



Heriot-Watt University
Research Gateway

Numerical analysis of a time-stepping method for the Westervelt equation with time-fractional damping

Citation for published version:

Baker, K, Banjai, L & Ptashnyk, M 2022 'Numerical analysis of a time-stepping method for the Westervelt equation with time-fractional damping' arXiv.

Link:

[Link to publication record in Heriot-Watt Research Portal](#)

Document Version:

Early version, also known as pre-print

General rights

Copyright for the publications made accessible via Heriot-Watt Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

Heriot-Watt University has made every reasonable effort to ensure that the content in Heriot-Watt Research Portal complies with UK legislation. If you believe that the public display of this file breaches copyright please contact open.access@hw.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

NUMERICAL ANALYSIS OF A TIME-STEPPING METHOD FOR THE WESTERVELT EQUATION WITH TIME-FRACTIONAL DAMPING

KATHERINE BAKER, LEHEL BANJAI, AND MARIYA PTASHNYK

ABSTRACT. We develop a numerical method for the Westervelt equation, an important equation in nonlinear acoustics, in the form where the attenuation is represented by a class of non-local in time operators. A semi-discretisation in time based on the trapezoidal rule and A-stable convolution quadrature is stated and analysed. Existence and regularity analysis of the continuous equations informs the stability and error analysis of the semi-discrete system. The error analysis includes the consideration of the singularity at $t = 0$ which is addressed by the use of a correction in the numerical scheme. Extensive numerical experiments confirm the theory.

1. INTRODUCTION

We consider the attenuated Westervelt equation modelling wave propagation through lossy media in cases where the wave propagation is poorly approximated by linear wave models. A typical application is in medical ultrasound, where the attenuation depends on a fractional power of the frequency and the fractional power determined by the type of tissue; see [29, Chapter 4]. This leads to models of the form

$$\partial_t^2 u - \Delta u + aLu = k\partial_t^2(u^2),$$

where a, k are positive constants and the attenuation is represented by a nonlocal differential operator L . In this paper we consider $Lv(t) = -\int_0^t \beta(t-s)\partial_t \Delta v(s)ds$ with β chosen as either

$$\beta_A(t) := \frac{1}{\Gamma(\mu)} t^{\mu-1} e^{-rt}, \quad \mu \in (0, 1), r \geq 0$$

or

$$\beta_B(t) := -\dot{e}_\mu(t), \quad e_\mu(t) := E_{\mu,1}(-t^\mu),$$

where, see [21], $E_{\mu,\gamma}$ is the Mittag-Leffler function

$$(1.1) \quad E_{\mu,\gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \gamma)}.$$

Note that, for $\beta = \beta_A$ and $r = 0$, $L = -\partial_t^{1-\mu} \Delta$, where $\partial_t^{1-\mu}$ is the Caputo fractional derivative of order $1-\mu$. The value of μ depends on the tissue [9, Chapter 4.3] and is used to model the frequency dependence of attenuation [2, Chapter 3]. See also the recent [15], which includes other choices of nonlocal attenuation

2020 *Mathematics Subject Classification.* 35L77, 65M06, 65M15, 35R11.

Key words and phrases. fractional time derivatives, nonlinear wave equations, time-discretisation, convolution quadrature.

operators L . The case $\beta = \beta_B$ is of interest in modelling viscoelastic materials [16, 25].

In this work we develop and analyse a numerical method for the time-discretisation of the attenuated Westervelt equation stated above. The time-discretisation of the non-local operator is done by convolution quadrature [17, 18] whose ability to translate a positivity property of the continuous operator to the discrete case allows a full stability and convergence analysis. It further allows for fast and memory efficient implementation [4] that is not addressed further in this paper. The full time discretisation is a variation of the discretisation used in [3] for a related linear model and is based on the trapezoidal (Newmark with $\gamma = 1/2$, $\beta = 1/4$) scheme.

There are several results on the well-posedness and regularity for quasilinear wave equations and for the Westervelt equations, see e.g. [8, 14, 23]. In [16, 27], semigroup techniques and the Galerkin method are used to prove well-posedness results for linear integro-differential equations modeling dynamics of fractional order viscoelasticity. For equations with fractional integrals the semigroup techniques can be applied in the same way as in the case of equations of linear viscoelasticity [10, 11]. However similar approach cannot be used to prove existence results for equations with nonlocal *differential* operators, which include fractional time derivative as a special case, considered in this work. The Galerkin method, together with the fixed point argument, is applied in [15] for the well-posedness analysis of fractional Westervelt equations. Galerkin approximation, together with the energy estimates, is also used in [25] to prove existence of weak solutions to the fractional Zener wave equations for heterogeneous viscoelastic materials. In the proof of existence and uniqueness results for the nonlocally attenuated Westervelt equation considered here we follow similar ideas as in [15], and hence include only the main steps of the proof.

The literature on the numerical methods for the case of local strong damping, i.e., $L = -\Delta\partial_t$ includes the semi-discretisation by continuous [24] and discontinuous [1] Galerkin finite element methods. Let us also mention the recent approach via semigroups to the analysis of the spatial discretisation of a large class of quasilinear wave equations [13]. Analysis of a fully discrete scheme for nonlinear elastic waves with the finite element method in space and rational approximation in time is presented in [22].

In the linear case ($k = 0$), the literature also includes the numerical analysis of full discretisations of non-local attenuations. Namely a weaker form of non-local attenuation than we are interested in ($Lv(t) = -\int_0^t \beta(t-s)\Delta v(s)ds$) is investigated in [16] where a continuous Galerkin semi-discretisation is analysed. In the already mentioned work [3] a fully discrete scheme is investigated with again weaker attenuation $L = \partial_t^\gamma$, $\gamma \in (0, 1)$. The fully discrete scheme in [3] consists of continuous Galerkin method in space and leapfrog combined with convolution quadrature in time. This was extended to the strongly damped nonlocal case (still with $k = 0$) in the thesis [2, Chapter 6] with the explicit leapfrog scheme replaced by the implicit trapezoidal time-stepping. This numerical approach we now extend to the non-linear case to give what we believe to be the first analysis of a time-discretisation of the nonlocally attenuated Westervelt equation. The analysis also includes realistic assumptions on the regularity of the solution including the possible lack of smoothness at $t = 0$.

The paper consists of six sections, the first being this introduction. In the next section we give the formulation of the mathematical model and prove an important property of L . Section 3 then briefly gives the well-posedness of the nonlinear system with some of the technical details of the proof relegated to the appendix. In Section 4 we state the numerical scheme and show its stability. This leads to the proof of convergence estimates in Section 5. Finally, the results are illustrated by numerical experiments in one and two spatial dimensions in Section 6.

2. FORMULATION OF MATHEMATICAL MODEL

We start with the formulation of the mathematical model. In the damping term we shall consider a class of convolution kernels which includes fractional time derivative as a special case.

Let $\Omega \subset \mathbb{R}^d$ be a smooth (C^2) domain, where $d \leq 3$. We consider

$$(2.1) \quad \partial_t^2 u - \Delta u - a\beta * \Delta \partial_t u = k\partial_t^2(u^2),$$

where $a, k > 0$ are constants,

$$f * g(t) := \int_0^t f(t-\tau)g(\tau)d\tau$$

denotes the one sided convolution, and β is chosen as either

$$(2.2) \quad \beta_A(t) := \frac{1}{\Gamma(\mu)} t^{\mu-1} e^{-rt}, \quad r \geq 0, \mu \in (0, 1),$$

or

$$(2.3) \quad \beta_B(t) := -\dot{e}_\mu(t), \quad e_\mu(t) = E_{\mu,1}(-t^\mu), \quad \mu \in (0, 1),$$

and $E_{\mu,\gamma}$ being the Mittag-Leffler function (1.1). In both cases

$$\beta(t) \sim \frac{1}{\Gamma(\mu)} t^{\mu-1} \quad \text{as } t \rightarrow 0^+.$$

When a result holds for both kernels, we will use β to denote either of the kernels. Note that for β_A and $r = 0$,

$$\beta_A * f = I_t^\mu f \quad \text{and} \quad \beta_A * \partial_t f = \partial_t^{1-\mu} f,$$

holds, where I_t^μ denotes the Riemann-Liouville fractional integral of order $\mu \in (0, 1)$ and $\partial_t^{1-\mu}$ the Caputo derivative of order $1 - \mu \in (0, 1)$ [26].

We will need two properties of β . First of all, denoting by $\hat{\beta} := \mathcal{L}\{\beta\}$ the Laplace transform of β we have that

$$(2.4) \quad \hat{\beta}_A(z) = (z+r)^{-\mu}, \quad \hat{\beta}_B(z) = \frac{1}{z^\mu + 1}.$$

The expression for $\hat{\beta}_B(z)$ is obtained from the fact that the Laplace transform of e_μ is given by $\frac{z^{\mu-1}}{z^\mu + 1}$, see [21], and the calculation

$$\mathcal{L}\{-\dot{e}_\mu\}(z) = -\left(\frac{z^\mu}{z^\mu + 1} - 1\right) = \frac{1}{z^\mu + 1},$$

where we used that $e_\mu(0) = 1$.

Thus, a property we will require later, follows:

$$(2.5) \quad \operatorname{Re} \frac{1}{\hat{\beta}_A(z)} \geq (\sigma+r)^\mu, \quad \operatorname{Re} \frac{1}{\hat{\beta}_B(z)} \geq 1 \quad \forall \operatorname{Re} z \geq \sigma > 0.$$

The second property is stated as a lemma.

Lemma 2.1. For any $v \in L^2(0, T)$ we have

$$\int_0^t [\beta * v](s) v(s) ds \geq \frac{1}{2} \min_{s \in [0, t]} (\gamma(t-s) + \gamma(s)) \int_0^t |\beta * v(s)|^2 ds, \quad t \in (0, T),$$

where for $\beta = \beta_A$

$$\gamma(t) = \frac{1}{\Gamma(1-\mu)} e^{-rt} t^{-\mu} + \frac{r}{\Gamma(1-\mu)} \int_0^t \tau^{-\mu} e^{-r\tau} d\tau$$

and for $\beta = \beta_B$

$$\gamma(t) = \frac{1}{\Gamma(1-\mu)} t^{-\mu} + 1.$$

Proof. Note that γ is chosen so that $\frac{1}{z} = \hat{\gamma}(z) \hat{\beta}(z)$ and hence

$$(2.6) \quad \int_0^t v(s) ds = \int_0^t \gamma(t-\tau) \int_0^\tau \beta(\tau-\eta) v(\eta) d\eta d\tau,$$

for any sufficiently smooth v .

Denoting $w = \beta * v$, we have, by differentiating (2.6), multiplying by w and integrating, that

$$\int_0^t w(s) v(s) ds = \int_0^t w(s) \frac{d}{ds} \int_0^s \gamma(s-\tau) w(\tau) d\tau ds.$$

We complete the proof by noticing that γ satisfies the conditions of the kernel k in [25, Lemma 3.1] with the lemma thus implying

$$\begin{aligned} \int_0^t \frac{d}{ds} \left[\int_0^s \gamma(s-\tau) w(\tau) d\tau \right] w(s) ds &\geq \frac{1}{2} \int_0^t [\gamma(t-s) + \gamma(s)] |w(s)|^2 ds \\ &\geq \frac{1}{2} \min_{s \in [0, t]} [\gamma(t-s) + \gamma(s)] \int_0^t |w(s)|^2 ds. \end{aligned}$$

Finally note that since $\beta \in L^1(0, T)$, Young's inequality for convolutions implies that both sides of the above inequality are well-defined for $v \in L^2(0, T)$ thus completing the proof. \square

Remark 2.2. An application of Plancherel's formula as in [5, Lemma 2.2], shows that from (2.5) it follows that

$$\int_0^\infty e^{-2\sigma s} [\beta * v](s) v(s) ds \geq C(\sigma) \int_0^\infty e^{-2\sigma s} |\beta * v(s)|^2 ds$$

for $\sigma > 0$ and $C(\sigma) = (\sigma + r)^\mu$ for $\beta = \beta_A$ and $C(\sigma) = 1$ for $\beta = \beta_B$. While it would be possible to develop the theory in the next section based on this inequality, it is easier to use Lemma 2.1.

We rewrite equation (2.1) as

$$(2.7) \quad \begin{aligned} (1 - 2ku) \partial_t^2 u - \Delta u - a\beta * \partial_t \Delta u &= 2k(\partial_t u)^2 && \text{in } (0, T) \times \Omega, \\ u &= u_D && \text{on } (0, T) \times \partial\Omega, \\ u(0) = u_0, \quad \partial_t u(0) &= v_0 && \text{in } \Omega \end{aligned}$$

and make the following assumptions on the smoothness of the data:

$$(2.8) \quad \begin{aligned} u_0 - u_D(0) &\in H^3(\Omega) \cap H_0^1(\Omega), \quad v_0 \in H^2(\Omega), \\ u_D &\in H^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^3(\Omega)). \end{aligned}$$

For the simplification of the presentation, we shall consider $u_D = 0$, however all results hold for non-zero Dirichlet boundary conditions by considering $\hat{u} = u - u_D$, resulting in

$$(2.9) \quad \begin{aligned} ((1 - 2ku_D) - 2k\hat{u})\partial_t^2\hat{u} - \Delta\hat{u} - a\beta * \partial_t\Delta\hat{u} &= 2k(\partial_t\hat{u})^2 + f(t, x) \\ &+ 4k\partial_t\hat{u}\partial_t u_D + 2k\hat{u}\partial_t^2 u_D, \end{aligned}$$

where $f(t, x) = 2k(\partial_t u_D)^2 + (2ku_D - 1)\partial_t^2 u_D + \Delta u_D + a\beta * \partial_t \Delta u_D$. For u_D independent of t , the difference between (2.7) and (2.9) is in the presence of function $f(t, x)$, which is regular for regular u_D and the analysis below holds for all sufficiently regular u_D with $\|u_D\|_{L^\infty(\Omega)} < 1/(2k)$. In case u_D depends on t we obtain additional linear terms, which can be treated in the same way as in the case $u_D = 0$.

3. EXISTENCE AND UNIQUENESS RESULTS

We shall apply the Banach fixed-point theorem and the Galerkin method to show existence and uniqueness of solutions of (2.7). Similar approach was considered in [15], however for completeness we present here the short outline of the main ideas. Also we have a more general convolution kernel, compared to the one considered in [15].

In the proofs below we shall use that for $w \in H_0^1(\Omega) \cap H^2(\Omega)$, and $\dim(\Omega) \leq 3$, we have the following estimate

$$(3.1) \quad \|w\|_{H^2(\Omega)} \leq C_\Omega \|\Delta w\|_{L^2(\Omega)},$$

for some positive constant C_Ω , depending on the domain Ω , see e.g. [12, Theorem 3.1.2.1, Remark 3.1.2.2], and the Sobolev embeddings

$$(3.2) \quad \begin{aligned} \|w\|_{L^\infty(\Omega)} &\leq C_\Omega \|w\|_{H^2(\Omega)} \leq C_\Omega \|\Delta w\|_{L^2(\Omega)}, \\ \|\nabla w\|_{L^4(\Omega)} &\leq C_\Omega \|w\|_{H^2(\Omega)} \leq C_\Omega \|\Delta w\|_{L^2(\Omega)}, \end{aligned}$$

where by C_Ω we denote the generic constant in the embedding inequalities.

By C we shall denote a generic constant that is allowed to change from line to line. For shortness of notation we denote $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k(\Omega)}$ by $\|\cdot\|_{H^k}$, with $2 \leq p \leq \infty$ and $k = 1, 2, 3$, and the L^2 -inner product is denoted by $\langle \cdot, \cdot \rangle$.

