N=2 quiver gauge theories on A-type ALE spaces

Citation for published version:
Bruzzo, U, Sala, F & Szabo, RJ 2014, 'N=2 quiver gauge theories on A-type ALE spaces', *Letters in Mathematical Physics*, vol. 105, no. 3, pp. 401-445. https://doi.org/10.1007/s11005-014-0734-x

Digital Object Identifier (DOI):
10.1007/s11005-014-0734-x

Link:
Link to publication record in Heriot-Watt Research Portal

Document Version:
Peer reviewed version

Published In:
Letters in Mathematical Physics

$\mathcal{N} = 2$ QUIVER GAUGE THEORIES ON A-TYPE ALE SPACES

UGO BRUZZO[§‡], FRANCESCO SALA[•] and RICHARD J. SZABO[¶⋆∘]

[§] Scuola Internazionale Superiore di Studi Avanzati (SISSA),
Via Bonomea 265, 34136 Trieste, Italia

[‡] Istituto Nazionale di Fisica Nucleare, Sezione di Trieste

[•] Department of Mathematics, The University of Western Ontario,
Middlesex College, London N6A 5B7, Ontario, Canada

[¶] Department of Mathematics, Heriot-Watt University,
Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, United Kingdom;

[⋆] Maxwell Institute for Mathematical Sciences, Edinburgh, United Kingdom

[∘] The Tait Institute, Edinburgh, United Kingdom

ABSTRACT. We survey and compare recent approaches to the computation of the partition functions and correlators of chiral BPS observables in $\mathcal{N} = 2$ gauge theories on ALE spaces based on quiver varieties and the minimal resolution X_k of the A_{k-1} toric singularity $\mathbb{C}^2/\mathbb{Z}_k$, in light of their recently conjectured duality with two-dimensional coset conformal field theories. We review and elucidate the rigorous constructions of gauge theories for a particular family of ALE spaces, using their relation to the cohomology of moduli spaces of framed torsion free sheaves on a suitable orbifold compactification of X_k. We extend these computations to generic $\mathcal{N} = 2$ superconformal quiver gauge theories, obtaining in these instances new constraints on fractional instanton charges, a rigorous proof of the Nekrasov master formula, and new quantizations of Hitchin systems based on the underlying Seiberg-Witten geometry.

Date: October 2014

2010 Mathematics Subject Classification: 14D20, 14D21, 14J80, 81T13, 81T60

Keywords: stacks, framed sheaves, ALE spaces, supersymmetric gauge theories, partition functions, blowup formulas

E-Mail: `bruzzo@sissa.it, salafra83@gmail.com, R.J.Szabo@hw.ac.uk`

CONTENTS

1. INTRODUCTION AND SUMMARY

Asymptotically locally Euclidean (ALE) spaces have received a great deal of attention over the years as examples of self-dual solutions to the Einstein equations in four dimensions with vanishing cosmological constant [29], and as spaces on which the construction of (anti-)selfdual solutions to the Yang-Mills equations (instantons) can be explicitly carried out in much the same way as on flat Euclidean space $\mathbb{R}^4$ [35]. They are diffeomorphic to resolutions of the complex quotient singularities $\mathbb{C}^2/\Gamma$ for Γ a finite subgroup of $\mathrm{SU}(2)$ [34]; in this paper we focus on the special case $\Gamma=\mathbb{Z}_k$. They became an important testing ground for Montonen-Olive S-duality conjectures with the realisation [52] that the partition functions of topologically twisted $\mathcal{N}=4$ supersymmetric Yang-Mills theory on ALE spaces, which are generating functions for the Euler characteristics of instanton moduli spaces, reproduce (modular invariant) characters of affine Lie algebras in accordance with Nakajima's geometric construction of highest weight representations of Kac-Moody algebras based on quiver varieties [38]. These instanton moduli spaces find a natural realisation in Type II string theory where they arise as Higgs branches of certain quiver gauge theories (with eight real supercharges) which appear as worldvolume field theories on Dp-branes in Dp-D$(p+4)$ systems with the D$(p+4)$-branes located at the fixed point of the orbifold $\mathbb{R}^4/\mathbb{Z}_k$ [20].

Gauge theories on ALE spaces with $\mathcal{N}=2$ supersymmetry have received renewed impetus in recent years as their partition functions extend the connection to Nakajima's construction beyond the level of affine characters, and hence provide extensions to curved manifolds of the duality between $\mathcal{N}=2$ supersymmetric Yang-Mills theory on $\mathbb{R}^4$ and two-dimensional conformal field theories [5, 48, 3, 51]. A crucial ingredient in all these correspondences is the role played by *fractional instantons*, which are those nonperturbative configurations of the gauge theory which are stuck at the orbifold singularity of $\mathbb{C}^2/\mathbb{Z}_k$; on the resolution of the singularity they correspond to magnetic monopoles on the exceptional divisors of the ALE space which carry fractional topological charge. For the topologically twisted $\mathcal{N}=4$ gauge theory their contributions have been addressed from various points of view [26, 31, 15]. For topologically twisted $\mathcal{N}=2$ gauge theories most of the literature has been concerned with their interpretation as gauge theories on the orbifold $\mathbb{R}^4/\mathbb{Z}_k$ [25, 27]. Until very recently [8, 12] there has been no first principles analysis of them from the perspective of moduli spaces of instantons directly on the resolutions.

It is by now customary to consider not a conventional gauge theory but rather a noncommutative deformation of it, with the noncommutativity related to the supergravity two-form B-field on the D-branes in the Type II string theory picture. Depending on the choice of stability parameters, the instantons of this gauge theory can correspond to torsion free sheaves [47]; the resulting moduli spaces then provide a smooth completion of the instanton moduli spaces. This point of view will be adopted in the present paper.

The purpose of this paper is threefold. Firstly, we shall survey the various approaches to computing the partition functions of $\mathcal{N}=2$ gauge theories on the A_{k-1}-type ALE spaces, comparing and contrasting the different results. Secondly, we present a summary of the main results from [12] which provided the first rigorous computations of $\mathcal{N}=2$ gauge theory partition functions on ALE spaces from a moduli theory of framed sheaves on the minimal resolution X_k and presented the first step in providing an interpretation of Nakajima quiver varieties with real stability parameter in the chamber associated with X_k in terms of sheaves on toric Deligne-Mumford stacks (in the complex analytic setting, a similar description by using sheaves on complex V-manifolds was provided in [39]). The main technical ingredient in this approach is the construction of moduli spaces of framed torsion free sheaves on a suitable orbifold compactification of X_k, which is the proper mathematical arena to incorporate the contributions from fractional instantons; in the following we shall present these constructions in a somewhat more informal manner that we hope will be more palatable to physicists. The duality between abelian $\mathcal{N}=2$ quiver gauge theories in this setting and two-dimensional conformal field theory was recently formulated and proved in [49]. Thirdly, we extend the computations of [12] to calculate the partition functions of generic $\mathcal{N}=2$ superconformal quiver gauge theories on X_k, obtaining many new results that are summarised below.

The outline of this paper is as follows. In Section 2 we review the relations between Nakajima quiver varieties and moduli spaces of instantons on ALE spaces. In Section 3 we review the calculations of [25, 27] which are based on the interpretation of moduli spaces of ALE instantons as a resolution of the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_k$. In Section 4 we review the calculations of [8] which are based on the minimal resolution X_k of the toric singularity $\mathbb{C}^2/\mathbb{Z}_k$ and use the conjectural Nekrasov master formula [43]. In Section 5 we review the constructions from [12] of moduli spaces parameterizing framed sheaves on an orbifold compactification of X_k; these moduli spaces are used to rigorously define the partition functions and correlators of chiral BPS operators for $\mathcal{N}=2$ gauge theories on X_k, thus proving the factorization formulas involving partition functions on the affine torus-invariant open subsets of X_k as conjectured by [6, 7, 8]. In Section 6 we extend the calculations of [12] to arbitrary $\mathcal{N}=2$

superconformal quiver gauge theories on X_k, obtaining new constraints on fractional instanton charges in these instances; we also present a new computation of the perturbative partition functions and hence provide a rigorous proof of the Nekrasov master formula for X_k. We further explore the Seiberg-Witten geometry and elaborate on the interpretations of these gauge theories as quantizations of certain Hitchin systems. Two appendices at the end of the paper summarise some technical details which are used in the main text: in Appendix A we discuss some aspects of equivariant cohomology which are used to compute instanton partition functions, while in Appendix B we list some of the factors which enter the explicit expressions for the partition functions.

Acknowledgements. This work was supported in part by PRIN "Geometria delle varietà algebriche," by GNSAGA-INdAM, by the Consolidated Grant ST/J000310/1 from the UK Science and Technology Facilities Council and by AG Laboratory SU-HSE, RF Government Grant ag.11.G34.31.0023. Most of this paper was written while the second author was staying at Laboratory of Algebraic Geometry and its Applications, SU-HSE, and at the Department of Mathematics, Heriot-Watt University; he thanks these institutions for hospitality and support.

2. Quiver varieties and instantons on ALE spaces

ALE spaces are hyper-Kähler four-manifolds with one end at infinity which resembles a quotient $\mathbb{R}^4/\Gamma$, for $\Gamma \subset \mathrm{SU}(2)$ a finite group acting isometrically on $\mathbb{R}^4 \simeq \mathbb{C}^2$. A large class of ALE spaces was discovered by Gibbons and Hawking [29]. Kronheimer [34] realized them as hyper-Kähler quotients. Kronheimer and Nakajima [35] constructed moduli spaces of $\mathrm{U}(r)$ instantons on ALE spaces with fixed Chern character and flat connection at infinity by means of an ADHM-type construction; they are also hyper-Kähler manifolds. These moduli spaces were completed by Nakajima [38] via a modification of the ADHM equations, obtaining the Nakajima quiver varieties. In this section we describe the relation between Nakajima quiver varieties of type $\widehat{A}_{k-1}$ and moduli spaces of instantons on ALE spaces of type A_{k-1} for an integer $k \geq 2$.

2.1. **Quiver varieties of type $\widehat{A}_{k-1}$.** Let $\mathtt{Q} = (\mathtt{Q}_0, \mathtt{Q}_1, \mathtt{s}, \mathtt{t})$ be a quiver, i.e. an oriented graph with a finite set of vertices $\mathtt{Q}_0$, a finite set of edges $\mathtt{Q}_1$, and two projection maps $\mathtt{s}, \mathtt{t} \colon \mathtt{Q}_1 \rightrightarrows \mathtt{Q}_0$ which assign to each oriented edge its source and target vertex respectively. We denote by $\mathtt{Q}^{\mathrm{op}}$ the opposite quiver obtained from $\mathtt{Q}$ by reversing the orientation of the edges, and by $\bar{\mathtt{Q}}$ the double of $\mathtt{Q}$, i.e. the quiver with the same vertex set as $\mathtt{Q}$ and whose edge set is the disjoint union of the edge sets of $\mathtt{Q}$ and of $\mathtt{Q}^{\mathrm{op}}$.

Let $k \geq 2$ be an integer. For the rest of this section $\mathtt{Q}$ is the quiver whose underlying graph is that of the affine Dynkin diagram of type $\widehat{A}_{k-1}$, i.e. for $k=2$

$$\begin{tikzcd} \overset{0}{\circ} \arrow[r, shift left] & \overset{1}{\circ} \arrow[l, shift left] \end{tikzcd}$$

and for $k \geq 3$

$$\begin{tikzcd} & & \overset{0}{\circ} \arrow[lld] \arrow[rrd] & & \\ \overset{1}{\circ} \arrow[r] & \overset{2}{\circ} \arrow[r] & \cdots \arrow[r] & \overset{k-2}{\circ} \arrow[r] & \overset{k-1}{\circ} \end{tikzcd}$$

Let $\vec{v}, \vec{w} \in (\mathbb{Z}_{\geq 0})^k$ and let $V := \bigoplus_{i \in \mathtt{Q}_0} V_i$, $W := \bigoplus_{i \in \mathtt{Q}_0} W_i$ be $\mathtt{Q}_0$-graded complex vector spaces with $\dim V_i = v_i$, $\dim W_i = w_i$. Let

$$M(\vec{v}, \vec{w}\,) := \Big(\bigoplus_{e \in \bar{\mathtt{Q}}_1} \mathrm{Hom}\big(V_{\mathtt{s}(e)}, V_{\mathtt{t}(e)}\big) \Big) \oplus \Big(\bigoplus_{i \in \bar{\mathtt{Q}}_0} \mathrm{Hom}\big(W_i, V_i\big) \oplus \mathrm{Hom}\big(V_i, W_i\big) \Big)$$

be a representation space for the double quiver $\bar{\mathsf{Q}}$. The group $\mathrm{GL}_{\vec{v}} := \prod_{i\in\mathsf{Q}_0} \mathrm{GL}(V_i)$ acts on the affine space $M(\vec{v},\vec{w}\,)$ by

$$(g_j)_{j\in\mathsf{Q}_0} \triangleright (B_e, a_i, b_i) := \big(g_{\mathtt{t}(e)}\, B_e\, g_{\mathtt{s}(e)}^{-1}\,,\ g_i\, a_i\,,\ b_i\, g_i^{-1}\big) \tag{2.1}$$

for $g_i \in \mathrm{GL}(V_i)$, $B_e \in \mathrm{Hom}\big(V_{\mathtt{s}(e)}, V_{\mathtt{t}(e)}\big)$, $a_i \in \mathrm{Hom}\big(W_i, V_i\big)$ and $b_i \in \mathrm{Hom}\big(V_i, W_i\big)$. One can define a symplectic form on $M(\vec{v},\vec{w}\,)$ which is preserved by this $\mathrm{GL}_{\vec{v}}$-action (see e.g. [39, Section 1(i)]). Then the corresponding moment map $\mu_{\mathbb{C}}\colon M(\vec{v},\vec{w}\,) \to \mathfrak{gl}_{\vec{v}} := \bigoplus_{i\in\mathsf{Q}_0} \mathrm{End}(V_i)$ which vanishes at the origin is given by

$$\mu_{\mathbb{C}}(B,a,b) := \Bigg(\sum_{\substack{e\in\bar{\mathsf{Q}}_1\\ \mathtt{t}(e)=i}} \epsilon(e)\, B_e\, B_{\bar{e}} + a_i\, b_i\Bigg)_{i\in\mathsf{Q}_0}\,,$$

where $\bar{e}\in\mathsf{Q}_1^{\rm op}$ is the reverse edge of e with $\epsilon(e)=1$ and $\epsilon(\bar{e})=-1$ for $e\in\mathsf{Q}_1$.

Let us fix $\xi=(\xi_{\mathbb{C}},\xi_{\mathbb{R}})\in\mathbb{C}^k\oplus\mathbb{R}^k$. We define an element corresponding to $\xi_{\mathbb{C}}$ in the center of $\mathfrak{gl}_{\vec{v}}$ by $\bigoplus_{i\in\mathsf{Q}_0} \xi_{\mathbb{C}}^i\, \mathrm{id}_{V_i}$, where we delete the summand corresponding to node i if $V_i=\{0\}$. One can introduce the notion of $\xi_{\mathbb{R}}$-(semi)stable points in $M(\vec{v},\vec{w}\,)$ (see e.g. [39, Definition 1.1]). One says that two $\xi_{\mathbb{R}}$-semistable points (B,a,b) and $(B',a',b'\,)$ are S-equivalent, and writes $(B,a,b)\sim(B',a',b'\,)$, when the closures of their $\mathrm{GL}_{\vec{v}}$-orbits intersect inside the $\xi_{\mathbb{R}}$-semistable locus $\big(\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})\big)^{\rm ss}$ of $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})$. Then we define the *Nakajima quiver variety* associated with $\vec{v}$, $\vec{w}$ and ξ as the quotient

$$\mathcal{M}_\xi(\vec{v},\vec{w}\,) := \big(\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})\big)^{\rm ss} \big/ \sim\ .$$

Below we also use the superscript 's' to denote $\xi_{\mathbb{R}}$-stable points.

The Cartan matrix of the affine Dynkin diagram of type $\widehat{A}_{k-1}$ is $\widehat{C} = 2\,\mathrm{id}_{\mathbb{C}^k} - A$, where $A=(a_{i,j})_{i,j\in\bar{\mathsf{Q}}_0}$ is the adjacency matrix of the double quiver $\bar{\mathsf{Q}}$, i.e. $a_{i,j}$ is the number of edges from i to j in $\bar{\mathsf{Q}}_1$. Explicitly, for $k=2$ the matrix $\widehat{C}$ is given by

$$\widehat{C} = \begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix}$$

and for $k\geq 3$

$$\widehat{C} = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1\\ -1 & 2 & -1 & \cdots & 0\\ 0 & -1 & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & 0 & 0 & \cdots & 2\end{pmatrix}\,.$$

Let $\widehat{\mathfrak{Q}}_+$ be the set of positive roots of the affine Dynkin diagram of type $\widehat{A}_{k-1}$, i.e. the set

$$\widehat{\mathfrak{Q}}_+ := \big\{\theta=(\theta_i)_{i\in\mathsf{Q}_0}\in(\mathbb{Z}_{\geq 0})^k \,\big|\, \theta\cdot\widehat{C}\theta\leq 2\big\}\ \setminus\ \{0\}\,.$$

Define the *wall* D_θ associated with the root $\theta\in\widehat{\mathfrak{Q}}_+$ by

$$\mathsf{D}_\theta := \Big\{x=\big(x^i\big)_{i\in\mathsf{Q}_0}\in\mathbb{R}^k\ \Big|\ \sum_{i\in\mathsf{Q}_0} x^i\,\theta_i = 0\Big\}\ \subset\ \mathbb{R}^k\,.$$

For $\vec{v}\in(\mathbb{Z}_{\geq 0})^k$ we set

$$\widehat{\mathfrak{Q}}_+(\vec{v}\,) := \big\{\theta\in\widehat{\mathfrak{Q}}_+ \,\big|\, \theta_i\leq v_i\quad \forall i\in\mathsf{Q}_0\big\}\,.$$

Although $\widehat{\mathfrak{Q}}_+$ is an infinite set, the set $\widehat{\mathfrak{Q}}_+(\vec{v}\,)$ is always finite.

An element $\xi\in\mathbb{C}^k\oplus\mathbb{R}^k$ is *generic* with respect to $\vec{v}\in(\mathbb{Z}_{\geq 0})^k$ if for any root $\theta\in\widehat{\mathfrak{Q}}_+(\vec{v}\,)$ one has

$$\xi\ \notin\ \mathsf{D}_\theta\otimes\mathbb{R}^3\subset\mathbb{R}^k\otimes\mathbb{R}^3\simeq\mathbb{C}^k\oplus\mathbb{R}^k\,.$$

By [38, Theorem 2.8], if ξ is generic with respect to $\vec{v} \in (\mathbb{Z}_{\geq 0})^k$ then $\mathcal{M}_\xi(\vec{v}, \vec{w}\,)$ is a smooth connected variety of dimension

$$\dim \mathcal{M}_\xi(\vec{v}, \vec{w}\,) = 2\vec{w} \cdot \vec{v} - \vec{v} \cdot \widehat{C}\vec{v}\ .$$

Fix a complex parameter $\xi_{\mathbb{C}}$. A connected component of the affine space

$$\mathbb{R}^k \setminus \bigcup_{\substack{\theta \in \widehat{\mathfrak{Q}}_+(\vec{v}) \\ \xi_{\mathbb{C}} \cdot \theta = 0}} \mathsf{D}_\theta \tag{2.2}$$

is called a *chamber*. It is known that [39, Section 1(i)]:

- If $\xi_{\mathbb{R}}, \xi'_{\mathbb{R}}$ lie in two distinct chambers such that $\xi = (\xi_{\mathbb{C}}, \xi_{\mathbb{R}})$ and $\xi' = (\xi_{\mathbb{C}}, \xi'_{\mathbb{R}})$ are generic with respect to $\vec{v} \in (\mathbb{Z}_{\geq 0})^k$, then the quiver varieties $\mathcal{M}_\xi(\vec{v}, \vec{w}\,)$ and $\mathcal{M}_{\xi'}(\vec{v}, \vec{w}\,)$ are diffeomorphic.
- If $\xi_{\mathbb{R}}, \xi'_{\mathbb{R}}$ lie in the same chamber such that $\xi = (\xi_{\mathbb{C}}, \xi_{\mathbb{R}})$ and $\xi' = (\xi_{\mathbb{C}}, \xi'_{\mathbb{R}})$ are generic with respect to $\vec{v} \in (\mathbb{Z}_{\geq 0})^k$, then the quiver varieties $\mathcal{M}_\xi(\vec{v}, \vec{w}\,)$ and $\mathcal{M}_{\xi'}(\vec{v}, \vec{w}\,)$ are isomorphic (as algebraic varieties).

2.2. **ALE spaces of type** A_{k-1}. Fix an integer $k \geq 2$ and let ω be a primitive k-th root of unity. We define an action of the cyclic group $\mathbb{Z}_k$ of order k on $\mathbb{R}^4 \simeq \mathbb{C}^2$ as $\omega \triangleright (z, w) := (\omega\, z, \omega^{-1}\, w)$; it has a unique fixed point at the origin. The quotient $\mathbb{C}^2/\mathbb{Z}_k$ is a normal affine toric surface, with a Kleinian singularity at the origin. If $\varphi_k \colon X_k \to \mathbb{C}^2/\mathbb{Z}_k$ is the minimal resolution of the singularity of $\mathbb{C}^2/\mathbb{Z}_k$, then X_k is a smooth toric surface.

Let $\delta = (1, \dots, 1) \in (\mathbb{Z}_{\geq 0})^k$. The wall $\mathsf{D}_\delta = \big\{x = (x^i) \in \mathbb{R}^k \,\big|\, \sum_{i \in \mathsf{Q}_0} x^i = 0\big\}$ is called the *level zero hyperplane*. Let $\xi^\circ = (\xi^\circ_{\mathbb{C}}, \xi^\circ_{\mathbb{R}})$ be an element in $\mathsf{D}_\delta \otimes \mathbb{R}^3$ which is not contained in any $\mathsf{D}_\theta \otimes \mathbb{R}^3$ for $\theta \in \widehat{\mathfrak{Q}}_+ \setminus \{\delta\}$; we shall say that ξ° is generic. Then we define the *ALE space X_{ξ° of type A_{k-1} with parameter ξ°* by the geometric quotient

$$X_{\xi^\circ} := \big(\mu_{\mathbb{C}}^{-1}(-\xi_{\mathbb{C}})\big)^{\mathrm{s}} \big/ \big(\mathrm{GL}_\delta/\mathbb{C}^*\big)\ ,$$

where we consider $(-\xi_{\mathbb{R}})$-stable points of $\mu_{\mathbb{C}}^{-1}(-\xi_{\mathbb{C}})$.

It is proven in [34] (see also [38, Proposition 2.12]) that X_{ξ° is a four-dimensional hyper-Kähler manifold which is diffeomorphic to X_k; it furthermore carries a hyper-Kähler ALE metric g of order four, i.e., there is a compact subset $K \subset X_{\xi^\circ}$ and a diffeomorphism $X_{\xi^\circ} \setminus K \to (\mathbb{R}^4 \setminus B_r(0))/\mathbb{Z}_k$ under which the metric g is approximated by the standard Euclidean metric on $\mathbb{R}^4/\mathbb{Z}_k$, and in local Euclidean coordinates $(x_\alpha)_{\alpha=1}^4$ one has

$$g_{\alpha\beta} = \delta_{\alpha\beta} + b_{\alpha\beta}$$

with $\partial^p b_{\alpha\beta} = O(r^{-4-p})$ for $p \gg 0$, where $r^2 = \sum_{\alpha=1}^4 x_\alpha^2$ and ∂ denotes differentiation with respect to the coordinates x_α. Here $B_r(0)$ is the ball of radius r centered at the origin of $\mathbb{R}^4$. We call $X_{\xi^\circ} \setminus K$ the *end* of X_{ξ°.

2.3. **Instanton moduli spaces.** A $\mathrm{U}(r)$ instanton on X_{ξ° is a pair (E, A) where E is a Hermitean vector bundle on X_{ξ° of rank r, and A is a unitary connection on E which is anti-selfdual and square-integrable. At the end of X_{ξ°, A is approximated by a flat unitary connection A_0, i.e.

$$A - A_0 = O\big(r^{-3}\big)\ , \qquad \nabla A - \nabla A_0 = O\big(r^{-4}\big)\ , \qquad \dots\ ,$$

where ∇ is the covariant derivative with respect to A_0. The connection A_0 is determined by its holonomy, and hence by a unitary representation of the fundamental group of the end of X_{ξ°, which is isomorphic to $\mathbb{Z}_k$.

