A novel sampling theorem on the rotation group

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Jason D. McEwen, Martin Büttner, Boris Leistedt, Hiranya V. Peiris and Yves Wiaux

Abstract—We develop a novel sampling theorem for functions defined on the three-dimensional rotation group $SO(3)$ by associating the rotation group with the three-torus through a periodic extension. Our sampling theorem requires $4L^3$ samples to capture all of the information content of a signal band-limited at L, reducing the number of required samples by a factor of two compared to other equiangular sampling theorems. We present fast algorithms to compute the associated Fourier transform on the rotation group, the so-called Wigner transform, which scale as $\mathcal{O}(L^4)$, compared to the naive scaling of $\mathcal{O}(L^6)$. For the common case of a low directional band-limit N, complexity is reduced to $\mathcal{O}(NL^3)$. Our fast algorithms will be of direct use in speeding up the computation of directional wavelet transforms on the sphere. We make our $SO3$ code implementing these algorithms publicly available.

Index Terms—Harmonic analysis, sampling, spheres, rotation group, Wigner transform.

I. INTRODUCTION

S HANNON established the theoretical foundations of sampling theory in Euclidean space over half a century ago, proving that the information content of a band-limited continuous signal could be captured completely in a finite number of samples [1]. The fast Fourier transform (FFT) [2] is one of the most important algorithmic developments of our era and has been instrumental in rendering the frequency content of signals accessible in practice. The combination of theoretical foundations and fast algorithms has led to the extensive use of Fourier methods to analyse data in myriad applications.

In many applications, however, data are acquired on non-Euclidean manifolds where Euclidean sampling theory is not applicable. Spherical manifolds are one of the most prevalent non-Euclidean domains. When observing over directions, data are acquired on the two-dimensional sphere. If distance information is also accessible, then data are acquired on the three-dimensional ball. For example, in cosmology observations are made on the celestial sphere, with the cosmic microwave background (CMB), observed on the sphere (*e.g.* [3]), while surveys of the distribution of galaxies are made on the ball (*e.g.* [4]). Other examples of data defined on spherical manifolds are found in many fields, including planetary science (*e.g.* [5], [6]) and geophysics (*e.g.* [7], [8]).

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Sampling theory and corresponding fast harmonic transforms on the sphere remain active areas of research. Until recently, the canonical equiangular sampling theorem on the sphere was that of Driscoll & Healy [9], requiring $\sim 4L^2$ samples on the sphere to capture the information content of a signal band-limited at L. A novel sampling theorem on the sphere was developed recently by two of the authors of the present article (McEwen & Wiaux), reducing the number of samples required to capture a band-limited signal to $\sim 2L^2$ [10], building on the developments of [11], [12] (see [10] for a review of sampling theory on the sphere). No existing sampling theorem on the sphere reaches the optimal number of samples given by the harmonic dimensionality of a band-limited signal of L^2. A new sampling scheme that achieves the optimal number of L^2 samples was developed recently by [13]. Whilst this scheme does not lead to a sampling theorem with theoretically exact spherical harmonic transforms, good numerical accuracy is achieved in practice and fast algorithms were developed. A new sampling theory on the ball was developed by [14] recently, augmenting the sampling theorem on the sphere of [10] with Gaussian quadrature on the radial line to recover an exact harmonic transform suitable for the analysis of large data-sets defined on the ball.

The analysis of data defined on the sphere or ball often leads to data defined on the rotation group $SO(3)$, the space of three-dimensional rotations. For example, directional wavelet transforms on the sphere (*e.g.* [15]–[23]) probe signal content not only in scale and position on the sphere, but also in orientation. The resulting wavelet coefficients thus live on the rotation group. Moreover, the wavelet transform can be computed via a Fourier transform on the rotation group [21], [22]. Data defined natively on the rotation group also arise in many applications, for example searching databases of objects over arbitrary rotations [24]. Sampling theorems on the rotation group with fast harmonic transforms are thus of both important theoretical interest and practical use.