Consider

$$\mathcal{K} = \left\{ \begin{aligned} u &\in L^\infty(0, T; H^2(\Omega)) \cap W^{1, \infty}(0, T; H_0^1(\Omega)) : \\ u &\in L^\infty(0, T; H^3(\Omega)), \partial_t u \in L^\infty(0, T; H^2(\Omega)), \\ \|\Delta u\|_{L^\infty(0, T; L^2(\Omega))} &\leq r, \quad \|u\|_{L^\infty(0, T; H^3(\Omega))}^2 + \kappa \|\partial_t u\|_{L^\infty(0, T; H^2(\Omega))}^2 \leq R^2 \end{aligned} \right\},$$

for some fixed $0 < C_\Omega r \leq (1 - \kappa)/2k$, with $0 < \kappa < 1$ and C_Ω being the constant in the embedding inequality of $H^2(\Omega)$ in $L^\infty(\Omega)$, and $R^2 = C[(1 + 2kC_\Omega r)\|v_0\|_{H^2}^2 + \|u_0\|_{H^3}^2]$ for some constant $C > 1$.

The map $\mathcal{T} : \tilde{u} \mapsto u = \mathcal{T}(\tilde{u})$, for $\tilde{u} \in \mathcal{K}$, is defined via the solution of the following linear problem

$$(3.3) \quad \begin{aligned} (1 - 2k\tilde{u})\partial_t^2 u - \Delta u - a\beta * \partial_t \Delta u &= 2k\partial_t \tilde{u} \partial_t u && \text{in } (0, T) \times \Omega, \\ u &= 0 && \text{on } (0, T) \times \partial\Omega, \\ u(0) = u_0, \quad \partial_t u(0) &= v_0 && \text{in } \Omega. \end{aligned}$$

First we show the existence of a unique solution of (3.3). Then by showing that the map \mathcal{T} , for some $T > 0$, is a contraction we obtain the existence of a unique solution of (2.7).

Theorem 3.1. For $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$, $v_0 \in H^2(\Omega)$ and $\tilde{u} \in \mathcal{K}$ there exists a unique solution $u \in L^\infty(0, T; H^3(\Omega))$ of (3.3), with $\partial_t u \in L^\infty(0, T; H^2(\Omega))$.

Proof. The existence of a unique solution of (3.3) can be shown using the Galerkin approximation

$$u^\ell(t, x) = \sum_{j=1}^{\ell} c_j^\ell(t) q_j(x),$$

where $\{q_j\}_{j \in \mathbb{N}}$ is the basis of eigenfunctions of $-\Delta$ on $H_0^1(\Omega)$ with eigenvalues $\{\lambda_j\}$. The coefficient vector $\mathbf{c}^\ell = (c_j^\ell)_{j=1}^\ell$ satisfies the following system of ODEs

$$(3.4) \quad \frac{d^2}{dt^2} \mathbf{c}^\ell + \Lambda(t) \mathbf{c}^\ell + a\Lambda(t)\beta * \frac{d}{dt} \mathbf{c}^\ell - A(t) \frac{d}{dt} \mathbf{c}^\ell = 0,$$

where

$$(\Lambda(t))_{ij} = \lambda_j \left\langle \frac{1}{1 - 2k\tilde{u}(\cdot, t)} q_j(\cdot), q_i(\cdot) \right\rangle, \quad (A(t))_{ij} = \left\langle \frac{2k\partial_t \tilde{u}(\cdot, t)}{1 - 2k\tilde{u}(\cdot, t)} q_j(\cdot), q_i(\cdot) \right\rangle.$$

Writing $\mathbf{v}^\ell = \frac{d^2}{dt^2} \mathbf{c}^\ell$ we have that

$$(3.5) \quad \mathbf{c}^\ell(t) = \int_0^t (t-s) \mathbf{v}^\ell(s) ds + t \frac{d}{dt} \mathbf{c}^\ell(0) + \mathbf{c}^\ell(0)$$

and

$$\frac{d}{dt} \mathbf{c}^\ell(t) = \int_0^t \mathbf{v}^\ell(s) ds + \frac{d}{dt} \mathbf{c}^\ell(0),$$

with the initial data $\mathbf{c}^\ell(0)$ and $\frac{d}{dt} \mathbf{c}^\ell(0)$ given by the orthogonal projections onto the basis $\{q_j\}$ of the initial data u_0 and v_0 respectively. Thus the original problem (3.4) is transformed to the Volterra integral equation

$$\mathbf{v}^\ell(t) + \Lambda(t) \int_0^t (t-s) \mathbf{v}^\ell(s) ds + a\Lambda(t)\beta_1 * \mathbf{v}^\ell(t) - A(t) \int_0^t \mathbf{v}^\ell(s) ds = g(t),$$

where β_1 is the inverse Laplace transform of

$$\hat{\beta}_1(z) := z^{-1} \hat{\beta}(z)$$

and

$$g(t) := -\Lambda(t) \mathbf{c}^\ell(0) - \frac{d}{dt} \mathbf{c}^\ell(0) - \left(t + a \int_0^t \beta(s) ds \right) \Lambda(t) \frac{d}{dt} \mathbf{c}^\ell(0).$$

Thus the Volterra integral equation can be written as

$$\mathbf{v}^\ell(t) + \int_0^t K(t, s) \mathbf{v}^\ell(s) ds = g(t)$$

with

$$K(t, s) = \Lambda(t)(t - s + a\beta_1(t - s)) - A(t).$$

From the behaviour of its Laplace transform, we know that β_1 is analytic for $t > 0$ with a singularity of the type t^μ at $t = 0$, thus the kernel $K(t, s)$ is continuous and so is the right-hand side g . The existence of unique continuous solution follows from [7, Theorem 2.1.7]. The $C^2[0, T]$ -solution \mathbf{c} of the original problem (3.4) is then obtained from \mathbf{v} and (3.5).

The existence of a solution of (3.3) is obtained by taking the limit as $k \rightarrow \infty$ in the Galerkin approximation and using a priori estimates, uniformly in k , similar to the ones in Lemma 3.2. \square

Lemma 3.2. For solution of (3.3) we have the following a priori estimates

$$(3.6) \quad \begin{aligned} & \|\Delta u\|_{L^\infty(0, T; L^2(\Omega))}^2 + \kappa \|\partial_t \nabla u\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq [\|\Delta u_0\|_{L^2(\Omega)}^2 + \xi \|\nabla v_0\|_{L^2(\Omega)}^2] \times \\ & \quad \times \exp \left\{ T \frac{C_\Omega}{\kappa} \left[\|\partial_t \tilde{u}\|_{L^\infty(\Omega_T)} + \|\partial_t \Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))} \right. \right. \\ & \quad \left. \left. + \frac{1}{\kappa} \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)} (\|\nabla \tilde{u}\|_{L^\infty(\Omega_T)} + \|\partial_t \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}) \right] \right\}, \\ & \|u\|_{L^\infty(0, T; H^3(\Omega))}^2 + \kappa \|\partial_t u\|_{L^\infty(0, T; H^2(\Omega))}^2 \leq C_\Omega [\|u_0\|_{H^3(\Omega)}^2 + \xi \|v_0\|_{H^2(\Omega)}^2] \times \\ & \quad \times \exp \left\{ T \frac{C_\Omega}{\kappa} \left[\|\partial_t \Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))} + \frac{1}{\kappa} [\|\Delta \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))} \right. \right. \\ & \quad \left. \left. + \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)} (1 + \|\Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}) \right] (1 + \|\partial_t \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}) \right. \\ & \quad \left. + \frac{1}{\kappa^2} \|\Delta \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}^2 (1 + \|\Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}^2) \right\}, \end{aligned}$$

together with

$$(3.7) \quad \begin{aligned} & \|\beta * \Delta \partial_t u\|_{L^2((0, T) \times \Omega)}^2 \leq [\|u_0\|_{H^2(\Omega)}^2 + \xi \|v_0\|_{H^1(\Omega)}^2] \left(T \frac{C_\Omega}{\kappa} \left[\frac{1}{\kappa} \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)} \right. \right. \\ & \quad \left. \left. + \|\partial_t \Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))} \right] \left[1 + \frac{1}{\kappa} \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)} \right] \exp \left\{ T \frac{C_\Omega}{\kappa} \left[\frac{1}{\kappa} \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)} \right. \right. \right. \\ & \quad \left. \left. \left. + \|\partial_t \Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))} \right] \left[1 + \frac{1}{\kappa} \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)} \right] \right\} + \frac{1}{a} \right), \end{aligned}$$

and

$$\begin{aligned} & \|\beta * \Delta \nabla \partial_t u\|_{L^2(\Omega_T)}^2 \leq [\|u_0\|_{H^3(\Omega)}^2 + \xi \|v_0\|_{H^2(\Omega)}^2] \left(T \frac{C_\Omega}{\kappa} \left[\|\partial_t \Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))} \right. \right. \\ & \quad \left. \left. + \frac{1}{\kappa} [\|\Delta \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))} + \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)} (1 + \|\Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))})] \right] [1 \right. \\ & \quad \left. + \|\partial_t \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}] + \frac{1}{\kappa^2} \|\Delta \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}^2 [1 + \|\Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}^2] \right) \times \\ & \quad \times \exp \left\{ T \frac{C_\Omega}{\kappa} \left[\|\partial_t \Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))} + \frac{1}{\kappa} [\|\Delta \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))} \right. \right. \\ & \quad \left. \left. + \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)} (1 + \|\Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}) \right] (1 + \|\partial_t \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}) \right. \\ & \quad \left. + \frac{1}{\kappa^2} \|\Delta \nabla \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}^2 (1 + \|\Delta \tilde{u}\|_{L^\infty(0, T; L^2(\Omega))}^2) \right\} + \frac{1}{a} \right), \end{aligned}$$

where $\xi = 1 + 2kC_\Omega r$, $\tilde{u} \in \mathcal{K}$, r and κ as in the definition of \mathcal{K} , $\Omega_T = (0, T) \times \Omega$, and the constant $C_\Omega > 0$ includes constants from the embedding inequalities and hence depend on the domain Ω .

Proof. Considering $\partial_t u$ as a test function in the weak formulation of (3.3) yields

$$\begin{aligned} & \kappa \|\partial_t u\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\nabla u\|_{L^\infty(0,T;L^2(\Omega))}^2 \\ & \leq [(1 + 2kC_\Omega r)\|v_0\|_{L^2(\Omega)}^2 + \|\nabla u_0\|_{L^2(\Omega)}^2] \exp\left\{T\frac{2k}{\kappa}\|\partial_t \tilde{u}\|_{L^\infty(\Omega_T)}\right\}, \end{aligned}$$

where $1 - 2k\|\tilde{u}\|_{L^\infty(0,T;L^\infty(\Omega))} \geq 1 - 2kC_\Omega r \geq \kappa$ and we used Lemma 2.1 in the simpler form $\int_0^t \beta * \partial_t \nabla u \cdot \partial_t \nabla u \, d\tau \geq 0$.

Considering $-\Delta \partial_t u$ as a test function in the weak formulation of (3.3) and using estimates in Lemma 2.1 we obtain

$$\begin{aligned} & \kappa \|\partial_t \nabla u\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\Delta u\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq [\xi \|\nabla v_0\|_{L^2(\Omega)}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2] \times \\ & \quad \times \exp\left\{T\frac{C_\Omega}{\kappa}\left[\|\partial_t \tilde{u}\|_{L^\infty(\Omega_T)} + \|\partial_t \nabla \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))}\right.\right. \\ & \quad \left.\left. + \frac{1}{\kappa}\|\nabla \tilde{u}\|_{L^\infty(\Omega_T)}\left(1 + \|\partial_t \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))} + \frac{1}{\kappa}\|\nabla \tilde{u}\|_{L^\infty(\Omega_T)}\right)\right]\right\} \end{aligned}$$

and

$$\begin{aligned} & \|\beta * \Delta \partial_t u\|_{L^2(\Omega_T)}^2 \leq [\xi \|\nabla v_0\|_{L^2(\Omega)}^2 + \|\Delta u_0\|_{L^2(\Omega)}^2] \left[T\frac{C_\Omega}{\kappa}\left(\|\partial_t \nabla \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))}\right.\right. \\ & \quad \left. + \|\partial_t \tilde{u}\|_{L^\infty(\Omega_T)} + \frac{1}{\kappa}\|\nabla \tilde{u}\|_{L^\infty(\Omega_T)}\left[1 + \|\partial_t \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))} + \frac{1}{\kappa}\|\nabla \tilde{u}\|_{L^\infty(\Omega_T)}\right]\right) \times \\ & \quad \times \exp\left\{T\frac{C_\Omega}{\kappa}\left(\|\partial_t \tilde{u}\|_{L^\infty(\Omega_T)} + \|\partial_t \nabla \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))}\right.\right. \\ & \quad \left. \left. + \frac{1}{\kappa}\|\nabla \tilde{u}\|_{L^\infty(\Omega_T)}\left[1 + \|\partial_t \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))} + \frac{1}{\kappa}\|\nabla \tilde{u}\|_{L^\infty(\Omega_T)}\right]\right)\right\} + \frac{1}{a}. \end{aligned}$$

Applying Δ to (3.3) and taking $\Delta \partial_t u$ as a test function in the weak formulation of the problem implies

$$\begin{aligned} & \kappa \|\partial_t \Delta u\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\Delta \nabla u\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq [\xi \|\Delta v_0\|_{L^2(\Omega)}^2 + \|\Delta \nabla u_0\|_{L^2(\Omega)}^2] \times \\ & \quad \times \exp\left\{T\frac{C_\Omega}{\kappa}\left(\|\partial_t \Delta \tilde{u}\|_{L^\infty(0,T;L^2(\Omega))} + \frac{1}{\kappa}\left[\|\Delta \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))}\left(\|\partial_t \nabla \tilde{u}\|_{L^\infty(0,T;L^2(\Omega))}\right.\right.\right.\right. \\ & \quad \left. \left. \left. + 1\right) + \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)}\left(\|\nabla \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))} + 1\right)\left(\|\partial_t \nabla \tilde{u}\|_{L^\infty(0,T;L^2(\Omega))} + 1\right)\right]\right. \\ & \quad \left. + \frac{1}{\kappa^2}\left[\|\Delta \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))}^2 + \|\nabla \tilde{u}\|_{L^\infty(\Omega_T)}^2\left(1 + \|\nabla \tilde{u}\|_{L^\infty(0,T;L^4(\Omega))}^2\right)\right]\right\}. \end{aligned}$$

Using the Sobolev embedding inequality yields the second estimate in (3.6). From those estimates, using the weak formulation of the problem, we also obtain the last estimate in the lemma. See appendix for more details on the derivation of a priori estimates. \square

Using a priori estimates proven in Lemma 3.2 and applying the Banach fixed point theorem yield local existence of a unique solution of nonlinear problem (2.7).

Theorem 3.3. For $u_0 \in H^3(\Omega) \cap H_0^1(\Omega)$ and $v_0 \in H^2(\Omega)$, with

$$\|\Delta u_0\|_{L^2(\Omega)}^2 + (1 + 2kC_\Omega r)\|\nabla v_0\|_{L^2(\Omega)}^2 \leq \eta r^2,$$

for any $\eta \in (0, 1)$, there exists time interval $T = T(R, r, \eta) > 0$ such that $u \in \mathcal{K}$ is a unique solution of (2.7).