Let ρ_i for $i = 0, 1, \dots, k-1$ be the one-dimensional representation $z \mapsto \omega^i\, z$ of $\mathbb{Z}_k$ with weight i, and let $\vec{w} := (w_0, w_1, \dots, w_{k-1}) \in (\mathbb{Z}_{\geq 0})^k$ with $\sum_{i=0}^{k-1} w_i = r$. The main result of [35] is that the moduli space of $\mathrm{U}(r)$ instantons on X_{ξ°, with fixed Chern character and flat connection at infinity characterised by the representation $\bigoplus_{i=0}^{k-1} \rho_i^{\oplus w_i}$, is isomorphic to the quotient space

$$\mathcal{M}^{\mathrm{reg}}_{\xi^\circ}(\vec v,\vec w\,) := \big(\mu_{\mathbb{C}}^{-1}(\xi^\circ_{\mathbb{C}})\big)^{\mathrm{s}} \big/ \mathrm{GL}_{\vec v} \subseteq \mathcal{M}_{\xi^\circ}(\vec v,\vec w\,)$$

for a suitable dimension vector $\vec v\in(\mathbb{Z}_{\geq 0})^k$. The quiver variety $\mathcal{M}_{\xi^\circ}(\vec v,\vec w\,)$ is a type of Uhlenbeck space of $\mathcal{M}^{\mathrm{reg}}_{\xi^\circ}(\vec v,\vec w\,)$, in the sense that one has (cf. [35, Proposition 9.2] and [39, Proposition 1.10])

$$\mathcal{M}_{\xi^\circ}(\vec v,\vec w\,) = \coprod_{p\geq 0} \mathcal{M}^{\mathrm{reg}}_{\xi^\circ}(\vec v - p\,\delta,\vec w\,) \times \mathrm{Sym}^p\, X_{\xi^\circ} ,$$

where $\mathrm{Sym}^p\, X_{\xi^\circ}$ is the p-th symmetric product of X_{ξ°; moreover, $\mathcal{M}^{\mathrm{reg}}_{\xi^\circ}(\vec v,\vec w\,)$ is the smooth locus of $\mathcal{M}_{\xi^\circ}(\vec v,\vec w\,)$.

Let us fix $\xi^\circ_{\mathbb{C}}$ and $\vec v\in(\mathbb{Z}_{\geq 0})^k$. Choose a parameter $\xi_{\mathbb{R}}$ in the chamber inside (2.2) with $\xi_{\mathbb{R}}\cdot\delta<0$ which contains $\xi^\circ_{\mathbb{R}}$ in its closure (this chamber is uniquely determined by these conditions and by $\vec v$). Then $\xi=(\xi^\circ_{\mathbb{C}},\xi_{\mathbb{R}})$ is generic with respect to $\vec v$, hence $\mathcal{M}_\xi(\vec v,\vec w\,)$ is smooth and there is a resolution of singularities (cf. [39, Section 2(i)] and [38, Section 4])

$$\mathcal{M}_\xi(\vec v,\vec w\,) \longrightarrow \mathcal{M}_{\xi^\circ}(\vec v,\vec w\,)$$

which is an isomorphism over $\mathcal{M}^{\mathrm{reg}}_{\xi^\circ}(\vec v,\vec w\,)$ for any choice of $\vec w\in(\mathbb{Z}_{\geq 0})^k$.

In the following we introduce two distinguished chambers and describe the partition functions of supersymmetric gauge theories on ALE spaces with parameters in these chambers.

3. $\mathcal{N}=2$ GAUGE THEORY FOR THE LEVEL ZERO CHAMBER

Following [37, Section 4.2], we define

$$\mathsf{C}_0 := \big\{\xi_{\mathbb{R}}=\big(\xi^i_{\mathbb{R}}\big)\in\mathbb{R}^k \bigm| \xi^i_{\mathbb{R}}<0 \quad \forall i\in\mathsf{Q}_0\big\} .$$

Then any parameter $\xi_0=(0,\xi_{\mathbb{R}})\in\mathbb{C}^k\oplus\mathbb{R}^k$ with $\xi_{\mathbb{R}}\in\mathsf{C}_0$ is generic for any choice of $\vec v\in(\mathbb{Z}_{\geq 0})^k$. Hence the Nakajima quiver variety $\mathcal{M}_{\xi_0}(\vec v,\vec w\,)$ is smooth for any choice of dimension vectors $\vec v,\vec w\in(\mathbb{Z}_{\geq 0})^k$. We call C_0 the *level zero chamber*.

By [53, Section 2.3], the quiver variety $\mathcal{M}_{\xi_0}(\vec v,\vec w\,)$ is isomorphic to the moduli space parameterizing isomorphism classes of framed sheaves $(E,\phi_E\colon E|_{\ell_\infty}\xrightarrow{\sim}\mathcal{O}_{\ell_\infty}\otimes W)$ on the projective plane $\mathbb{P}^2$, where E is a $\mathbb{Z}_k$-equivariant torsion free sheaf on $\mathbb{P}^2$ with a $\mathbb{Z}_k$-invariant isomorphism $H^1(\mathbb{P}^2;E\otimes\mathcal{O}_{\mathbb{P}^2}(-\ell_\infty))\simeq V$ and ϕ_E is a $\mathbb{Z}_k$-invariant isomorphism; here ℓ_∞ is a line at infinity. In this sense $\mathcal{M}_{\xi_0}(\vec v,\vec w\,)$ can be regarded as a resolution of the Uhlenbeck space of instantons on $\mathbb{C}^2/\mathbb{Z}_k$; see [33] for an analysis of explicit ALE instanton solutions in this context. In this section we first introduce the instanton part of Nekrasov's partition function for pure $\mathcal{N}=2$ gauge theory on $\mathbb{C}^2$, and then consider its $\mathbb{Z}_k$-invariant projection.

3.1. Instanton counting on $\mathbb{R}^4$.

Nekrasov's instanton partition function [42] is the generating function for integrals of equivariant cohomology classes over the moduli space $\mathcal{M}_{r,n}$ of framed torsion free sheaves $(E,\phi_E: E|_{\ell_\infty}\xrightarrow{\sim}\mathcal{O}_{\ell_\infty}^{\oplus r})$ over $\mathbb{P}^2$ of rank r and second Chern class $\mathrm{c}_2(E)=n$. The moduli space $\mathcal{M}_{r,n}$ is a smooth quasi-projective variety of dimension $2\,r\,n$ which can be regarded as a resolution of the Uhlenbeck space of $\mathrm{U}(r)$ instantons on $\mathbb{R}^4$; in the rank one case $r=1$ it is isomorphic to the Hilbert scheme of n points on $\mathbb{C}^2$: $\mathcal{M}_{1,n}\simeq\mathrm{Hilb}^n(\mathbb{C}^2)$. There is an action of the torus $T:=(\mathbb{C}^*)^2\times(\mathbb{C}^*)^r$ on $\mathcal{M}_{r,n}$, where the first factor of T acts by pullback on E from the natural $(\mathbb{C}^*)^2$-action on $\mathbb{P}^2$ and the second factor acts by diagonal multiplication on ϕ_E, for any point (E,ϕ_E) of the moduli space. Let $\mathcal{P}_{r,n}$ be the set of r-tuples of Young tableaux $\boldsymbol{Y}=(Y_1,\dots,Y_r)$ of total weight $|\boldsymbol{Y}|:=\sum_{\alpha=1}^r|Y_\alpha|=n$, where

$|Y_\alpha|$ is the number of boxes in the Young tableau Y_α. Then the T-fixed points in $\mathcal{M}_{r,n}$ are parameterized by $\mathcal{P}_{r,n}$ [41].

Denote by $\varepsilon_1,\varepsilon_2,a_1,\dots,a_r$ the generators of $H_T^*(\mathrm{pt})$ (see Appendix A). Then the instanton partition function for pure $\mathcal{N}=2$ supersymmetric $\mathrm{U}(r)$ gauge theory on $\mathbb{R}^4$ is given by

$$\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\mathsf{q}) := \sum_{n=0}^\infty \mathsf{q}^n \int_{\mathcal{M}_{r,n}} \big[\mathcal{M}_{r,n}\big]_T ,$$

where q is a formal variable which weighs the different topological sectors. The localization theorem in T-equivariant cohomology then gives the combinatorial expansion [42, 24, 11]

$$\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\mathsf{q}) = \sum_{n=0}^\infty \mathsf{q}^n \sum_{\boldsymbol{Y}\in\mathcal{P}_{r,n}} \prod_{\alpha,\beta=1}^r \frac{1}{m_{Y_\alpha,Y_\beta}(\varepsilon_1,\varepsilon_2,a_{\beta\alpha})} ,$$

where $a_{\beta\alpha}:=a_\beta-a_\alpha$ and

$$m_{Y_\alpha,Y_\beta}(\varepsilon_1,\varepsilon_2,a) = \prod_{s\in Y_\alpha}\Big(a-L_{Y_\beta}(s)\,\varepsilon_1+\big(A_{Y_\alpha}(s)+1\big)\,\varepsilon_2\Big) \times \prod_{s'\in Y_\beta}\Big(a+\big(L_{Y_\alpha}(s')+1\big)\,\varepsilon_1-A_{Y_\beta}(s')\,\varepsilon_2\Big) \tag{3.1}$$

with $A_{Y_\alpha}(s)$ the number of boxes to the right of box s in the tableau Y_α (the *arm length* of the box) and $L_{Y_\alpha}(s)$ the number of boxes on top of it (the *leg length* of the box).

3.2. $\mathbb{Z}_k$**-projection.** After fixing a lift of the $\mathbb{Z}_k$-action to $\mathcal{O}_{\ell_\infty}^{\oplus r}$ by $\vec{w}\in(\mathbb{Z}_{\geq 0})^k$, with $r=\sum_{i=0}^{k-1} w_i$, there is a natural $\mathbb{Z}_k$-action on $\mathcal{M}_{r,n}$ induced by the $\mathbb{Z}_k$-action on $\mathbb{P}^2$; we write $W=\bigoplus_{\alpha=1}^r \rho_{p_\alpha}$ so that w_i is the number of times that $i\in\{0,1,\dots,k-1\}$ appears in the vector $\boldsymbol{p}=(p_1,\dots,p_r)$ with $p_\alpha\in\mathbb{Z}_k$ for $\alpha=1,\dots,r$. Then the fixed point set $(\mathcal{M}_{r,n})^{\mathbb{Z}_k}$ is a disjoint union of quiver varieties $\mathcal{M}_{\xi_0}(\vec{v},\vec{w}\,)$ over all dimension vectors $\vec{v}\in(\mathbb{Z}_{\geq 0})^k$ with $\sum_{i=0}^{k-1} v_i=n$; in the rank one case $\mathcal{M}_{\xi_0}(\vec{v},\vec{w}_0)$ is isomorphic to the moduli space $\mathrm{Hilb}^n(\mathbb{C}^2)^{\mathbb{Z}_k,\vec{v}}$ parameterizing $\mathbb{Z}_k$-invariant zero-dimensional subschemes Z of $\mathbb{C}^2$ of length n such that v_i is the multiplicity of the representation ρ_i in $H^0(Z;\mathcal{O}_Z)$, for $\vec{w}_0:=(1,0,\dots,0)$ and $i=0,1,\dots,k-1$, where $\mathcal{O}_Z$ is the sheaf of regular functions on Z. To define the instanton partition function, we introduce an action of the torus T on $\mathcal{M}_{\xi_0}(\vec{v},\vec{w}\,)$ in exactly the same way that we defined the T-action on $\mathcal{M}_{r,n}$. The resulting fixed point locus consists of r-tuples of *coloured* Young tableaux. A colouring by $\gamma\in\{0,1,\dots,k-1\}$ of a Young tableau $Y\subset\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ is an assignment to each box $s=(i,j)$ of Y of an element $\mathrm{res}(s):=\gamma-i+j\in\mathbb{Z}_k$, called the k-residue of s. An r-tuple of Young tableaux $\boldsymbol{Y}=(Y_1,\dots,Y_r)$ is said to be coloured by $\vec{w}$ if Y_α is coloured by $i\in\{0,1,\dots,k-1\}$ for $\sum_{j=0}^{i-1} w_j<\alpha\leq\sum_{j=0}^{i} w_j$. From a geometric perspective, the colours and the k-residues $\mathrm{res}(s)$, as s varies over boxes in Y_α for $\alpha=1,\dots,r$, specify the representations of $\mathbb{Z}_k$ acting on the fixed point corresponding to $\boldsymbol{Y}$, so that the colouring encodes the holonomy at infinity of the instanton corresponding to the fixed point.

Denote by $\mathcal{P}(\vec{v},\vec{w}\,)$ the set of r-tuples of coloured Young tableaux $\boldsymbol{Y}$ corresponding to the torus-fixed points of $\mathcal{M}_{\xi_0}(\vec{v},\vec{w}\,)$. Then the instanton partition function for pure $\mathcal{N}=2$ supersymmetric $\mathrm{U}(r)$ gauge theory on the ALE space $X_{\xi_0^\circ}$ with stability parameter $\xi_0^\circ=(0,\xi_{\mathbb{R}}^\circ)$, where $\xi_{\mathbb{R}}^\circ$ lies in the closure of C_0, is given by

$$\mathcal{Z}^{\rm inst}_{X_{\xi_0^\circ}}\big(\varepsilon_1,\varepsilon_2;\boldsymbol{a};\mathsf{z},\vec{y}\,\big)_{\vec{w}} := \sum_{\vec{v}\in(\mathbb{Z}_{\geq 0})^k} \mathsf{z}^v \prod_{l=1}^{k-1} y_l^{u_l} \int_{\mathcal{M}_{\xi_0}(\vec{v},\vec{w}\,)} \big[\mathcal{M}_{\xi_0}(\vec{v},\vec{w}\,)\big]_T$$

where z and $\vec{\xi}=(\xi_1,\dots,\xi_{k-1})$ are formal variables that weigh the topological invariants and the holonomies at infinity; here we have set

$$u_l=w_l+v_{l+1}+v_{l-1}-2v_l \qquad \text{and} \qquad v=\sum_{l=0}^{k-1}\Big(v_l+\frac{1}{2}\,(k-l)\,l\,u_l\Big)\ ,$$

where indices are read modulo k. By the localization theorem one obtains

$$\mathcal{Z}^{\rm inst}_{X_{\xi_0^\circ}}\big(\varepsilon_1,\varepsilon_2;\boldsymbol{a};\mathsf{z},\vec{y}\,\big)_{\vec{w}}=\sum_{\vec{v}\in(\mathbb{Z}_{\geq0})^k}\mathsf{z}^v\prod_{l=1}^{k-1}y_l^{u_l}\sum_{\boldsymbol{Y}\in\mathcal{P}(\vec{v},\vec{w}\,)}\ \prod_{\alpha,\beta=1}^{r}\frac{1}{m^{\mathbb{Z}_k}_{Y_\alpha,Y_\beta}(\varepsilon_1,\varepsilon_2,a_{\beta\alpha})}\ ,$$

where

$$\begin{aligned} m^{\mathbb{Z}_k}_{Y_\alpha,Y_\beta}(\varepsilon_1,\varepsilon_2,a)=&\prod_{s\in Y_\alpha}\Big(a-L_{Y_\beta}(s)\,\varepsilon_1+\big(A_{Y_\alpha}(s)+1\big)\,\varepsilon_2\Big)\,\delta^{(k)}_{h_{Y_\beta,Y_\alpha}(s),p_\alpha-p_\beta}\\ &\times\prod_{s'\in Y_\beta}\Big(a+\big(L_{Y_\alpha}(s'\,)+1\big)\,\varepsilon_1-A_{Y_\beta}(s'\,)\,\varepsilon_2\Big)\,\delta^{(k)}_{h_{Y_\alpha,Y_\beta}(s'\,),p_\beta-p_\alpha} \end{aligned} \tag{3.2}$$

with $h_{Y_\beta,Y_\alpha}(s)=L_{Y_\alpha}(s)+A_{Y_\beta}(s)+1$ the *hook length* of the box s in the tableau Y_α, and

$$\delta^{(k)}_{i,j}:=\begin{cases}1 & \text{for} \quad i=j \mod k\ ,\\ 0 & \text{otherwise}\ .\end{cases}$$

The delta function factors in (3.2) select weight zero assignments modulo k with respect to the induced action of the orbifold group $\mathbb{Z}_k$ on the torus T. This partition function was first computed by Fucito, Morales and Poghossian in [25], where however only the case $r=2$, $k=2$ was considered and they wrote down the contributions involving coloured Young tableaux with up to four boxes. A complete and more systematic analysis was developed in [27] (see also [55, Section 2]).

In an analogous way one can define partition functions for supersymmetric gauge theories with matter fields in fundamental or adjoint representations of the gauge group $\mathrm{U}(r)$. In particular, by taking the zero mass limit of the gauge theory with a single adjoint hypermultiplet, one obtains the Vafa-Witten partition function $\mathcal{Z}^{\rm VW}_{X_{\xi_0^\circ}}\big(\mathsf{z},\vec{y}\,\big)_{\vec{w}}$ [52] for topologically twisted $\mathcal{N}=4$ supersymmetric Yang-Mills theory on the ALE space $X_{\xi_0^\circ}$, which is the generating function for the Euler characteristics of instanton moduli spaces [27, Section 4.2.2]. In [27] it is also verified that the resulting partition function is the character of a highest weight representation of the affine Lie algebra $\widehat{\mathfrak{gl}}(k)$.

4. $\mathcal{N}=2$ GAUGE THEORY FOR THE LEVEL INFINITY CHAMBER

Define

$$\mathsf{C}=\big\{\xi_{\mathbb{R}}=\big(\xi^i_{\mathbb{R}}\big)\in\mathsf{D}_\delta\ \big|\ \xi^i_{\mathbb{R}}>0\quad\forall i\in\mathsf{Q}_0\setminus\{0\}\big\}\ \subset\ \mathsf{D}_\delta\ .$$

For any $\vec{v}\in(\mathbb{Z}_{\geq0})^k$, let $\mathsf{C}_\infty(\vec{v}\,)$ be the unique chamber inside (2.2) which is contained in the set $\{\xi_{\mathbb{R}}=(\xi^i_{\mathbb{R}})\in\mathbb{R}^k\mid\sum_{i\in\mathsf{Q}_0}\xi^i_{\mathbb{R}}>0\}$ and has C as its face. Then any parameter $\xi_\infty(\vec{v}\,)=(0,\xi_{\mathbb{R}})\in\mathbb{C}^k\oplus\mathbb{R}^k$ with $\xi_{\mathbb{R}}\in\mathsf{C}_\infty(\vec{v}\,)$ is generic with respect to $\vec{v}$. By [36] (see also [37, Section 4.4]), for any $\xi_{\mathbb{R}}\in\mathsf{C}_\infty(\delta)$ there is an isomorphism of ALE spaces $\mathcal{M}_{\xi_\infty(\delta)}(\delta,\vec{w}_0\,)\simeq X_k$ for $\vec{w}_0:=(1,0,\dots,0)$. We call $\mathsf{C}_\infty(\vec{v}\,)$ the *level infinity chamber*.

The first approach to the study of $\mathcal{N}=2$ gauge theories in the level infinity chamber is due to Bonelli, Maruyoshi, Tanzini and Yagi [8]. It does not rely on the formalism of the Nakajima quiver varieties $\mathcal{M}_{\xi_\infty(\vec{v}\,)}(\vec{v},\vec{w}\,)$, but is instead based on the conjectural *master formula* due to Nekrasov [43].

4.1. **Nekrasov master formula.** Let X be a smooth toric surface with torus $T_t := \mathbb{C}^* \times \mathbb{C}^*$. One can associate with X a smooth fan Σ_X in $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, where N is the lattice of one-parameter subgroups of T_t [17, Chapter 3]. By the orbit-cone correspondence, the two-dimensional cones of Σ_X correspond to the T_t-fixed points of X and the one-dimensional cones correspond to the T_t-invariant lines of X. If $X^{T_t} = \{p_1, \ldots, p_k\}$ with $k \geq 1$, then for each fixed point p_i there exists a T_t-invariant affine open neighbourhood $U_i \simeq \mathbb{C}^2$ of p_i for $i = 1, \ldots, k$. On the other hand, the Picard group $\mathrm{Pic}(X)$ is a free abelian group of finite rank (say m) [17, Proposition 4.25]; denote by $\mathcal{L}_j$ a set of generators, with $j = 1, \ldots, m$.

Let $\varepsilon_1, \varepsilon_2$ be the generators of $H^*_{T_t}(\mathrm{pt})$ (see Appendix A). For $i = 1, \ldots, k$ the tangent space to X at a fixed point p_i is a representation of the torus T_t, hence it decomposes into two irreducible representations of T_t. Denote by $\varepsilon_1^{(i)}(\varepsilon_1, \varepsilon_2)$ and $\varepsilon_2^{(i)}(\varepsilon_1, \varepsilon_2)$ the weights of the characters corresponding to these representations. The fibre of the line bundle $\mathcal{L}_j \to X$ over the point p_i is also a representation of T_t for any $j = 1, \ldots, m$, and we denote by $\phi_j^{(i)}(\varepsilon_1, \varepsilon_2)$ the weight of the associated character.

The Ω-deformation of $\mathcal{N} = 2$ supersymmetric $\mathrm{U}(r)$ gauge theory on X is obtained as a reduction of six-dimensional $\mathcal{N} = 1$ gauge theory on a flat X-bundle M over $\mathbb{T}^2$ in the limit where the torus $\mathbb{T}^2$ collapses to a point [43, Section 3.1]. The bundle M can be realised as the quotient of $\mathbb{C} \times X$ by the $\mathbb{Z}^2$-action

$$(n_1, n_2) \triangleright (w, x) = \big(w + (n_1 + \sigma\, n_2)\,,\, g_1^{n_1}\, g_2^{n_2}(x)\big)\ ,$$

where $x \in X$, $w \in \mathbb{C}$, $(n_1, n_2) \in \mathbb{Z}^2$, g_1, g_2 are two commuting isometries of X and σ is the complex structure modulus of $\mathbb{T}^2$. In the collapsing limit, fields of the gauge theory which are charged under the R-symmetry group are sections of the pullback to M of a flat T_t-bundle over $\mathbb{T}^2$. As pointed out in [44, Section 2.2.2], the chiral observables of the Ω-deformed $\mathcal{N} = 2$ gauge theory become closed forms on the moduli spaces of framed instantons which are equivariant with respect to the action of the torus $T := T_t \times T_{\boldsymbol{a}}$, where $T_{\boldsymbol{a}}$ is the maximal torus of the group $\mathrm{GL}(r, \mathbb{C})$ of constant gauge transformations which rotates the framing. Thus correlation functions of chiral BPS operators become integrals of equivariant Chern classes of natural bundles over the moduli spaces.