The canonical equiangular sampling theorem on the rotation group $SO(3)$ is that of Kostelec *et al.* [25], which relies on the Driscoll & Healy sampling theorem on the sphere [9], and thus requires $\sim 8L^3$ samples to capture a signal on the rotation group that is band-limited at L. In this article we develop a novel sampling theorem on the rotation group (extending our recent sampling theorem on the sphere [10]), reducing the number of samples required to capture a band-limited signal to $\sim 4L^3$. Furthermore, we present fast algorithms to compute the associated harmonic transform on the rotation group (often called the Wigner transform). No existing sampling theorem on the rotation group reaches the optimal number of samples given by the $\sim 4L^2/3$ degrees of freedom in harmonic space, although our approach comes closest to this bound.

The remainder of this article is structured as follows. After

concisely reviewing harmonic analysis on the rotation group we present our novel sampling theorem, before defining explicitly an exact quadrature rule on the rotation group. We then discuss and evaluate fast algorithms for computing the Wigner transform associated with our sampling theorem, presenting a new public code, followed by concluding remarks.

II. HARMONIC ANALYSIS ON THE ROTATION GROUP

We consider the space of square integrable functions on the rotation group $\mathrm{L}^2(\mathrm{SO}(3))$, with inner product $\langle f, g \rangle = \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) f(\rho) g^*(\rho)$ for $f, g \in \mathrm{L}^2(\mathrm{SO}(3))$, where $\mathrm{d}\varrho(\rho) = \sin\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma$ is the usual invariant measure on $\mathrm{SO}(3)$, which is parameterised by the Euler angles $\rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3)$, with $\alpha \in [0, 2\pi)$, $\beta \in [0, \pi]$ and $\gamma \in [0, 2\pi)$. We adopt the zyz Euler convention corresponding to the rotation of a physical body in a *fixed* coordinate system about the z, y and z axes by γ, β and α, respectively.

The Wigner D-functions D^ℓ_{mn}, with natural $\ell \in \mathbb{N}$ and integer $m, n \in \mathbb{Z}$, $|m|, |n| \leq \ell$, are the matrix elements of the irreducible unitary representation of the rotation group $\mathrm{SO}(3)$. Consequently, the $D^{\ell*}_{mn}$ also form an orthogonal basis in $\mathrm{L}^2(\mathrm{SO}(3))$.[1] The orthogonality and completeness relations for the Wigner D-functions read, respectively, $\langle D^\ell_{mn}, D^{\ell'}_{m'n'} \rangle = 8\pi^2 \delta_{\ell\ell'} \delta_{mm'} \delta_{nn'}/(2\ell+1)$ and $\sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell \sum_{n=-\ell}^\ell D^\ell_{mn}(\alpha, \beta, \gamma) D^{\ell*}_{mn}(\alpha', \beta', \gamma') = \delta(\alpha - \alpha')\delta(\cos\beta - \cos\beta')\delta(\gamma - \gamma')$, where δ_{ij} is the Kronecker delta symbol and $\delta(x)$ is the Dirac delta function. The Wigner functions may be decomposed as [26]

$$D^\ell_{mn}(\alpha, \beta, \gamma) = \mathrm{e}^{-\mathrm{i}m\alpha} d^\ell_{mn}(\beta) \, \mathrm{e}^{-\mathrm{i}n\gamma}, \quad (1)$$

where the real polar d-functions can be computed by recursion (*e.g.* [27], [28]). The Wigner d-functions have the following Fourier series decomposition [29]:

$$d^\ell_{mn}(\beta) = \mathrm{i}^{n-m} \sum_{m'=-\ell}^\ell \Delta^\ell_{m'm} \Delta^\ell_{m'n} \, \mathrm{e}^{\mathrm{i}m'\beta}, \quad (2)$$

where $\Delta^\ell_{mn} \equiv d^\ell_{mn}(\pi/2)$. This expression follows from a factoring of rotations as highlighted by Risbo [27]. We also note that the Wigner functions are related to the spin spherical harmonics by [30]

$${}_sY_{\ell m}(\theta, \varphi) = (-1)^s \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell*}_{m,-s}(\varphi, \theta, 0). \quad (3)$$