Proof. For $R^2 = C[\|u_0\|_{H^3}^2 + (1 + 2kC_\Omega r)\|v_0\|_{H^2}^2]$ and $T = T(R, r, \eta)$ such that

$$\exp\{TC_\Omega(R + R^2)\} \leq 1/\eta \quad \text{and} \quad C_\Omega \exp\{TC_\Omega(R + R^2 + R^3 + R^4)\} \leq C,$$

estimates in (3.6) imply $u = \mathcal{T}(\tilde{u}) \in \mathcal{K}$.

To show that $\mathcal{T}: \mathcal{K} \rightarrow \mathcal{K}$ is a contraction we consider (3.3) for \tilde{u}_1 and \tilde{u}_2 in \mathcal{K} and, taking $-\Delta \partial_t(u_1 - u_2)$ as a test function in the weak formulation of the difference of the corresponding equations, obtain

$$\begin{aligned}
& \kappa \|\nabla \partial_t(u_1 - u_2)\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\Delta(u_1 - u_2)\|_{L^\infty(0,T;L^2(\Omega))}^2 \\
& \leq C_\Omega \left(\left[\|\partial_t \Delta u_2\|_{L^2(\Omega_T)}^2 + \frac{1}{\kappa} \|\partial_t \nabla u_2\|_{L^2(\Omega_T)}^2 \right] \|\nabla \partial_t(\tilde{u}_1 - \tilde{u}_2)\|_{L^\infty(0,T;L^2(\Omega))}^2 \right. \\
& + \frac{1}{\kappa} \left[\|\Delta \nabla u_2\|_{L^2(\Omega_T)}^2 + \|\partial_t \nabla \tilde{u}_2\|_{L^2(0,T;H^1(\Omega))}^2 + \|\Delta u_2\|_{H^1(0,T;L^2(\Omega))}^2 \right. \\
& + \|\beta * \Delta \partial_t u_2\|_{L^1(0,T;H^1(\Omega))} (1 + \|\nabla \tilde{u}_2\|_{L^\infty(\Omega_T)}) \left. \right] \|\tilde{u}_1 - \tilde{u}_2\|_{L^\infty(0,T;H^2(\Omega))}^2 \Big) \times \\
& \times \exp \left\{ \frac{C_\Omega}{\kappa} \left(\|\partial_t \Delta \tilde{u}_1\|_{L^1(0,T;L^2(\Omega))} + \frac{1}{\kappa} \left[\|\nabla \tilde{u}_2\|_{L^\infty(\Omega_T)}^2 \|\partial_t \nabla u_2\|_{L^2(\Omega_T)}^2 \right. \right. \right. \\
& + \|\beta * \Delta \partial_t u_2\|_{L^1(0,T;H^1(\Omega))} (1 + \|\nabla \tilde{u}_2\|_{L^\infty(\Omega_T)}) + T(1 + \|\nabla \tilde{u}_1\|_{L^\infty(\Omega_T)}^2) \\
& + \|\tilde{u}_2\|_{W^{1,\infty}(\Omega_T)}^2 + \|\partial_t u_2\|_{L^\infty(\Omega_T)}^2 + \|\nabla \tilde{u}_1\|_{L^\infty(\Omega_T)} \|\partial_t \nabla \tilde{u}_1\|_{L^1(0,T;L^2(\Omega))} \left. \right] \\
& + \frac{1}{\kappa^2} \|\nabla \tilde{u}_1\|_{L^2(0,T;L^\infty(\Omega))}^2 \Big) \Big\} \leq C_\Omega [TR^2 + T^{\frac{1}{2}}(R + R^2)] \times \\
& \times \exp \left\{ C_\Omega [T(1 + R + R^2 + R^4) + T^{\frac{1}{2}}(R^3 + R^4) \exp\{C_\Omega T(R + R^2)\}] \right\} \times \\
& \quad \times \left(\kappa \|\nabla \partial_t(\tilde{u}_1 - \tilde{u}_2)\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\Delta(\tilde{u}_1 - \tilde{u}_2)\|_{L^\infty(0,T;L^2(\Omega))}^2 \right).
\end{aligned}$$

Then for T such that $C_\Omega [TR^2 + T^{\frac{1}{2}}(R + R^2)] \exp \{C_\Omega [T(1 + R + R^2 + R^4) + T^{\frac{1}{2}}(R^3 + R^4) \exp\{C_\Omega T(R + R^2)\}]\} < 1$ we have that $\mathcal{T}: \mathcal{K} \rightarrow \mathcal{K}$ is a contraction. Thus applying the Banach fixed point theorem, and iterating over time, yields existence of a unique solution of the nonlinear problem (2.7). \square

4. TRAPEZOIDAL DISCRETIZATION

In this section we present analysis for the numerical scheme for problem (2.7). The time semi-discretization considered here is based on trapezoidal time-stepping with uniform time-step $\Delta t > 0$, $n = 1, 2, \dots$,

$$(4.1) \quad (1 - 2k\{u\}_n) D^2 u_n - \Delta \{u\}_n - a\beta *_{\Delta t} D \Delta u_n = 2k(Du_n)^2,$$

where $u_n \in H_0^1(\Omega) \cap H^2(\Omega)$ and

$$\begin{aligned}
Du_n &= \frac{1}{2\Delta t}(u_{n+1} - u_{n-1}), & D^2 u_n &= \frac{1}{\Delta t^2}(u_{n+1} - 2u_n + u_{n-1}), \\
\{u\}_n &= \frac{1}{4}(u_{n+1} + 2u_n + u_{n-1}), & \tilde{D}u_n &= \frac{1}{\Delta t}(u_{n+1} - u_n),
\end{aligned}$$

with $Du_0 := v_0$, and $[\beta *_{\Delta t} g]_n$ (with the square brackets in most places left-out) a convolution quadrature approximation of $\int_0^{t_n} \beta(t_n - \tau)g(\tau)d\tau$. We will use convolution quadrature based on the second order backward difference formula (BDF2) [17, 18, 19] which results in the discrete convolution

$$[\beta *_{\Delta t} v]_n = \sum_{j=0}^n \omega_{n-j} v_j,$$

with convolution weights ω_j given by the generating function

$$\hat{\beta} \left(\frac{\delta(\zeta)}{\Delta t} \right) = \sum_{j=0}^{\infty} \omega_j \zeta^j, \quad \delta(\zeta) = (1 - \zeta) + \frac{1}{2}(1 - \zeta)^2.$$

For v that is sufficiently smooth and with sufficiently many zero derivatives at $t = 0$, we have that $[\beta *_{\Delta t} v]_n = \beta * v(t_n) + \mathcal{O}(\Delta t^2)$, whereas for $v(t) = t^\alpha$ and real $\alpha > -1$

$$(4.2) \quad \left| [\beta *_{\Delta t} v]_n - \beta * v(t_n) \right| \leq \begin{cases} C t_n^{\mu-1} \Delta t^{\alpha+1} & \text{for } -1 < \alpha \leq 1, \\ C t_n^{\mu+\alpha-2} \Delta t^2 & \text{for } \alpha \geq 1, \end{cases}$$

for $n = 1, \dots$; see [20, Theorem 2.2].

Alternatively, we can use the corrected CQ formula

$$(4.3) \quad [\beta \tilde{*}_{\Delta t} v]_n := [\beta *_{\Delta t} v]_n + \omega_{n,0} v_0,$$

where $\omega_{n,0}$ is chosen so that the formula is exact for constant function, i.e.,

$$\omega_{n,0} := \int_0^{t_n} \beta(\tau) d\tau - [\beta *_{\Delta t} 1]_n = \int_0^{t_n} \beta(\tau) d\tau - \sum_{j=0}^n \omega_j.$$

Note the trivial but useful fact that $\beta \tilde{*}_{\Delta t} v \equiv \beta *_{\Delta t} v$ if $v_0 = 0$. From (4.2) and the definition of $\omega_{n,0}$ it follows that we have the stability bound

$$(4.4) \quad |\omega_{n,0}| \leq C t_n^{\mu-1} \Delta t$$

for $n \geq 1$. The first correction weight is $\omega_{0,0} = -\omega_0$, where $\omega_0 = \hat{\beta}(\delta(0)/\Delta t) = \hat{\beta}(\frac{3}{2\Delta t}) \sim (2/3)^\mu \Delta t^\mu$ as $\Delta t \rightarrow 0$. The semi-discretisation with the corrected CQ formula reads

$$(4.5) \quad (1 - 2k\{u\}_n) D^2 u_n - \Delta \{u\}_n - a \beta \tilde{*}_{\Delta t} D \Delta u_n = 2k(Du_n)^2.$$

A crucial property of (the non-corrected) convolution quadrature, see [5, Lemma 2.1] and [6, Theorem 2.25], is that (2.5) implies

$$(4.6) \quad \sum_{j=0}^{\infty} \varrho^{2j} \langle v_j, [\beta *_{\Delta t} v]_j \rangle \geq C_\beta \sum_{j=0}^{\infty} \varrho^{2j} \|[\beta *_{\Delta t} v]_j\|_{L^2}^2,$$

for $\varrho = e^{-\sigma \Delta t}$, with $\sigma > 0$, and

$$C_\beta := (\tilde{\sigma} + r)^\mu \quad (\text{if } \beta = \beta_A), \quad C_\beta := 1 \quad (\text{if } \beta = \beta_B),$$

where $\tilde{\sigma} = C_2 \min(1, \sigma)$. Thus, if $\beta = \beta_A$ with $r > 0$ or $\beta = \beta_B$, we can set $\sigma = 0$ (i.e., $\varrho = 1$ and $\tilde{\sigma} = 0$) and still obtain positivity of the left hand side in (4.6).

For the corrected version, all we can say is that

$$(4.7) \quad \sum_{j=0}^{\infty} \varrho^{2j} \langle v_j, [\beta \tilde{*}_{\Delta t} v]_j \rangle \geq C_\beta \sum_{j=0}^{\infty} \varrho^{2j} \|[\beta *_{\Delta t} v]_j\|_{L^2}^2 + \sum_{j=0}^{\infty} \varrho^{2j} \omega_{j,0} \langle v_j, v_0 \rangle.$$

To initiate the iterations in (4.1) we set

$$(4.8) \quad u_1 = u_0 + \Delta t v_0 + \frac{1}{2} \Delta t^2 \partial_t^2 u(0),$$

where we can determine $\partial_t^2 u(0)$ from the equation (2.7)

$$(4.9) \quad \partial_t^2 u(0) = \frac{1}{1 - 2k u_0} (\Delta u_0 + 2k(v_0)^2).$$

Lemma 4.1. Under the assumptions on the initial data (2.8) and

$$\left\| \Delta \frac{u_0 + u_1}{2} \right\|_{L^2(\Omega)}^2 + (1 + 2krC_\Omega) \left\| \nabla \frac{u_1 - u_0}{\Delta t} \right\|_{L^2(\Omega)}^2 \leq \eta r^2,$$

where $0 < C_\Omega r \leq (1 - \kappa)/2k$, with $0 < \kappa < 1$, and the constant C_Ω is the constant in (3.1) and (3.2), we have the following stability estimates for the scheme (4.1)

$$(4.10) \quad \begin{aligned} \kappa \sup_{1 \leq n \leq N-1} \|\tilde{D}\nabla u_n\|_{L^2(\Omega)}^2 + \sup_{1 \leq n \leq N-1} \|\Delta u_n\|_{L^2(\Omega)}^2 &\leq r^2, \\ \kappa \sup_{1 \leq n \leq N-1} \|\tilde{D}u_n\|_{H^2(\Omega)}^2 + \sup_{1 \leq n \leq N-1} \|u_n\|_{H^3(\Omega)}^2 &\leq R. \end{aligned}$$

Proof. In order to analyse the system we need that $1 - 2k\{u\}_n \geq \kappa > 0$ for some (fixed) $\kappa \in (0, 1)$. Thus similarly to the continuous case, we consider the fixed-point iteration

$$(4.11) \quad (1 - 2kd_n)D^2u_n - \Delta\{u\}_n - a\beta *_{\Delta t} D\Delta u_n = 2kv_n Du_n,$$

where $d_n = \{\tilde{u}\}_n$ and $v_n = D\tilde{u}_n$ for $\tilde{u}_n \in H_0^1(\Omega)$ satisfying

$$(4.12) \quad \begin{aligned} \kappa \sup_{1 \leq n \leq N-1} \|D\tilde{u}_n\|_{H^2(\Omega)}^2 + \sup_{1 \leq n \leq N-1} \|\{\tilde{u}\}_n\|_{H^3(\Omega)}^2 &\leq R^2, \\ \kappa \sup_{1 \leq n \leq N-1} \|\tilde{D}\nabla\tilde{u}_n\|_{L^2(\Omega)}^2 + \sup_{1 \leq n \leq N-1} \|\Delta(\tilde{u})_n\|_{L^2(\Omega)}^2 &\leq r^2. \end{aligned}$$

To derive the stability estimates we first test (4.11) with $\varrho^{2n} Du_n$, $\varrho = e^{-\Delta t/T}$, and estimate each term separately. For the first term we have

$$\begin{aligned} &\Delta t \sum_{n=1}^{N-1} \varrho^{2n} \langle (1 - 2kd_n)D^2u_n, Du_n \rangle \\ &\geq \frac{1}{2} \int_{\Omega} \varrho^{2(N-1)} (1 - 2kd_{N-1}) \left(\frac{u_N - u_{N-1}}{\Delta t} \right)^2 dx - \frac{1}{2} \int_{\Omega} (1 - 2kd_1) \left(\frac{u_1 - u_0}{\Delta t} \right)^2 dx \\ &\quad + k \int_{\Omega} \sum_{n=1}^{N-2} \varrho^{2(n+1)} (d_{n+1} - d_n) \left(\frac{u_{n+1} - u_n}{\Delta t} \right)^2 dx \\ &= \frac{1}{2} \int_{\Omega} \varrho^{2(N-1)} (1 - 2kd_{N-1}) \left(\frac{u_N - u_{N-1}}{\Delta t} \right)^2 dx - \frac{1}{2} \int_{\Omega} (1 - 2kd_1) \left(\frac{u_1 - u_0}{\Delta t} \right)^2 dx \\ &\quad + k\Delta t \int_{\Omega} \sum_{n=1}^{N-2} \varrho^{2(n+1)} \left(\frac{(\tilde{u})_{n+1} - (\tilde{u})_{n-1}}{2\Delta t} \right) (\tilde{D}u_n)^2 dx, \end{aligned}$$

where $(w)_n = (w_{n+1} + w_n)/2$. The last term in the estimate above can be bounded by

$$k\Delta t \sum_{n=1}^{N-2} \varrho^{2(n+1)} \|D(\tilde{u})_n\|_{L^\infty} \|\tilde{D}u_n\|_{L^2}^2.$$

Using $1 - 2kd_n \geq \kappa > 0$, together with the estimates for the convolution quadrature, yields

$$\begin{aligned} E_{N-1} &\leq E_0 - C_\beta \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\beta *_{\Delta t} Du_n\|_{H^1}^2 \\ &\quad + 3k\Delta t \sum_{n=1}^{N-2} \varrho^{2(n+1)} \|D(\tilde{u})_n\|_{L^\infty} \|\tilde{D}u_n\|_{L^2}^2, \end{aligned}$$

where for $n \geq 1$

$$E_n = \frac{1}{2} \varrho^{2n} \left| \frac{u_{n+1} + u_n}{2} \right|_{H^1}^2 + \frac{1}{2} \int_{\Omega} \varrho^{2n} (1 - 2kd_n) |\tilde{D}u_n|^2 dx$$

and

$$E_0 = \frac{1}{2} \left| \frac{u_1 + u_0}{2} \right|_{H^1}^2 + \frac{1}{2} \int_{\Omega} (1 - 2kd_1) |\tilde{D}u_0|^2 dx.$$

Then the discrete Gronwall inequality ensures

$$E_{N-1} \leq E_0 \exp \left\{ 3k\Delta t \sum_{n=1}^{N-2} \varrho^{2(n+1)} \left\| \frac{1}{1 - 2kd_n} \right\|_{L^\infty} \|D(\tilde{u})_n\|_{L^\infty} \right\}.$$

When considering the corrected convolution quadrature, we will have the additional term

$$\begin{aligned} & \Delta t \sum_{n=1}^{N-1} \varrho^{2n} |\omega_{n,0} \langle \nabla D u_n, \nabla D u_0 \rangle| \\ (4.13) \quad & \leq C \Delta t \sum_{n=1}^{N-1} \varrho^{2n} t_n^{\mu-1} \left[\frac{\delta}{2} (\|\nabla(u)_{n+1}\|_{L^2}^2 + \|\nabla(u)_n\|_{L^2}^2) + C_\delta \|v_0\|_{H^1}^2 \right] \\ & \leq \delta \sup_n \|\nabla(u)_n\|_{L^2}^2 + C, \end{aligned}$$

for any fixed $\delta > 0$. Then the first term can be subtracted from the corresponding term on the left-hand side.