Denote again by $\varepsilon_1, \varepsilon_2, a_1, \ldots, a_r$ the generators of $H^*_T(\mathrm{pt})$; in gauge theory, $a_1, \ldots, a_r$ are the expectation values of the complex scalar field ϕ of the $\mathcal{N} = 2$ vector multiplet and $\varepsilon_1, \varepsilon_2$ parameterize the holonomy of a flat connection on the T_t-bundle over $\mathbb{T}^2$ used to define the Ω-deformation. Nekrasov's conjecture for the master formula is stated in [43, Section 4] in the presence of 2-observables and by considering instead of T_t the torus associated with the *Cox ring* of X, which is the polynomial ring with variables associated to the rays of Σ_X. Here we state the conjecture for gauge theories without 2-observables and in a form which will be suitable for describing the results of [8]: the full partition function $\mathcal{Z}_X^{\mathrm{full}}(\varepsilon_1, \varepsilon_2, \boldsymbol{a}; \mathsf{q})$ of the Ω-deformed pure $\mathcal{N} = 2$ $\mathrm{U}(r)$ gauge theory on X factorises into a product of partition functions on the open affine subsets U_i of X as

$$\mathcal{Z}_X^{\mathrm{full}}(\varepsilon_1, \varepsilon_2, \boldsymbol{a}; \mathsf{q}) = \sum_{(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_m) \in (\mathbb{Z}^r)^m}\ \prod_{i=1}^{k} \mathcal{Z}_{\mathbb{C}^2}^{\mathrm{full}}\Big(\varepsilon_1^{(i)}\,,\, \varepsilon_2^{(i)}\,,\, \boldsymbol{a} + \sum_{j=1}^{m} \phi_j^{(i)}\, \boldsymbol{h}_j\,;\, \mathsf{q}\Big)\ . \tag{4.1}$$

4.2. **Master formula for ALE spaces.** The minimal resolution X_k is a smooth toric surface with k torus-fixed points $p_1, \ldots, p_k$, and $k+1$ torus-invariant divisors $D_0, D_1, \ldots, D_k$ which are smooth projective curves of genus zero. For $i = 1, \ldots, k$ the divisors D_{i-1} and D_i intersect at the point p_i. The curves $D_1, \ldots, D_{k-1}$ are the irreducible components of the exceptional divisor $\varphi_k^{-1}(0)$, where $\varphi_k \colon X_k \to \mathbb{C}^2/\mathbb{Z}_k$ is the resolution of singularities morphism. By the McKay correspondence, there is a one-to-one correspondence between the irreducible representations of $\mathbb{Z}_k$ and the divisors $D_1, \ldots, D_{k-1}$ [17, Corollary 10.3.11]. By [17, Equation (10.4.3)], the intersection matrix $(D_i \cdot D_j)_{1 \leq i,j \leq k-1}$ is given by minus the Cartan matrix C of type A_{k-1}, i.e. one has

$$(D_i\cdot D_j)_{1\leq i,j\leq k-1}=-C=\begin{pmatrix}-2 & 1 & \cdots & 0\\ 1 & -2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -2\end{pmatrix}\ .$$

The coordinate ring of $\mathbb{C}^2/\mathbb{Z}_k$ is $\mathbb{C}\big[\mathbb{C}^2/\mathbb{Z}_k\big]:=\mathbb{C}\big[T_1,T_1^{k-1}\,T_2^k\big]$. On the other hand $\mathbb{C}\big[\mathbb{C}^2/\mathbb{Z}_k\big]=\mathbb{C}[t_1,t_2]^{\mathbb{Z}_k}=\mathbb{C}\big[t_1^k,t_2^k,t_1\,t_2\big]$. These two rings are isomorphic under the change of variables

$$T_1=t_1^k\qquad\text{and}\qquad T_2=t_2\,t_1^{1-k}\ . \tag{4.2}$$

Let U_i be the torus-invariant affine open subset of X_k which is a neighbourhood of the torus-fixed point p_i for $i=1,\dots,k$; its coordinate ring is given by $\mathbb{C}[U_i]:=\mathbb{C}\big[T_1^{2-i}\,T_2^{1-i},T_1^{i-1}\,T_2^{i}\big]$. By the relations (4.2) we have

$$\mathbb{C}[U_i]=\mathbb{C}\big[t_1^{k-i+1}\,t_2^{1-i}\,,\,t_1^{i-k}\,t_2^{i}\big]\ .$$

Thus in this case

$$\varepsilon_1^{(i)}(\varepsilon_1,\varepsilon_2)=(k-i+1)\,\varepsilon_1-(i-1)\,\varepsilon_2\qquad\text{and}\qquad\varepsilon_2^{(i)}(\varepsilon_1,\varepsilon_2)=-(k-i)\,\varepsilon_1+i\,\varepsilon_2\ .$$

The master formula (4.1) for pure $\mathcal{N}=2$ $\mathrm{U}(r)$ gauge theory on X_k reads (with some slight modification)

$$\mathcal{Z}_{X_k}^{\mathrm{full}}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\mathsf{q},\vec{\zeta}\,\big)=\sum_{(\vec{h}_1,\dots,\vec{h}_r)\in(\frac{1}{k}\,\mathbb{Z}^{k-1})^r}\ \prod_{i=1}^{k}\mathcal{Z}_{\mathbb{C}^2}^{\mathrm{full}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{a}^{(i)};\mathsf{q}\big)\ \prod_{l=1}^{k-1}\zeta_l^{\mathrm{c}_1^{(l)}}\ , \tag{4.3}$$

where $\boldsymbol{a}^{(i)}:=\big(a_1^{(i)},\dots,a_r^{(i)}\big)$ is defined by

$$a_\alpha^{(i)}:=a_\alpha+(\vec{h}_\alpha)_i\,\varepsilon_1^{(i)}+(\vec{h}_\alpha)_{i-1}\,\varepsilon_2^{(i)}\qquad\text{for}\quad\alpha=1,\dots,r\ , \tag{4.4}$$

and we set $(\vec{h}_\alpha)_0=(\vec{h}_\alpha)_k=0$. Equation (4.4) is superficially different from [8, Equation (2.2)] because there the T_t-fixed points of X_k are labelled $0,1,\dots,k-1$, while in this paper we label them as $1,\dots,k$; this implies that our weights in (4.4) are equivalent to those of [8] after a suitable reparameterization.

By localization, a T-invariant $\mathrm{U}(r)$ instanton on X_k decomposes into T_t-invariant point-like "regular" instantons over each chart U_i for $i=1,\dots,k$, together with "fractional" instantons which carry the magnetic fluxes of the gauge fields on the exceptional curves $D_1,\dots,D_{k-1}$ and keep track of the first Chern class of the instanton (see e.g. [26, 31]). The vectors $(\vec{h}_1,\dots,\vec{h}_r)$, which appear in (4.3), fix the first Chern class. For this, let us denote by R_l the l-th tautological line bundle on X_k for $l=1,\dots,k-1$ introduced in [30]. Then the first Chern classes $\mathrm{c}_1(R_l)$ for $l=1,\dots,k-1$ form a basis of $H^2(X_k;\mathbb{Z})$ [35, Proposition 2.2]. Therefore, any first Chern class of a T-invariant $\mathrm{U}(r)$ instanton is of the form

$$\mathrm{c}_1=\sum_{\alpha=1}^{r}\sum_{l=1}^{k-1}(\vec{u}_\alpha)_l\ \mathrm{c}_1(R_l)$$

for $\vec{u}_\alpha\in\mathbb{Z}^{k-1}$ and $\alpha=1,\dots,r$. We set $C^{-1}\,\vec{u}_\alpha=\vec{h}_\alpha$ for $\alpha=1,\dots,r$, where the inverse of the Cartan matrix C is given by

$$\big(C^{-1}\big)^{ij}=\frac{i\,(k-j)}{k}\qquad\text{for}\quad i\leq j\ .$$

One further introduces the chemical potentials ζ_l for $l=1,\dots,k-1$ for the fractional instantons to keep track of the first Chern classes

$$\mathrm{c}_1^{(l)}:=\sum_{\alpha=1}^{r}(\vec{u}_\alpha)_l=\sum_{\alpha=1}^{r}\sum_{m=1}^{k-1}C_{lm}\,(\vec{h}_\alpha)_m\ .$$

In order to determine the shifts $a_\alpha^{(i)}$ explicitly, one computes the weights $\phi_l^{(i)}$ from the local Chern character of R_l over U_i for $i=1,\dots,k$ and $l=1,\dots,k-1$; using Equation (6.3) below one immediately arrives at Equation (4.4). In [8, Section 2], the shifts (4.4) are interpreted heuristically by studying the patching together of non-trivial magnetic fluxes of gauge fields through the exceptional curves $D_1,\dots,D_{k-1}$ of X_k.

By subdividing the full partition function $\mathcal{Z}^{\rm full}_{X_k}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\mathsf{q},\vec\zeta\,\big)$ with respect to the possible holonomies at infinity of the fractional $\mathrm{U}(r)$ instantons one finds

$$\mathcal{Z}^{\rm full}_{X_k}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\mathsf{q},\vec\zeta\,\big)=\mathcal{Z}^{\rm cl}_{X_k}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\tau_{\rm cl}\big)\sum_{\boldsymbol{I}=(I_1,\dots,I_r)}\mathcal{Z}^{\rm pert}_{X_k}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a}\big)_{\boldsymbol{I}}\,\mathcal{Z}^{\rm inst}_{X_k}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\mathsf{q},\vec\zeta\,\big)_{\boldsymbol{I}}\ ,$$

where $I_\alpha\in\{0,1,\dots,k-1\}$ for $\alpha=1,\dots,r$ and the vector $\boldsymbol{I}$ parameterizes the holonomy class of a T-invariant $\mathrm{U}(r)$ instanton on X_k. Explicit expressions for the classical, perturbative and instanton contributions $\mathcal{Z}^{\rm cl}_{X_k}$, $\mathcal{Z}^{\rm pert}_{X_k}$ and $\mathcal{Z}^{\rm inst}_{X_k}$ are derived in [8] in terms of the corresponding partition functions $\mathcal{Z}^{\rm cl}_{\mathbb{C}^2}$, $\mathcal{Z}^{\rm pert}_{\mathbb{C}^2}$ and $\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}$. The perturbative partition function is described below, while the classical contribution is given by

$$\mathcal{Z}^{\rm cl}_{X_k}(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\tau_{\rm cl})=\mathcal{Z}^{\rm cl}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\tau_{\rm cl})^{\frac1k}=\exp\Big(-\frac{\tau_{\rm cl}}{2\,k\,\varepsilon_1\,\varepsilon_2}\,\sum_{\alpha=1}^r a_\alpha^2\Big)\ .$$

The explicit form of the instanton contributions is given by

$$\begin{aligned}\mathcal{Z}^{\rm inst}_{X_k}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\mathsf{q},\vec\zeta\,\big)_{\boldsymbol{I}}:=&\sum_{\substack{(\vec h_1,\dots,\vec h_r)\in(\frac1k\,\mathbb{Z}^{k-1})^r\\ k\,(\vec h_\alpha)_1=I_\alpha\ \mathrm{mod}\ k}}\mathsf{q}^{\frac12\sum_{\alpha=1}^r\sum_{l,m=1}^{k-1}(\vec h_\alpha)_l\,C_{lm}\,(\vec h_\alpha)_m}\ \prod_{l=1}^{k-1}\zeta_l^{\sum_{\alpha=1}^r(\vec u_\alpha)_l}\\ &\times\prod_{n=1}^{k-1}\ \prod_{\alpha\neq\beta}g^{(n)}\big(a^{(n)}_{\alpha\beta},\varepsilon_1^{(n)},\varepsilon_2^{(n)},(\vec h_{\alpha\beta})_n,(\vec h_{\alpha\beta})_{n+1}\big)^{-1}\ \prod_{i=1}^k\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{a}^{(i)};\mathsf{q}\big)\ ,\end{aligned}$$

where $a^{(n)}_{\alpha\beta}=a^{(n)}_\alpha-a^{(n)}_\beta$, $\vec h_{\alpha\beta}=\vec h_\alpha-\vec h_\beta$ for $\alpha,\beta=1,\dots,r$, and

$$g^{(n)}(a,e_1,e_2,\mu,\nu):=\begin{cases}\displaystyle\prod_{\substack{m_1\geq0\,,\,m_2\leq-1\\ n\,(\nu+m_1)\leq(n+1)\,(\mu+m_2)}}(a+m_1\,e_1+m_2\,e_2)\ , & n\,\nu<(n+1)\,\mu\ ,\\ 1\ , & n\,\nu=(n+1)\,\mu\ ,\\ \displaystyle\prod_{\substack{m_1\leq-1\,,\,m_2\geq0\\ n\,(\nu+m_1)>(n+1)\,(\mu+m_2)}}(a+m_1\,e_1+m_2\,e_2)\ , & n\,\nu>(n+1)\,\mu\ .\end{cases}$$

By fixing the holonomy at infinity $\boldsymbol{I}$, we have in addition the constraint

$$k\,(\vec h_\alpha)_1=I_\alpha\mod k \tag{4.5}$$

on the allowed first Chern classes of the T-invariant $\mathrm{U}(r)$ instantons on X_k.

Following [44, Appendix A], we denote by $\Gamma_2(x|-\varepsilon_1,-\varepsilon_2)$ the Barnes double gamma-function [2] which is the double zeta-function regularization of the infinite product

$$\prod_{i,j=0}^\infty\big(x-i\,\varepsilon_1-j\,\varepsilon_2\big)\ .$$

Then the perturbative partition function for $\mathbb{C}^2$ is

$$\mathcal{Z}^{\rm pert}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a}):=\prod_{\alpha\neq\beta}\frac{1}{\Gamma_2(a_{\alpha\beta}|\varepsilon_1,\varepsilon_2)}\ . \tag{4.6}$$

The "edge factor" $g^{(n)}$ depends only on the combinatorial data of the fan Σ_{X_k} of the toric variety X_k and is induced by the difference between k copies of the perturbative partition functions for $\mathbb{C}^2$

$$\prod_{i=1}^{k} \mathcal{Z}_{\mathbb{C}^2}^{\mathrm{pert}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{a}^{(i)}\big)$$

and the perturbative partition function $\mathcal{Z}_{X_k}^{\mathrm{pert}}$ for X_k [8, Equation (3.13)]. The partition function $\mathcal{Z}_{X_k}^{\mathrm{pert}}$ is defined via a slight modification of (4.6), obtained by replacing the Barnes double gamma function $\Gamma_2(a_{\alpha\beta}|\varepsilon_1,\varepsilon_2)$ with a suitable $\mathbb{Z}_k$-invariant version and imposing the dependence on the holonomy $\boldsymbol{I}$ via (4.5) [8, Equation (3.12)]; explicitly one finds

$$\mathcal{Z}_{X_k}^{\mathrm{pert}}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a}\big)_{\boldsymbol{I}} = \sum_{\substack{(\vec{h}_1,\dots,\vec{h}_r)\in(\frac{1}{k}\,\mathbb{Z}^{k-1})^r \\ k\,(\vec{h}_\alpha)_1=I_\alpha \bmod k}} \ \prod_{n=1}^{k-1}\prod_{\alpha\neq\beta} g^{(n)}\big(a_{\alpha\beta}^{(n)},\varepsilon_1^{(n)},\varepsilon_2^{(n)},(\vec{h}_{\alpha\beta})_n,(\vec{h}_{\alpha\beta})_{n+1}\big) \times \prod_{i=1}^{k} \mathcal{Z}_{\mathbb{C}^2}^{\mathrm{pert}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{a}^{(i)}\big)\ .$$

Comparison of the instanton partition functions $\mathcal{Z}_{X_k}^{\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\mathsf{q},\vec{\zeta}\,\big)_{\boldsymbol{I}}$ with those of Section 3.2 is not straightforward. Since the chambers C_0 and $\mathsf{C}_\infty(\vec{v})$ are distinct, the ALE spaces $X_{\xi_0^\circ}$ and X_k are diffeomorphic, but not isomorphic as algebraic varieties; hence one expects a non-trivial equivalence between the corresponding partition functions. Although some explicit low order comparisons of the combinatorial expansions confirm these expectations (see e.g. [4, 8, 32, 1]), at present there is no general relationship established between the two partition functions.

5. Sheaves on root stacks and fractional instantons on X_k

In this section we summarise the main results of [12], which provided the first attempt at putting the calculations of Section 4 on a rigorous footing and elucidating the geometry behind these constructions. The key development is a moduli theory of framed sheaves on a suitable *orbifold compactification* of the ALE space X_k. A related but somewhat different construction in the rank one case for unframed sheaves is provided by [15].

5.1. **Stacks.** The geometric construction of $\mathcal{N}=2$ gauge theories on the ALE space X_k relies on the formalism of Deligne-Mumford stacks; see e.g. [54, Section 7] and [21] for an overview. The main class of Deligne-Mumford stacks that we will encounter in the following are global quotient stacks $[X/G]$ where X is a variety, and G is a reductive linear algebraic group acting on X properly and with finite stabilizers. The stack $[X/G]$ is associated with the action groupoid $G\times X \rightrightarrows X$, where one arrow is given by the G-action and the other arrow by projection to the second factor. If in addition X is smooth and the generic stabilizer of the G-action is trivial (equivalently $[X/G]$ contains a variety as a dense open subset), we say that $[X/G]$ is an (effective) orbifold. If $f\colon X\to Y$ is the geometric quotient of X by G, then Y is called the coarse moduli space of $[X/G]$; for example, when the action of G on X is free then Y is the orbit space X/G, i.e. Y is the set of isomorphism classes of objects of the associated action groupoid. If the variety X has trivial Picard group, then line bundles on the global quotient stack $[X/G]$ are associated with characters of the group G.

A nice family of global quotient stacks are toric stacks. The theory of Deligne-Mumford tori and toric Deligne-Mumford stacks is developed in [23] (see also [9] for an equivalent approach based on stacky fans). Here we just recall the relevant definitions. A Deligne-Mumford torus $\mathscr{T}$ is the product of an ordinary torus T and a global quotient stack of the form $\mathscr{B}H := [\mathrm{pt}/H]$, where H is a finite group. One can define a group-like structure on $\mathscr{T}$ and a notion of $\mathscr{T}$-action by using the theory of Picard stacks, but we do not enter into such details here. For us a toric Deligne-Mumford stack is a global quotient stack $[X/G]$ as before, with a Deligne-Mumford torus $\mathscr{T}$ embedded in it as a dense open substack such that the action of $\mathscr{T}$ on itself extends to an action on the whole stack $[X/G]$. A toric Deligne-Mumford stack with a projective coarse moduli space is said to be projective.

5.2. **Orbifold compactification of X_k.** We define a normal projective compactification $\bar{X}_k = X_k \cup D_\infty$ of the minimal resolution X_k by adding a smooth rational curve D_∞ in such a way that $\bar{X}_2$ is the second Hirzebruch surface $\mathbb{F}_2$. For $k \geq 3$, $\bar{X}_k$ is only normal: it has two singular points, which are the two T_t-fixed points $0, \infty$ of $D_\infty \simeq \mathbb{P}^1$; the affine toric open neighbourhoods of these points in $\bar{X}_k$ are isomorphic to $\mathbb{C}^2/\mathbb{Z}_{\tilde{k}}$, where $\tilde{k} = \frac{k}{2}$ if k is even while $\tilde{k} = k$ if k is odd. Since $\bar{X}_k$ is not smooth for $k \geq 3$, we replace it with its *canonical* orbifold; as $\bar{X}_k$ is toric, such an orbifold also naturally has a toric structure. Thus the canonical orbifold of $\bar{X}_k$ is a two-dimensional projective toric orbifold $\mathscr{X}_k^{\rm can}$ with Deligne-Mumford torus T_t and coarse moduli space $\pi_k^{\rm can}\colon \mathscr{X}_k^{\rm can} \to \bar{X}_k$. The morphism $\pi_k^{\rm can}$ is an isomorphism over the smooth locus of $\bar{X}_k$; hence, for $k=2$, we find that $\mathscr{X}_2^{\rm can}$ is isomorphic to the Hirzebruch surface $\mathbb{F}_2$. The orbifold $\mathscr{X}_k^{\rm can}$ is realised as the global quotient stack of an open subset Z of $\mathbb{C}^{k+2}$ by an action of the torus $(\mathbb{C}^*)^k$. The ALE space X_k is a dense open subset $\imath_k^{\rm can}\colon X_k \xrightarrow{\sim} \mathscr{X}_k^{\rm can} \setminus \tilde{\mathscr{D}}_\infty$, where $\tilde{\mathscr{D}}_\infty$ is the (reduced) preimage of the divisor D_∞ in $\mathscr{X}_k^{\rm can}$ through $\pi_k^{\rm can}$. The stack $\tilde{\mathscr{D}}_\infty$ is an orbifold curve over the projective line $\mathbb{P}^1$ with two orbifold points (a “football”).

By performing a k-th root construction on this orbifold along $\tilde{\mathscr{D}}_\infty$, one extends the automorphism group of a generic point on $\tilde{\mathscr{D}}_\infty$ by the cyclic group $\mathbb{Z}_k$ and obtains a two-dimensional projective toric orbifold $\mathscr{X}_k$ with coarse moduli space $\pi_k\colon \mathscr{X}_k \to \bar{X}_k$. The orbifold $\mathscr{X}_k$ is obtained as the fibre product

$$\begin{tikzcd} \mathscr{X}_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathbb{A}^1/\mathbb{C}^*]} \arrow[d] \\ \mathscr{X}_k^{\rm can} \arrow[r] & {[\mathbb{A}^1/\mathbb{C}^*]} \end{tikzcd} ,$$

where the bottom horizontal arrow is determined by the line bundle $\mathcal{O}_{\mathscr{X}_k^{\rm can}}(\tilde{\mathscr{D}}_\infty)$ together with its tautological global section and the right vertical arrow is induced by the morphism $z \mapsto z^k$. The orbifold $\mathscr{X}_k$ can also be realised as a global quotient stack of the same variety Z as before by an action of the torus $(\mathbb{C}^*)^k$; the difference between the stacks $\mathscr{X}_k$ and $\mathscr{X}_k^{\rm can}$ lies in the different $(\mathbb{C}^*)^k$-actions on Z. The ALE space X_k is a dense open subset $\imath_k\colon X_k \xrightarrow{\sim} \mathscr{X}_k \setminus \mathscr{D}_\infty$, where the (reduced) preimage $\mathscr{D}_\infty$ of D_∞ in $\mathscr{X}_k$ though π_k is a smooth Cartier divisor which has the structure of a $\mathbb{Z}_k$-gerbe over $\tilde{\mathscr{D}}_\infty$.

The orbifold structure of $\mathscr{X}_k$ is concentrated on the compactification gerbe $\mathscr{D}_\infty$, which will enable the rigorous incorporation of contributions from fractional instantons. The stack $\mathscr{D}_\infty$ is isomorphic to the global quotient stack

$$\mathscr{D}_\infty \simeq \left[\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^* \times \mathbb{Z}_k} \right] ,$$

where the group action in given in [12, Equation (3.28)]; this characterises $\mathscr{D}_\infty$ as a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathscr{T} = \mathbb{C}^* \times \mathscr{B}\mathbb{Z}_k$ and coarse moduli space D_∞. Hence the Picard group $\mathrm{Pic}(\mathscr{D}_\infty)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_k$, and it is generated by the line bundles $\mathcal{L}_{\rm free}, \mathcal{L}_{\rm tor}$ corresponding to the characters $\chi_{\rm free}, \chi_{\rm tor}\colon \mathbb{C}^* \times \mathbb{Z}_k \to \mathbb{C}^*$ given respectively by the projections $(t,\omega) \mapsto t$ and $(t,\omega) \mapsto \omega$, where $t \in \mathbb{C}^*$ and ω is a primitive k-th root of unity. In particular, $\mathcal{L}_{\rm tor}^{\otimes k}$ is trivial. Define the degree zero line bundles $\mathcal{O}_{\mathscr{D}_\infty}(i) := \mathcal{L}_{\rm tor}^{\otimes i}$ for k even and $\mathcal{O}_{\mathscr{D}_\infty}(i) := \mathcal{L}_{\rm tor}^{\otimes i\,(k+1)/2}$ for k odd. As pointed out in [22], the fundamental group of the underlying topological stack of $\mathscr{D}_\infty$ is isomorphic to $\mathbb{Z}_k$, and for any $i=0,1,\dots,k-1$ the line bundle $\mathcal{O}_{\mathscr{D}_\infty}(i)$ inherits a unitary flat connection associated with the i-th irreducible unitary representation ρ_i of $\mathbb{Z}_k$. Hence by [22, Theorem 6.9] line bundles on $\mathscr{X}_k$ which are isomorphic along $\mathscr{D}_\infty$ to $\mathcal{O}_{\mathscr{D}_\infty}(i)$ correspond to $\mathrm{U}(1)$ instantons on X_k with holonomy at infinity given by ρ_i. One can also consider more general line bundles by tensoring $\mathcal{O}_{\mathscr{D}_\infty}(i)$ with a power $\mathcal{L}_{\mathrm{free}}^{\otimes s}$ to get line bundles of degree a rational multiple of s [12, Lemma 3.39]; however, here we shall set $s=0$ as we wish to concentrate on the line bundles corresponding to fractional instantons.

For $i=1,\dots,k-1$ let $\mathscr{D}_i$ be the preimages of the exceptional divisors D_i equipped with the reduced scheme structure; their intersection form is given by $-C$. Let $\mathcal{R}_i=\mathcal{O}_{\mathscr{X}_k}(\omega_i)$ be the line bundle associated to the dual class

$$\omega_i=-\sum_{j=1}^{k-1}\left(C^{-1}\right)^{ij}\mathscr{D}_j\ ,$$

which is an integral class in the Picard group $\mathrm{Pic}(\mathscr{X}_k)$ for $i=1,\dots,k-1$ (cf. [12, Lemma 3.21]). The restrictions $\mathcal{R}_i|_{X_k}$ coincide with the Kronheimer-Nakajima tautological line bundles R_i, while $\mathcal{R}_i|_{\mathscr{D}_\infty}\simeq\mathcal{O}_{\mathscr{D}_\infty}(i)$. The line bundles $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)$ and $\mathcal{R}_i$ for $i=1,\dots,k-1$ freely generate $\mathrm{Pic}(\mathscr{X}_k)$ over $\mathbb{Z}$.