A square integrable function defined on the rotation group may thus be represented by its Fourier expansion

$$f(\rho) = \sum_{\ell=0}^\infty \frac{2\ell+1}{8\pi^2} \sum_{m=-\ell}^\ell \sum_{n=-\ell}^\ell f^\ell_{mn} D^{\ell*}_{mn}(\rho), \quad (4)$$

where the Fourier coefficients are given by

$$f^\ell_{mn} = \langle f, D^{\ell*}_{mn} \rangle = \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) f(\rho) D^\ell_{mn}(\rho). \quad (5)$$

The Fourier transform on the rotation group $\mathrm{SO}(3)$ is often called the Wigner transform, while the Fourier coefficients f^ℓ_{mn} are often called Wigner coefficients.

[1]We adopt the conjugate D-functions as basis elements since this convention simplifies connections to wavelet transforms on the sphere.

III. SAMPLING THEOREM

We develop a novel sampling theorem on the rotation group based on an extension of the rotation group to the three-torus, extending similar approaches on the sphere [10]–[12] (following closely the approach of our sampling theorem on the sphere [10]).[2] Band-limited signals $f \in \mathrm{L}^2(\mathrm{SO}(3))$ are considered, with $f^\ell_{mn} = 0$ for all $\ell \geq L$, $m \geq M$ and $n \geq N$ (with $M, N \leq L$). Our sampling theorem is encapsulated in an exact computation of the Wigner transform of f from a finite set of samples.

We adopt an equiangular sampling of the rotation group with sample positions given by

$$\alpha_a = \frac{2\pi a}{2M-1}, \quad \text{where } a \in \{0, 1, \ldots, 2M-2\}, \quad (6)$$

$$\beta_b = \frac{\pi(2b+1)}{2L-1}, \quad \text{where } b \in \{0, 1, \ldots, L-1\}, \quad (7)$$

and

$$\gamma_g = \frac{2\pi g}{2N-1}, \quad \text{where } g \in \{0, 1, \ldots, 2N-2\}. \quad (8)$$

To make the connection with the three-torus, the β domain is extended to $[0, 2\pi)$ by simply extending the domain of the b index to include $\{L, L+1, \ldots, 2L-1\}$. The number of required samples is thus $[(L-1)(2M-1)+1](2N-1) \sim 4LMN$, or $\sim 4L^3$ samples when $L = M = N$.

By noting the Wigner decomposition of Eqn. (1) and the Fourier series representation of Eqn. (2), the forward Wigner transform of Eqn. (5) may be written

$$f^\ell_{mn} = \mathrm{i}^{m-n} \sum_{m'=-(L-1)}^{L-1} \Delta^\ell_{m'm} \Delta^\ell_{m'n} G_{mnm'}, \quad (9)$$

where

$$G_{mnm'} = \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) f(\rho) \, \mathrm{e}^{-\mathrm{i}(m\alpha + m'\beta + n\gamma)} \quad (10)$$

$$= \int_0^\pi \mathrm{d}\beta \sin\beta \, G_{mn}(\beta) \, \mathrm{e}^{-\mathrm{i}m'\beta}, \quad (11)$$

and

$$G_{mn}(\beta) = \int_0^{2\pi} \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\gamma f(\rho) \mathrm{e}^{-\mathrm{i}(m\alpha + n\gamma)} \quad (12)$$

$$= (2\pi)^2 \sum_{a=-(M-1)}^{M-1} \sum_{g=-(N-1)}^{N-1} f(\alpha_a, \beta, \gamma_g)$$

$$\times \mathrm{e}^{-\mathrm{i}(m\alpha_a + n\gamma_g)} / [(2M-1)(2N-1)]. \quad (13)$$

Since Wigner coefficients are not defined for $|m|, |n| > \ell$, we set them to zero to enforce the contraint $|m|, |n| \leq \ell$ when the order of summations and integrals are interchanged. The final expression of Eqn. (13) follows by appealing to the discrete and continuous orthogonality of the complex exponentials.