Taking $-\varrho^{2n} \Delta D u_n$ as a test function in (4.11) and integrating by parts in the first term on the left-hand side and in the right-hand side yield

$$\begin{aligned} & \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[\langle (1 - 2kd_n) D^2 \nabla u_n, D \nabla u_n \rangle + \langle \Delta \{u\}_n, \Delta D u_n \rangle \right. \\ (4.14) \quad & \left. + \langle a\beta *_{\Delta t} D \Delta u_n, D \Delta u_n \rangle \right] \\ & = 2k\Delta t \sum_{n=1}^{N-1} \varrho^{2n} \langle D u_n \nabla v_n + D \nabla u_n v_n + \nabla d_n D^2 u_n, D \nabla u_n \rangle. \end{aligned}$$

Using equation (4.11) we rewrite the last term on the right-hand side as

$$\begin{aligned} \langle \nabla d_n D^2 u_n, D \nabla u_n \rangle & = \left\langle \frac{1}{1 - 2kd_n} \nabla d_n \left(\Delta \{u\}_n + a\beta *_{\Delta t} D \Delta u_n \right), D \nabla u_n \right\rangle \\ & \quad + 2k \left\langle \frac{1}{1 - 2kd_n} \nabla d_n D u_n v_n, D \nabla u_n \right\rangle. \end{aligned}$$

Similar as above for the first term in (4.14) we have

$$\begin{aligned} & \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[\langle (1 - 2kd_n) D^2 \nabla u_n, D \nabla u_n \rangle \right. \\ & \geq \frac{1}{2} \varrho^{2(N-1)} \int_{\Omega} (1 - 2kd_{N-1}) |\nabla \tilde{D} u_{N-1}|^2 dx - \frac{1}{2} \int_{\Omega} (1 - 2kd_1) |\nabla \tilde{D} u_0|^2 dx \\ & \quad \left. + k\Delta t \sum_{n=1}^{N-2} \varrho^{2(n+1)} \int_{\Omega} D(\tilde{u})_n |\tilde{D} \nabla u_n|^2 dx. \right. \end{aligned}$$

For the second and third terms in (4.14) we have

$$\begin{aligned} & \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[\langle \Delta \{u\}_n, \Delta D u_n \rangle + \langle a \beta *_{\Delta t} D \Delta u_n, D \Delta u_n \rangle \right] \\ & \geq \frac{1}{2} \varrho^{2(N-1)} \left\| \frac{\Delta(u_N + u_{N-1})}{2} \right\|_{L^2}^2 - \frac{1}{2} \left\| \frac{\Delta(u_1 + u_0)}{2} \right\|_{L^2}^2 \\ & \quad + C_\beta \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\beta *_{\Delta t} D \Delta u_n\|_{L^2}^2. \end{aligned}$$

The terms on the right-hand side are estimated as

$$\begin{aligned} & \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[\left\langle \frac{\nabla d_n}{1-2kd_n} \Delta \{u\}_n, D \nabla u_n \right\rangle \right. \\ & \quad \left. + a \left\langle \frac{\nabla d_n}{1-2kd_n} \beta *_{\Delta t} D \Delta u_n, D \nabla u_n \right\rangle + 2k \left\langle \frac{\nabla d_n}{1-2kd_n} D u_n v_n, D \nabla u_n \right\rangle \right] \\ & \leq \varsigma \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\beta *_{\Delta t} D \Delta u_n\|_{L^2}^2 + C \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left\| \frac{1}{1-2kd_n} \right\|_{L^\infty} \|\nabla d_n\|_{L^\infty} \times \\ & \quad \times \left[\|\Delta \{u\}_n\|_{L^2}^2 + \|D \nabla u_n\|_{L^2}^2 \left(1 + \|v_n\|_{L^4} + C \left\| \frac{1}{1-2kd_n} \right\|_{L^\infty} \|\nabla d_n\|_{L^\infty} \right) \right], \end{aligned}$$

where we assume $\varsigma \leq C_\beta/2$, and

$$\begin{aligned} & 2k \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[\langle D u_n \nabla v_n, D \nabla u_n \rangle + \langle D \nabla u_n v_n, D \nabla u_n \rangle \right] \\ & \leq C \Delta t \sum_{n=1}^{N-1} \varrho^{2n} (\|v_n\|_{L^\infty} + \|\nabla v_n\|_{L^4}) \|D \nabla u_n\|_{L^2}^2. \end{aligned}$$

Applying the discrete Grönwall inequality we obtain

$$\begin{aligned} & \int_{\Omega} \varrho^{2(N-1)} (1-2kd_{N-1}) |\tilde{D} \nabla u_{N-1}|^2 dx + \varrho^{2(N-1)} \|\Delta(u)_{N-1}\|_{L^2}^2 \\ & \quad + C_\beta \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\beta *_{\Delta t} D \Delta u_n\|_{L^2}^2 \leq [\nu \|\tilde{D} u_0\|_{L^2}^2 + \|\Delta(u_0 + u_1)/2\|_{L^2}^2] \times \\ (4.15) \quad & \times \exp \left\{ C \Delta t \sum_{n=1}^{N-1} \left\| \frac{1}{1-2kd_n} \right\|_{L^\infty} \left[\|\nabla v_n\|_{L^4} + \|v_n\|_{L^\infty} \right. \right. \\ & \quad \left. \left. + \frac{\|\nabla d_n\|_{L^\infty}}{\|1-2kd_n\|_{L^\infty}} \left(\left\| \frac{1}{1-2kd_n} \right\|_{L^\infty} \|\nabla d_n\|_{L^\infty} + 1 + \|v_n\|_{L^4} \right) \right] \right\} \\ & \leq [\nu \|\nabla \tilde{D} u_0\|_{L^2}^2 + \|\Delta(u_0 + u_1)/2\|_{L^2}^2] \exp \left\{ \frac{C_\Omega}{\kappa} \Delta t \sum_{n=1}^{N-1} \left[\|\Delta v_n\|_{L^2} \right. \right. \\ & \quad \left. \left. + \frac{1}{\kappa^2} \|\nabla \Delta d_n\|_{L^2}^2 + \frac{1}{\kappa} (\|\nabla \Delta d_n\|_{L^2} + \|\nabla \Delta d_n\|_{L^2}^2 + \|\nabla v_n\|_{L^2}^2) \right] \right\}, \end{aligned}$$

where $\nu = 1 + k(\|\tilde{u}_0\|_{L^\infty} + 2\|\tilde{u}_1\|_{L^\infty} + \|\tilde{u}_2\|_{L^\infty})/2$. Thus for

$$\kappa \sup_{1 \leq n \leq N-1} \|v_n\|_{H^2}^2 + \sup_{1 \leq n \leq N-1} \|d_n\|_{H^3}^2 \leq R^2,$$

and initial conditions

$$\|\Delta(u_0 + u_1)/2\|_{L^2}^2 + (1 + 2krC_\Omega)\|\nabla v_0 + \Delta t \nabla \partial_t^2 u(0)\|_{L^2}^2 \leq \eta r^2,$$

and appropriate $T > 0$, such that

$$\exp\{C_\Omega T(R/\kappa + (R + 2R^2)/\kappa^2 + R^2/\kappa^3)\} \leq \frac{\min_n \rho^{2(n-1)}}{\eta},$$

we obtain

$$\kappa \sup_{1 \leq n \leq N-1} \|\tilde{D}\nabla u_n\|_{L^2}^2 + \sup_{1 \leq n \leq N-1} \|\Delta(u)_n\|_{L^2}^2 \leq r^2,$$

where $0 < \kappa \leq 1 - 2kC_\Omega r$ and C_Ω is the constant in (3.1) and (3.2). This ensures

$$\begin{aligned} \|\{u\}_n\|_{L^\infty} &\leq \frac{1}{2}C_\Omega(\|(u)_n\|_{H^2} + \|(u)_{n-1}\|_{H^2}) \\ &\leq \frac{1}{2}C_\Omega(\|\Delta(u)_n\|_{L^2} + \|\Delta(u)_{n-1}\|_{L^2}) \leq C_\Omega r. \end{aligned}$$

In the case of the corrected convolution quadrature the additional term is estimated in the same way as in (4.13), with ΔDu_n and Δv_0 instead of ∇Du_n and ∇v_0 .

Applying the Laplace operator to (4.11) yields

$$(4.16) \quad \begin{aligned} (1 - 2kd_n)D^2\Delta u_n - 2k\Delta d_n D^2u_n - 4k\nabla d_n D^2\nabla u_n - \Delta^2\{u\}_n \\ - a\beta *_{\Delta t} D\Delta^2u_n = 2k(Du_n\Delta v_n + D\Delta u_n v_n) + 4k\nabla Du_n \nabla v_n. \end{aligned}$$

Considering $\varrho^{2n}D\Delta u_n$ as a test function in (4.16), we obtain

$$(4.17) \quad \begin{aligned} \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[\langle (1 - 2kd_n)D^2\Delta u_n, D\Delta u_n \rangle + \langle \Delta \nabla \{u\}_n, \Delta \nabla Du_n \rangle \right. \\ \left. + a \langle \beta *_{\Delta t} D\Delta \nabla u_n, \Delta \nabla Du_n \rangle \right] \\ = 2k\Delta t \sum_{n=1}^{N-1} \varrho^{2n} \langle \Delta d_n D^2u_n + 2\nabla d_n D^2\nabla u_n, D\Delta u_n \rangle \\ + 2k\Delta t \sum_{n=1}^{N-1} \varrho^{2n} \langle Du_n \Delta v_n + D\Delta u_n v_n + 2\nabla Du_n \nabla v_n, D\Delta u_n \rangle. \end{aligned}$$

For the first term in the same way as above we obtain

$$\begin{aligned} \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \langle (1 - 2kd_n)D^2\Delta u_n, D\Delta u_n \rangle &\geq k\Delta t \int_{\Omega} \sum_{n=1}^{N-2} \varrho^{2(n+1)} D(\tilde{u})_n |\tilde{D}\Delta u_n|^2 dx \\ &+ \frac{1}{2} \int_{\Omega} \varrho^{2(N-1)} (1 - 2kd_{N-1}) |\tilde{D}\Delta u_{N-1}|^2 dx - \frac{1}{2} \int_{\Omega} (1 - 2kd_1) |\tilde{D}\Delta u_0|^2 dx. \end{aligned}$$

The second and third terms in (4.17) are estimates in the same way as above and we have

$$\begin{aligned} & \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[\langle \Delta \nabla \{u\}_n, \Delta \nabla D u_n \rangle + \langle \beta *_{\Delta t} D \Delta \nabla D u_n, \Delta \nabla D u_n \rangle \right] \\ & \geq \frac{1}{2} \varrho^{2(N-1)} \left\| \frac{\Delta \nabla (u_N + u_{N-1})}{2} \right\|_{L^2}^2 - \frac{1}{2} \left\| \frac{\Delta \nabla (u_1 + u_0)}{2} \right\|_{L^2}^2 \\ & \quad + C_\beta \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\beta *_{\Delta t} D \Delta \nabla u_n\|_{L^2}^2. \end{aligned}$$

For the last term on the right-hand side of (4.17) we obtain

$$\begin{aligned} & \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left| \langle D u_n \Delta v_n + D \Delta u_n v_n + 2 \nabla D u_n \nabla v_n, D \Delta u_n \rangle \right| \\ & \leq C_\Omega \Delta t \sum_{n=1}^{N-1} \varrho^{2n} (\|\Delta v_n\|_{L^2} + \|v_n\|_{L^\infty} + \|\nabla v_n\|_{L^4}) \|D \Delta u_n\|_{L^2}^2. \end{aligned}$$

Here we used estimates (3.1) and (3.2). To estimate the first term on the right-hand side of (4.17) we first use the equation (4.11) to write

$$D^2 u_n = \frac{1}{(1 - 2k d_n)} \left(\Delta \{u\}_n + \beta *_{\Delta t} D \Delta u_n + 2k D u_n v_n \right)$$

and

$$\begin{aligned} D^2 \nabla u_n &= \frac{1}{(1 - 2k d_n)} \left(\Delta \nabla \{u\}_n + \beta *_{\Delta t} D \Delta \nabla u_n + 2k (D u_n \nabla v_n + D \nabla u_n v_n) \right) \\ & \quad + 2k \nabla d_n \frac{1}{(1 - 2k d_n)^2} \left(\Delta \{u\}_n + \beta *_{\Delta t} D \Delta u_n + 2k D u_n v_n \right). \end{aligned}$$

Then, using the Sobolev embedding theorem, yields

$$\begin{aligned} & \Delta t \sum_{n=1}^{N-1} \varrho^{2n} 2k \langle \Delta d_n D^2 u_n, D \Delta u_n \rangle \\ & \leq C \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \frac{\|\Delta d_n\|_{L^4}}{\|1 - 2k d_n\|_{L^\infty}} \left[\|v_n\|_{L^8} \|D u_n\|_{L^8} + \|\Delta \{u\}_n\|_{L^4} \right] \|D \Delta u_n\|_{L^2} \\ & \quad + \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[C \left\| \frac{1}{1 - 2k d_n} \right\|_{L^\infty}^2 \|\Delta d_n\|_{L^4}^2 \|D \Delta u_n\|_{L^2}^2 + \varsigma \|\beta *_{\Delta t} D \Delta u_n\|_{L^4}^2 \right] \\ & \leq \Delta t \sum_{n=1}^{N-1} \varrho^{2n} C \left\| \frac{1}{1 - 2k d_n} \right\|_{L^\infty}^2 \left(\|\Delta d_n\|_{L^4}^2 + \|v_n\|_{L^8}^2 \right) \|D \Delta u_n\|_{L^2}^2 \\ & \quad + C \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\Delta \nabla \{u\}_n\|_{L^2}^2 + \varsigma \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\beta *_{\Delta t} D \Delta \nabla u_n\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned}
\Delta t \sum_{n=1}^{N-1} \varrho^{2n} |\langle \nabla d_n D^2 \nabla u_n, D \Delta u_n \rangle| &\leq C \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[\frac{\|\nabla d_n\|_{L^\infty}}{\|1 - 2kd_n\|_{L^\infty}} \left[\|\Delta \nabla \{u\}_n\|_{L^2} \right. \right. \\
&\quad \left. \left. + \|Du_n\|_{L^\infty} \|\nabla v_n\|_{L^2} + \|D \nabla u_n\|_{L^4} \|v_n\|_{L^4} + \|\beta *_{\Delta t} D \Delta \nabla u_n\|_{L^2} \right] \right. \\
&\quad \left. + \frac{\|\nabla d_n\|_{L^6}^2}{\|1 - 2kd_n\|_{L^\infty}^2} \left[\|\Delta \{u\}_n\|_{L^6} + \|\beta *_{\Delta t} D \Delta u_n\|_{L^6} + \|Du_n\|_{L^\infty} \|v_n\|_{L^6} \right] \right] \|D \Delta u_n\|_{L^2} \\
&\leq C_\Omega \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left\| \frac{1}{1 - 2kd_n} \right\|_{L^\infty}^2 \left(\left\| \frac{1}{1 - 2kd_n} \right\|_{L^\infty}^2 \|\nabla d_n\|_{L^6}^4 + \|\nabla d_n\|_{L^6}^4 + \|\nabla d_n\|_{L^\infty}^2 \right. \\
&\quad \left. + \|\nabla v_n\|_{L^2}^2 \right) \|D \Delta u_n\|_{L^2}^2 + \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left(\|\Delta \nabla \{u\}_n\|_{L^2}^2 + \varsigma \|\beta *_{\Delta t} D \Delta \nabla u_n\|_{L^2}^2 \right).
\end{aligned}$$