5.3. **Framed sheaves.** In order to construct moduli spaces of framed sheaves on the orbifold $\mathscr{X}_k$ which are needed for the formulation of supersymmetric gauge theories on X_k, we first have to choose a suitable framing sheaf, which will encode the holonomy at infinity of the fractional instantons. In light of the discussion of Section 5.2, we choose as framing sheaf the locally free sheaf

$$\mathcal{F}_\infty^{\vec{w}}:=\bigoplus_{i=0}^{k-1}\mathcal{O}_{\mathscr{D}_\infty}(i)^{\oplus w_i}$$

for a fixed vector $\vec{w}:=(w_0,w_1,\dots,w_{k-1})\in(\mathbb{Z}_{\geq 0})^k$. By applying the general theory of framed sheaves on projective Deligne-Mumford stacks developed in [14], one can construct a fine moduli space $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ parameterizing isomorphism classes of $(\mathscr{D}_\infty,\mathcal{F}_\infty^{\vec{w}})$-framed sheaves $(\mathcal{E},\phi_{\mathcal{E}}\colon\mathcal{E}|_{\mathscr{D}_\infty}\xrightarrow{\sim}\mathcal{F}_\infty^{\vec{w}})$ on $\mathscr{X}_k$ with fixed rank $r:=\sum_{i=0}^{k-1}w_i$, first Chern class $\mathrm{c}_1(\mathcal{E})=\sum_{l=1}^{k-1}u_l\,\omega_l$ and discriminant

$$\Delta(\mathcal{E}):=\int_{\mathscr{X}_k}\Big(\mathrm{c}_2(\mathcal{E})-\frac{r-1}{r}\,\mathrm{c}_1(\mathcal{E})^2\Big)=\Delta\ .$$

Due to the framing the vector $\vec{u}=(u_1,\dots,u_{k-1})\in\mathbb{Z}^{k-1}$ satisfies the constraint

$$\sum_{i=1}^{k-1}i\,u_i=\sum_{i=1}^{k-1}i\,w_i\mod k\ , \tag{5.1}$$

or equivalently

$$k\,h_{k-1}=\sum_{i=1}^{k-1}i\,w_i\mod k$$

where $\vec{h}:=C^{-1}\vec{u}$. As we now explain, this constraint has a representation theory meaning. For this, let us denote by $\vec{e}_i$ the i-th coordinate vector in $\mathbb{Z}^k$. Define the elements $\vec{\gamma}_i:=\vec{e}_i-\vec{e}_{i+1}$ for $i=1,\dots,k-1$. Then $\vec{\gamma}_i\cdot\vec{\gamma}_j=C_{ij}$, where C is the Cartan matrix of the Dynkin diagram of type A_{k-1} which has root lattice $\mathfrak{Q}:=\bigoplus_{i=1}^{k-1}\mathbb{Z}\vec{\gamma}_i$ with a nondegenerate symmetric bilinear form $\langle-,-\rangle_{\mathfrak{Q}}$ induced by C. The elements of $\mathfrak{Q}$ are called roots, and $\vec{\gamma}_i$ is called the i-th simple root for $i=1,\dots,k-1$. The fundamental weights $\vec{\omega}_i$ of type A_{k-1} are the vectors in $\mathbb{Z}^k$ given by

$$\vec{\omega}_i:=\sum_{l=1}^{i}\vec{e}_l-\frac{i}{k}\sum_{l=1}^{k}\vec{e}_l$$

for $i=1,\dots,k-1$. Let $\mathfrak{P}:=\bigoplus_{i=1}^{k-1}\mathbb{Z}\vec{\omega}_i$ be the weight lattice; then $\mathfrak{Q}\subset\mathfrak{P}$, as $\vec{\gamma}_i=\sum_{j=1}^{k-1}C_{ij}\,\vec{\omega}_j$. We subdivide the vectors $\vec{u}\in\mathbb{Z}^{k-1}$ according to Equation (5.1) as

$$\mathfrak{U}_{\vec{w}}:=\Big\{\vec{u}\in\mathbb{Z}^{k-1}\ \Big|\ \sum_{i=1}^{k-1}i\,u_i=\sum_{i=1}^{k-1}i\,w_i \mod k\Big\}\ .$$

Define now a bijective map by

$$\psi:\ \mathbb{Z}^{k-1}\longmapsto\mathfrak{P}\ ,\qquad \vec{u}\longmapsto\sum_{i=1}^{k-1}u_i\,\vec{\omega}_i\ .$$

Then $\psi^{-1}\big(\mathfrak{Q}+\sum_{i=1}^{k-1}w_i\,\vec{\omega}_i\big)=\mathfrak{U}_{\vec{w}}$, which implies that $\psi(\vec{u}\,)$ for $\vec{u}\in\mathfrak{U}_{\vec{w}}$ is naturally written as a sum of the weight $\sum_{i=1}^{k-1}w_i\,\vec{\omega}_i$ and the root $\vec{\gamma}_{\vec{u}}$ given by

$$\vec{\gamma}_{\vec{u}}:=\sum_{i,j=1}^{k-1}\big(C^{-1}\big)^{ij}\,(u_j-w_j)\,\vec{\gamma}_i\ .$$

Since the moduli space $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ is fine, there exists a universal framed sheaf

$$\big(\boldsymbol{\mathcal{E}}_{\vec{u},\Delta,\vec{w}}\ ,\ \boldsymbol{\phi}_{\boldsymbol{\mathcal{E}}_{\vec{u},\Delta,\vec{w}}}:\boldsymbol{\mathcal{E}}_{\vec{u},\Delta,\vec{w}}\big|_{\mathcal{M}_{\vec{u},\Delta,\vec{w}}\times\mathscr{D}_\infty}\xrightarrow{\ \sim\ }p_2^*(\mathcal{F}_\infty^{\vec{w}})\big)\ ,$$

where $\boldsymbol{\mathcal{E}}_{\vec{u},\Delta,\vec{w}}$ is a coherent sheaf on $\mathcal{M}_{\vec{u},\Delta,\vec{w}}\times\mathscr{X}_k$ which is flat over $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$; here and in the following we use the notation p_i for the projection of a product of varieties to the i-th factor. The fibre over $[(\mathcal{E},\phi_{\mathcal{E}})]\in\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ is itself the $(\mathscr{D}_\infty,\mathcal{F}_\infty^{\vec{w}}\,)$-framed sheaf $(\mathcal{E},\phi_{\mathcal{E}})$ on $\mathscr{X}_k$.

The moduli space $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ is a smooth quasi-projective variety of dimension

$$\dim\mathcal{M}_{\vec{u},\Delta,\vec{w}}=2\,r\,\Delta-\frac12\sum_{j=1}^{k-1}\big(C^{-1}\big)^{jj}\,\vec{w}\cdot\vec{w}(j)\ ,$$

where for $j=1,\dots,k-1$ we defined the vector $\vec{w}(j):=(w_j,\dots,w_{k-1},w_0,w_1,\dots,w_{j-1})$; the Zariski tangent space of $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ at a point $[(\mathcal{E},\phi_{\mathcal{E}})]$ is $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$. By [22, Theorem 6.9], it contains as an open subset the moduli space of $\mathrm{U}(r)$ instantons on X_k with first Chern class $\mathrm{c}_1=\sum_{l=1}^{k-1}u_l\ \mathrm{c}_1(R_l)$, discriminant Δ and holonomy at infinity associated with the unitary representation $\rho=\bigoplus_{i=0}^{k-1}\rho_i^{\oplus w_i}$ of the cyclic group $\mathbb{Z}_k$.

In the rank one case $r=1$, there is a non-canonical isomorphism of fine moduli spaces

$$\mathcal{M}_{\vec{u},n,\vec{w}_j}\simeq\mathrm{Hilb}^n(X_k)\ ,$$

where $j\in\{0,1,\dots,k-1\}$ is defined by $(\vec{w}_j)_i=\delta_{ij}$, and $\mathrm{Hilb}^n(X_k)$ is the Hilbert scheme of n points on X_k, which is a smooth quasi-projective variety of dimension $2n$. Kuznetsov shows in [36] that this Hilbert scheme is isomorphic to the Nakajima quiver variety $\mathcal{M}_{\xi_\infty(n\,\delta)}(n\,\delta,\vec{w}_0\,)$ for $\vec{w}_0=(1,0,\dots,0)$, $\xi_\infty(n\,\delta)=(0,\xi_{\mathbb{R}})$ and $\xi_{\mathbb{R}}\in\mathsf{C}_\infty(n\,\delta\,)$, and hence our rank one moduli spaces are also non-canonically isomorphic to quiver varieties. It is natural to anticipate that this property generalises to the higher rank case, and hence we conjecture that the moduli spaces $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ are isomorphic to Nakajima quiver varieties $\mathcal{M}_{\xi_\infty(\vec{v})}(\vec{v},\vec{w}\,)$ for a suitable choice of dimension vector $\vec{v}\in(\mathbb{Z}_{\geq0})^k$.

5.4. $\mathcal{N}=2$ **gauge theory on** X_k**.** We can now define the instanton partition functions for the Ω-deformed pure $\mathcal{N}=2$ gauge theory on X_k as generating functions for T-equivariant integrals over the moduli spaces $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$. There is a natural T-action on $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ defined on a framed sheaf by pullback of the underlying torsion-free sheaf via the T_t-action on $\mathscr{X}_k$ and by rotation of the framing by a constant gauge transformation from $T_{\boldsymbol{a}}$. Then a torus-fixed point $[(\mathcal{E},\phi_{\mathcal{E}})]\in(\mathcal{M}_{\vec{u},\Delta,\vec{w}})^T$ decomposes as a direct sum of rank one framed sheaves

$$(\mathcal{E},\phi_{\mathcal{E}})=\bigoplus_{\alpha=1}^{r}(\mathcal{E}_\alpha,\phi_\alpha)$$

parameterized by combinatorial data $(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{u}}\,)$ consisting of collections of vectors of Young tableaux $\vec{\boldsymbol{Y}} = (\vec{Y}_1, \dots, \vec{Y}_r)$, $\vec{Y}_\alpha = \{Y_\alpha^i\}_{i=1,\dots,k}$, and vectors of integers $\vec{\boldsymbol{u}} = (\vec{u}_1, \dots, \vec{u}_r)$ such that for each $i = 0, 1, \dots, k-1$ and $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j$:

- $\mathcal{E}_\alpha = \imath_{k*}(I_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}$, where I_α is the ideal sheaf of a zero-dimensional subscheme Z_α of X_k with length $n_\alpha = \sum_{i=1}^k \big|Y_\alpha^i\big|$ supported at the T_t-fixed points $p_1, \dots, p_k$, while $\vec{u}_\alpha \in \mathbb{Z}^{k-1}$ obeys $\sum_{\alpha=1}^r \vec{u}_\alpha = \vec{u}$ and $\vec{h}_\alpha := C^{-1}\vec{u}_\alpha$ satisfies
$$k\,(\vec{h}_\alpha)_{k-1} = i \mod k\ ;$$
- $\phi_\alpha \colon \mathcal{E}_\alpha\big|_{\mathscr{D}_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathscr{D}_\infty}(i)$ is induced by the canonical isomorphism $\mathcal{R}^{\vec{u}_\alpha}\big|_{\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{D}_\infty}(i)$;
- $\displaystyle \Delta = \sum_{\alpha=1}^r n_\alpha + \frac{1}{2}\sum_{\alpha=1}^r \vec{h}_\alpha \cdot C\vec{h}_\alpha - \frac{1}{2r}\sum_{\alpha,\beta=1}^r \vec{h}_\alpha \cdot C\vec{h}_\beta \ \in\ \tfrac{1}{2\,r\,k}\,\mathbb{Z}\ .$

Here we introduced the shorthand $\mathcal{R}^{\vec{u}_\alpha} := \bigotimes_{l=1}^{k-1} \mathcal{R}_l^{\otimes(\vec{u}_\alpha)_l}$.

Introduce topological couplings $\mathsf{q} \in \mathbb{C}^*$ with $|\mathsf{q}| < 1$ and $\vec{\xi} = (\xi_1, \dots, \xi_{k-1}) \in (\mathbb{C}^*)^{k-1}$ with $|\xi_i| < 1$. The instanton partition function in the topological sector labelled by $\vec{h} = C^{-1}\vec{u}$ is defined by
$$\mathcal{Z}_{\vec{h}}^{\text{inst}}\big(\varepsilon_1, \varepsilon_2, \boldsymbol{a}; \mathsf{q}\big)_{\vec{w}} = \sum_{\Delta \in \frac{1}{2\,r\,k}\,\mathbb{Z}} \mathsf{q}^{\Delta + \frac{1}{2r}\,\vec{h}\cdot C\vec{h}} \int_{\mathcal{M}_{C\vec{h},\Delta,\vec{w}}} \big[\mathcal{M}_{C\vec{h},\Delta,\vec{w}}\big]_T\ ,$$
while the full instanton partition function is given by a weighted sum over fractional instantons with chemical potentials $\vec{\xi}$ as
$$\mathcal{Z}_{X_k}^{\text{inst}}\big(\varepsilon_1, \varepsilon_2, \boldsymbol{a}; \mathsf{q}, \vec{\xi}\,\big)_{\vec{w}} = \sum_{C\vec{h} \in \mathfrak{U}_{\vec{w}}} \vec{\xi}^{\,\vec{h}}\ \mathcal{Z}_{\vec{h}}^{\text{inst}}(\varepsilon_1, \varepsilon_2, \boldsymbol{a}; \mathsf{q})_{\vec{w}}$$
where $\vec{\xi}^{\,\vec{h}} := \prod_{l=1}^{k-1} \xi_l^{h_l}$. The T-equivariant Euler class of the tangent bundle $T\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ at a T-fixed point with combinatorial data $(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{u}})$ is given by
$$\text{eu}_T\left(T_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{u}})}\mathcal{M}_{\vec{u},\Delta,\vec{w}}\right) = \prod_{\alpha,\beta=1}^r\ \prod_{i=1}^k m_{Y_\alpha^i, Y_\beta^i}\big(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, a_{\beta\alpha}^{(i)}\big) \prod_{n=1}^{k-1} \ell_{\vec{h}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_{\beta\alpha}\big)$$
where $a_{\beta\alpha} = a_\beta - a_\alpha$ and $\vec{h}_{\beta\alpha} = \vec{h}_\beta - \vec{h}_\alpha$. The contributions $m_{Y_\alpha^i, Y_\beta^i}$ from the open affine neighbourhoods U_i are given by (3.1). The explicit expressions for the "edge factors" $\ell_{\vec{h}_{\beta\alpha}}^{(n)}$ are rather complicated and can be found in Appendix B; they depend explicitly on the Cartan matrix C. By using the localization theorem in equivariant cohomology (see Appendix A) we thus obtain the factorization formula
$$\begin{aligned}\mathcal{Z}_{X_k}^{\text{inst}}\big(\varepsilon_1, \varepsilon_2, \boldsymbol{a}; \mathsf{q}, \vec{\xi}\,\big)_{\vec{w}} = \sum_{C\vec{h} \in \mathfrak{U}_{\vec{w}}} \vec{\xi}^{\,\vec{h}} \sum_{\substack{\vec{\boldsymbol{h}} = (\vec{h}_1, \dots, \vec{h}_r) \\ \vec{h} = \sum_{\alpha=1}^r \vec{h}_\alpha}} \mathsf{q}^{\frac{1}{2}\sum_{\alpha=1}^r \vec{h}_\alpha \cdot C\vec{h}_\alpha} \\ \times \prod_{\alpha,\beta=1}^r\ \prod_{n=1}^{k-1} \ell_{\vec{h}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_{\beta\alpha}\big)^{-1} \prod_{i=1}^k \mathcal{Z}_{\mathbb{C}^2}^{\text{inst}}\big(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \boldsymbol{a}^{(i)}; \mathsf{q}\big)\ .\end{aligned}$$

Let us compare this result explicitly with the result of Section 4.2. First note that our $\vec{u} \in \mathbb{Z}^{k-1}$ are the same as in [8] while $\vec{I} = (k-1, \dots, k-1, k-2, \dots, k-2, \dots, 0, \dots, 0)$, where $k-i-1$ appears with multiplicity w_i for $i = 0, 1, \dots, k-1$; hence the constraint (4.5) is equivalent to (5.1) because $k\,(\vec{h}_\alpha)_{k-1} = k\,(\vec{h}_\alpha)_1 \mod k$. The "fugacities" are related by $\xi_l = \zeta_{l-1}\,\zeta_{l+1} / \zeta_l$ for $l = 1, \dots, k-1$ (where we set $\zeta_0 = \zeta_k = 1$).

For $k=2$ our instanton partition function assumes the form

$$\mathcal{Z}^{\rm inst}_{X_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a}\,;\mathsf{q},\xi)_{\vec w} = \sum_{\substack{h\in\frac12\mathbb{Z}\\ 2h=w_1 \bmod 2}} \xi^h \sum_{\substack{\boldsymbol{h}=(h_1,\dots,h_r)\\ h=\sum_{\alpha=1}^r h_\alpha}} \mathsf{q}^{\sum_{\alpha=1}^r h_\alpha^2} \prod_{\alpha,\beta=1}^r \ell_{h_{\beta\alpha}}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha}\big)^{-1}$$
$$\times\ \mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,\boldsymbol{a}+2\varepsilon_1\,\boldsymbol{h};\mathsf{q}\big)\ \mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1-\varepsilon_2,2\varepsilon_2,\boldsymbol{a}+2\varepsilon_2\,\boldsymbol{h};\mathsf{q}\big)\ ,$$

where from Appendix B the edge factors are given by

$$\ell_h(e_1,e_2,a)=\begin{cases} \displaystyle\prod_{m_1=0}^{\lfloor h\rfloor-1}\ \prod_{m_2=0}^{2m_1+2\{h\}} \big(a+m_1\,e_1+m_2\,e_2\big)\ , & \lfloor h\rfloor>0\ ,\\ 1\ , & \lfloor h\rfloor=0\ ,\\ \displaystyle\prod_{m_1=1}^{-\lfloor h\rfloor}\ \prod_{m_2=1}^{2m_1-2\{h\}-1} \big(a+(2\{h\}-m_1)\,e_1-m_2\,e_2\big)\ , & \lfloor h\rfloor<0\ . \end{cases}$$

For $\{h\}=0$ these formulas coincide with the edge factors obtained in [28] up to a redefinition of the equivariant parameters (see also [16]), as they should, since the computation of the edge factors in this case is equivalent to that carried out for the Hirzebruch surface $\mathbb{F}_2$ in [13, Section 4.2]. Moreover, for $\lfloor h\rfloor>0$ they can be easily written in the form

$$\ell_h(e_1,e_2,a)=\prod_{\substack{m_1,m_2\geq1\,,\,m_1+m_2\leq 2\lfloor h\rfloor\\ m_1+m_2\ \mathrm{even}}} \big(a+(m_1-1)\,\tilde e_1+(m_2-1)\,\tilde e_2\big)$$

with $\tilde e_1=\frac{e_1}{2}$ and $\tilde e_2=\frac{e_1}{2}+e_2$, which coincide with the edge factors of [6, 7, 4] (similarly for $\lfloor h\rfloor<0$ and/or $\{h\}=\frac12$).

However, for $k\geq3$ the edge factors $\ell^{(n)}_{\vec h}$ appear to be drastically different from $g^{(n)}$, and although some explicit checks for $r=2$, $k=3$ in [12, Appendix D] demonstrate that our partition functions agree with those in [8, Appendix C] at leading orders in the q-expansion, a complete proof of the equivalence between the two partition functions is currently lacking.

6. $\mathcal{N}=2$ SUPERCONFORMAL QUIVER GAUGE THEORIES ON X_k

Most of the material in this section is new. We consider Ω-deformed $\mathcal{N}=2$ superconformal quiver gauge theories on the ALE space X_k, generalising the analysis of [49] which considered only the abelian cases. We shall also present a new computation of the perturbative partition functions for $\mathcal{N}=2$ gauge theories on X_k using our formalism, and hence provide a rigorous proof of the Nekrasov master formula for X_k. We also describe some aspects of the Seiberg-Witten geometry of these gauge theories.

6.1. **Hypermultiplet bundles.** Let $\mathtt{Q}=(\mathtt{Q}_0,\mathtt{Q}_1,\mathtt{s},\mathtt{t})$ be a quiver. The vertices $\mathtt{Q}_0$ label vector multiplets in the corresponding $\mathcal{N}=2$ quiver gauge theory on X_k, while the matter hypermultiplets are associated with representations of $\mathtt{Q}$. We fix integer vectors $\boldsymbol{r},\boldsymbol{m},\bar{\boldsymbol{m}}\in(\mathbb{Z}_{\geq0})^{\mathtt{Q}_0}$. Let $E:=\bigoplus_{v\in\mathtt{Q}_0}E_v$, $V:=\bigoplus_{v\in\mathtt{Q}_0}V_v$ and $\bar V:=\bigoplus_{v\in\mathtt{Q}_0}\bar V_v$ be $\mathtt{Q}_0$-graded complex vector spaces such that $\dim E_v=r_v$, $\dim V_v=m_v$ and $\dim\bar V_v=\bar m_v$ for each vertex $v\in\mathtt{Q}_0$. Set

$$M_{\mathtt{Q}}(\boldsymbol{r},\boldsymbol{m},\bar{\boldsymbol{m}}):=\Big(\bigoplus_{v\in\mathtt{Q}_0}\mathrm{Hom}\big(V_v,E_v\big)\oplus\mathrm{Hom}\big(E_v,\bar V_v\big)\Big)\ \oplus\ \Big(\bigoplus_{e\in\mathtt{Q}_1}\mathrm{Hom}\big(E_{\mathtt{s}(e)},E_{\mathtt{t}(e)}\big)\Big)\ .$$

The gauge group

$$G_{\boldsymbol{r}}:=\prod_{v\in\mathtt{Q}_0}\mathrm{GL}(r_v,\mathbb{C})$$

acts on $M_{\mathtt{Q}}(\boldsymbol{r},\boldsymbol{m},\bar{\boldsymbol{m}})$ in a way analogous to the action defined in (2.1). The hypermultiplet space also carries a natural action of the total flavour symmetry group

$$G_{\boldsymbol{m},\bar{\boldsymbol{m}}} := \prod_{v\in\mathtt{Q}_0} \mathrm{GL}(m_v,\mathbb{C})\times\mathrm{GL}(\bar{m}_v,\mathbb{C}) \ \times \ \prod_{e\in\mathtt{Q}_1}\mathbb{C}^* \ .$$

The matter bundle associated to the quiver representation $M_{\mathtt{Q}}(\boldsymbol{r},\boldsymbol{m},\bar{\boldsymbol{m}})$ is constructed as follows. Let $T_\mu=\mathbb{C}^*$ and $H^*_{T_\mu}(\mathrm{pt})=\mathbb{C}[\mu]$. Denote by $\mathcal{O}_{\mathscr{X}_k}(\mu)$ the trivial line bundle on the orbifold $\mathscr{X}_k$ on which T_μ acts by scaling the fibres with weight μ.