[2]Gauss-Legendre quadrature may also be used to define an efficient sampling theorem on the rotation group by a similar extension of the approach outlined in [10] and is developed in [31], where corresponding fast Wigner transforms are constructed by a separation of variables. Although the asymptotic number of samples is $4L^3$ in both cases, the approach described in this article requires fewer samples than Gauss-Legendre quadrature, which for small band-limits can be significant.

To recover a sampling theorem on the rotation group it remains to develop an exact quadrature to compute Eqn. (11). This is achieved by extending $G_{mn}(\beta)$ to the domain $[0, 2\pi)$ through the construction

$$\tilde{G}_{mn}(\beta) = \begin{cases} G_{mn}(\beta), & \beta \in [0, \pi] \\ (-1)^{m+n} G_{mn}(2\pi - \beta), & \beta \in (\pi, 2\pi) \end{cases} \quad (14)$$

(*cf.* [10], [12]). $\tilde{G}_{mn}(\beta)$ may then be represented by its Fourier decomposition

$$\tilde{G}_{mn}(\beta) = (2\pi)^2 \sum_{m'=-(L-1)}^{L-1} \tilde{G}_{mnm'} \, \mathrm{e}^{\mathrm{i}m'\beta_b} . \quad (15)$$

Substituting Eqn. (15) into Eqn. (11), one recovers the exact quadrature

$$G_{mnm'} = (2\pi)^2 \sum_{m''=-(L-1)}^{L-1} \tilde{G}_{mnm''} \, w(m'' - m'), \quad (16)$$

where the weights are given by $w(m') = \int_0^\pi \mathrm{d}\beta \sin \beta \, \mathrm{e}^{\mathrm{i}m'\beta}$, which can be evaluated analytically [10].

The inverse Wigner transform may be computed in an analogous manner, again noting Eqn. (1) and Eqn. (2). The Fourier coefficients of the extension of f to the three-torus are first computed by

$$F_{mnm'} = \mathrm{i}^{n-m} \sum_{\ell=0}^{L-1} \frac{2\ell+1}{8\pi^2} \, \Delta_{m'm}^\ell \, \Delta_{m'n}^\ell \, f_{mn}^\ell , \quad (17)$$

followed by the three-dimensional Fourier transform

$$f(\alpha_a, \beta_b, \gamma_g) = \sum_{m=-(M-1)}^{M-1} \sum_{n=-(N-1)}^{N-1} \sum_{m'=-(L-1)}^{L-1} F_{mnm'}$$
$$\times \mathrm{e}^{\mathrm{i}(m\alpha_a + m'\beta_b + n\gamma_g)} . \quad (18)$$

The samples of f computed over $\beta \in (\pi, 2\pi)$ are discarded.

IV. EXACT QUADRATURE

Our sampling theorem on the rotation group can be used to define an explicit quadrature rule for the integration of a band-limited function. Approximately a quarter of the number of samples of the sampling theorem are required for the quadrature rule since integrating a band-limited signal corresponds to computing f_{00}^0 only (aliasing in higher order coefficients can be tolerated). The exact quadrature reads:

$$I = \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) \, f(\rho) \quad (19)$$

$$= \sum_{a=0}^{M-1} \sum_{b=0}^{L-1} \sum_{g=0}^{N-1} f(\alpha_a', \beta_b, \gamma_g') \, q(\beta_b), \quad (20)$$

where $\alpha_a' = 2\pi a/M$ and $\beta_b' = 2\pi b/N$ (noting the reduced domain of a and b). The number of samples required is $[(L-1)M+1]N \sim LMN$, or $\sim L^3$ samples when $L = M = N$. The quadrature weights are defined by

$$q(\beta_b) = \frac{(2\pi)^2}{MN} \Big[v(\beta_b) + (1 - \delta_{b,L-1}) \, v(\beta_{2L-2-b}) \Big], \quad (21)$$

where [10]

$$v(\beta_b) = \frac{1}{2L-1} \sum_{m'=-(L-1)}^{L-1} w(-m') \, \mathrm{e}^{\mathrm{i}m'\beta_b} . \quad (22)$$