Here we used (3.1) and

$$\|\Delta w\|_{L^4} \leq C_\Omega \|\Delta \nabla w\|_{L^2}, \quad \|\nabla w\|_{L^\infty} \leq C_\Omega \|\Delta \nabla w\|_{L^2}.$$

Choosing $\varsigma > 0$ sufficiently small yields

$$\begin{aligned}
&\varrho^{2(N-1)} \kappa \|\tilde{D} \Delta u_{N-1}\|_{L^2}^2 + \varrho^{2(N-1)} \|\Delta \nabla (u)_{N-1}\|_{L^2}^2 \\
&\quad + C_\beta \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\beta *_{\Delta t} D \Delta \nabla u_n\|_{L^2}^2 \leq \nu \|\tilde{D} \Delta u_0\|_{L^2}^2 + \|\Delta \nabla (u)_0\|_{L^2}^2 \\
&\quad + C_\Omega \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left(\|\Delta \nabla \{u\}_n\|_{L^2}^2 + \left\| \frac{1}{1 - 2kd_n} \right\|_{L^\infty}^2 \left[\|\Delta v_n\|_{L^2}^2 + \|\Delta \nabla d_n\|_{L^2}^2 \right. \right. \\
&\quad \left. \left. + \left(1 + \left\| \frac{1}{1 - 2kd_n} \right\|_{L^\infty}^2 \right) \|\Delta d_n\|_{L^2}^4 \right] \|D \Delta u_n\|_{L^2}^2 \right).
\end{aligned}$$

Considering T such that

$$\begin{aligned}
&\left[\nu \|\tilde{D} \Delta u_0\|_{L^2}^2 + \|\Delta \nabla (u_0 + u_1)/2\|_{L^2}^2 \right] \exp \left\{ C_\Omega T \max \left\{ 1, \frac{R^2 + R^4}{\kappa^3} + \frac{R^4}{\kappa^5} \right\} \right\} \\
&\leq \min_n \varrho^{2(n-1)} R^2,
\end{aligned}$$

and using the discrete Grönwall inequality, we obtain

$$\varrho^{2n} \kappa \|\tilde{D} \Delta u_n\|_{L^2}^2 + \varrho^{2n} \|\Delta \nabla (u)_n\|_{L^2}^2 \leq \varrho^{2n} R^2 \quad \text{for all } n = 1, \dots, N-1,$$

and then also

$$\Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\beta *_{\Delta t} D \Delta \nabla u_n\|_{L^2}^2 \leq C.$$

In the case of the corrected convolution quadrature the additional term is estimated in the same way as in (4.13), with $\nabla \Delta Du_n$ and $\nabla \Delta v_0$ instead of ∇Du_n and ∇v_0 .

As next we have to show the contraction property for the map $\tilde{u}_n \rightarrow u_n$. Considering the equation for $w_n = u_n^1 - u_n^2$, where u_n^j satisfies (4.11) for \tilde{u}_n^j , with $j = 1, 2$,

taking $-\varrho^{2n}\tilde{D}w_n$ as a test function and summing over n , yields

$$\begin{aligned}
& \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left[\langle (1 - 2kd_n^1) D^2 \nabla w_n, D \nabla w_n \rangle + \langle \Delta \{w\}_n, \Delta D w_n \rangle \right. \\
& \quad \left. + \langle a\beta *_{\Delta t} D \Delta w_n, D \Delta w_n \rangle \right] \\
(4.18) \quad & = 2k\Delta t \sum_{n=1}^{N-1} \left[\varrho^{2n} \langle D w_n \nabla v_n^1 + D \nabla w_n v_n^1 + \nabla d_n^1 D^2 w_n, D \nabla w_n \rangle \right. \\
& \quad + \varrho^{2n} \langle (d_n^1 - d_n^2) D^2 \nabla u_n^2 + \nabla (d_n^1 - d_n^2) D^2 u_n^2, D \nabla w_n \rangle \\
& \quad \left. + \varrho^{2n} \langle (v_n^1 - v_n^2) D \nabla u_n^2 + \nabla (v_n^1 - v_n^2) D u_n^2, D \nabla w_n \rangle \right],
\end{aligned}$$

where $d_n^j = \{\tilde{u}^j\}_n$ and $v_n^j = D\tilde{u}^j$ for $j = 1, 2$. Performing estimates similar as above we obtain

$$\begin{aligned}
& \rho^{2(N-1)} \kappa \|\nabla \tilde{D}(u_{N-1}^1 - u_{N-1}^2)\|_{L^2}^2 + \rho^{2(N-1)} \|\Delta(u^1 - u^2)_{N-1}\|_{L^2}^2 \\
& \quad + (2C_\beta - \varsigma) \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \|\beta *_{\Delta t} D \Delta (u_n^1 - u_n^2)\|_{L^2}^2 \\
& \leq C_\Omega \Delta t \sum_{n=1}^{N-1} \varrho^{2n} (\|D \Delta u_n^2\|_{L^2}^2 + \|D^2 \nabla u_n^2\|_{L^2}^2) \left[\|\Delta(d_n^1 - d_n^2)\|_{L^2}^2 + \|\nabla(v_n^1 - v_n^2)\|_{L^2}^2 \right] \\
& \quad + C_\Omega \Delta t \sum_{n=1}^{N-1} \varrho^{2n} \left(\|\Delta v_n^1\|_{L^2} + \|\Delta \nabla d_n^1\|_{L^2} + \|\Delta \nabla d_n^1\|_{L^2}^2 \right) \|\tilde{D} \nabla (u_n^1 - u_n^2)\|_{L^2}^2 \\
& \quad + C_\Omega \Delta t \sum_{n=1}^{N-2} \varrho^{2n} \left(\left[1 + \frac{1}{\kappa} \|\nabla d_n^1\|_{L^\infty} \|\nabla v_n^1\|_{L^2} + \frac{1}{\kappa^2} \|\nabla d_n^1\|_{L^\infty}^2 (1 + \|D^2 \nabla u_n^2\|_{L^2}) \right. \right. \\
& \quad \left. \left. + \|D(\tilde{u}^1)_n\|_{L^\infty} \right] \|\tilde{D} \nabla (u_n^1 - u_n^2)\|_{L^2}^2 + \|\Delta(u^1 - u^2)_n\|_{L^2}^2 \right).
\end{aligned}$$

Applying the discrete Grönwall inequality we obtain the contraction property for an appropriate $\tau = N\Delta t$. Iterating over the time interval we obtain that there exists a fixed point of the map given by equation (4.11) and the corresponding stability conditions. \square

5. ERROR ESTIMATE

As next we derive the error estimates for the time-discretization scheme (4.11).

5.1. Expected smoothness at $t = 0$. Consider first the linear equation

$$(5.1) \quad \frac{1}{c^2} \partial_t^2 u - \Delta u - a\beta * \Delta \partial_t u = f,$$

with smooth f . The choice of β (either β_A or β_B) implies that for $k \in \mathbb{N}_0$

$$\int_0^t \beta(t-\tau) \tau^k d\tau \sim \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \tau^k d\tau = \frac{\Gamma(k+1)}{\Gamma(k+1+\mu)} t^{k+\mu} \quad \text{for } t \rightarrow 0^+.$$

Considering similar arguments as in [3, Remark 2.10], we thus expect that the behaviour at $t = 0$ of the solution to the linear problem to be given by

$$\partial_t^2 u(t) = c^2 \left(f + \Delta u_0 + \frac{1}{\Gamma(1+\mu)} t^\mu \Delta v_0 + \mathcal{O}(t) \right).$$

In the nonlinear case, as long as no singularity develops and $2ku < 1$ continues to hold, we expect the singularity at $t = 0$ to be of the same type

$$\partial_t^2 u(t) \sim w_0 + z_0 t^\mu + \mathcal{O}(t),$$

where $\mu \in (0, 1)$. This motivates the following assumption on the smoothness of the solution that will allow us to develop realistic error estimates.

Assumption 5.1. There exist constants $c_k \geq 0$, for $k = 0, \dots, 4$, such that

$$(5.2) \quad \|\partial_t^k u\|_{L^2(\Omega)} \leq C(1 + c_k t^{2+\mu-k}).$$

In what follows, for $u \in C[0, T]$ we shall use the following notation

$$\begin{aligned} Du(t_n) &= \frac{u(t_{n+1}) - u(t_{n-1}))}{2\Delta t}, & \tilde{D}u(t_n) &= \frac{u(t_{n+1}) - u(t_n)}{\Delta t}, \\ D^2u(t_n) &= \frac{u(t_{n+1}) - 2u(t_n) + u(t_{n-1}))}{\Delta t^2}, & \text{for } t_n \in [\Delta t, T - \Delta t], \\ \{u\}(t_n) &= \frac{1}{4}(u(t_{n+1}) + 2u(t_n) + u(t_{n-1})), & \text{for } t_n \in [\Delta t, T - \Delta t]. \end{aligned}$$

We will require the following lemma proved in [3] for $r = 0$. The extension to $r > 0$ is straightforward.

Lemma 5.2. For $u \in C^3(0, T]$ and any $t_n \in [\Delta t, T]$

(a)

$$\begin{aligned} |\beta * \partial_t u(t_n) - \beta *_{\Delta t} Du(t_n)| &\leq C \left\{ t_n^{\mu-1} \partial_t u(\Delta t) \Delta t \right. \\ &\quad \left. + \Delta t^2 \left(\partial_t^2 u(\Delta t) t_n^{\mu-1} + \int_{\Delta t}^{t_n} (t_n - \tau)^{\mu-1} |\partial_t^3 u(\tau)| d\tau \right) \right\}. \end{aligned}$$

(b)

$$|\beta * \partial_t u(t_n) - \beta \tilde{*}_{\Delta t} Du(t_n)| \leq C \Delta t^2 \left(\partial_t^2 u(\Delta t) t_n^{\mu-1} + \int_{\Delta t}^{t_n} (t_n - \tau)^{\mu-1} |\partial_t^3 u(\tau)| d\tau \right).$$

Proof. The result for $r = 0$, i.e., $\beta = \frac{1}{\Gamma(\mu)} t^{\mu-1}$ is shown in [3, Lemma 4.4] and [3, Lemma 4.5] respectively. For $r > 0$, the proof needs to be slightly modified. Namely, [3, Lemma 4.2] is straightforward to extend which then in turn implies the results following the same steps as in [3, Lemma 4.4]. \square

Theorem 5.3. Let u be the solution of (4.1) satisfying Assumption 5.1 and with initial data satisfying the conditions of Lemma 4.1. Let u_n be the solution of the semi-discrete scheme. If the uncorrected CQ is used, the error $w_n = u_n - u(t_n)$ satisfies the estimate

$$(5.3) \quad \sup_{1 \leq n \leq N-1} \|\tilde{D}w_n\|_{L^2(\Omega)}^2 + \sup_{1 \leq n \leq N-1} \|\nabla\{w\}_n\|_{L^2(\Omega)}^2 = \mathcal{O}(\Delta t).$$

If the corrected CQ is used the error bound becomes

$$(5.4) \quad \sup_{1 \leq n \leq N-1} \|\tilde{D}w_n\|_{L^2(\Omega)}^2 + \sup_{1 \leq n \leq N-1} \|\nabla\{w\}_n\|_{L^2(\Omega)}^2 = \mathcal{O}(\Delta t^{1+\mu}).$$

Proof. Consider the difference between the solution and approximation denoted by $w_n = u_n - u(t_n)$ to obtain

$$(5.5) \quad \begin{aligned} (1 - 2k\{u\}_n)D^2w_n - \Delta\{w_n\}_n - \beta *_{\Delta t} \Delta Dw_n &= 2kDw_n(Du_n + Du(t_n)) \\ &+ 2k\{w\}_n D^2u(t_n) + \varepsilon_n + \sigma_n + \delta_n + \kappa_n + \theta_n, \end{aligned}$$

where

$$\begin{aligned} \varepsilon_n &= (1 - 2ku(t_n))(D^2u(t_n) - \partial_t^2u(t_n)) \\ \sigma_n &= 2k(\{u\}(t_n) - u(t_n))D^2u(t_n), \\ \delta_n &= \beta *_{\Delta t} \Delta Du(t_n) - \beta * \Delta \partial_t u(t_n), \\ \kappa_n &= \Delta(\{u\}(t_n) - u(t_n)), \\ \theta_n &= 2k[(Du(t_n))^2 - (\partial_t u(t_n))^2]. \end{aligned}$$

Considering Dw_n as a test function in a weak formulation of (5.5) and summing over $n = 1, \dots, N-1$, yield

$$(5.6) \quad \begin{aligned} &\Delta t \sum_{n=1}^{N-1} \left[\langle (1 - 2k\{u\}_n)D^2w_n, Dw_n \rangle + \langle \nabla\{w_n\}_n, \nabla Dw_n \rangle \right] \\ &+ \Delta t \sum_{n=1}^{N-1} a \langle \beta *_{\Delta t} \nabla Dw_n, \nabla Dw_n \rangle = 2k\Delta t \sum_{n=1}^{N-1} \langle Dw_n [Du_n + Du(t_n)], Dw_n \rangle \\ &+ 2k\Delta t \sum_{n=1}^{N-1} \langle \{w\}_n D^2u(t_n), Dw_n \rangle + \Delta t \sum_{n=1}^{N-1} \langle \varepsilon_n + \sigma_n + \delta_n + \kappa_n + \theta_n, Dw_n \rangle. \end{aligned}$$

Using the boundedness of Du_n and $Du(t_n)$ we obtain

$$\langle Dw_n (Du_n + Du(t_n)), Dw_n \rangle \leq C(\|Du_n\|_{L^\infty} + \|\partial_t u\|_{L^\infty}) \|Dw_n\|_{L^2}^2.$$

Similarly using that $D^2u(t_n) \in L^4(\Omega)$, together with the Sobolev embedding inequality, we have

$$\begin{aligned} \langle \{w\}_n D^2u(t_n), Dw_n \rangle &\leq C\|D^2u(t_n)\|_{L^4} (\|\{w\}_n\|_{L^4}^2 + \|Dw_n\|_{L^2}^2) \\ &\leq C\|\partial_t^2 u\|_{L^4} (\|\nabla(w)_n\|_{L^2}^2 + \|\nabla(w)_{n-1}\|_{L^2}^2 + \|Dw_n\|_{L^2}^2). \end{aligned}$$

The last term in (5.6) we can write as

$$\begin{aligned} \Delta t \sum_{n=1}^{N-1} \left| \langle \varepsilon_n + \sigma_n + \delta_n + \kappa_n + \theta_n, Dw_n \rangle \right| &\leq C \left[\left(\Delta t \sum_{n=1}^{N-1} \|\varepsilon_n\|_{L^2} \right)^2 + \left(\Delta t \sum_{n=1}^{N-1} \|\sigma_n\|_{L^2} \right)^2 \right. \\ &\left. + \left(\Delta t \sum_{n=1}^{N-1} \|\delta_n\|_{L^2} \right)^2 + \left(\Delta t \sum_{n=1}^{N-1} \|\kappa_n\|_{L^2} \right)^2 + \left(\Delta t \sum_{n=1}^{N-1} \|\theta_n\|_{L^2} \right)^2 \right] + \zeta \sup_{1 \leq n \leq N-1} \|Dw_n\|_{L^2}^2. \end{aligned}$$

Assumption 5.1 allows us to bound the perturbation terms as in [3] by

$$\begin{aligned}
\Delta t \sum_{n=1}^{N-1} \|\varepsilon_n\|_{L^2} &= \Delta t \sum_{n=1}^{N-1} \left\| (1 - 2ku(t_n)) [D^2u(t_n) - \partial_t^2u(t_n)] \right\|_{L^2} \\
&\leq C\Delta t \int_0^{2\Delta t} \|\partial_t^3u\|_{L^2} dt + C\Delta t^2 \int_{\Delta t}^T \|\partial_t^4u\|_{L^2} dt \leq C\Delta t^2 (1 + (c_3 + c_4)\Delta t^{\mu-1}), \\
\Delta t \sum_{n=1}^{N-1} \|\sigma_n\|_{L^2} &= \Delta t \sum_{n=1}^{N-1} \left\| [\{u\}(t_n) - u(t_n)] D^2u(t_n) \right\|_{L^2} = \Delta t \sum_{n=1}^{N-1} \frac{\Delta t^2}{4} \|D^2u(t_n)\|_{L^4}^2 \\
&\leq C\Delta t^2 \|\partial_t^2u\|_{L^4(\Omega_T)}^2 \leq C\Delta t^2,
\end{aligned}$$

where we used that $\partial_t^2u \in L^\infty(0, T; H^1(\Omega))$.