As described in [12, Section 4.5], one can define Carlsson-Okounkov type Ext-bundles as the elements $\boldsymbol{E}_\mu^{\vec{u}_1,\Delta_1,\vec{w}_1;\vec{u}_2,\Delta_2,\vec{w}_2}$ in the K-theory group $K\big(\mathcal{M}_{\vec{u}_1,\Delta_1,\vec{w}_1}\times\mathcal{M}_{\vec{u}_2,\Delta_2,\vec{w}_2}\big)$ by

$$\boldsymbol{E}_\mu^{\vec{u}_1,\Delta_1,\vec{w}_1;\vec{u}_2,\Delta_2,\vec{w}_2} := p_{12*}\big(-\boldsymbol{\mathcal{E}}_1^\vee\otimes\boldsymbol{\mathcal{E}}_2\otimes p_3^*(\mathcal{O}_{\mathscr{X}_k}(\mu)\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))\big) \ ,$$

where $\boldsymbol{\mathcal{E}}_i := p_{i3}^*\big(\boldsymbol{\mathcal{E}}_{\vec{u}_i,\Delta_i,\vec{w}_i}\big)$ in $K\big(\mathcal{M}_{\vec{u}_1,\Delta_1,\vec{w}_1}\times\mathcal{M}_{\vec{u}_2,\Delta_2,\vec{w}_2}\times\mathscr{X}_k\big)$ with $\boldsymbol{\mathcal{E}}_{\vec{u},\Delta,\vec{w}}$ the universal sheaf on $\mathcal{M}_{\vec{u},\Delta,\vec{w}}\times\mathscr{X}_k$; here p_{ij} is the projection onto the product of the i-th and j-th factors. Its fibre over $\big([(\mathcal{E}_1,\phi_{\mathcal{E}_1})]\,,\,[(\mathcal{E}_2,\phi_{\mathcal{E}_2})]\big)$ is

$$\boldsymbol{E}_\mu^{\vec{u}_1,\Delta_1,\vec{w}_1;\vec{u}_2,\Delta_2,\vec{w}_2}\big|_{([(\mathcal{E}_1,\phi_{\mathcal{E}_1})]\,,\,[(\mathcal{E}_2,\phi_{\mathcal{E}_2})])} = \mathrm{Ext}^1\big(\mathcal{E}_1\,,\,\mathcal{E}_2\otimes\mathcal{O}_{\mathscr{X}_k}(\mu)\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \ .$$

One can compute the dimension of this vector space by a straightforward generalization of the computations of [12, Appendix A] to get the rank

$$\begin{aligned}\mathrm{rk}\big(\boldsymbol{E}_\mu^{\vec{u}_1,\Delta_1,\vec{w}_1;\vec{u}_2,\Delta_2,\vec{w}_2}\big) &= r_1\,\Delta_2+r_2\,\Delta_1+\frac{r_2}{2r_1}\,\vec{h}_1\cdot C\vec{h}_1+\frac{r_1}{2r_2}\,\vec{h}_2\cdot C\vec{h}_2-\vec{h}_1\cdot C\vec{h}_2\\ &\quad -\frac12\sum_{j=1}^{k-1}\big(C^{-1}\big)^{jj}\,\vec{w}_1\cdot\vec{w}_2(j) \ ,\end{aligned}$$

where $r_l=\sum_{j=0}^{k-1}(\vec{w}_l)_j$ and $\vec{h}_l:=C^{-1}\vec{u}_l$ for $l=1,2$.

Let $\boldsymbol{V}_\mu^{\vec{u},\Delta,\vec{w}}$ be the natural bundle on $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ defined by the derived pushforward (cf. [12, Section 4.4])

$$\boldsymbol{V}_\mu^{\vec{u},\Delta,\vec{w}} := R^1p_{1*}\big(\boldsymbol{\mathcal{E}}_{\vec{u},\Delta,\vec{w}}\otimes p_2^*(\mathcal{O}_{\mathscr{X}_k}(\mu)\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))\big) \ .$$

It is a vector bundle on $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ whose fibre over $[(\mathcal{E},\phi_{\mathcal{E}})]$ is given by

$$\boldsymbol{V}_\mu^{\vec{u},\Delta,\vec{w}}\big|_{[(\mathcal{E},\phi_{\mathcal{E}})]} = H^1\big(\mathscr{X}_k\,;\,\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(\mu)\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \ .$$

Its rank was computed in [12, Proposition 4.18] to be

$$\mathrm{rk}\big(\boldsymbol{V}_\mu^{\vec{u},\Delta,\vec{w}}\big)=\Delta+\frac{1}{2r}\,\vec{h}\cdot C\vec{h}-\frac12\sum_{j=1}^{k-1}\big(C^{-1}\big)^{jj}\,w_j \ .$$

We denote by $\bar{\boldsymbol{V}}_\mu^{\vec{u},\Delta,\vec{w}}$ the analogous vector bundle on $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ defined using the dual $\boldsymbol{\mathcal{E}}^\vee_{\vec{u},\Delta,\vec{w}}$ of the universal sheaf.

Let us now fix topological data $(\vec{u}_v,\Delta_v,\vec{w}_v)_{v\in\mathtt{Q}_0}$ associated with the moduli spaces $\mathcal{M}_{\vec{u}_v,\Delta_v,\vec{w}_v}$ at the vertices $\mathtt{Q}_0$ with $\vec{u}_v\in\mathbb{Z}^{k-1}$ satisfying the constraint (5.1), $\Delta_v\in\frac{1}{2\,r_v\,k}\,\mathbb{Z}$, and $\vec{w}_v\in(\mathbb{Z}_{\geq0})^k$. The fundamental (resp. antifundamental) hypermultiplets of masses μ_v^s, $s=1,\dots,m_v$ (resp. $\bar{\mu}_v^{\bar{s}}$, $\bar{s}=1,\dots,\bar{m}_v$) at the nodes $v\in\mathtt{Q}_0$ correspond to the vector bundles $\boldsymbol{V}_{\mu_v^s}^{\vec{u}_v,\Delta_v,\vec{w}_v}$ (resp. $\bar{\boldsymbol{V}}_{\bar{\mu}_v^{\bar{s}}}^{\vec{u}_v,\Delta_v,\vec{w}_v}$) on $\mathcal{M}_{\vec{u}_v,\Delta_v,\vec{w}_v}$. The bifundamental hypermultiplets of masses μ_e at the edges $e\in\mathtt{Q}_1$ correspond to the vector bundles $\boldsymbol{E}_{\mu_e}^{\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)};\vec{u}_{\mathtt{t}(e)},\Delta_{\mathtt{t}(e)},\vec{w}_{\mathtt{t}(e)}}$ on $\mathcal{M}_{\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)}}\times\mathcal{M}_{\vec{u}_{\mathtt{t}(e)},\Delta_{\mathtt{t}(e)},\vec{w}_{\mathtt{t}(e)}}$; in particular, for a vertex loop edge e, i.e. $\mathtt{s}(e)=\mathtt{t}(e)$, the restriction of the bundle $\boldsymbol{E}_{\mu_e}^{\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)};\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)}}$ to the diagonal of $\mathcal{M}_{\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)}}\times\mathcal{M}_{\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)}}$ describes an adjoint hypermultiplet of mass μ_e. The total matter field content of the $\mathcal{N}=2$ quiver gauge theory on X_k associated to Q in the sector labelled by $(\vec{u}_v,\Delta_v,\vec{w}_v)_{v\in\mathsf{Q}_0}$ is then described by the bundle on $\prod_{v\in\mathsf{Q}_0}\mathcal{M}_{\vec{u}_v,\Delta_v,\vec{w}_v}$ given by

$$\boldsymbol{M}_{\boldsymbol{\mu}}^{\vec{\boldsymbol{u}},\boldsymbol{\Delta},\vec{\boldsymbol{w}}} := \bigoplus_{v\in\mathsf{Q}_0} p_v^*\Big(\bigoplus_{s=1}^{m_v}\boldsymbol{V}_{\mu_v^s}^{\vec{u}_v,\Delta_v,\vec{w}_v}\ \oplus\ \bigoplus_{\bar{s}=1}^{\bar{m}_v}\bar{\boldsymbol{V}}_{\bar{\mu}_v^{\bar{s}}}^{\vec{u}_v,\Delta_v,\vec{w}_v}\Big) \\ \oplus\ \bigoplus_{e\in\mathsf{Q}_1} p_e^*\boldsymbol{E}_{\mu_e}^{\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)};\vec{u}_{\mathtt{t}(e)},\Delta_{\mathtt{t}(e)},\vec{w}_{\mathtt{t}(e)}}\ ,$$

where p_v is the projection of $\prod_{v\in\mathsf{Q}_0}\mathcal{M}_{\vec{u}_v,\Delta_v,\vec{w}_v}$ to the v-th factor while p_e is the projection to the product $\mathcal{M}_{\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)}}\times\mathcal{M}_{\vec{u}_{\mathtt{t}(e)},\Delta_{\mathtt{t}(e)},\vec{w}_{\mathtt{t}(e)}}$; for vertex loop edges p_e is understood as projection to the diagonal of $\mathcal{M}_{\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)}}\times\mathcal{M}_{\vec{u}_{\mathtt{s}(e)},\Delta_{\mathtt{s}(e)},\vec{w}_{\mathtt{s}(e)}}$.

In this paper we consider only $\mathcal{N}=2$ superconformal quiver gauge theories. On $\mathbb{R}^4$, this implies that $M_{\mathsf{Q}}(\boldsymbol{r},\boldsymbol{m},\bar{\boldsymbol{m}})$ is in the conformal class of quiver representations whose dimension vectors satisfy the constraint

$$m_v+\bar{m}_v=2r_v-\sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{s}(e)=v}} r_{\mathtt{t}(e)}-\sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{t}(e)=v}} r_{\mathtt{s}(e)} \tag{6.1}$$

at any vertex $v\in\mathsf{Q}_0$ (see e.g. [45, Chapter 3] and [46, Section 2]). In an analogous way to the pure $\mathcal{N}=2$ gauge theory on X_k, we shall wish to relate our quiver gauge theory partition functions to the corresponding partition functions on $\mathbb{R}^4$ via suitable factorization formulas. Hence we shall impose the same general conformal constraints (6.1). When $\bar{\boldsymbol{m}}=\boldsymbol{0}$, this restricts the admissible quivers into three classes according to the ADE classification of [45, Chapter 3] in the following way:

- When $\boldsymbol{m}\neq\boldsymbol{0}$, the underlying graph of Q coincides with the Dynkin diagram of a finite-dimensional simply-laced Lie algebra of ADE type; these theories are called 'Class I' in [45].
- When $\boldsymbol{m}=\boldsymbol{0}$ and $\mu_e=0$ for all $e\in\mathsf{Q}_1$, the underlying graph of Q is a simply-laced affine Dynkin diagram of type $\widehat{A}_n$, $\widehat{D}_n$ or $\widehat{E}_n$; these theories are called 'Class II' in [45]. In this case the dimension vectors $\boldsymbol{r}$ are uniquely determined by a single integer N as

$$r_v=N\,d_v\ ,$$

where d_v are the Dynkin indices of the affine roots at the vertices Q_0.
- When $\boldsymbol{m}=\boldsymbol{0}$ and $\mu_e\neq 0$ for some $e\in\mathsf{Q}_1$, the underlying graph of Q is an affine Dynkin diagram of type $\widehat{A}_n$; these theories are called 'Class II*' in [45].

To formulate the analogous conformal constraint on X_k, we compute the degree of the Euler class of the hypermultiplet bundle $\boldsymbol{M}_{\boldsymbol{\mu}}^{\vec{\boldsymbol{u}},\boldsymbol{\Delta},\vec{\boldsymbol{w}}}$ which is given by

$$\begin{aligned}
&\deg\,\mathrm{eu}\big(\boldsymbol{M}_{\boldsymbol{\mu}}^{\vec{\boldsymbol{u}},\boldsymbol{\Delta},\vec{\boldsymbol{w}}}\big) := \sum_{v\in\mathsf{Q}_0}\dim\mathcal{M}_{\vec{u}_v,\Delta_v,\vec{w}_v}-\mathrm{rk}\big(\boldsymbol{M}_{\boldsymbol{\mu}}^{\vec{\boldsymbol{u}},\boldsymbol{\Delta},\vec{\boldsymbol{w}}}\big)\\
&=\sum_{v\in\mathsf{Q}_0}\Big(2r_v\,\Delta_v-\frac{1}{2}\sum_{j=1}^{k-1}\big(C^{-1}\big)^{jj}\,\vec{w}_v\cdot\vec{w}_v(j)\Big)\\
&-\sum_{v\in\mathsf{Q}_0}(m_v+\bar{m}_v)\Big(\Delta_v+\frac{1}{2r_v}\vec{h}_v\cdot C\vec{h}_v-\frac{1}{2}\sum_{j=1}^{k-1}\big(C^{-1}\big)^{jj}\,w_{v,j}\Big)-\sum_{e\in\mathsf{Q}_1}\Big(r_{\mathtt{s}(e)}\,\Delta_{\mathtt{t}(e)}+r_{\mathtt{t}(e)}\,\Delta_{\mathtt{s}(e)}\\
&+\frac{r_{\mathtt{t}(e)}}{2r_{\mathtt{s}(e)}}\vec{h}_{\mathtt{s}(e)}\cdot C\vec{h}_{\mathtt{s}(e)}+\frac{r_{\mathtt{s}(e)}}{2r_{\mathtt{t}(e)}}\vec{h}_{\mathtt{t}(e)}\cdot C\vec{h}_{\mathtt{t}(e)}-\vec{h}_{\mathtt{s}(e)}\cdot C\vec{h}_{\mathtt{t}(e)}-\frac{1}{2}\sum_{j=1}^{k-1}\big(C^{-1}\big)^{jj}\,\vec{w}_{\mathtt{s}(e)}\cdot\vec{w}_{\mathtt{t}(e)}(j)\Big)\ ,
\end{aligned}$$

where $r_\upsilon = \sum_{j=0}^{k-1} (\vec{w}_\upsilon)_j$ and $\vec{h}_\upsilon := C^{-1}\vec{u}_\upsilon$. Using (6.1) the degree becomes

$$\deg \mathrm{eu}\left(\boldsymbol{M}_{\boldsymbol{\mu}}^{\vec{\boldsymbol{u}},\boldsymbol{\Delta},\vec{\boldsymbol{w}}}\right) = \sum_{\upsilon\in\mathsf{Q}_0} d_\upsilon^{X_k}(\vec{\boldsymbol{h}}_\upsilon, \vec{\boldsymbol{w}}_\upsilon) ,$$

where we defined

$$\begin{aligned} d_\upsilon^{X_k}(\vec{\boldsymbol{h}}_\upsilon, \vec{\boldsymbol{w}}_\upsilon) := \frac{1}{2}\sum_{j=1}^{k-1} \left(C^{-1}\right)^{jj} w_{\upsilon,j} \Big(2r_\upsilon - \sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{s}(e)=\upsilon}} r_{\mathtt{t}(e)} - \sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{t}(e)=\upsilon}} r_{\mathtt{s}(e)}\Big) \\ - \frac{1}{4}\sum_{j=1}^{k-1} \left(C^{-1}\right)^{jj} \Big(2\vec{w}_\upsilon\cdot\vec{w}_\upsilon(j) - \sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{s}(e)=\upsilon}} \vec{w}_\upsilon\cdot\vec{w}_{\mathtt{t}(e)}(j) - \sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{t}(e)=\upsilon}} \vec{w}_{\mathtt{s}(e)}\cdot\vec{w}_\upsilon(j)\Big) \\ - \vec{h}_\upsilon\cdot C\vec{h}_\upsilon + \frac{1}{2}\sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{s}(e)=\upsilon}} \vec{h}_\upsilon\cdot C\vec{h}_{\mathtt{t}(e)} + \frac{1}{2}\sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{t}(e)=\upsilon}} \vec{h}_\upsilon\cdot C\vec{h}_{\mathtt{s}(e)} \end{aligned}$$

for each vertex $\upsilon\in\mathsf{Q}_0$; here $\vec{\boldsymbol{h}}_\upsilon := \big(\vec{h}_\upsilon, (\vec{h}_{\mathtt{t}(e)})_{e\in\mathsf{Q}_1\,:\,\mathtt{s}(e)=\upsilon}, (\vec{h}_{\mathtt{s}(e)})_{e\in\mathsf{Q}_1\,:\,\mathtt{t}(e)=\upsilon}\big)$ and similarly for $\vec{\boldsymbol{w}}_\upsilon$. By analogy with the case of gauge theories on $\mathbb{R}^4$, we say that the $\mathcal{N}=2$ quiver gauge theory on X_k is conformal if $d_\upsilon^{X_k}(\vec{\boldsymbol{h}}_\upsilon, \vec{\boldsymbol{w}}_\upsilon)=0$ for all $\upsilon\in\mathsf{Q}_0$; this is formally the requirement of vanishing beta-function for the running of the υ-th gauge coupling constant. For any vertex $\upsilon\in\mathsf{Q}_0$, define the set of conformal fractional instanton charges by

$$\mathfrak{U}_{\vec{\boldsymbol{w}}_\upsilon}^{\mathrm{conf}} := \big\{\vec{\boldsymbol{h}}_\upsilon \bigm| d_\upsilon^{X_k}(\vec{\boldsymbol{h}}_\upsilon, \vec{\boldsymbol{w}}_\upsilon)=0\big\} .$$

These charge sets generalise the conformal restrictions derived in [12, Section 5.4] for the A_0-theory with fundamental matter, whose quiver $\mathsf{Q}=\circ$ consists of a single node with no arrows.

6.2. **Instanton partition functions.** Introduce topological couplings $\mathsf{q}_\upsilon\in\mathbb{C}^*$ with $|\mathsf{q}_\upsilon|<1$ and $\vec{\xi}_\upsilon = \big((\xi_\upsilon)_1,\dots,(\xi_\upsilon)_{k-1}\big)\in(\mathbb{C}^*)^{k-1}$ with $|(\xi_\upsilon)_i|<1$ at each vertex $\upsilon\in\mathsf{Q}_0$. Let $T:=T_t\times T_{\vec{\boldsymbol{a}}}$, where $T_{\vec{\boldsymbol{a}}}$ is the maximal torus of the gauge group $G_{\boldsymbol{r}}$ with $H^*_{T_{\vec{\boldsymbol{a}}}}(\mathrm{pt}) = \mathbb{C}\big[(a_1^\upsilon,\dots,a_{r_\upsilon}^\upsilon)_{\upsilon\in\mathsf{Q}_0}\big]$. For a vertex $\upsilon\in\mathsf{Q}_0$ we write $T_\upsilon := T_t\times T_{\boldsymbol{a}^\upsilon}$, where $T_{\boldsymbol{a}^\upsilon}$ is the maximal torus of the gauge group $\mathrm{GL}(r_\upsilon,\mathbb{C})$ at the node with $H^*_{T_{\boldsymbol{a}^\upsilon}}(\mathrm{pt}) = \mathbb{C}[a_1^\upsilon,\dots,a_{r_\upsilon}^\upsilon]$, and for an arrow $e\in\mathsf{Q}_1$ we write $T_e := T_t\times T_{\boldsymbol{a}^{\mathtt{s}(e)}}\times T_{\boldsymbol{a}^{\mathtt{t}(e)}}$; for a vertex loop edge we set $T_e := T_t\times T_{\boldsymbol{a}^{\mathtt{s}(e)}}$. Let $T_{\boldsymbol{\mu}}$ be the maximal torus of the flavour symmetry group $G_{\boldsymbol{m},\bar{\boldsymbol{m}}}$ with $H^*_{T_{\boldsymbol{\mu}}}(\mathrm{pt}) = \mathbb{C}\big[(\mu_e)_{e\in\mathsf{Q}_1}\,,\,(\mu_\upsilon^1,\dots,\mu_\upsilon^{m_\upsilon})_{\upsilon\in\mathsf{Q}_0}\,,\,(\bar{\mu}_\upsilon^1,\dots,\bar{\mu}_\upsilon^{\bar{m}_\upsilon})_{\upsilon\in\mathsf{Q}_0}\big]$.

The instanton partition function for the $\mathcal{N}=2$ superconformal quiver gauge theory on X_k is then defined by the generating function

$$\begin{aligned} \mathcal{Z}_{X_k}^{\mathsf{Q},\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q},\vec{\boldsymbol{\xi}}\,\big)_{\vec{\boldsymbol{w}}} &:= \sum_{\substack{\vec{\boldsymbol{h}}\,,\,\boldsymbol{\Delta}\\ C\vec{h}_\upsilon\in\mathfrak{U}_{\vec{w}_\upsilon}\,,\,\vec{\boldsymbol{h}}_\upsilon\in\mathfrak{U}^{\mathrm{conf}}_{\vec{\boldsymbol{w}}_\upsilon}}} \vec{\boldsymbol{\xi}}^{\,\vec{\boldsymbol{h}}}\ \mathbf{q}^{\boldsymbol{\Delta}+\frac{1}{2r}\,\vec{\boldsymbol{h}}\cdot C\vec{\boldsymbol{h}}} \\ &\qquad\times \int_{\prod_{\upsilon\in\mathsf{Q}_0}\mathcal{M}_{C\vec{h}_\upsilon,\Delta_\upsilon,\vec{w}_\upsilon}} \mathrm{eu}_{T\times T_{\boldsymbol{\mu}}}\left(\boldsymbol{M}_{\boldsymbol{\mu}}^{C\vec{\boldsymbol{h}},\boldsymbol{\Delta},\vec{\boldsymbol{w}}}\right) \\ &= \sum_{\substack{\vec{\boldsymbol{h}}\,,\,\boldsymbol{\Delta}\\ C\vec{h}_\upsilon\in\mathfrak{U}_{\vec{w}_\upsilon}\,,\,\vec{\boldsymbol{h}}_\upsilon\in\mathfrak{U}^{\mathrm{conf}}_{\vec{\boldsymbol{w}}_\upsilon}}} \vec{\boldsymbol{\xi}}^{\,\vec{\boldsymbol{h}}}\ \mathbf{q}^{\boldsymbol{\Delta}+\frac{1}{2r}\,\vec{\boldsymbol{h}}\cdot C\vec{\boldsymbol{h}}} \\ &\qquad\times \int_{\prod_{\upsilon\in\mathsf{Q}_0}\mathcal{M}_{C\vec{h}_\upsilon,\Delta_\upsilon,\vec{w}_\upsilon}} \prod_{\upsilon\in\mathsf{Q}_0} p_\upsilon^*\Big(\prod_{s=1}^{m_\upsilon} \mathrm{eu}_{T_\upsilon\times T_{\mu_\upsilon^s}}\big(\boldsymbol{V}_{\mu_\upsilon^s}^{C\vec{h}_\upsilon,\Delta_\upsilon,\vec{w}_\upsilon}\big) \prod_{\bar{s}=1}^{\bar{m}_\upsilon} \mathrm{eu}_{T_\upsilon\times T_{\bar{\mu}_\upsilon^{\bar{s}}}}\big(\bar{\boldsymbol{V}}_{\bar{\mu}_\upsilon^{\bar{s}}}^{C\vec{h}_\upsilon,\Delta_\upsilon,\vec{w}_\upsilon}\big)\Big) \end{aligned}$$

$$\times \prod_{e\in\mathsf{Q}_1} p_e^*\, \mathrm{eu}_{T_e\times T_{\mu_e}}\big(\boldsymbol{E}_{\mu_e}^{C\vec h_{\mathsf{s}(e)},\Delta_{\mathsf{s}(e)},\vec w_{\mathsf{s}(e)};C\vec h_{\mathsf{t}(e)},\Delta_{\mathsf{t}(e)},\vec w_{\mathsf{t}(e)}}\big)\ ,$$

where $\mathsf{q}^{\boldsymbol{\Delta}+\frac{1}{2\boldsymbol{r}}\,\vec{\boldsymbol{h}}\cdot C\vec{\boldsymbol{h}}} := \prod_{v\in\mathsf{Q}_0} \mathsf{q}_v^{\Delta_v+\frac{1}{2r_v}\,\vec h_v\cdot C\vec h_v}$ and $\vec{\boldsymbol{\xi}}^{\,\vec{\boldsymbol{h}}} := \prod_{v\in\mathsf{Q}_0} \vec\xi_v^{\,\vec h_v}$. The equivariant Euler class of the Ext-bundle at a fixed point with combinatorial data $\big((\vec{\boldsymbol{Y}},\vec{\boldsymbol{u}}\,)\,,\,(\vec{\boldsymbol{Y}}',\vec{\boldsymbol{u}}\,')\big)$ is computed in [12, Section 4.7] with the result

$$\begin{aligned}&\mathrm{eu}_{T_t\times T_{\vec a}\times T_{\vec a\,'}\times T_\mu}\big(\boldsymbol{E}_\mu^{\vec u,\Delta,\vec w;\vec u\,',\Delta',\vec w\,'}\big|_{((\vec{\boldsymbol{Y}},\vec{\boldsymbol{u}}\,)\,,\,(\vec{\boldsymbol{Y}}',\vec{\boldsymbol{u}}\,'))}\big)\\ &\qquad = \prod_{\alpha=1}^{r}\prod_{\alpha'=1}^{r'}\prod_{i=1}^{k} m_{Y_\alpha^i,Y_{\alpha'}^{\prime\,i}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_{\alpha'\alpha}^{(i)}+\mu\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec h_{\alpha'\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\alpha'\alpha}^{(n)}+\mu\big)\ ,\end{aligned}$$