V. FAST ALGORITHMS

Fast algorithms to compute the Wigner transforms associated with our sampling theorem can be implemented through a separation of variables, as described in Sec. III, using FFTs throughout to compute Fourier transforms efficiently. Furthermore, Eqn. (16) can be computed more efficiently in Fourier space, allowing aliasing in terms $|m'| > L - 1$ so that only L samples are needed in β (as described in [10]). Rather than implement this algorithm from scratch, however, we instead make the connection to spin spherical harmonic transforms to leverage our existing SSHT[3] code [10].

The forward Wigner transform may be expressed as

$$f_n(\alpha, \beta) = \frac{2\pi}{2N-1} \sum_{g=-(N-1)}^{N-1} f(\alpha, \beta, \gamma_g) \, \mathrm{e}^{-\mathrm{i}n\gamma_g}, \quad (23)$$

where

$$f_{mn}^\ell = (-1)^n \sqrt{\frac{4\pi}{2\ell+1}} \int_0^\pi \mathrm{d}\beta \sin \beta \int_0^{2\pi} \mathrm{d}\alpha$$
$$\times f_n(\alpha, \beta) \, {}_{-n}Y_{\ell m}^*(\beta, \alpha) . \quad (24)$$

Eqn. (24) is simply a spin spherical harmonic transform with spin number $-n$, which may be computed for each n with asymptotic complexity $\mathcal{O}(L^3)$ by SSHT. The Fourier transform of Eqn. (23) can be computed by an FFT in $\mathcal{O}(L^2 N \log_2 N)$. The forward Wigner transform can thus be computed with overall complexity $\mathcal{O}(NL^3)$.

The inverse Wigner transform may be expressed as

$$f_n(\alpha, \beta) = (-1)^n \sum_{\ell=0}^{L-1} \sum_{m=-\min(\ell,M-1)}^{\min(\ell,M-1)} \sqrt{\frac{2\ell+1}{16\pi^3}}$$
$$\times f_{mn}^\ell \, {}_{-n}Y_{\ell m}(\beta, \alpha), \quad (25)$$

where

$$f(\alpha, \beta, \gamma_g) = \sum_{n=-(N-1)}^{N-1} f_n(\alpha, \beta) \, \mathrm{e}^{\mathrm{i}n\gamma_g} . \quad (26)$$

Eqn. (26) is simply an inverse spin spherical harmonic transform with spin number $-n$, which may be computed for each n with asymptotic complexity $\mathcal{O}(L^3)$ by SSHT. The Fourier transform of Eqn. (26) can be computed by an FFT in $\mathcal{O}(L^2 N \log_2 N)$. The inverse Wigner transform can thus be computed with overall complexity $\mathcal{O}(NL^3)$.

For real functions on the rotation group we exploit the conjugate symmetry relation $f_{mn}^{\ell*} = (-1)^{m+n} f_{(-m)(-n)}^\ell$ to reduce memory requirements and computational time by a factor of two.

The fast Wigner transforms implementing our novel sampling theorem on the rotation group are implemented in our new SO3[4] code, which uses SSHT to compute spin spherical harmonics transforms and FFTW[5] to compute Fourier trans-
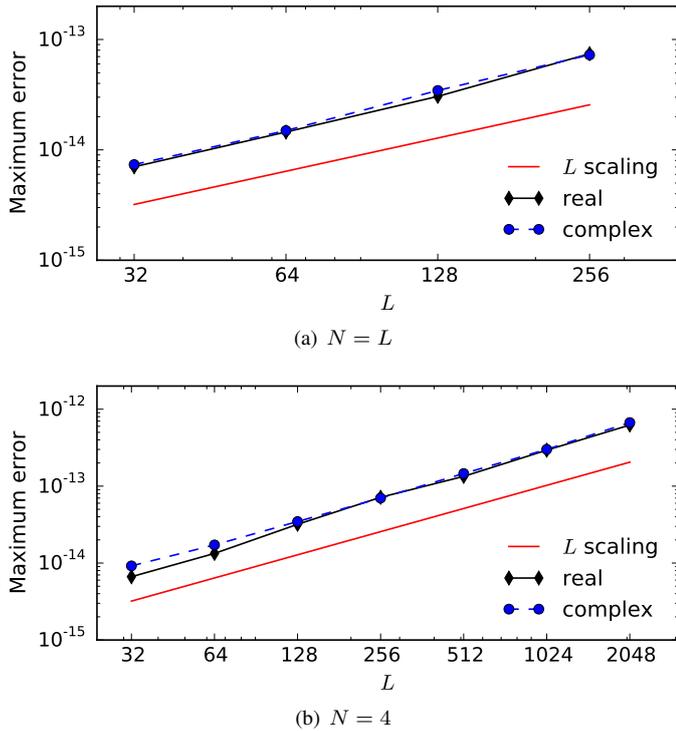
(a) $N = L$