Assuming that $\|\Delta \partial_t^2u\|_{L^2(\Omega_T)}^2$ is bounded, which can be shown assuming higher regularity of initial data, i.e. $u_0 \in H^4(\Omega)$, $v_0 \in H^3(\Omega)$, yields

$$\Delta t \sum_{n=1}^{N-1} \|\kappa_n\|_{L^2} = \Delta t \sum_{n=1}^{N-1} \|\Delta(\{u\}(t_n) - u(t_n))\|_{L^2} \leq C\Delta t^2 \|\Delta \partial_t^2u\|_{L^2(\Omega_T)} \leq C\Delta t^2.$$

To bound the CQ approximation error we use Lemma 5.2 to conclude

$$(5.7) \quad \Delta t \sum_{n=1}^{N-1} \|\delta_n\|_{L^2} = \Delta t \sum_{n=1}^{N-1} \left\| \Delta(\beta *_{\Delta t} Du(t_n) - \beta * \partial_t u(t_n)) \right\|_{L^2} \leq C\Delta t.$$

Instead if we are using the correction, then

$$(5.8) \quad \Delta t \sum_{n=1}^{N-1} \|\delta_n\|_{L^2} = \Delta t \sum_{n=1}^{N-1} \left\| \Delta(\beta \tilde{*}_{\Delta t} Du(t_n) - \beta * \partial_t u(t_n)) \right\|_{L^2} \leq C\Delta t^2.$$

The last error term is estimated as

$$\begin{aligned}
\Delta t \sum_{n=1}^{N-1} \|\theta_n\|_{L^2} &= 2k\Delta t \sum_{n=1}^{N-1} \left\| (Du(t_n))^2 - (\partial_t u(t_n))^2 \right\|_{L^2} \\
(5.9) \quad &\leq C\Delta t \sum_{n=1}^{N-1} \|Du(t_n) - \partial_t u(t_n)\|_{L^2} (\|\partial_t u(t_n)\|_{L^\infty} + \|Du(t_n)\|_{L^\infty}) \\
&\leq C\Delta t \sum_{n=1}^{N-1} \Delta t^2 \left(1 + c_3 \int_0^T t^{\mu-1} dt \right) \leq C\Delta t^2.
\end{aligned}$$

Hence combining all estimates and applying Grönwall inequality yields

$$\begin{aligned}
(5.10) \quad &\kappa \sup_{1 \leq n \leq N-1} \|\tilde{D}w_n\|_{L^2}^2 + \sup_{1 \leq n \leq N} \left\| \nabla \frac{w_n + w_{n-1}}{2} \right\|_{L^2}^2 \leq \left(\left\| \nabla \frac{w_0 + w_1}{2} \right\|_{L^2}^2 \right. \\
&+ (1 + 2krC_\Omega) \|\tilde{D}w_0\|_{L^2}^2 + C\Delta t^4 [1 + (c_3 + c_4)\Delta t^{2(\mu-1)}] + C_1\Delta t^2 \Big) \times \\
&\times \exp \left\{ C(\|\partial_t u\|_{L^\infty} + \|\partial_t^2u\|_{L^4} + \sup_{1 \leq n \leq N-1} \|Du_n\|_{L^\infty}) \right\},
\end{aligned}$$

where $C_1 = 0$ when using the corrected CQ scheme and $0 < \kappa \leq 1 - 2k\|u\|_{L^\infty}$.

As $w_0 = 0$, it remains to estimate $\|\nabla w_1\|$ and $\|\tilde{D}w_0\|_{L^2} = \Delta t^{-1} \|u_1 - u(t_1)\|_{L^2}$. The choice of u_1 in (4.8), Taylor expansion and Assumption 5.1 ensures that the two terms are of size $\mathcal{O}(\Delta t^{2+\mu})$ and $\mathcal{O}(\Delta t^{1+\mu})$ respectively. \square

Remark 5.4. Notice that $Dw_0 = 0$, thus we do not have any additional contribution for corrected CQ.

6. NUMERICAL EXPERIMENTS

Coupling the time discretization (4.1) with the piecewise linear Galerkin finite element space discretization we obtain a fully discrete scheme. Denoting by $V^h \subset H_0^1(\Omega)$ the space of piecewise linear finite element functions we have that the fully discrete solution $u_n^h \in V^h$ satisfies

$$(6.1) \quad \begin{aligned} \langle (1 - 2k\{u^h\}_n)D^2u_n^h, v \rangle + \langle \nabla\{u^h\}_n, \nabla v \rangle + a\langle \beta *_{\Delta t} D\nabla u_n^h, \nabla v \rangle \\ = 2k\langle (Du_n^h)^2, v \rangle, \end{aligned}$$

for all $v \in V^h$, $n = 1, \dots$. The initial data is set to

$$u_0^h = P_h u_0, \quad u_1^h = u_0^h + \Delta t P_h v_0 + \frac{1}{2} \Delta t^2 P_h \partial_t^2 u(0),$$

where $P_h: L^2(\Omega) \rightarrow V_h$ is the L^2 orthogonal projection and $P_h \partial_t^2 u(0)$ is obtained from (4.9). Throughout the numerical experiments we set $\beta = \beta_A$.

6.1. 1D Experiments. We first report on a series of experiments in 1D. To solve (6.1) at each time step for u_{n+1} we use a Newton iteration as described in [2, Chapter 7.1.2]. In space we use a uniform mesh with spatial meshwidth $h > 0$.

6.1.1. Test 1: Convergence. In this numerical experiment we solve (2.1) on $\Omega = (-1, 1)$ with initial data given by

$$u_0 = \sin(\pi x) \quad \text{and} \quad v_0 = \sin(\pi x),$$

choose the parameters $k = 0.09$, $r = 0$ and $a = 30$ and set the final time to $T = 1/2$. As error measure we use the maximum over time of the discrete energy error:

$$(6.2) \quad \begin{aligned} \text{error} = \max_{1 \leq n \leq N} \left\| \frac{u_n^{\text{ex}} - u_{n-1}^{\text{ex}}}{\Delta t} - \frac{u_n^h - u_{n-1}^h}{\Delta t} \right\|_{L^2} \\ + \max_{1 \leq n \leq N} \left\| \nabla \frac{u_n^{\text{ex}} + u_{n-1}^{\text{ex}}}{2} - \nabla \frac{u_n^h + u_{n-1}^h}{2} \right\|_{L^2}. \end{aligned}$$

As the exact solution is not available, for u^{ex} we use a numerical solution on a fine mesh with spatial mesh-width $h = 1.7 \times 10^{-3}$ and $\Delta t^{\text{ex}} = 3.1 \times 10^{-4}$. The same spatial-mesh is used for u_n^h with a range of time-steps Δt . Theorem 5.3 predicts $\mathcal{O}(\Delta t)$ convergence of the error if no correction is used and $\mathcal{O}(\Delta t^{1+\mu})$ for the scheme with the correction.

The numerical results for the version without the correction is shown in Figure 1. We see that similar rate of convergence is seen for both values of μ shown and that it is close to the predicted linear convergence. In Figure 2, where the convergence of the corrected scheme is shown, we see better convergence for larger μ closely following the predicted order $\mathcal{O}(\Delta t^{1+\mu})$.

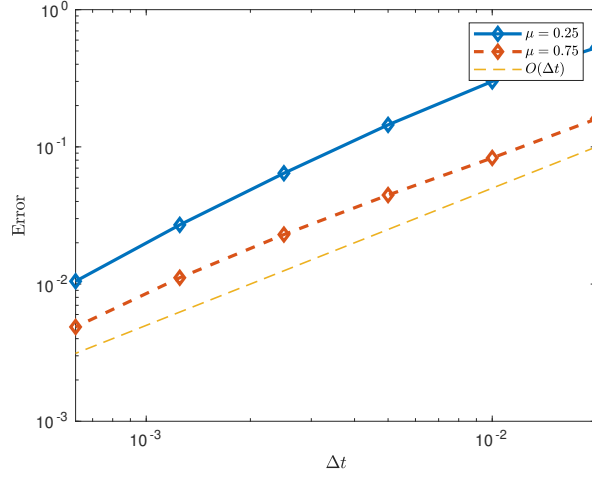


FIGURE 1. Convergence of the maximum energy error for the numerical scheme without the correction term for $\mu = 0.25$ and $\mu = 0.75$. Predicted convergence order of $\mathcal{O}(\Delta t)$ is also shown.

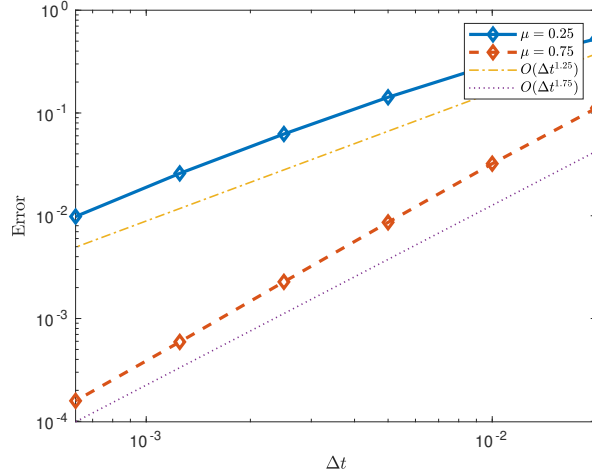


FIGURE 2. Convergence of the maximum energy error for the numerical scheme with the correction term included for $\mu = 0.25$ and $\mu = 0.75$. Predicted convergence order of $\mathcal{O}(\Delta t^{1+\mu})$ is also shown.

6.1.2. *Test 2: Changing k .* For the remaining 1D experiments we solve (2.1) on $\Omega = (0, 20)$ and, unless otherwise stated, consider the example with initial data given by a Gaussian

$$(6.3) \quad u_0 = 5e^{-\frac{(x-10)^2}{2}} \quad \text{and} \quad v_0 = 0.$$

Note that while strictly speaking $u_0 \notin H_0^1(\Omega)$, u_0 is zero close to machine precision on the boundary of Ω .

In Figure 3 we show the solution of (2.1) without the fractional derivative ($a = 0$) at time $T = 4$ for various choices of k . This figure shows that as k gets larger and $(1 - 2ku) \rightarrow 0$ the nonlinearity has a stronger effect on the wave form, resulting in the formation of a sawtooth shape.

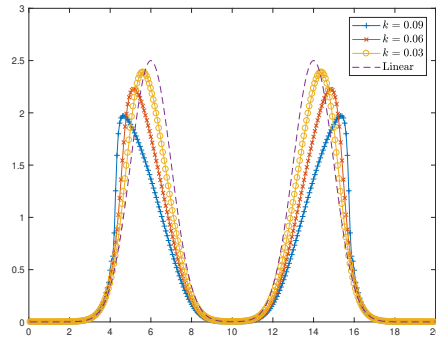


FIGURE 3. Solution of (2.1) at $T = 4$ approximated with the scheme (6.1) with $a = 0$ for various values of k .

When we reincorporate the fractional derivative, choosing $a = 1$, $r = 0$ and $\mu = 0.5$, we still see the damping from the nonlinearity but no longer observe the sawtooth formation. Instead the strong fractional damping term controls the form of the solution, causing more parabolic-like behaviour, i.e, the solution is trying to disperse rather than form a travelling wave; see Figure 4.

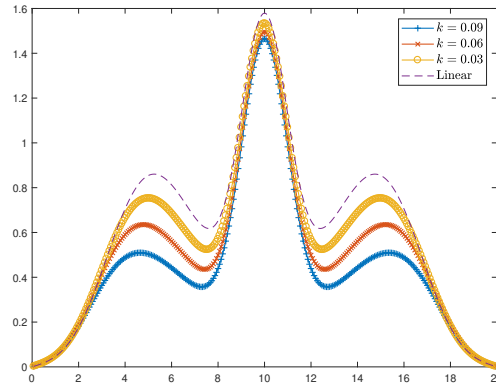


FIGURE 4. Solution of (2.1) at $T = 4$ approximated using the scheme (6.1) with $a = 1$, $\beta = \beta_A$, $\mu = 0.5$, and $r = 0$, for various values of k .

6.1.3. *Test 3: Changing a .* This test investigates how changing the size of the coefficient scaling the fractional derivative, with $\mu = 0.5$ and $r = 0$, affects the form of the solution over time. We let $k = 0.09$, since the previous experiment has shown

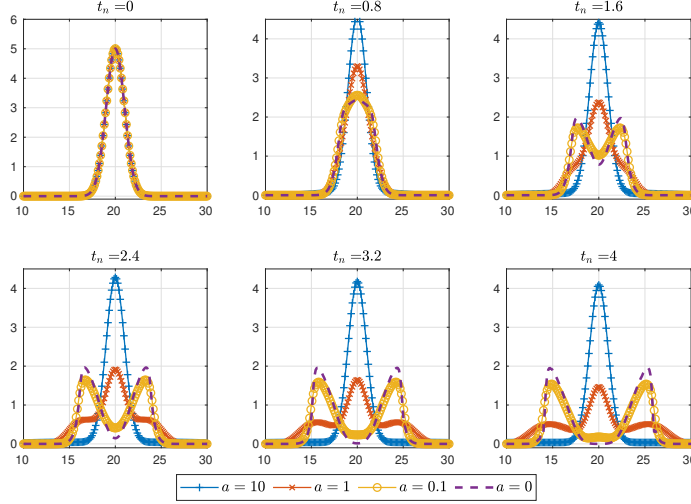


FIGURE 5. Solution of (2.1) at various time points up to $T = 4$ approximated with the scheme (6.1) with $k = 0.09$, $\mu = 0.5$, and $r = 0$, for various values of constant a .

that this will give a strong effect from the nonlinearity without causing shocks to form, at least up to time $T = 4$.

