des}(\zeta) \subseteq S} \frac{(-1)^{|S|}}{|S| + 1}.$$

Suppose that ζ has quasi-descent indices p_1, \dots, p_k so that $d(\zeta) = k$. To compute this coefficient we therefore have to determine the number of subsets $S \subseteq \{1, \dots, p-1\}$ which contain p_1, \dots, p_k . Note the coefficient itself only depends on the size of such sets. These subsets have possible size $|S| = k$ through to $|S| = p-1$. Starting with the case $|S| = k$ there is of course only one set of this size containing p_1, \dots, p_k , the set of these integers themselves. Now consider the case $|S| = k+1$. Then an extra “quasi-descent” can be placed in total of $p-1-k$ possible positions, or equivalently in $p-1-k$ choose 1 ways. When $|S| = k+2$, there are $p-1-k$ choose 2 ways, and so forth so that in general, when $|S| = k+i$, there are $p-1-i$ choose i possible ways. Hence the coefficient above equals

$$\sum_{i=0}^{p-1-d(\zeta)} \binom{p-1-d(\zeta)}{i} \frac{(-1)^{d(\zeta)+i}}{d(\zeta)+i+1}.$$

This form of the coefficient is equal to

$$\begin{aligned} \int_{-1}^0 \sum_{i=0}^{p-1-d(\zeta)} \binom{p-1-d(\zeta)}{i} x^{d(\zeta)+i} dx &= \int_{-1}^0 x^{d(\zeta)} (1+x)^{p-1-d(\zeta)} dx \\ &= (-1)^{d(\zeta)} \frac{d(\zeta)!(p-1-d(\zeta))!}{p!} \\ &= \frac{(-1)^{d(\zeta)}}{p} \binom{p-1}{d(\zeta)}^{-1}, \end{aligned}$$

using the integral identity preceding the corollary. □

Remark 5.12. This is equivalent to the quasi-shuffle logarithm given in Ebrahimi-Fard *et al.* [11, Theorem 6.2]. We included an explicit derivation here for completeness.

6. Concluding remarks

There has been a recent surge in the development of quasi-shuffle algebras and stochastic Taylor solution formulae in the context of semimartingales, on the theoretical and practical level. See Platen and Bruti-Liberati [32], and Marcus [29], Friz and Shekhar [15] and Hairer and Kelly [18] for contemporary references. For example Li and Liu [24] considered systems driven by both Wiener and Poisson processes. We have shown that the Chen–Strichartz flowmap solution formula which is well-known for Stratonovich stochastic differential systems driven by Wiener processes extends to systems driven by general continuous semimartingales. We demonstrated it is in fact a Lie series, and this property holds irrespective of whether we consider the system in the Itô or Stratonovich sense. We also give and prove an explicit formula for the Lie series coefficients. Curry, Ebrahimi-Fard, Malham and Wiese [7] have developed so-called efficient simulation schemes for such systems driven by Lévy processes. This involves the antisymmetric sign reverse endomorphism rather than the quasi-shuffle logarithm endomorphism.

Ethics

No research on humans or animals was conducted.

Data accessibility

The paper is self-contained, there is no supporting data.

Competing interests

We have no competing interests.

Author's contributions

The research presented herein was a joint and equal effort by all authors.

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