(b) $N = 4$

Fig. 1. Numerical accuracy of the algorithms implementing our sampling theorem for real and complex signals for $N = L$ (panel a) and $N = 4$ (panel b). Accuracy is found empirically to scale approximately linearly with increasing band-limit.
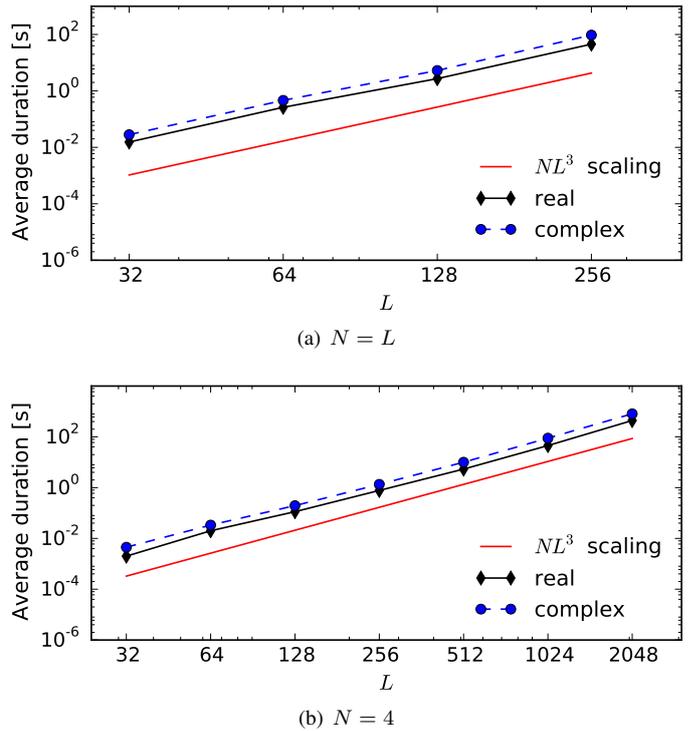

(a) $N = L$

(b) $N = 4$

Fig. 2. Computation time of the algorithms implementing our sampling theorem for real and complex signals for $N = L$ (panel a) and $N = 4$ (panel b). As predicted, computation time scales as $\mathcal{O}(NL^3)$ and real transforms are approximately twice as fast as complex transforms (since additional symmetries are exploited).

forms. The core code of SO3 is written in C, while Matlab interfaces are also exposed.

VI. EVALUATION

In order to assess the numerical accuracy and computation time of our algorithms we perform the following numerical experiment. We generate band-limited test signals defined by uniformly random Wigner coefficients with real and imaginary parts distributed in the interval $[-1, 1]$. An inverse transform is performed to synthesise the test signal on the rotation group from its Wigner coefficients, followed by a forward transform to recompute Wigner coefficients. Numerical accuracy is measured by the maximum absolute error between the original Wigner coefficients and the recomputed values. Computation time is measured by the average of the times taken to perform the inverse and forward transform.