In these experiments we consider $\Omega = (0, 40)$ and the initial data

$$u_0 = 5e^{-\frac{(x-20)^2}{2}} \quad \text{and} \quad v_0 = 0.$$

Figure 5 shows the progression of the wave up to time $T = 4$ over regular intervals for different values of a . For $a = 10$ a travelling wave does not form, rather the solution attempts to disperse, and after the initial damping from $t_n = 0$ to $t_n = 0.8$ the solution is minimally damped. For $a = 0.1$ the nonlinearity has more control, since it appears almost identical to the solution with no involvement of the fractional derivative ($a = 0$). Lastly, the case $a = 1$ shows a balance of effects from the strong damping and nonlinear terms. Finally, for $a = 0$, letting the experiment run until $T = 8$, a shock seems to begin to form; see Figure 6.

6.1.4. *Test 4: Changing μ .* These experiments demonstrate the effect the order of the fractional operator has on the solution over time. In Figure 7 we show the solution at $T = 4$ with different values of μ , with and without the nonlinearity.

6.1.5. *Test 5: Changing r .* In this experiment we vary the value of r , recalling that we choose $\beta = \beta_A$ and that $\hat{\beta}_A(z) = (z + r)^{-\mu}$.

Considering the previously stated initial conditions (6.3) and fixing $a = 1$, $\mu = 0.5$, we compute the approximate solution up to end time $T = 4$. The solutions at various points up to the end time for the nonlinear ($k = 0.09$) and linear ($k = 0$) cases are presented in Figure 8 where we seen that as r gets larger the fractional operator displays weaker dispersive behaviour and the solution looks more like a travelling wave solution.

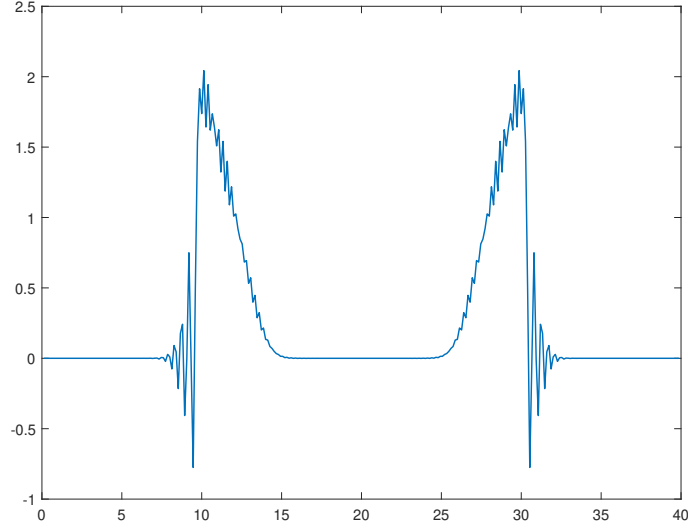


FIGURE 6. Solution of (2.1) at $T = 8$ approximated with the scheme (6.1) with $k = 0.09$ and fixing $a = 0$ to remove the fractional derivative.

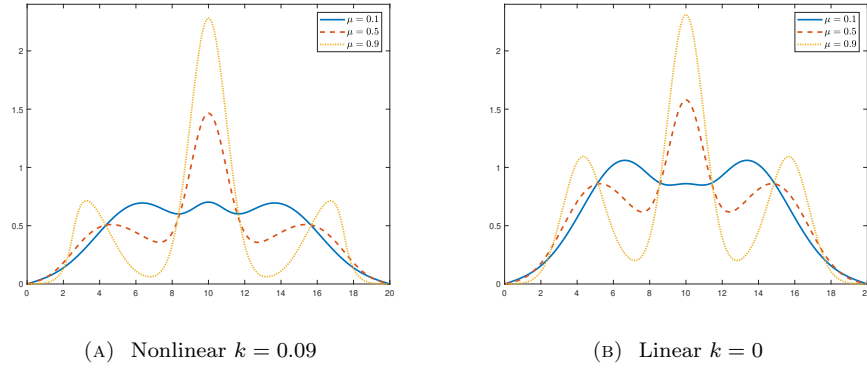


FIGURE 7. Solution of (2.1) at the end time $T = 4$ approximated with the scheme (6.1) with $a = 1$, $r = 0$ and varying values of $\mu \in (0, 1)$.

6.2. 2D Experiments. In this section we present results of numerical simulations for (2.1) in 2D, where $\Omega = (-1, 1)^2$ is a square. We let the exact solution be

$$(6.4) \quad u(x, t) = (\sin(24t) + \cos(12t)) \sin(\pi x) \sin(\pi y),$$

and choose a corresponding source term f so as to obtain the fully discrete system

$$(6.5) \quad \begin{aligned} \langle (1 - 2k\{u^h\}_n) D^2 u_n^h, v \rangle + \langle \nabla \{u^h\}_n, \nabla v \rangle + a \langle \beta *_{\Delta t} D \nabla u_n^h, \nabla v \rangle \\ = 2k \langle (Du_n^h)^2, v \rangle + \langle f^h, v \rangle, \end{aligned}$$

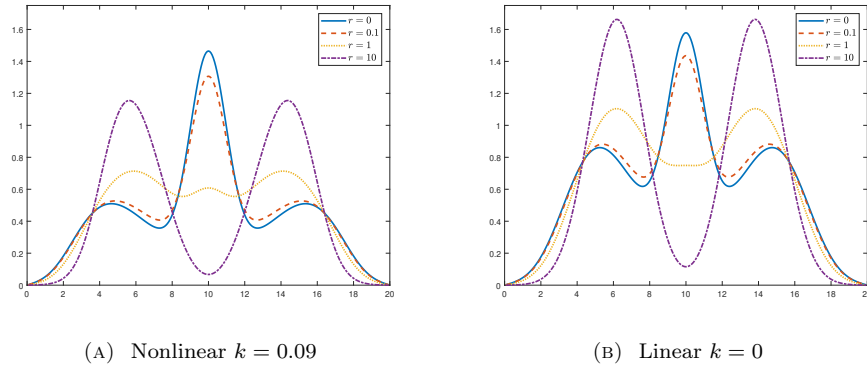


FIGURE 8. Solution of (2.1) at the end time $T = 4$ approximated with the scheme (6.1) with $a = 1$, $\mu = 0.5$ and varying values of r .

where standard CQ without the correction term is used. We again use $\beta = \beta_A$ and the various parameters are set to

$$a = 1, \quad k = 0.09, \quad \mu = 1/2, \quad r = 0.$$

The experiments were performed using the finite element library Netgen/NGSolve package [28]. In Figure 9, we show the projection of the initial data u_0 onto the finite element space, as well as the underlying automatically constructed triangular mesh.

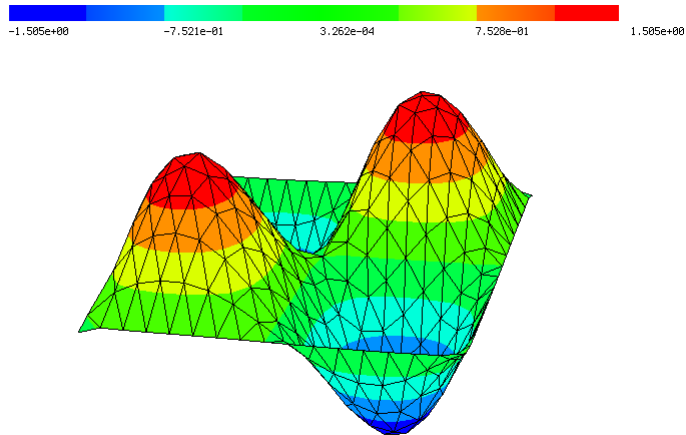


FIGURE 9. Projection of the initial data on the space of piecewise linear finite elements. The triangulation constructed by Netgen/NGSolve [28] is also seen.

To examine the convergence rate we compute the maximum L^2 -error, given by

$$(6.6) \quad \max_{1 \leq n \leq N} \|u_n - u(t_n)\|_{L^2}$$

on increasingly finer meshes. In Figure 10 we see, as expected, $\mathcal{O}(\Delta t)$ convergence.

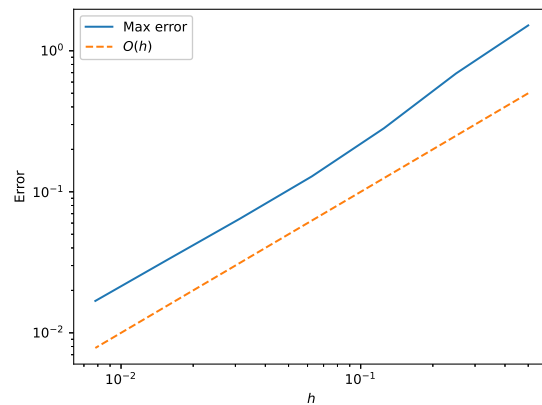


FIGURE 10. Maximum L^2 -error of the approximated solution to (2.1) generated using the scheme (6.5).

REFERENCES

1. Paola F. Antonietti, Ilario Mazziari, Markus Muhr, Vanja Nikolić, and Barbara Wohlmuth, *A high-order discontinuous Galerkin method for nonlinear sound waves*, *Journal of Computational Physics* **415** (2020), 109484.
2. Katherine Baker, *Linear and nonlinear wave equation models with power law attenuation*, Ph.D. thesis, Heriot-Watt University, 2022.
3. Katherine Baker and Lehel Banjai, *Numerical analysis of a wave equation for lossy media obeying a frequency power law*, *IMA Journal of Numerical Analysis* **42** (2021), no. 3, 2083–2117.
4. L Banjai and M López-Fernández, *Efficient high order algorithms for fractional integrals and fractional differential equations*, *Numerische Mathematik* **141** (2019), no. 2, 289–317.
5. Lehel Banjai, Christian Lubich, and Francisco-Javier Sayas, *Stable numerical coupling of exterior and interior problems for the wave equation*, *Numer. Math.* **129** (2015), no. 4, 611–646.
6. Lehel Banjai and Francisco-Javier Sayas, *Integral equation methods for evolutionary PDE*, *Springer Series in Computational Mathematics*, Springer, To be published in 2022.
7. Hermann Brunner, *Collocation methods for Volterra integral and related functional differential equations*, *Cambridge Monographs on Applied and Computational Mathematics*, vol. 15, Cambridge University Press, Cambridge, 2004.
8. W Dörfler, H Gerner, and Schnaubelt R, *Local well-posedness of a quasilinear wave equation*, *Applicable Analysis* **95** (2016), 2110–2123.
9. Francis Duck, *Physical properties of tissue: a comprehensive reference book*, Academic Press, 1990.
10. RH Fabiano and K Ito, *Semigroup theory and numerical approximation for equations in linear viscoelasticity*, *SIAM J Math Anal* **21** (1990), 374–393.
11. M Fabrizio, C Giorgi, and V Pata, *A new approach to equations with memory*, *Arch. Rational Mech. Anal.* **198** (2010), 189–232.
12. P Grisvard, *Elliptic problems in nonsmooth domains*, Pitman Publishing, 1985.
13. Marlis Hochbruck and Bernhard Maier, *Error analysis for space discretizations of quasilinear wave-type equations*, *IMA J. Numer. Anal.* **42** (2022), no. 3, 1963–1990.
14. B Kaltenbacher and I Lasiecka, *Global existence and exponential decay rates for the Westervelt equation*, *Discrete and Continuous Dynamical Systems - S* **2** (2009), no. 3, 503–523.
15. B Kaltenbacher and W Rundell, *On an inverse problem of nonlinear imaging with fractional damping*, *Mathematics of Computation* **91** (2022), 245–276.
16. S Larsson and F Saedpanah, *The continuous Galerkin method for an integro-differential equation modeling dynamic fractional order viscoelasticity*, *IMA J Numer Analysis* **30** (2010), 964–986.

17. C. Lubich, *Discretized fractional calculus*, SIAM Journal on Mathematical Analysis **17** (1986), no. 3, 704–719.
18. ———, *Convolution quadrature and discretized operational calculus. I*, Numerische Mathematik **52** (1988), no. 2, 129–145.
19. ———, *Convolution quadrature and discretized operational calculus. II*, Numer. Math. **52** (1988), no. 4, 413–425.
20. Christian Lubich, *Convolution quadrature revisited*, BIT **44** (2004), no. 3, 503–514.
21. Francesco Mainardi, *On some properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$, completely monotone for $t > 0$ with $0 < \alpha < 1$* , Discrete Contin. Dyn. Syst. Ser. B **19** (2014), no. 7, 2267–2278.
22. Charalambos G. Makridakis, *Finite element approximations of nonlinear elastic waves*, Math. Comp. **61** (1993), no. 204, 569–594. MR 1195426
23. Vanja Nikolić and Barbara Kaltenbacher, *On higher regularity for the Westervelt equation with strong nonlinear damping*, Applicable Analysis **95** (2016), no. 12, 2824–2840.
24. Vanja Nikolić and Barbara Wohlmuth, *A priori error estimates for the finite element approximation of Westervelt’s quasi-linear acoustic wave equation*, SIAM Journal on Numerical Analysis **57** (2019), no. 4, 1897–1918.
25. Lj. Oparnica and E. Süli, *Well-posedness of the fractional Zener wave equation for heterogeneous viscoelastic materials*, Fract. Calc. Appl. Anal. **23** (2020), no. 1, 126–166.
26. Igor Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, vol. 198, Academic Press, Inc., San Diego, CA, 1999, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.
27. F Saedpanah, *Well-posedness of an integro-differential equation with positive type kernels modeling fractional order viscoelasticity*, European J Mechanics -A/Solids **44** (2014), 201–211.
28. J. Schöberl, *NETGEN An advancing front 2D/3D-mesh generator based on abstract rules*, J. Comput. Visual. Sci. **1** (1997), 41–52.
29. Thomas Szabo, *Diagnostic ultrasound imaging: Inside out, second edition*, Academic Press, 2014.

APPENDIX: ESTIMATION DETAILS FOR THE WELL-POSEDNESS PROOF

To derive the first estimate in Lemma 3.2, we consider $\partial_t u$ as a test function in the weak formulation of (3.3) to obtain

$$\begin{aligned} \int_{\Omega_\tau} \left[\partial_t \left((1 - 2k\tilde{u}) |\partial_t u|^2 \right) + \partial_t |\nabla u|^2 + 2a\beta * \partial_t \nabla u \partial_t \nabla u \right] dxdt \\ \leq \int_{\Omega_\tau} 2k |\partial_t \tilde{u}| |\partial_t u|^2 dxdt, \end{aligned}$$

for $\tau \in (0, T]$, where $\Omega_\tau = (0, \tau) \times \Omega$. Using the nonnegativity of the third term on the left-hand side and the regularity of \tilde{u} , and applying the Grönwall inequality we obtain the first estimate in the proof of the lemma.