where

$$a_{\alpha'\alpha}^{(i)} := a_{\alpha'\alpha} + (\vec h_{\alpha'\alpha})_i\,\varepsilon_1^{(i)} + (\vec h_{\alpha'\alpha})_{i-1}\,\varepsilon_2^{(i)}$$

for $i=1,\dots,k$ and for $\alpha=1,\dots,r$, $\alpha'=1,\dots,r'$, and $a_{\alpha'\alpha} := a'_{\alpha'}-a_\alpha$, $\vec h_{\alpha'\alpha} := \vec h'_{\alpha'}-\vec h_\alpha$ (we set $(\vec h_{\alpha'\alpha})_0 = (\vec h_{\alpha'\alpha})_k = 0$). The equivariant Euler class of the bundle $\boldsymbol{V}_\mu^{\vec u,\Delta,\vec w}$ (resp. $\bar{\boldsymbol{V}}_\mu^{\vec u,\Delta,\vec w}$) is obtained from this formula by setting $(\vec{\boldsymbol{Y}},\vec{\boldsymbol{u}}\,)$ (resp. $(\vec{\boldsymbol{Y}}',\vec{\boldsymbol{u}}\,')$) to $(\emptyset,\vec{\boldsymbol{0}}\,)$. By applying the localization theorem we obtain

$$\begin{aligned}\mathcal{Z}_{X_k}^{\mathsf{Q},\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}}\,\big)_{\vec{\boldsymbol{w}}} &= \sum_{\substack{\vec{\boldsymbol{h}}\,,\,\vec{\boldsymbol{Y}}\\ C\vec h_v\in\mathfrak{U}_{\vec w_v}\,,\,\vec{\boldsymbol{h}}_v\in\mathfrak{U}^{\mathrm{conf}}_{\vec{\boldsymbol{w}}_v}}} \vec{\boldsymbol{\xi}}^{\,\vec{\boldsymbol{h}}}\ \mathsf{q}^{\boldsymbol{\Delta}+\frac{1}{2\boldsymbol{r}}\,\vec{\boldsymbol{h}}\cdot C\vec{\boldsymbol{h}}}\\ &\qquad\times \sum_{\substack{(\vec h_{v,1},\dots,\vec h_{v,r_v})_{v\in\mathsf{Q}_0}\\ \vec h_v=\sum_{\alpha=1}^{r_v}\vec h_{v,\alpha}}} m_{\vec{\boldsymbol{Y}},(\vec h_{v,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu})\,\ell_{(\vec h_{v,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu})\ ,\end{aligned}$$

where

$$\begin{aligned} m_{\vec{\boldsymbol{Y}},(\vec h_{v,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu}) &:= \prod_{i=1}^{k}\frac{\prod_{e\in\mathsf{Q}_1}\prod_{\alpha=1}^{r_{\mathsf{s}(e)}}\prod_{\beta=1}^{r_{\mathsf{t}(e)}} m_{Y^i_{\mathsf{s}(e),\alpha},Y^i_{\mathsf{t}(e),\beta}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{e;\beta\alpha}+\mu_e\big)}{\prod_{v\in\mathsf{Q}_0}\prod_{\alpha,\beta=1}^{r_v} m_{Y^i_{v,\alpha},Y^i_{v,\beta}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{v;\beta\alpha}\big)}\\ &\quad\times\prod_{v\in\mathsf{Q}_0}\prod_{\alpha=1}^{r_v}\prod_{i=1}^{k}\prod_{s=1}^{m_v} m_{Y^i_{v,\alpha}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{v;\alpha}+\mu_v^s\big)\prod_{\bar s=1}^{\bar m_v} m_{Y^i_{v,\alpha}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{v;\alpha}+\bar\mu_v^{\bar s}\big)\end{aligned}$$

and

$$\begin{aligned}\ell_{(\vec h_{v,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu}) &:= \prod_{n=1}^{k-1}\frac{\prod_{e\in\mathsf{Q}_1}\prod_{\alpha=1}^{r_{\mathsf{s}(e)}}\prod_{\beta=1}^{r_{\mathsf{t}(e)}} \ell^{(n)}_{\vec h_{e;\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a^e_{\beta\alpha}+\mu_e\big)}{\prod_{v\in\mathsf{Q}_0}\prod_{\alpha,\beta=1}^{r_v}\ell^{(n)}_{\vec h_{v;\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a^v_{\beta\alpha}\big)}\\ &\quad\times\prod_{v\in\mathsf{Q}_0}\prod_{\alpha=1}^{r_v}\prod_{n=1}^{k-1}\prod_{s=1}^{m_v}\ell^{(n)}_{\vec h_{v,\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a^v_\alpha+\mu_v^s\big)\prod_{\bar s=1}^{\bar m_v}\ell^{(n)}_{\vec h_{v,\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a^v_\alpha+\bar\mu_v^{\bar s}\big)\ .\end{aligned}$$

Here

$$m_Y(e_1,e_2,a) := \prod_{(s_1,s_2)\in Y}\big(a-(s_1-1)\,e_1-(s_2-1)\,e_2\big)$$

for a Young tableau $Y\subset\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$, while

$$a^{(i)}_{v;\alpha} = a^v_\alpha + (\vec h_{v,\alpha})_i\,\varepsilon_1^{(i)} + (\vec h_{v,\alpha})_{i-1}\,\varepsilon_1^{(i)} \qquad\text{and}\qquad a^{(i)}_{e;\beta\alpha} = a^e_{\beta\alpha} + (\vec h_{e;\beta\alpha})_i\,\varepsilon_1^{(i)} + (\vec h_{e;\beta\alpha})_{i-1}\,\varepsilon_1^{(i)}\ ,$$

with $\vec{h}_{v,\alpha}:=C^{-1}\vec{u}_{v,\alpha}$, $\vec{h}_{e;\beta\alpha}:=C^{-1}(\vec{u}_{\mathtt{t}(e),\beta}-\vec{u}_{\mathtt{s}(e),\alpha})$ and $a^e_{\beta\alpha}=a^{\mathtt{t}(e)}_\beta-a^{\mathtt{s}(e)}_\alpha$; for vertex loop edges with $\mathtt{t}(e)=\mathtt{s}(e)=v$ we also set $a^{(i)}_{v;\beta\alpha}:=a^{(i)}_{e;\beta\alpha}$, $\vec{h}_{v;\beta\alpha}:=\vec{h}_{e;\beta\alpha}$ and $a^v_{\beta\alpha}:=a^e_{\beta\alpha}$.

It follows that the instanton partition function $\mathcal{Z}^{\mathtt{Q},\mathrm{inst}}_{X_k}$ factorises in terms of the corresponding partition functions $\mathcal{Z}^{\mathtt{Q},\mathrm{inst}}_{\mathbb{C}^2}$ for the $\mathcal{N}=2$ superconformal quiver gauge theory on $\mathbb{R}^4$ (cf. [46, Section 2]) as

$$\begin{aligned}\mathcal{Z}^{\mathtt{Q},\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q},\vec{\boldsymbol{\xi}}\,\big)_{\vec{\boldsymbol{w}}}=&\sum_{\substack{\vec{\boldsymbol{h}}\\ C\vec{h}_v\in\mathfrak{U}_{\vec{w}_v}\,,\,\vec{\boldsymbol{h}}_v\in\mathfrak{U}^{\mathrm{conf}}_{\vec{\boldsymbol{w}}_v}}}\vec{\boldsymbol{\xi}}^{\,\vec{\boldsymbol{h}}}\sum_{\substack{(\vec{h}_{v,1},\dots,\vec{h}_{v,r_v})_{v\in\mathtt{Q}_0}\\ \vec{h}_v=\sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}}}\ \prod_{v\in\mathtt{Q}_0}\mathsf{q}_v^{\frac12\sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}\cdot C\vec{h}_{v,\alpha}}\\ &\times\ \ell_{(\vec{h}_{v,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu})\prod_{i=1}^k\mathcal{Z}^{\mathtt{Q},\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{\boldsymbol{a}}^{(i)},\boldsymbol{\mu};\mathbf{q}\big)\ .\end{aligned}$$

6.3. **Perturbative partition functions.** The perturbative partition function for the pure $\mathcal{N}=2$ gauge theory on X_k, which is independent of the topological couplings q and $\vec{\xi}$, is also defined in [12, Section 6]. Here we define a version of it for the $\mathcal{N}=2$ superconformal quiver gauge theories on X_k.

The perturbative partition function is determined by defining the perturbative part of the equivariant Chern character of the vector bundle $\boldsymbol{M}^{\vec{\boldsymbol{u}},\boldsymbol{\Delta},\vec{\boldsymbol{w}}}_{\boldsymbol{\mu}}$ at a fixed point

$$\big([(\mathcal{E}_v,\phi_{\mathcal{E}_v})]\big)_{v\in\mathtt{Q}_0}=\bigoplus_{\alpha=1}^r\big([(\imath_{k*}(I_{v,\alpha})\otimes\mathcal{R}^{\vec{u}_{v,\alpha}},\phi_{v,\alpha})]\big)_{v\in\mathtt{Q}_0}$$

of $\prod_{v\in\mathtt{Q}_0}\mathcal{M}_{\vec{u}_v,\Delta_v,\vec{w}_v}$ as

$$\begin{aligned}\mathrm{ch}^{\mathrm{pert}}_{T\times T_{\boldsymbol{\mu}}}\boldsymbol{M}^{\vec{\boldsymbol{u}},\boldsymbol{\Delta},\vec{\boldsymbol{w}}}_{\boldsymbol{\mu}}\big|_{([(\mathcal{E}_v,\phi_{\mathcal{E}_v})])_{v\in\mathtt{Q}_0}}:=&\sum_{v\in\mathtt{Q}_0}\Big(\sum_{s=1}^{m_v}\mathrm{e}^{\mu^s_v}+\sum_{\bar{s}=1}^{\bar{m}_v}\mathrm{e}^{\bar{\mu}^{\bar{s}}_v}\Big)\\ &\times\sum_{\alpha=1}^{r_v}\mathrm{e}^{a^v_\alpha}\Big(\chi_{T_t}\big(\bar{X}_k\,,\,\pi_{k*}(\mathcal{R}^{C\vec{h}_{v,\alpha}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))\big)-\chi_{T_t}\big(X_k\,,\,R^{C\vec{h}_{v,\alpha}}\big)\Big)\\ +\sum_{e\in\mathtt{Q}_1}\sum_{\alpha=1}^{r_{\mathtt{s}(e)}}\sum_{\beta=1}^{r_{\mathtt{t}(e)}}&\mathrm{e}^{a^e_{\beta\alpha}+\mu_e}\Big(\chi_{T_t}\big(\bar{X}_k\,,\,\pi_{k*}(\mathcal{R}^{C\vec{h}_{e;\beta\alpha}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))\big)-\chi_{T_t}\big(X_k\,,\,R^{C\vec{h}_{e;\beta\alpha}}\big)\Big)\end{aligned}\tag{6.2}$$

where $\pi_k:\mathscr{X}_k\to\bar{X}_k$ is the coarse moduli space morphism and $R^{\vec{u}}:=\mathcal{R}^{\vec{u}}\big|_{X_k}$. One way to compute it is to cancel the common factors between the two equivariant Euler characteristics, as was done in [12, Section 6]. In this way $\mathrm{ch}^{\mathrm{pert}}_{T\times T_{\boldsymbol{\mu}}}\boldsymbol{M}^{\vec{\boldsymbol{u}},\boldsymbol{\Delta},\vec{\boldsymbol{w}}}_{\boldsymbol{\mu}}\big|_{([(\mathcal{E}_v,\phi_{\mathcal{E}_v})])_{v\in\mathtt{Q}_0}}$ reduces to a sum depending on the equivariant Euler characteristics, over the two open affine toric neighbourhoods of the two fixed points $0,\infty$ of the compactification divisor D_∞, of a Weil divisor on $\bar{X}_k$ which is a linear combination of D_∞, D_0 and D_k with integer coefficients. The result depends only on the holonomies at infinity, i.e. on the framing vectors $\vec{w}_v$ for $v\in\mathtt{Q}_0$.

In this paper we explicitly compute $\chi_{T_t}\big(X_k\,,\,R^{\vec{u}}\big)$ for any $\vec{u}\in\mathbb{Z}^{k-1}$ in (6.2) and arrive at an equivalent yet more transparent formulation of the perturbative partition function. By using some of the arguments from the proof of [10, Theorem 4.1], we have

$$\chi_{T_t}\big(X_k\,,\,R^{\vec{u}}\big)=\sum_{i=1}^k\mathrm{ch}^i_{T_t}\big(R^{\vec{u}}\big)\ \big(1-\mathrm{e}^{\varepsilon_1^{(i)}}\big)^{-1}\big(1-\mathrm{e}^{\varepsilon_2^{(i)}}\big)^{-1}\ ,$$

where $\mathrm{ch}^i_{T_t}\big(R^{\vec{u}}\big)$ is the local Chern character of $R^{\vec{u}}$ over the open affine toric neighbourhood of the fixed point p_i for $i=1,\dots,k$ (cf. [10, Section 2.1]). Since X_k is smooth, by [10, Section 2.3] for any Weil divisor $D=\sum_{j=0}^k d_j\, D_j$ on X_k one has

$$\mathrm{ch}^i_{T_t}\big(\mathcal{O}_{X_k}(D)\big) = \mathrm{e}^{-d_i\,\varepsilon_1^{(i)}-d_{i-1}\,\varepsilon_2^{(i)}} \ .$$

It follows that in our case for $i=1,\dots,k$ we have

$$\mathrm{ch}^i_{T_t}\big(R^{\vec u}\big) = \mathrm{e}^{h_i\,\varepsilon_1^{(i)}+h_{i-1}\,\varepsilon_2^{(i)}} \ , \tag{6.3}$$

where $\vec h := C^{-1}\vec u$.

To write down the perturbative partition function, we follow the same arguments as in [28, Section 6]: the perturbative partition function is given by a sum, over the torus-fixed points, of the Euler classes obtained formally from the perturbative characters. Since the Euler class associated with a term $\chi_{T_t}\big(X_k\,,\,R^{\vec u}\big)$ involves infinite products, we regularise it by using Barnes' double gamma function Γ_2. Then we define the perturbative partition function for the $\mathcal{N}=2$ superconformal quiver gauge theory on X_k as

$$\mathcal{Z}^{\mathsf{Q},\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol a},\boldsymbol\mu)_{\vec{\boldsymbol w}} := \sum_{\substack{\vec{\boldsymbol h}\\ C\vec h_\upsilon\in\mathfrak{U}_{\vec w_\upsilon}\,,\,\vec{\boldsymbol h}_\upsilon\in\mathfrak{U}^{\mathrm{conf}}_{\vec{\boldsymbol w}_\upsilon}}} \ \sum_{\substack{(\vec h_{\upsilon,1},\dots,\vec h_{\upsilon,r_\upsilon})_{\upsilon\in\mathsf{Q}_0}\\ \vec h_\upsilon=\sum_{\alpha=1}^{r_\upsilon}\vec h_{\upsilon,\alpha}}} \mathcal{Z}^{\mathsf{Q},\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol a},\boldsymbol\mu)_{\vec{\boldsymbol w},(\vec h_{\upsilon,\alpha})} \ ,$$

where

$$\begin{aligned}\mathcal{Z}^{\mathsf{Q},\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol a},\boldsymbol\mu)_{\vec{\boldsymbol w},(\vec h_{\upsilon,\alpha})} &:= \ell_{(\vec h_{\upsilon,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol a},\boldsymbol\mu)^{-1}\\ &\qquad\times\prod_{i=1}^k \frac{\prod\limits_{e\in\mathsf{Q}_1}\prod\limits_{\alpha=1}^{r_{\mathsf{s}(e)}}\prod\limits_{\beta=1}^{r_{\mathsf{t}(e)}}\Gamma_2\big(a^{(i)}_{e;\beta\alpha}+\mu_e\,\big|\,\varepsilon_1^{(i)},\varepsilon_2^{(i)}\big)}{\prod\limits_{\upsilon\in\mathsf{Q}_0}\prod\limits_{\alpha\neq\beta}\Gamma_2\big(a^{(i)}_{\upsilon;\beta\alpha}\,\big|\,\varepsilon_1^{(i)},\varepsilon_2^{(i)}\big)}\\ &\times\prod_{\upsilon\in\mathsf{Q}_0}\prod_{\alpha=1}^{r_\upsilon}\prod_{i=1}^k\prod_{s=1}^{m_\upsilon}\Gamma_2\big(a^{(i)}_{\upsilon,\alpha}+\mu^s_\upsilon\,\big|\,\varepsilon_1^{(i)},\varepsilon_2^{(i)}\big)\prod_{\bar s=1}^{\bar m_\upsilon}\Gamma_2\big(a^{(i)}_{\upsilon,\alpha}+\bar\mu^{\bar s}_\upsilon\,\big|\,\varepsilon_1^{(i)},\varepsilon_2^{(i)}\big)\ ,\end{aligned}$$

and we observe once again the ensuing factorization formula

$$\mathcal{Z}^{\mathsf{Q},\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol a},\boldsymbol\mu)_{\vec{\boldsymbol w},(\vec h_{\upsilon,\alpha})} = \ell_{(\vec h_{\upsilon,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol a},\boldsymbol\mu)^{-1}\prod_{i=1}^k \mathcal{Z}^{\mathsf{Q},\mathrm{pert}}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{\boldsymbol a}^{(i)},\boldsymbol\mu\big)\ .$$

6.4. **Proof of the master formula.** Let us define the generating function for correlators of p-observables ($p=0,2$) in the $\mathcal{N}=2$ superconformal quiver gauge theory on X_k in terms of integrals of equivariant cohomology classes over the moduli spaces as

$$\begin{aligned}&\mathcal{Z}^{\mathsf{Q}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol a},\boldsymbol\mu;\mathsf{q},\vec{\boldsymbol\xi},\vec{\boldsymbol\tau},\vec{\boldsymbol t}^{\,(1)},\dots,\vec{\boldsymbol t}^{\,(k-1)}\big)_{\vec{\boldsymbol w}}\\ &:= \sum_{\substack{\vec{\boldsymbol h}\,,\,\boldsymbol\Delta\\ C\vec h_\upsilon\in\mathfrak{U}_{\vec w_\upsilon}\,,\,\vec{\boldsymbol h}_\upsilon\in\mathfrak{U}^{\mathrm{conf}}_{\vec{\boldsymbol w}_\upsilon}}} \vec{\boldsymbol\xi}^{\,\vec{\boldsymbol h}}\,\mathsf{q}^{\boldsymbol\Delta+\frac{1}{2r}\vec{\boldsymbol h}\cdot C\vec{\boldsymbol h}}\int_{\prod_{\upsilon\in\mathsf{Q}_0}\mathcal{M}_{C\vec h_\upsilon,\Delta_\upsilon,\vec w_\upsilon}}\mathrm{eu}_{T\times T_{\boldsymbol\mu}}\big(\boldsymbol M^{C\vec{\boldsymbol h},\boldsymbol\Delta,\vec{\boldsymbol w}}_{\boldsymbol\mu}\big)\\ &\times\prod_{\upsilon\in\mathsf{Q}_0}p_\upsilon^*\exp\Big(\sum_{s=0}^\infty\Big(\sum_{i=1}^{k-1}t_s^{\upsilon,(i)}\,\big[\mathrm{ch}_T(\boldsymbol{\mathcal{E}}_{C\vec h_\upsilon,\Delta_\upsilon,\vec w_\upsilon})/[\mathscr{D}_i]\big]_s+\tau_s^\upsilon\,\big[\mathrm{ch}_T(\boldsymbol{\mathcal{E}}_{C\vec h_\upsilon,\Delta_\upsilon,\vec w_\upsilon})/[X_k]\big]_{s-1}\Big)\Big)\ ,\end{aligned}$$

where $\tau_0^\upsilon=\frac{1}{2\pi\,\mathrm{i}}\,\log\mathsf{q}_\upsilon$ is the υ-th bare gauge coupling. Here $\mathrm{ch}_T(\boldsymbol{\mathcal{E}}_{C\vec h,\Delta,\vec w})/[\mathscr{D}_i]$ is the slant product between the equivariant Chern character of the universal sheaf $\boldsymbol{\mathcal{E}}_{C\vec h,\Delta,\vec w}$ of $\mathcal{M}_{C\vec h,\Delta,\vec w}$ and the divisor class $[\mathscr{D}_i]$, and the class $\mathrm{ch}_T(\boldsymbol{\mathcal{E}}_{C\vec{h},\Delta,\vec{w}})/[X_k]$ is defined by localization as

$$\mathrm{ch}_T(\boldsymbol{\mathcal{E}}_{C\vec{h},\Delta,\vec{w}})/[X_k] := \sum_{i=1}^{k} \frac{1}{\mathrm{eu}_{T_t}(T_{p_i}X_k)}\, \imath^*_{\mathcal{M}_{C\vec{h},\Delta,\vec{w}}\times\{p_i\}}\, \mathrm{ch}_T(\boldsymbol{\mathcal{E}}_{C\vec{h},\Delta,\vec{w}})$$

with $\imath_{\mathcal{M}_{C\vec{h},\Delta,\vec{w}}\times\{p_i\}}$ the inclusion map of $\mathcal{M}_{C\vec{h},\Delta,\vec{w}}\times\{p_i\}$ in $\mathcal{M}_{C\vec{h},\Delta,\vec{w}}\times X_k$. The brackets $[-]_s$ indicate the degree s part. For further discussion about these kinds of partition functions and their physical origins, see [12, Section 1.2]. By applying the localization theorem, it is straightforward to generalise the computations leading to [12, Proposition 5.9] for the pure $\mathcal{N}=2$ gauge theory on X_k to the present case and obtain

$$\begin{aligned}
&\mathcal{Z}^{\mathsf{Q}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}},\vec{\boldsymbol{\tau}},\vec{\boldsymbol{t}}^{\,(1)},\dots,\vec{\boldsymbol{t}}^{\,(k-1)}\big)_{\vec{\boldsymbol{w}}} = \sum_{\substack{\vec{\boldsymbol{h}}\,,\,\vec{\boldsymbol{Y}}\\ C\vec{h}_v\in\mathfrak{U}_{\vec{w}_v}\,,\,\vec{\boldsymbol{h}}_v\in\mathfrak{U}^{\mathrm{conf}}_{\vec{\boldsymbol{w}}_v}}} \vec{\boldsymbol{\xi}}^{\,\vec{\boldsymbol{h}}}\ \mathsf{q}^{\boldsymbol{\Delta}+\frac{1}{2\boldsymbol{r}}\,\vec{\boldsymbol{h}}\cdot C\vec{\boldsymbol{h}}}\\
&\qquad\times \sum_{\substack{(\vec{h}_{v,1},\dots,\vec{h}_{v,r_v})_{v\in\mathsf{Q}_0}\\ \vec{h}_v=\sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}}} m_{\vec{\boldsymbol{Y}},(\vec{h}_{v,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu})\,\ell_{(\vec{h}_{v,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu})\\
&\qquad\times\prod_{v\in\mathsf{Q}_0}\prod_{i=1}^{k}\exp\Big(\sum_{s=0}^{\infty}\Big(\big(t_s^{v,(i)}\,\varepsilon_1^{(i)}+t_s^{v,(i-1)}\,\varepsilon_2^{(i)}+\tau_s^v\big)\Big[\mathrm{ch}_{\boldsymbol{Y}^i_v}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{a}_v^{(i)}\big)\Big]_{s-1}\Big)\Big)\\
&\times\prod_{v\in\mathsf{Q}_0}\exp\Big(\sum_{s=0}^{\infty}\Big(\sum_{l=1}^{k}\Big(\sum_{i=1}^{l-2}t_s^{v,(i)}+\sum_{i=l+1}^{k-1}t_s^{v,(i)}\Big)\Big[\mathrm{ch}_{\boldsymbol{Y}^l_v}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\boldsymbol{a}_v^{(l)}\big)\Big]_s\\
&\qquad+\sum_{i=1}^{k-1}t_s^{v,(i)}\sum_{l=2}^{k-1}\Big(\Big[\big(\varepsilon_1^{(l)}\big)^{\delta_{l,i}}\ \mathrm{ch}_{\boldsymbol{Y}^l_v}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\boldsymbol{a}_v^{(l)}-\big(\vec{h}_v\big)_{l-1}\,\varepsilon_2^{(l)}\big)\Big]_s\\
&\qquad\qquad+\Big[\big(\varepsilon_2^{(l)}\big)^{\delta_{l-1,i}}\ \mathrm{ch}_{\boldsymbol{Y}^l_v}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\boldsymbol{a}_v^{(l)}-\big(\vec{h}_v\big)_l\,\varepsilon_1^{(l)}\big)\Big]_s\Big)\Big)\Big)
\end{aligned}$$

where we set $t_s^{v,(0)}=t_s^{v,(k)}=0$ for any $s\geq 0$ and $v\in\mathsf{Q}_0$, and we introduced the notation

$$\begin{aligned}
\mathrm{ch}_{\boldsymbol{Y}^i_v}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{a}_v^{(i)}\big) := \sum_{\alpha=1}^{r_v}&\frac{\mathrm{e}^{\,a^{(i)}_{v;\alpha}}}{\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}}\\
&\times\Big(1-\big(1-\mathrm{e}^{-\varepsilon_1^{(i)}}\big)\big(1-\mathrm{e}^{-\varepsilon_2^{(i)}}\big)\sum_{(s_1,s_2)\in Y^i_{v,\alpha}}\mathrm{e}^{-(s_1-1)\,\varepsilon_1^{(i)}-(s_2-1)\,\varepsilon_2^{(i)}}\Big)
\end{aligned}$$

for $i=1,\dots,k$. Apart from the case $k=2$, generically there seems to be no nice factorizations of this expression into corresponding generating functions for 0-observables on $\mathbb{R}^4$; for $k\geq 3$ the apparent lack of a factorization property for $\mathcal{Z}^{\mathsf{Q}}_{X_k}$ is due to the fact that it involves terms which depend on pairs of exceptional divisors intersecting at the fixed points of X_k, and such terms do not split into terms each depending on a single affine toric subset of X_k. In the following we consider various specializations of this generating function.