All numerical experiments are performed on a single core of a 2.67 GHz Intel(R) Xeon(R) CPU X5650 machine with 100 GB of RAM and are averaged over ten random test signals. We consider two modes of operation: (i) $N = L$ and (ii) a small constant N, in this example $N = 4$. The latter mode of operation is a common use-case, for example when computing directional wavelet transforms on the sphere for wavelets with low azimuthal band-limit, and allows one to consider very large harmonic band-limits L. Note that the results presented here are obtained using version 1.1b1 of SSHT, which contains many computational optimisations compared to previous versions. The Risbo recursion [27] for computing Wigner d-functions is adopted.

The maximum absolute error is plotted against the band-limit in Fig. 1. High numerical accuracy is achieved, with errors on the order of machine precision and found empirically to increase approximately linearly with band-limit.

The computation time for both real and complex signals on the rotation group is plotted against the band-limit in Fig. 2. Computation time evolves as $\mathcal{O}(NL^3)$ as predicted. Computation time for real signals is approximately twice as fast as for complex signals, also as predicted.

VII. CONCLUSIONS

We have presented a new sampling theorem on the rotation group SO(3) and fast algorithms to compute the associated Wigner transform. Our sampling theorem requires $4L^3$ samples to capture the full information content of a signal band-limited at L, reducing the number of required samples by a factor of two compared to other equiangular sampling theorems on the rotation group [25]. Our fast algorithms are theoretically exact and achieve accuracy close to machine precision, while scaling as $\mathcal{O}(L^4)$, compared to the naive scaling of $\mathcal{O}(L^6)$. For the common case of a low directional band-limit N the scaling is reduced to $\mathcal{O}(NL^3)$. In a separate article [23] we apply these fast algorithms to improve the efficiency of a directional wavelet transform on the sphere [21], [22] in order to support the analysis of large data-sets.

REFERENCES

[1] C. E. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 37, pp. 10–21, 1949.

[2] J. W. Cooley and J. W. Tukey, "An algorithm for the machine calculation of complex fourier series," *Math. Comp.*, vol. 19, pp. 297–301, 1965.

[3] Planck Collaboration I, "*Planck* 2013 results. I. Overview of products and scientific results," *Astron. & Astrophys.*, vol. 571, no. A1, 2014.

[4] C. P. Ahn, R. Alexandroff, C. Allende Prieto, S. F. Anderson, T. Anderton, B. H. Andrews, É. Aubourg, S. Bailey, E. Balbinot, R. Barnes, and et al., "The Ninth Data Release of the Sloan Digital Sky Survey: First Spectroscopic Data from the SDSS-III Baryon Oscillation Spectroscopic Survey," *Astrophys. J. Supp.*, vol. 203, p. 21, Dec. 2012.

[5] P. Audet, "Directional wavelet analysis on the sphere: Application to gravity and topography of the terrestrial planets," *J. Geophys. Res.*, vol. 116, no. E1, 2011.

[6] ——, "Toward mapping the effective elastic thickness of planetary lithospheres from a spherical wavelet analysis of gravity and topography," *Phys. Earth Planet In.*, vol. 226, no. 0, pp. 48–82, 2014.

[7] F. J. Simons, F. A. Dahlen, and M. A. Wieczorek, "Spatiospectral concentration on a sphere," *SIAM Rev.*, vol. 48, no. 3, pp. 504–536, 2006.

[8] S. Swenson and J. Wahr, "Methods for inferring regional surface-mass anomalies from GRACE measurements of time-variable gravity," *J. Geophys. Res.*, vol. 107, pp. 2193–278, 2002.

[9] J. R. Driscoll and D. M. J. Healy, "Computing Fourier transforms and convolutions on the sphere," *Adv. Appl. Math.*, vol. 15, pp. 202–250, 1994.

[10] J. D. McEwen and Y. Wiaux, "A novel sampling theorem on the sphere," *IEEE Trans. Sig. Proc.*, vol. 59, no. 12, pp. 5876–5887, 2011.

[11] J. D. McEwen, "Fast, exact (but unstable) spin spherical harmonic transforms," *All Res. J. Phys.*, vol. 1, no. 1, 2011.