Considering $-\Delta \partial_t u$ as a test function in the weak formulation of (3.3) and integrating by parts in the first term we obtain

$$\begin{aligned} \int_{\Omega_\tau} \left[\partial_t \left((1 - 2k\tilde{u}) |\partial_t \nabla u|^2 \right) + \partial_t |\Delta u|^2 + 2a\beta * \partial_t \Delta u \partial_t \Delta u \right] dxdt \\ = \int_{\Omega_\tau} \left[4k \nabla \tilde{u} \partial_t^2 u \partial_t \nabla u + 2k \partial_t \tilde{u} |\partial_t \nabla u|^2 + 4k \partial_t \nabla \tilde{u} \partial_t u \partial_t \nabla u \right] dxdt, \quad \text{for } \tau \in (0, T]. \end{aligned}$$

Using that

$$(6.7) \quad \partial_t^2 u = \frac{1}{1 - 2k\tilde{u}} \left(\Delta u + a\beta * \partial_t \Delta u + 2k \partial_t \tilde{u} \partial_t u \right),$$

we can estimate the terms on the right-hand side as

$$\begin{aligned} \int_{\Omega_\tau} |\nabla \tilde{u} \partial_t^2 u \partial_t \nabla u| dx dt &\leq \frac{1}{2\kappa} \int_{\Omega_\tau} |\nabla \tilde{u}| \left(|\Delta u|^2 + |\partial_t \nabla u|^2 + 4k |\partial_t \tilde{u}| |\partial_t u| |\partial_t \nabla u| \right) dx dt \\ &\quad + \frac{1}{2\kappa^2} \frac{1}{\varsigma} \int_{\Omega_\tau} |\nabla \tilde{u}|^2 |\partial_t \nabla u|^2 dx dt + \frac{\varsigma a^2}{2} \int_{\Omega_\tau} |\beta * \partial_t \Delta u|^2 dx dt \\ &\leq \frac{1}{2\kappa} \int_0^\tau \|\nabla \tilde{u}\|_{L^\infty} \left(\|\Delta u\|_{L^2}^2 + \left[1 + \frac{1}{\kappa \varsigma} \|\nabla \tilde{u}\|_{L^\infty} + 4k C_\Omega \|\nabla \partial_t \tilde{u}\|_{L^2}^2 \right] \|\partial_t \nabla u\|_{L^2}^2 \right) dt \\ &\quad + \frac{\varsigma a^2}{2} \int_{\Omega_\tau} |\beta * \partial_t \Delta u|^2 dx dt \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega_\tau} \left[2k \partial_t \tilde{u} |\partial_t \nabla u|^2 + 4k \partial_t \nabla \tilde{u} \partial_t u \partial_t \nabla u \right] dx dt &\leq 2k \int_0^\tau \left[\|\partial_t \tilde{u}\|_{L^\infty} \|\partial_t \nabla u\|_{L^2}^2 \right. \\ &\quad \left. + 2C_\Omega \|\nabla \partial_t \tilde{u}\|_{L^4} \|\nabla \partial_t u\|_{L^2}^2 \right] dt. \end{aligned}$$

Then using estimate in Lemma 2.1 and applying the Grönwall inequalities yields the second estimate in the proof of Lemma 3.2.

Applying Δ to (3.3) and taking $\Delta \partial_t u$ as a test function in the weak formulation of the problem imply

$$\begin{aligned} &\int_{\Omega_\tau} \left(\partial_t \left((1 - 2k\tilde{u}) |\partial_t \Delta u|^2 \right) + \partial_t |\Delta \nabla u|^2 + 2a \beta * \partial_t \Delta \nabla u \partial_t \Delta \nabla u \right) dx dt \\ (6.8) \quad &= \int_{\Omega_\tau} \left[4k \Delta \tilde{u} \partial_t^2 u \partial_t \Delta u + 8k \nabla \tilde{u} \partial_t^2 \nabla u \partial_t \Delta u + 2k \partial_t \tilde{u} |\partial_t \Delta u|^2 \right. \\ &\quad \left. + 4k \partial_t \Delta \tilde{u} \partial_t u \partial_t \Delta u + 8k \partial_t \nabla \tilde{u} \partial_t \nabla u \partial_t \Delta u \right] dx dt = I_1 + I_2 + I_3. \end{aligned}$$

Using (6.7) together with

$$\begin{aligned} (6.9) \quad \partial_t^2 \nabla u &= \frac{1}{1 - 2k\tilde{u}} \left(\Delta \nabla u + a\beta * \partial_t \Delta \nabla u + 2k \partial_t \nabla \tilde{u} \partial_t u + 2k \partial_t \tilde{u} \partial_t \nabla u \right) \\ &\quad + \frac{2k \nabla \tilde{u}}{(1 - 2k\tilde{u})^2} \left(\Delta u + a\beta * \partial_t \Delta u + 2k \partial_t \tilde{u} \partial_t u \right), \end{aligned}$$

the terms on the right-hand side in (6.8) can be estimated as

$$\begin{aligned} |I_1| &\leq \frac{2k}{\kappa} \int_{\Omega_\tau} \left(2|\Delta \tilde{u}| |\Delta u| |\partial_t \Delta u| + 4k |\Delta \tilde{u}| |\partial_t \tilde{u}| |\partial_t u| |\partial_t \Delta u| \right) dx dt \\ &\quad + 2k^2 / (\kappa^2 \varsigma) \int_0^\tau \|\Delta \tilde{u}\|_{L^4}^2 \|\partial_t \Delta u\|_{L^2}^2 dt + 2\varsigma a^2 \int_0^\tau \|\beta * \partial_t \Delta u\|_{L^4}^2 dt \\ &\leq \frac{2k}{\kappa} \int_0^\tau \left[\|\Delta \tilde{u}\|_{L^4} (C_\Omega \|\Delta \nabla u\|_{L^2}^2 + \|\partial_t \Delta u\|_{L^2}^2) + 2k \left(4C_\Omega \|\Delta \tilde{u}\|_{L^4} \|\partial_t \nabla \tilde{u}\|_{L^2} \right. \right. \\ &\quad \left. \left. + \frac{1}{\kappa \varsigma} \|\Delta \tilde{u}\|_{L^4}^2 \right) \|\partial_t \Delta u\|_{L^2}^2 \right] dt + 2\varsigma a^2 C_\Omega \int_0^\tau \|\beta * \partial_t \Delta \nabla u\|_{L^2}^2 dt \end{aligned}$$

and

$$\begin{aligned}
|I_2| &\leq \frac{4k}{\kappa} \int_{\Omega_\tau} |\nabla \tilde{u}| \left[|\Delta \nabla u|^2 + 4k(|\nabla \partial_t \tilde{u}| |\partial_t u| + |\partial_t \tilde{u}| |\nabla \partial_t u| + |\partial_t \Delta u|) |\partial_t \Delta u| \right] dxdt \\
&\quad + \frac{8k^2}{\kappa^2 \varsigma} \int_0^\tau \|\nabla \tilde{u}\|_{L^\infty}^2 \|\partial_t \Delta u\|_{L^2}^2 dt + \varsigma a^2 \int_{\Omega_\tau} |\beta * \partial_t \Delta \nabla u|^2 dxdt \\
&\quad + \frac{16k^2}{\kappa} \int_{\Omega_\tau} \left(|\nabla \tilde{u}|^2 |\Delta u| |\partial_t \Delta u| + 2k |\nabla \tilde{u}|^2 |\partial_t \tilde{u}| |\partial_t u| |\partial_t \Delta u| \right) dxdt \\
&\quad + \frac{16k^2}{\kappa} a \int_{\Omega_\tau} |\nabla \tilde{u}|^2 |\beta * \partial_t \Delta u| |\partial_t \Delta u| dxdt \leq 8\varsigma a^2 \int_0^\tau \|\beta * \partial_t \Delta \nabla u\|_{L^2}^2 dt \\
&\quad + \frac{4k}{\kappa} \int_0^\tau \|\nabla \tilde{u}\|_{L^\infty} \left[(1 + 2kC_\Omega \|\nabla \tilde{u}\|_{L^4}) \|\Delta \nabla u\|_{L^2}^2 + 1 + 8k \|\nabla \partial_t \tilde{u}\|_{L^2} \right. \\
&\quad \left. + \frac{2k}{\kappa \varsigma} \|\nabla \tilde{u}\|_{L^\infty} + 2kC_\Omega \|\nabla \tilde{u}\|_{L^4} (1 + 8k \|\partial_t \nabla \tilde{u}\|_{L^2}) + \frac{2k^3}{\kappa \varsigma} \|\nabla \tilde{u}\|_{L^4}^3 \right] \|\partial_t \Delta u\|_{L^2}^2 dt.
\end{aligned}$$

The last three terms on the right-hand side in (6.8) are estimated as

$$\begin{aligned}
|I_3| &\leq 2k \int_0^\tau \left[\|\partial_t \tilde{u}\|_{L^\infty} \|\partial_t \Delta u\|_{L^2}^2 + 2 \|\partial_t \Delta \tilde{u}\|_{L^2} \|\partial_t u\|_{L^\infty} \|\partial_t \Delta u\|_{L^2} \right. \\
&\quad \left. + 4 \|\partial_t \nabla \tilde{u}\|_{L^4} \|\partial_t \nabla u\|_{L^4} \|\partial_t \Delta u\|_{L^2} \right] dt \leq 2k \int_0^\tau (C_\Omega + 2) \|\partial_t \Delta \tilde{u}\|_{L^2} \|\partial_t \Delta u\|_{L^2}^2 dt.
\end{aligned}$$

Collecting the estimates from above and applying the Grönwall inequality yields the third estimate in the proof of Lemma 3.2.

To show that $\mathcal{T}: \mathcal{K} \rightarrow \mathcal{K}$ is a contraction in the proof of Theorem 3.3 we consider (3.3) for \tilde{u}_1 and \tilde{u}_2 in \mathcal{K} and, taking $\Delta \partial_t(u_1 - u_2)$ as a test function in the weak formulation of the difference of the corresponding equations, obtain

$$\begin{aligned}
&\int_{\Omega_T} \left[\partial_t \left[(1 - 2k\tilde{u}_1) |\nabla \partial_t(u_1 - u_2)|^2 \right] + \partial_t |\Delta(u_1 - u_2)|^2 \right. \\
&\quad \left. + 2a\beta * \Delta \partial_t(u_1 - u_2) \Delta \partial_t(u_1 - u_2) \right] dxdt \\
&= 4k \int_{\Omega_T} \left[\nabla(\tilde{u}_1 - \tilde{u}_2) \partial_t^2 u_2 \nabla \partial_t(u_1 - u_2) + (\tilde{u}_1 - \tilde{u}_2) \partial_t^2 \nabla u_2 \nabla \partial_t(u_1 - u_2) \right. \\
&\quad \left. + \nabla \tilde{u}_1 \partial_t^2(u_1 - u_2) \nabla \partial_t(u_1 - u_2) \right] dxdt \\
&+ 2k \int_{\Omega_T} \left[\partial_t \tilde{u}_1 |\nabla \partial_t(u_1 - u_2)|^2 + 2\partial_t \nabla \tilde{u}_1 \partial_t(u_1 - u_2) \nabla \partial_t(u_1 - u_2) \right. \\
&\quad \left. + 2\partial_t(\tilde{u}_1 - \tilde{u}_2) \nabla \partial_t u_2 \nabla \partial_t(u_1 - u_2) + 2\partial_t \nabla(\tilde{u}_1 - \tilde{u}_2) \partial_t u_2 \nabla \partial_t(u_1 - u_2) \right] dxdt,
\end{aligned}$$

where $\Omega_T = (0, T) \times \Omega$. Using (6.7) and (6.9), the terms on the right-hand side of the last equality are estimated as

$$\begin{aligned}
|I_1| &\leq \int_{\Omega_T} |\nabla(\tilde{u}_1 - \tilde{u}_2) \partial_t^2 u_2 \nabla \partial_t(u_1 - u_2)| dxdt \leq \frac{1}{\kappa} \int_0^T \|\nabla(\tilde{u}_1 - \tilde{u}_2)\|_{L^4} \left(\|\Delta u_2\|_{L^4} \right. \\
&\quad \left. + a \|\beta * \partial_t \Delta u_2\|_{L^4} + 2k \|\partial_t u_2\|_{L^8} \|\partial_t \tilde{u}_2\|_{L^8} \right) \|\nabla \partial_t(u_1 - u_2)\|_{L^2} dt,
\end{aligned}$$

$$\begin{aligned}
|I_2| &\leq \int_{\Omega_T} |(\tilde{u}_1 - \tilde{u}_2) \partial_t^2 \nabla u_2 \nabla \partial_t(u_1 - u_2)| dx dt \leq \frac{1}{\kappa} \int_0^T \|\tilde{u}_{1,2}\|_{L^\infty} \left(a \|\beta * \Delta \nabla \partial_t u_2\|_{L^2} \right. \\
&\quad \left. + \|\Delta \nabla u_2\|_{L^2} + 2k \|\nabla \tilde{u}_2\|_{L^\infty} \left[a \|\beta * \Delta \partial_t u_2\|_{L^2} + 2k \|\partial_t u_2\|_{L^4} \|\partial_t \tilde{u}_2\|_{L^4} + \|\Delta u_2\|_{L^2} \right] \right. \\
&\quad \left. + 2k \|\partial_t \nabla \tilde{u}_2\|_{L^4} \|\partial_t u_2\|_{L^4} + 2k \|\partial_t \nabla u_2\|_{L^4} \|\partial_t \tilde{u}_2\|_{L^4} \right) \|\nabla \partial_t(u_1 - u_2)\|_{L^2} dt,
\end{aligned}$$

where $\tilde{u}_{1,2} := \tilde{u}_1 - \tilde{u}_2$,

$$\begin{aligned}
|I_3| &\leq \int_{\Omega_T} |\nabla \tilde{u}_1 \partial_t^2(u_1 - u_2) \nabla \partial_t(u_1 - u_2)| dx dt \\
&\leq \frac{1}{\kappa} \int_0^T \|\nabla \tilde{u}_1\|_{L^\infty} \left(\left[2k \|\partial_t u_2\|_{L^4} \|\nabla \partial_t(\tilde{u}_1 - \tilde{u}_2)\|_{L^2} + a \|\beta * \partial_t \Delta(u_1 - u_2)\|_{L^2} \right. \right. \\
&\quad \left. \left. + \|\Delta(u_1 - u_2)\|_{L^2} \right] \|\nabla \partial_t(u_1 - u_2)\|_{L^2} + 2k \|\partial_t \tilde{u}_1\|_{L^4} \|\nabla \partial_t(u_1 - u_2)\|_{L^2}^2 \right) dt,
\end{aligned}$$

and

$$\begin{aligned}
|I_4| &\leq 2k \int_0^T \left[\left(\|\partial_t \tilde{u}_1\|_{L^\infty} + 2 \|\nabla \partial_t \tilde{u}_1\|_{L^4} \right) \|\nabla \partial_t(u_1 - u_2)\|_{L^2}^2 \right. \\
&\quad \left. + 2 \left(\|\nabla \partial_t u_2\|_{L^4} + \|\partial_t u_2\|_{L^\infty} \right) \|\nabla \partial_t(\tilde{u}_1 - \tilde{u}_2)\|_{L^2} \|\nabla \partial_t(u_1 - u_2)\|_{L^2} \right] dt.
\end{aligned}$$

Then using that $\tilde{u}_j \in \mathcal{K}$, for $j = 1, 2$, and estimates for u_j in Lemma 3.2 we obtain the contraction inequality in Theorem 3.3, which, by applying the Banach fixed point theorem, ensures the existence of a unique fixed of the map \mathcal{T} and hence the existence of a unique solution of the nonlinear problem (2.7).

DEPARTMENT OF MATHEMATICS, HERIOT-WATT UNIVERSITY
Email address: k.baker@hw.ac.uk

DEPARTMENT OF MATHEMATICS, HERIOT-WATT UNIVERSITY
Email address: l.banjai@hw.ac.uk

DEPARTMENT OF MATHEMATICS, HERIOT-WATT UNIVERSITY
Email address: m.ptashnyk@hw.ac.uk