Here we specialise to the generating function for correlators of quadratic 0-observables on X_k, and hence provide a rigorous derivation of the Nekrasov master formula for all $\mathcal{N}=2$ quiver gauge theories on X_k; a similar computation is done in [40, Section 4.4] in the context of pure $\mathcal{N}=2$ gauge theory on the blowup of $\mathbb{C}^2$ at a point. For this, we set $\vec{\boldsymbol{\tau}}=(0,-\tau^v_{\mathrm{cl}},0,\dots)_{v\in\mathsf{Q}_0}$ and $\vec{\boldsymbol{t}}^{\,(1)}=\cdots=\vec{\boldsymbol{t}}^{\,(k-1)}=\vec{\boldsymbol{0}}$; we denote the resulting partition function by $\mathcal{Z}^{\mathsf{Q},\circ}_{X_k}$. Then one finds

$$\mathcal{Z}^{\mathsf{Q},\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathsf{q},\boldsymbol{\tau}_{\mathrm{cl}},\vec{\boldsymbol{\xi}}\,\big)_{\vec{\boldsymbol{w}}}$$

$$= \sum_{\substack{\vec{\boldsymbol{h}} \\ C\vec{h}_v\in\mathfrak{U}_{\vec{w}_v}\,,\,\vec{\boldsymbol{h}}_v\in\mathfrak{U}^{\rm conf}_{\vec{\boldsymbol{w}}_v}}} \ \sum_{\substack{(\vec{h}_{v,1},\dots,\vec{h}_{v,r_v})_{v\in\mathsf{Q}_0} \\ \vec{h}_v=\sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}}} \mathcal{Z}^{\mathsf{Q},\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q},\boldsymbol{\tau}_{\rm cl},\vec{\boldsymbol{\xi}}\,\big)_{\vec{\boldsymbol{w}},(\vec{h}_{v,\alpha})}\ ,$$

where

$$\begin{aligned}\mathcal{Z}^{\mathsf{Q},\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q},\boldsymbol{\tau}_{\rm cl},\vec{\boldsymbol{\xi}}\,\big)_{\vec{\boldsymbol{w}},(\vec{h}_{v,\alpha})} &:= \prod_{v\in\mathsf{Q}_0} \vec{\xi}_v^{\ \sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}}\ \mathsf{q}_v^{\ \frac12\sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}\cdot C\vec{h}_{v,\alpha}}\ \ell_{(\vec{h}_{v,\alpha})}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu}) \\ &\quad\times \prod_{i=1}^{k} \mathcal{Z}^{\mathsf{Q},\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{\boldsymbol{a}}^{(i)},\boldsymbol{\mu};\mathbf{q}_{\rm eff}\big)\ \mathcal{Z}^{\mathsf{Q},\rm cl}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{\boldsymbol{a}}^{(i)};\boldsymbol{\tau}_{\rm cl}\big)\ ,\end{aligned}$$

with $\mathbf{q}_{\rm eff}:=(\mathsf{q}_v\ {\rm e}^{\,\tau^v_{\rm cl}})_{v\in\mathsf{Q}_0}$ and

$$\mathcal{Z}^{\mathsf{Q},\rm cl}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{\boldsymbol{a}}^{(i)};\boldsymbol{\tau}_{\rm cl}\big) := \prod_{v\in\mathsf{Q}_0}\mathcal{Z}^{\rm cl}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{a}_v^{(i)};\tau^v_{\rm cl}\big)\ .$$

Define now the full generating function for correlators of quadratic 0-observables for the $\mathcal{N}=2$ superconformal quiver gauge theory on X_k as

$$\begin{aligned}&\mathcal{Z}^{\mathsf{Q},\rm full}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q},\boldsymbol{\tau}_{\rm cl},\vec{\boldsymbol{\xi}}\,\big)_{\vec{\boldsymbol{w}}} \\ &\quad:= \sum_{\substack{\vec{\boldsymbol{h}} \\ C\vec{h}_v\in\mathfrak{U}_{\vec{w}_v}\,,\,\vec{\boldsymbol{h}}_v\in\mathfrak{U}^{\rm conf}_{\vec{\boldsymbol{w}}_v}}} \ \sum_{\substack{(\vec{h}_{v,1},\dots,\vec{h}_{v,r_v})_{v\in\mathsf{Q}_0} \\ \vec{h}_v=\sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}}} \mathcal{Z}^{\mathsf{Q},\rm pert}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu}\,)_{\vec{\boldsymbol{w}},(\vec{h}_{v,\alpha})} \\ &\qquad\qquad\times\ \mathcal{Z}^{\mathsf{Q},\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q},\boldsymbol{\tau}_{\rm cl},\vec{\boldsymbol{\xi}}\,\big)_{\vec{\boldsymbol{w}},(\vec{h}_{v,\alpha})}\ .\end{aligned}$$

Then

$$\begin{aligned}&\mathcal{Z}^{\mathsf{Q},\rm full}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q},\boldsymbol{\tau}_{\rm cl},\vec{\boldsymbol{\xi}}\,\big)_{\vec{\boldsymbol{w}}} \\ &\quad= \sum_{\substack{\vec{\boldsymbol{h}} \\ C\vec{h}_v\in\mathfrak{U}_{\vec{w}_v}\,,\,\vec{\boldsymbol{h}}_v\in\mathfrak{U}^{\rm conf}_{\vec{\boldsymbol{w}}_v}}} \ \sum_{\substack{(\vec{h}_{v,1},\dots,\vec{h}_{v,r_v})_{v\in\mathsf{Q}_0} \\ \vec{h}_v=\sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}}} \prod_{v\in\mathsf{Q}_0} \vec{\xi}_v^{\ \sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}}\ \mathsf{q}_v^{\ \frac12\sum_{\alpha=1}^{r_v}\vec{h}_{v,\alpha}\cdot C\vec{h}_{v,\alpha}} \\ &\qquad\qquad\times\prod_{i=1}^{k}\mathcal{Z}^{\mathsf{Q},\rm full}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{\boldsymbol{a}}^{(i)},\boldsymbol{\mu};\mathbf{q}_{\rm eff}\big)\ .\end{aligned}$$

Therefore we obtain a factorization of the full generating function $\mathcal{Z}^{\mathsf{Q},\rm full}_{X_k}$ in terms of k copies of $\mathcal{Z}^{\mathsf{Q},\rm full}_{\mathbb{C}^2}$. This proves the Nekrasov master formula for $\mathcal{N}=2$ quiver gauge theories on X_k.

The master formula suggests in particular that the $\mathcal{N}=2$ quiver gauge theories on the ALE space X_k, like their counterparts on $\mathbb{R}^4$, are some sort of quantizations of Hitchin systems; we confirm this expectation in an explicit example below. Recall that the low energy limit of $\mathcal{N}=2$ gauge theories on $\mathbb{R}^4$ is completely characterised by the (punctured) Seiberg-Witten curve Σ of genus $r:=\sum_{v\in\mathsf{Q}_0} r_v$ [50]. The curve Σ is equiped with a meromorphic differential $\lambda_{\rm SW}$, called the Seiberg-Witten differential, and its periods determine the Seiberg-Witten prepotential $\mathcal{F}^{\mathsf{Q}}_{\mathbb{C}^2}(\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q})$ which is a holomorphic function of all parameters. Setting $\bar{\boldsymbol{m}}=\boldsymbol{0}$ for brevity, in a symplectic basis $\{A_1^v,B_1^v,\dots,A^v_{r_v},B^v_{r_v},S_1^v,\dots,S^v_{m_v}\}_{v\in\mathsf{Q}_0}\cup\{T_e\}_{e\in\mathsf{Q}_1}$ for the homology group $H_1(\Sigma;\mathbb{Z})$, the periods of the Seiberg-Witten differential determine the quantities

$$a^v_\alpha=\oint_{A^v_\alpha}\lambda_{\rm SW} \qquad\text{and}\qquad \frac{\partial\mathcal{F}^{\mathsf{Q}}_{\mathbb{C}^2}}{\partial a^v_\alpha}(\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q})=-2\pi\,{\rm i}\ \oint_{B^v_\alpha}\lambda_{\rm SW}\ ,$$

together with the mass parameters

$$\mu_v^s = \oint_{S_s^v} \lambda_{\rm SW} \qquad \text{and} \qquad \mu_e = \oint_{T_e} \lambda_{\rm SW} \ .$$

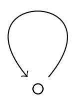
It follows that the period matrix $\boldsymbol{\tau} = (\tau_{\alpha\beta}^{vv'})$ of the Seiberg-Witten curve Σ is related to the prepotential by

$$\tau_{\alpha\beta}^{vv'} = -\frac{1}{2\pi\,\mathrm{i}}\,\frac{\partial^2 \mathcal{F}_{\mathbb{C}^2}^{\mathtt{Q}}}{\partial a_\alpha^v\,\partial a_\beta^{v'}}(\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q}) \ ,$$

and it determines the infrared effective gauge couplings. The prepotential is conjecturally recovered from the partition function $\mathcal{Z}_{\mathbb{C}^2}^{\mathtt{Q},\mathrm{full}}(\varepsilon_1,\varepsilon_2,\vec{\boldsymbol{a}},\boldsymbol{\mu};\mathbf{q})$ in the low energy limit in which the equivariant parameters $\varepsilon_1,\varepsilon_2$ vanish. Recall that Σ is also the spectral curve of an algebraic integrable system, the Donagi-Witten integrable system [19], which can be described as a particular Hitchin system on a punctured Riemann surface C such that Σ is a branched cover of C. For the ADE quivers $\mathtt{Q}$ we label the vertices $\mathtt{Q}_0$ of the Dynkin diagram by $v = 0,1,\dots,n$, where n is the rank of the corresponding simply-laced finite-dimensional Lie algebra. Then for the various theories in the ADE classification the spectral data can be described in the following way [45] (see also [49, Section 8]):

- For Class I theories Σ has $n+1$ punctures and is the spectral curve of a Hitchin system on the Riemann sphere with $n+4$ punctures at $\infty,1,z_1,\dots,z_{n+1},0$, where $z_v := \mathsf{q}_0\,\mathsf{q}_1\cdots\mathsf{q}_{v-1}$ for $v = 1,\dots,n+1$.
- For Class II theories Σ has no punctures and is the spectral curve of a $\mathrm{U}(N)$-Hitchin system on an elliptic curve with nome $\mathsf{q} := \mathsf{q}_0\,\mathsf{q}_1\cdots\mathsf{q}_n$.
- For Class II* theories Σ has $n+1$ punctures and is the spectral curve of a $\mathrm{U}(N)$-Hitchin system on an elliptic curve with nome q as above and $n+1$ punctures at $0,z_1,\dots,z_n$, where $z_v := \mathsf{q}_1\cdots\mathsf{q}_v$ for $v = 1,\dots,n$.

6.5. $\widehat{A}_0$**-theory.** Let us consider the quiver $\mathtt{Q}$ consisting of a single node with a vertex loop edge



The corresponding quiver gauge theory is the $\mathcal{N} = 2$ gauge theory with an adjoint hypermultiplet of mass μ, also known as the $\mathcal{N} = 2^*$ gauge theory. In this case the superconformal constraint is trivially satisfied by any charge vector $\vec{h}$ for a fixed choice of framing vector $\vec{w}$.

In the following we characterise the low energy limit of the $\mathcal{N} = 2^*$ $\mathrm{U}(r)$ gauge theory on X_k. The spectral curve Σ in this case has genus r and one puncture, and its period matrix $\tau = (\tau_{\alpha\beta})$ is related to the Seiberg-Witten prepotential $\mathcal{F}_{\mathbb{C}^2}^{\widehat{A}_0}(\boldsymbol{a},\mu;\mathsf{q})$ by

$$\tau_{\alpha\beta} = -\frac{1}{2\pi\,\mathrm{i}}\,\frac{\partial^2 \mathcal{F}_{\mathbb{C}^2}^{\widehat{A}_0}}{\partial a_\alpha\,\partial a_\beta}(\boldsymbol{a},\mu;\mathsf{q}) \ .$$

The Donagi-Witten integrable system is given by the elliptic Calogero-Moser model of type A_r, which can also be described as a $\mathrm{U}(r)$-Hitchin system on an elliptic curve with nome q and one puncture at $z = 0$. The prepotential can be recovered from the partition function for the Ω-deformed $\mathcal{N} = 2^*$ gauge theory on $\mathbb{R}^4$ in the low energy limit in which the equivariant parameters $\varepsilon_1,\varepsilon_2$ vanish; this result was originally conjectured by Nekrasov [42] and subsequently proven in [41, 44]. In [12, Section 7] we prove analogous results for gauge theory on X_k. For this, let us define

$$F_{X_k}^{\widehat{A}_0,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\vec{\xi}\,)_{\vec{w}} := -\tilde{k}\,\varepsilon_1\,\varepsilon_2\,\log \mathcal{Z}_{X_k}^{\widehat{A}_0,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\vec{\xi}\,)_{\vec{w}} \ .$$

Then $F^{\widehat{A}_0,\mathrm{inst}}_{X_k}$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$ and

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} F^{\widehat{A}_0,\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\vec{\xi}\,\big)_{\vec{w}}=\frac{1}{k}\,\mathcal{F}^{\widehat{A}_0,\mathrm{inst}}_{\mathbb{C}^2}(\boldsymbol{a},\mu;\mathsf{q})\ ,$$

where $\mathcal{F}^{\widehat{A}_0,\mathrm{inst}}_{\mathbb{C}^2}(\boldsymbol{a},\mu;\mathsf{q})$ is the instanton part of the Seiberg-Witten prepotential of $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$; the same result is true of the perturbative contributions. We can also define a prepotential associated with the generating function for correlators of quadratic 0-observables by letting

$$F^{\widehat{A}_0,\circ}_{X_k}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\vec{\xi},\tau_{\mathrm{cl}})_{\vec{w}}:=-\tilde{k}\,\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}^{\widehat{A}_0,\circ}_{X_k}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\vec{\xi},\tau_{\mathrm{cl}})_{\vec{w}}\ .$$

Then $F^{\widehat{A}_0,\circ}_{X_k}$ is also analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$ and

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} F^{\widehat{A}_0,\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a};\mathsf{q},\vec{\xi},\tau_{\mathrm{cl}})_{\vec{w}}=\frac{1}{k}\Big(\frac{\tilde{k}\,\tau_{\mathrm{cl}}}{2}\sum_{\alpha=1}^r a_\alpha^2+\mathcal{F}^{\widehat{A}_0,\mathrm{inst}}_{\mathbb{C}^2}(\boldsymbol{a},\mu;\mathsf{q}_{\mathrm{eff}})\Big)\ .$$

These results confirm that the $\mathcal{N}=2^*$ gauge theory on the ALE space X_k is a quantization of a Hitchin system with spectral curve Σ.

Let us focus now on the case $k=2$, and denote by $\mathcal{Z}^{\widehat{A}_0,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\vec{\xi},t)_{\vec{w}}$ the generating function for correlators of quadratic 2-observables of $\mathcal{N}=2^*$ gauge theory on X_2 which is obtained from the generating function $\mathcal{Z}^{\widehat{A}_0}_{X_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}\,)_{\vec{w}}$ specialised at $\vec{\tau}=\vec{0}$ and $\vec{t}:=(0,-t,0,\dots)$. Then we have the factorization formula (cf. [12, Example 5.11 and Section 7.3])

$$\begin{aligned}\mathcal{Z}^{\widehat{A}_0,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\xi,t)_{\vec{w}}&=\sum_{\substack{h\in\frac12\mathbb{Z}\\ 2h=w_1 \bmod 2}}\xi^h\\ &\times\sum_{\substack{\boldsymbol{h}=(h_1,\dots,h_r)\\ h=\sum_{\alpha=1}^r h_\alpha}}\big(\mathsf{q}\ \mathrm{e}^{\,2t\,(\varepsilon_1+\varepsilon_2)}\big)^{\sum_{\alpha=1}^r h_\alpha^2}\ \mathrm{e}^{\,-2t\,\sum_{\alpha=1}^r h_\alpha\,a_\alpha}\prod_{\alpha,\beta=1}^r\frac{\ell_{h_{\beta\alpha}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha}+\mu)}{\ell_{h_{\beta\alpha}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha})}\\ &\times\ \mathcal{Z}^{\widehat{A}_0,\mathrm{inst}}_{\mathbb{C}^2}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,\boldsymbol{a}-2\varepsilon_1\,\boldsymbol{h},\mu;\mathsf{q}\ \mathrm{e}^{\,2\varepsilon_1\,t}\big)\,\mathcal{Z}^{\widehat{A}_0,\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1-\varepsilon_2,2\varepsilon_2,\boldsymbol{a}-2\varepsilon_2\,\boldsymbol{h},\mu;\mathsf{q}\ \mathrm{e}^{\,2\varepsilon_2\,t}\big)\ .\end{aligned}$$

We shall describe blowup equations which relate $\mathcal{Z}^{\widehat{A}_0,\bullet}_{X_2}$ to the instanton partition function $\mathcal{Z}^{\widehat{A}_0,\mathrm{inst}}_{X_2}$ in the low energy limit. Let $\Theta\big[\begin{smallmatrix}\boldsymbol{\mu}\\ \boldsymbol{\nu}\end{smallmatrix}\big](\boldsymbol{\zeta}\,|\,\tau)$ be the Riemann theta-function with characteristic $\big[\begin{smallmatrix}\boldsymbol{\mu}\\ \boldsymbol{\nu}\end{smallmatrix}\big]$ on the Seiberg-Witten curve Σ for $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$. Then the ratio

$$\mathcal{Z}^{\widehat{A}_0,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\xi,t)_{\vec{w}}\,\big/\,\mathcal{Z}^{\widehat{A}_0,\mathrm{inst}}_{X_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\xi)_{\vec{w}}$$

is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$, and

$$\begin{aligned}&\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{\mathcal{Z}^{\widehat{A}_0,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\xi,t)_{\vec{w}}}{\mathcal{Z}^{\widehat{A}_0,\mathrm{inst}}_{X_2}(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\xi)_{\vec{w}}}\\ &\qquad=\exp\Big(\Big(\mathsf{q}\,\frac{\partial}{\partial\mathsf{q}}\Big)^2\mathcal{F}^{\widehat{A}_0,\mathrm{inst}}_{\mathbb{C}^2}(\boldsymbol{a},\mu;\mathsf{q})\,t^2+2\pi\,\mathrm{i}\sum_{\alpha=w_0+1}^r\zeta_\alpha\Big)\,\frac{\Theta\big[\begin{smallmatrix}0\\ C\boldsymbol{\nu}\end{smallmatrix}\big](C\,(\boldsymbol{\zeta}+\boldsymbol{\kappa})\,|\,C\,\tau)}{\Theta\big[\begin{smallmatrix}0\\ C\boldsymbol{\nu}\end{smallmatrix}\big](C\,\boldsymbol{\kappa}\,|\,C\,\tau)}\ ,\end{aligned}$$

where

$$\zeta_\alpha:=-\frac{t}{2\pi\,\mathrm{i}}\Big(a_\alpha+\mathsf{q}\,\frac{\partial^2\mathcal{F}^{\widehat{A}_0,\mathrm{inst}}_{\mathbb{C}^2}}{\partial\mathsf{q}\,\partial a_\alpha}(\boldsymbol{a},\mu;\mathsf{q})\Big)\ ,$$

while $\kappa_\alpha := \frac{1}{4\pi\,\mathrm{i}}\,\log(\xi)$ for $\alpha=1,\dots,r$ and

$$\nu_\alpha := \begin{cases} \displaystyle\sum_{\beta=w_0+1}^{r} \log\big((a_\beta-a_\alpha)^2-\mu^2\big) - \frac{2\pi\,\mathrm{i}\,w_1}{r}\,\tau_0 \\ \displaystyle\qquad\qquad + \sum_{\beta=w_0+1}^{r} \frac{\partial^2\mathcal{F}_{\mathbb{C}^2}^{\widehat{A}_0,\mathrm{inst}}}{\partial a_\alpha\,\partial a_\beta}(\boldsymbol{a},\mu;\mathsf{q}) & \text{for } \alpha=1,\dots,w_0\ , \\ \displaystyle -\sum_{\beta=1}^{w_0} \log\big((a_\beta-a_\alpha)^2-\mu^2\big) + \frac{4\pi\,\mathrm{i}\,w_0}{r}\,\tau_0 & \text{for } \alpha=w_0+1,\dots,r\ . \end{cases}$$

If the fixed holonomy at infinity is trivial, i.e. $\vec{w}=(w_0,w_1)=(r,0)$, the characteristic vector $\boldsymbol{\nu}\in\mathbb{C}^r$ vanishes and this result resembles [41, Theorem 8.1] and [8, Equation (2.25)]. In general, the nontrivial holonomy at infinity is encoded in $\boldsymbol{\nu}$. It would be interesting to extend this analysis to the low energy limit of all $\mathcal{N}=2$ superconformal quiver gauge theories on ALE spaces in the spirit of [45].

This blowup equation underlies the $\mathrm{Sp}(2r,\mathbb{Z})$ modularity properties of the partition function and of the correlators of quadratic 2-observables on the Seiberg-Witten curve for $\mathcal{N}=2^*$ gauge theory on X_2 with period matrix τ twisted by the A_1 Cartan matrix C. It generalises the representation of the Vafa-Witten partition function for the $\mathcal{N}=4$ superconformal gauge theory on X_k [52] at $\mu=0$ in terms of modular forms as (cf. [12, Section 5.3])

$$\mathcal{Z}_{X_k}^{\mathrm{VW}}\big(\mathsf{q},\vec{\xi}\,\big)_{\vec{w}} := \lim_{\mu\to0}\,\mathcal{Z}_{X_k}^{\widehat{A}_0,\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{a},\mu;\mathsf{q},\vec{\xi}\,\big)_{\vec{w}} = \mathsf{q}^{\frac{r\,k}{24}}\,\prod_{j=0}^{k-1}\Big(\frac{\chi^{\widehat{\omega}_j}(\zeta,\tau_0)}{\eta(\tau_0)}\Big)^{w_j}\ ,$$

Here $\chi^{\widehat{\omega}_j}(\zeta,\tau_0)$ is the character of the integrable highest weight representation of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}(k)$ at level one with highest weight the i-th fundamental weight $\widehat{\omega}_i$ of type $\widehat{A}_{k-1}$ for $i=0,1,\dots,k-1$, which can be expressed as a combination of string functions and theta functions as [18, Section 14.4]

$$\chi^{\widehat{\omega}_j}(\zeta,\tau_0) = \eta(\tau_0)^{-k+1}\sum_{\vec{\gamma}^{\,\vee}\in\mathfrak{Q}^\vee} \mathsf{q}^{\frac12\,|\vec{\gamma}^{\,\vee}+\vec{\omega}_i|^2}\ \mathrm{e}^{2\pi\,\mathrm{i}\,(\vec{\gamma}^{\,\vee}+\vec{\omega}_i)\cdot\zeta}\ ,$$

where $\xi_j=\exp(2\pi\,\mathrm{i}\,(2\zeta_j-\zeta_{j-1}-\zeta_{j+1}))$ (we set $\zeta_0=\zeta_k=0$) and $\mathfrak{Q}^\vee := \bigoplus_{i=1}^{k-1}\mathbb{Z}\vec{\gamma}_i^{\,\vee}$ is the coroot lattice of the Dynkin diagram of type A_{k-1} with $\vec{\gamma}_i^{\,\vee}$ the i-th simple coroot defined by $\vec{\gamma}_i\cdot\vec{\gamma}_j^{\,\vee}=\delta_{ij}$ for $i,j=1,\dots,k-1$. The inverse of the Dedekind eta function

$$\eta(\tau_0)=\mathsf{q}^{\frac{1}{24}}\,\prod_{n=1}^{\infty}\big(1-\mathsf{q}^n\big)$$

is the character of the Heisenberg algebra $\mathfrak{h}$. Thus in this limit we correctly reproduce the character of the representation of the affine Lie algebra $\widehat{\mathfrak{gl}}(k)_r$, and hence confirm the $\mathrm{SL}(2,\mathbb{Z})$ modularity (S-duality) of the partition function. Note that in this case the prepotential is $\frac12\,\tau_0\,\sum_{\alpha=1}^r a_\alpha^2$ and there are no quantum corrections to the Hitchin system.