[12] K. M. Huffenberger and B. D. Wandelt, "Fast and exact spin-s spherical harmonic transforms," *Astrophys. J. Supp.*, vol. 189, pp. 255–260, 2010.

[13] Z. Khalid, R. A. Kennedy, and J. D. McEwen, "An optimal-dimensionality sampling scheme on the sphere with fast spherical harmonic transforms," *IEEE Trans. Sig. Proc.*, vol. 62, no. 17, pp. 4597–4610, 2014.

[14] B. Leistedt and J. D. McEwen, "Exact wavelets on the ball," *IEEE Trans. Sig. Proc.*, vol. 60, no. 12, pp. 6257–6269, 2012.

[15] J.-P. Antoine and P. Vandergheynst, "Wavelets on the n-sphere and related manifolds," *J. Math. Phys.*, vol. 39, no. 8, pp. 3987–4008, 1998.

[16] Y. Wiaux, L. Jacques, and P. Vandergheynst, "Correspondence principle between spherical and Euclidean wavelets," *Astrophys. J.*, vol. 632, pp. 15–28, 2005.

[17] Y. Wiaux, L. Jacques, P. Vielva, and P. Vandergheynst, "Fast directional correlation on the sphere with steerable filters," *Astrophys. J.*, vol. 652, pp. 820–832, 2006.

[18] Y. Wiaux, L. Jacques, and P. Vandergheynst, "Fast spin ± 2 spherical harmonics transforms," *J. Comput. Phys.*, vol. 226, p. 2359, 2005.

[19] J. D. McEwen, M. P. Hobson, D. J. Mortlock, and A. N. Lasenby, "Fast directional continuous spherical wavelet transform algorithms," *IEEE Trans. Sig. Proc.*, vol. 55, no. 2, pp. 520–529, 2007.

[20] J. D. McEwen, M. P. Hobson, and A. N. Lasenby, "A directional continuous wavelet transform on the sphere," *ArXiv*, 2006.

[21] Y. Wiaux, J. D. McEwen, P. Vandergheynst, and O. Blanc, "Exact reconstruction with directional wavelets on the sphere," *Mon. Not. Roy. Astron. Soc.*, vol. 388, no. 2, pp. 770–788, 2008.

[22] J. D. McEwen, P. Vandergheynst, and Y. Wiaux, "On the computation of directional scale-discretized wavelet transforms on the sphere," in *SPIE Wavelets and Sparsity XV*, 2013.

[23] J. D. McEwen, B. Leistedt, M. Büttner, H. V. Peiris, and Y. Wiaux, "Directional spin wavelets on the sphere," *IEEE Trans. Sig. Proc.*, submitted, 2015.

[24] J. García, J. Valles, and C. Ferreira, "Detection of three-dimensional objects under arbitrary rotations based on range images," *Opt. Express*, vol. 11, no. 25, pp. 3352–3358, Dec 2003.

[25] P. Kostelec and D. Rockmore, "FFTs on the rotation group," *J. Fourier Anal. and Appl.*, vol. 14, pp. 145–179, 2008.

[26] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum theory of angular momentum*. Singapore: World Scientific, 1989.

[27] T. Risbo, "Fourier transform summation of Legendre series and *D*-functions," *J. Geodesy*, vol. 70, no. 7, pp. 383–396, 1996.

[28] S. Trapani and J. Navaza, "Calculation of spherical harmonics and Wigner *d* functions by FFT. Applications to fast rotational matching in molecular replacement and implementation into *AMoRe*," *Acta Crystallographica Section A*, vol. 62, no. 4, pp. 262–269, 2006.

[29] A. F. Nikiforov and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*. Berlin: Springer-Verlag, 1991.

[30] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudarshan, "Spin-s spherical harmonics and ð," *J. Math. Phys.*, vol. 8, no. 11, pp. 2155–2161, 1967.

[31] Z. Khalid, S. Durrani, R. A. Kennedy, Y. Wiaux, and J. D. McEwen, "Gauss-Legendre sampling on the rotation group," *IEEE Sig. Proc. Let.*, submitted, 2015.