APPENDIX A. EQUIVARIANT COHOMOLOGY

In this appendix we summarise various aspects of equivariant cohomology that have been used to compute instanton partition functions in the main text.

A.1. **Definitions.** Let us fix a complex algebraic torus $T := \left(\mathbb{C}^*\right)^n$. Denote by $M := \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{C}^*)$ the lattice of characters $\chi\colon T \to \mathbb{C}^*$ and by $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ the dual lattice of one-parameter subgroups of T.

Let E_T be a contractible space on which T acts freely and set $BT := E_T/T$; in the following we shall choose $E_T = \left(\mathbb{C}^\infty \setminus \{0\}\right)^n$ and $BT = \left(\mathbb{P}^\infty\right)^n$. Suppose that T acts on a topological space X. The Borel equivariant cohomology of X is defined as

$$H_T^*(X) := H^*(X \times_T E_T; \mathbb{C}) .$$

If X is a point, $H_T^*(\mathrm{pt}) = H^*(BT;\mathbb{C})$. By using the pullback via the collapsing map $X \to \mathrm{pt}$, one makes $H_T^*(X)$ into a module over the ring $H_T^*(\mathrm{pt}) = H^*(BT;\mathbb{C})$.

Let us denote by L_χ the line bundle on $\left(\mathbb{P}^\infty\right)^n$ associated to the principal T-bundle $\left(\mathbb{C}^\infty \setminus \{0\}\right)^n \to \left(\mathbb{P}^\infty\right)^n$ via a character χ. The assignment $\chi \mapsto -\mathrm{c}_1(L_\chi)$ defines an isomorphism $\psi\colon M \xrightarrow{\sim} H^2(BT;\mathbb{C})$ which induces a ring isomorphism $\mathrm{Sym}(M) \simeq H^*(BT;\mathbb{C})$. For a character χ, we call $\psi(\chi)$ the weight of χ. In particular, if χ_i is the character of T defined by $(t_1,\dots,t_n) \mapsto t_i$ for $i=1,\dots,n$, then

$$H_T^*(\mathrm{pt}) = H^*(BT;\mathbb{C}) = \mathbb{C}\big[\psi(\chi_1),\dots,\psi(\chi_n)\big] .$$

Let V be an equivariant vector bundle on X, i.e. a vector bundle over X such that the action of T on X lifts to an action on V which is linear on its fibres. Then $V_T := V \times_T E_T$ is a vector bundle over $X_T := X \times_T E_T$ and the equivariant Chern classes $(\mathrm{c}_l)_T(V) \in H_T^*(X)$ are defined to be the ordinary Chern classes $\mathrm{c}_l(V_T) \in H^*(X_T;\mathbb{C})$. The top Chern class of V_T is called the equivariant Euler class of V and is denoted by $\mathrm{eu}_T(V)$.

A.2. **Localization theorem.** Assume that the fixed point locus $X^T := \{p_1,\dots,p_k\}$ of X consists of a finite number of points with $k \geq 1$. The equivariant inclusion $\imath_l\colon \{p_l\} \hookrightarrow X$ induces a pullback $\imath_l^* : H_T^*(X) \to H_T^*(p_l)$ and a Gysin map $\imath_{l!}\colon H_T^*(p_l) \to H_T^*(X)$.

Let $\mathrm{Frac}\big(H_T^*(\mathrm{pt})\big)$ be the field of fractions of the ring $H_T^*(\mathrm{pt})$. For $l=1,\dots,k$ the equivariant Euler class $\mathrm{eu}_T(T_{p_l}X)$ of the tangent space to X at the fixed point p_l is invertible in $H_T^*(p_l) \otimes_{H_T^*(\mathrm{pt})} \mathrm{Frac}\big(H_T^*(\mathrm{pt})\big)$. There is an isomorphism

$$H_T^*(X) \otimes_{H_T^*(\mathrm{pt})} \mathrm{Frac}\big(H_T^*(\mathrm{pt})\big) \xrightarrow{\ \sim\ } \bigoplus_{l=1}^k H_T^*(p_l) \otimes_{H_T^*(\mathrm{pt})} \mathrm{Frac}\big(H_T^*(\mathrm{pt})\big)$$

induced by the map

$$\alpha \longmapsto \Big(\frac{\imath_l^*(\alpha)}{\mathrm{eu}_T(T_{p_l}X)}\Big)_{l=1,\dots,k} .$$

The inverse map is induced by $(\alpha_l)_{l=1,\dots,k} \mapsto \sum_{l=1}^k \imath_{l!}(\alpha_l)$.

APPENDIX B. EDGE CONTRIBUTIONS

In this appendix we list the edge contributions $\ell_{\vec h}^{(n)}\big(e_1,e_2,a\big)$ to the T-equivariant Euler class of the tangent bundle $T\mathcal{M}_{\vec u,\Delta,\vec w}$ which were derived in [12, Section 4.7 and Appendix C]. For this, we first introduce some notation. Let $c \in \{0,1,\dots,k-1\}$ be the equivalence class of $k\,h_{k-1}$ modulo k. Set $(C^{-1})^{n,0}=0$ for $n \in \{1,\dots,k-1\}$ and $(C^{-1})^{k,c}=0$. We denote by $\lfloor x\rfloor \in \mathbb{Z}$ the integer part and by $\{x\} := x - \lfloor x \rfloor \in [0,1)$ the fractional part of a rational number x.

If $h_n - (C^{-1})^{n,c} > 0$ for $n \in \{1,\dots,k-1\}$, consider the equation

$$i^2 - i\,\Big(\vec h - \sum_{p=1}^{n-1}\,\big(h_p - (C^{-1})^{p,c}\big)\,\vec e_p\Big)\cdot C\vec e_n + \frac{1}{2}\,\Big(\Big(\vec h - \sum_{p=1}^{n-1}\,\big(h_p - (C^{-1})^{p,c}\big)\,\vec e_p\Big)$$

$$\cdot\, C\Big(\vec{h}-\sum_{p=1}^{n-1}\left(h_p-(C^{-1})^{p,c}\right)\vec{e}_p\Big)-(C^{-1})^{c,c}\Big)=0\ , \quad \text{(B.1)}$$

and define the set

$$S_n^+ := \big\{ i\in\mathbb{Z}_{>0}\ \big|\ i\leq h_n-(C^{-1})^{n,c} \text{ is a solution of Equation (B.1)}\big\}\ .$$

Let $d_n^+=\min(S_n^+)$ if $S_n^+\neq\emptyset$, otherwise $d_n^+ := h_n-(C^{-1})^{n,c}$.

If $h_n-(C^{-1})^{n,c}<0$, consider the equation

$$i^2+i\Big(\vec{h}-\sum_{p=1}^{n-1}\left(h_p-(C^{-1})^{p,c}\right)\vec{e}_p\Big)\cdot C\vec{e}_n+\frac{1}{2}\Big(\Big(\vec{h}-\sum_{p=1}^{n-1}\left(h_p-(C^{-1})^{p,c}\right)\vec{e}_p\Big)$$
$$\cdot\, C\Big(\vec{h}-\sum_{p=1}^{n-1}\left(h_p-(C^{-1})^{p,c}\right)\vec{e}_p\Big)-(C^{-1})^{c,c}\Big)=0\ , \quad \text{(B.2)}$$

and define the set

$$S_n^- := \big\{ i\in\mathbb{Z}_{>0}\ \big|\ i\leq -h_n+(C^{-1})^{n,c} \text{ is a solution of Equation (B.2)}\big\}\ .$$

Let $d_n^-=\min(S_n^-)$ if $S_n^-\neq\emptyset$, otherwise $d_n^- := -h_n+(C^{-1})^{n,c}$.

Define m to be the smallest index $n\in\{1,\dots,k-1\}$ such that S_n^+ or S_n^- is nonempty, otherwise $m:=k-1$.

Then for fixed $n=1,\dots,m$ we set:

- If $h_n-(C^{-1})^{n,c}>0$:
  - For $\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}+2(h_n-(C^{-1})^{n,c}-d_n^+)\geq 0$:

$$\ell_{\vec{h}}^{(n)}(e_1,e_2,a)=\prod_{i=h_n-(C^{-1})^{n,c}-d_n^+}^{h_n-(C^{-1})^{n,c}-1}\ \prod_{j=0}^{2i+\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}\Big(a+\Big(i+\Big\lfloor\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\Big\rfloor\Big)\,e_1+j\,e_2\Big)\ .$$

  - For $2\leq\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}+2(h_n-(C^{-1})^{n,c})<2d_n^+$:

$$\ell_{\vec{h}}^{(n)}(e_1,e_2,a)=\prod_{i=h_n-(C^{-1})^{n,c}-d_n^+}^{-\lfloor\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\rfloor-1}\ \prod_{j=1}^{2i-(\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c})-1}\Big(a+\Big(i-\Big\lfloor-\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\Big\rfloor\Big)\,e_1-j\,e_2\Big)^{-1}\times$$
$$\prod_{i=-\lfloor\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\rfloor}^{2(h_n-(C^{-1})^{n,c})+\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}-2}\ \prod_{j=0}^{2i+\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}\Big(a+\Big(i+\Big\lfloor\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\Big\rfloor\Big)\,e_1-j\,e_2\Big)\ .$$

- For $\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}<2-2(h_n-(C^{-1})^{n,c})$:

$$\ell^{(n)}_{\vec{h}}(e_1,e_2,a)=\prod_{i=h_n-(C^{-1})^{n,c}-d_n^+}^{h_n-(C^{-1})^{n,c}-1}\ \prod_{j=1}^{-2i-\delta_{n,c}+h_{n+1}-(C^{-1})^{n+1,c}-1}\left(a+\left(i-\left\lfloor-\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\right\rfloor\right)e_1-j\,e_2\right)^{-1} .$$

■ If $h_n-(C^{-1})^{n,c}=0$:

$$\ell^{(n)}_{\vec{h}}(e_1,e_2,a)=1 .$$

■ If $h_n-(C^{-1})^{n,c}<0$:

- For $\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}+2h_n-2(C^{-1})^{n,c}<2-2d_n^-$:

$$\ell^{(n)}_{\vec{h}}(e_1,e_2,a)=\prod_{i=1-h_n+(C^{-1})^{n,c}-d_n^-}^{-h_n+(C^{-1})^{n,c}}\ \prod_{j=1}^{2i-(\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c})-1}\left(a-\left(i+\left\lfloor-\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\right\rfloor\right)e_1-j\,e_2\right) .$$

- For $2-2d_n^-\leq\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}+2h_n-2(C^{-1})^{n,c}<0$:

$$\ell^{(n)}_{\vec{h}}(e_1,e_2,a)=\prod_{i=\left\lfloor\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\right\rfloor+1}^{-h_n+(C^{-1})^{n,c}}\ \prod_{j=1}^{2i-(\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c})-1}\left(a-\left(i+\left\lfloor-\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\right\rfloor\right)e_1-j\,e_2\right)\times$$

$$\prod_{i=1-h_n+(C^{-1})^{n,c}-d_n^-}^{\left\lfloor\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\right\rfloor}\ \prod_{j=0}^{-2i+\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}\left(a+\left(-i+\left\lfloor\frac{\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}}{2}\right\rfloor\right)e_1+j\,e_2\right)^{-1} .$$

- For $\delta_{n,c}-h_{n+1}+(C^{-1})^{n+1,c}\geq-2h_n+2(C^{-1})^{n,c}$:

$$\ell^{(n)}_{\vec{h}}(e_1,e_2,a)=\prod_{i=1-n+(C^{-1})^{n,c}-d_n^-}^{-(\vec{h})_n+(C^{-1})^{n,c}}\ \prod_{j=0}^{-2i+\delta_{n,c}-(\vec{h})_{n+1}+(C^{-1})^{n+1,c}}\left(a+\left(-i+\left\lfloor\frac{\delta_{n,c}-(\vec{h})_{n+1}+(C^{-1})^{n+1,c}}{2}\right\rfloor\right)e_1+j\,e_2\right)^{-1} .$$

For $n=m+1,\dots,k-1$ we set

$$\ell^{(n)}_{\vec{h}}(e_1,e_2,a)=1 .$$

Note that for any fixed $n\in\{1,\dots,k-1\}$, $d_n^{\pm}=0$ implies $\ell^{(n)}_{\vec{h}}(e_1,e_2,a)=1$.

REFERENCES

[1] M. ALFIMOV, A. BELAVIN, AND G. TARNOPOLSKY, *Coset conformal field theory and instanton counting on $\mathbb{C}^2/\mathbb{Z}_p$*, J. High Energy Phys., 1308 (2013), p. 134. arXiv:1306.3938.

[2] E. W. BARNES, *The theory of the double gamma function*, Phil. Trans. Royal Soc. London Ser. A, 196 (1901), pp. 265–387.

[3] A. BELAVIN, V. BELAVIN, AND M. BERSHTEIN, *Instantons and $2D$ superconformal field theory*, J. High Energy Phys., 1109 (2011), p. 117. arXiv:1106.4001.

[4] A. BELAVIN, M. BERSHTEIN, B. FEIGIN, A. LITVINOV, AND G. TARNOPOLSKY, *Instanton moduli spaces and bases in coset conformal field theory*, Comm. Math. Phys., 319 (2013), pp. 269–301. arXiv:1111.2803.

[5] V. BELAVIN AND B. FEIGIN, *Super Liouville conformal blocks from $\mathcal{N} = 2$ $SU(2)$ quiver gauge theories*, J. High Energy Phys., 1107 (2011), p. 079. arXiv:1105.5800.

[6] G. BONELLI, K. MARUYOSHI, AND A. TANZINI, *Instantons on ALE spaces and super Liouville conformal field theories*, J. High Energy Phys., 1108 (2011), p. 056. arXiv:1106.2505.

[7] ———, *Gauge theories on ALE space and super Liouville correlation functions*, Lett. Math. Phys., 101 (2012), pp. 103–124. arXiv:1107.4609.

[8] G. BONELLI, K. MARUYOSHI, A. TANZINI, AND F. YAGI, *$\mathcal{N} = 2$ gauge theories on toric singularities, blow-up formulae and W-algebrae*, J. High Energy Phys., 1301 (2013), p. 014. arXiv:1208.0790.

[9] L. A. BORISOV, L. CHEN, AND G. G. SMITH, *The orbifold Chow ring of toric Deligne-Mumford stacks*, J. Amer. Math. Soc., 18 (2005), pp. 193–215. arXiv:math.AG/0309229.

[10] M. BRION AND M. VERGNE, *An equivariant Riemann-Roch theorem for complete, simplicial toric varieties*, J. Reine Angew. Math., 482 (1997), pp. 67–92. Online version.

[11] U. BRUZZO, F. FUCITO, J. F. MORALES, AND A. TANZINI, *Multi-instanton calculus and equivariant cohomology*, J. High Energy Phys., 0305 (2003), p. 054. arXiv:hep-th/0211108.

[12] U. BRUZZO, M. PEDRINI, F. SALA, AND R. J. SZABO, *Framed sheaves on root stacks and supersymmetric gauge theories on ALE spaces*. arXiv:1312.5554.

[13] U. BRUZZO, R. POGHOSSIAN, AND A. TANZINI, *Poincaré polynomial of moduli spaces of framed sheaves on (stacky) Hirzebruch surfaces*, Comm. Math. Phys., 304 (2011), pp. 395–409. arXiv:0909.1458.

[14] U. BRUZZO AND F. SALA, *Framed sheaves on projective stacks*. arXiv:1311.2861.

[15] M. CIRAFICI, A.-K. KASHANI-POOR, AND R. J. SZABO, *Crystal melting on toric surfaces*, J. Geom. Phys., 61 (2011), pp. 2199–2218. arXiv:0912.0737.

[16] M. CIRAFICI AND R. J. SZABO, *Curve counting, instantons and McKay correspondences*, J. Geom. Phys., 72 (2013), pp. 54–109. arXiv:1209.1486.

[17] D. A. COX, J. B. LITTLE, AND H. K. SCHENCK, *Toric Varieties*, vol. 124 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2011.

[18] P. DI FRANCESCO, P. MATHIEU, AND D. SÉNÉCHAL, *Conformal Field Theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 1997.

[19] R. DONAGI AND E. WITTEN, *Supersymmetric Yang-Mills theory and integrable systems*, Nucl. Phys. B, 460 (1996), pp. 299–334. arXiv:hep-th/9510101.

[20] M. R. DOUGLAS AND G. W. MOORE, *D-branes, quivers, and ALE instantons*. arXiv:hep-th/9603167.

[21] D. EDIDIN, *Notes on the construction of the moduli space of curves*, in Recent Progress in Intersection Theory (Bologna, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 2000, pp. 85–113.

[22] P. EYSSIDIEUX AND F. SALA, *Instantons and framed sheaves on Kähler Deligne-Mumford stacks*. arXiv:1404.3504.

[23] B. FANTECHI, E. MANN, AND F. NIRONI, *Smooth toric Deligne-Mumford stacks*, J. Reine Angew. Math., 648 (2010), pp. 201–244. arXiv:0708.1254.

[24] R. FLUME AND R. POGHOSSIAN, *An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential*, Int. J. Mod. Phys. A, 18 (2003), pp. 2541–2563. arXiv:hep-th/0208176.

[25] F. FUCITO, J. F. MORALES, AND R. POGHOSSIAN, *Multi-instanton calculus on ALE spaces*, Nucl. Phys. B, 703 (2004), pp. 518–536. arXiv:hep-th/0406243.

[26] ———, *Instanton on toric singularities and black hole countings*, J. High Energy Phys., 0612 (2006), p. 073. arXiv:hep-th/0610154.

[27] S. FUJII AND S. MINABE, *A combinatorial study on quiver varieties*. arXiv:math.AG/0510455.

[28] E. GASPARIM AND C.-C. M. LIU, *The Nekrasov conjecture for toric surfaces*, Comm. Math. Phys., 293 (2010), pp. 661–700. arXiv:0808.0884.

[29] G. W. GIBBONS AND S. W. HAWKING, *Gravitational multi-instantons*, Phys. Lett. B, 78 (1978), pp. 430–432.

[30] T. GOCHO AND H. NAKAJIMA, *Einstein-Hermitian connections on hyper-Kähler quotients*, J. Math. Soc. Japan, 44 (1992), pp. 43–51.

[31] L. GRIGUOLO, D. SEMINARA, R. J. SZABO, AND A. TANZINI, *Black holes, instanton counting on toric singularities and q-deformed two-dimensional Yang-Mills theory*, Nucl. Phys. B, 772 (2007), pp. 1–24. arXiv:hep-th/0610155.

[32] Y. ITO, K. MARUYOSHI, AND T. OKUDA, *Scheme dependence of instanton counting in ALE spaces*, J. High Energy Phys., 1305 (2013), p. 045. arXiv:1303.5765.

[33] T. A. IVANOVA, O. LECHTENFELD, A. D. POPOV, AND R. J. SZABO, *Orbifold instantons, moment maps and Yang-Mills theory with sources*, Phys. Rev. D, 88 (2013), p. 105026. arXiv:1310.3028.

[34] P. B. KRONHEIMER, *The construction of ALE spaces as hyper-Kähler quotients*, J. Diff. Geom., 29 (1989), pp. 665–683.

[35] P. B. KRONHEIMER AND H. NAKAJIMA, *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann., 288 (1990), pp. 263–307.

[36] A. KUZNETSOV, *Quiver varieties and Hilbert schemes*, Moscow Math. J., 7 (2007), pp. 673–697. arXiv:math.AG/0111092.

[37] K. NAGAO, *Quiver varieties and Frenkel-Kac construction*, J. Algebra, 321 (2009), pp. 3764–3789. arXiv:math.RT/0703107.

[38] H. NAKAJIMA, *Instantons on ALE spaces, quiver varieties and Kac-Moody algebras*, Duke Math. J., 76 (1994), pp. 365–416.

[39] ———, *Sheaves on ALE spaces and quiver varieties*, Moscow Math. J., 7 (2007), pp. 699–722.

[40] H. NAKAJIMA AND K. YOSHIOKA, *Lectures on instanton counting*, in Algebraic Structures and Moduli Spaces, vol. 38 of CRM Proc. Lecture Notes, Amer. Math. Soc., Providence, RI, 2004, pp. 31–101. arXiv:math.AG/0311058.

[41] ———, *Instanton counting on blowup. I. 4-dimensional pure gauge theory*, Invent. Math., 162 (2005), pp. 313–355. arXiv:math.AG/0306198.

[42] N. A. NEKRASOV, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys., 7 (2003), pp. 831–864. arXiv:hep-th/0206161.

[43] ———, *Localizing gauge theories*, in XIVth International Congress on Mathematical Physics, vol. 1, 2006, pp. 645–654. Online version.

[44] N. A. NEKRASOV AND A. OKOUNKOV, *Seiberg-Witten theory and random partitions*, in The Unity of Mathematics, vol. 244 of Progr. Math., Birkhäuser Boston, Boston, MA, 2006, pp. 525–596. arXiv:hep-th/0306238.

[45] N. A. NEKRASOV AND V. PESTUN, *Seiberg-Witten geometry of four-dimensional $\mathcal{N}=2$ quiver gauge theories*. arXiv:1211.2240.

[46] N. A. NEKRASOV, V. PESTUN, AND S. L. SHATASHVILI, *Quantum geometry and quiver gauge theories*. arXiv:1312.6689.

[47] N. A. NEKRASOV AND A. S. SCHWARZ, *Instantons on noncommutative $\mathbb{R}^4$ and $(2,0)$ superconformal six-dimensional theory*, Comm. Math. Phys., 198 (1998), pp. 689–703. arXiv:hep-th/9802068.

[48] T. NISHIOKA AND Y. TACHIKAWA, *Central charges of para-Liouville and Toda theories from M5-branes*, Phys. Rev., D 84 (2011), p. 046009. arXiv:1106.1172.

[49] M. PEDRINI, F. SALA, AND R. J. SZABO, *AGT relations for abelian quiver gauge theories on ALE spaces*. arXiv:1405.6992.

[50] N. SEIBERG AND E. WITTEN, *Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N}=2$ supersymmetric Yang-Mills theory*, Nucl. Phys. B, 426 (1994), pp. 19–52. arXiv:hep-th/9407087.

[51] M.-C. TAN, *M-theoretic derivations of 4d-2d dualities: From a geometric Langlands duality for surfaces, to the AGT correspondence, to integrable systems*, J. High Energy Phys., 1307 (2013), p. 171. arXiv:1301.1977.

[52] C. VAFA AND E. WITTEN, *A strong coupling test of S-duality*, Nucl. Phys., B 431 (1994), pp. 3–77. arXiv:hep-th/9408074.

[53] M. VARAGNOLO AND E. VASSEROT, *On the K-theory of the cyclic quiver variety*, Int. Math. Res. Not., 18 (1999), pp. 1005–1028. arXiv:math.AG/9902091.

[54] A. VISTOLI, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math., 97 (1989), pp. 613–670.

[55] N. WYLLARD, *Coset conformal blocks and $\mathcal{N}=2$ gauge theories*. arXiv:1109.4264.