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Transverse laser modes in Bose-Einstein condensates

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Bose-Einstein condensates (BECs) are in many ways similar to laser light. In Fox-Li-type simulations, which involve free evolution of a two-dimensional BEC in combination with periodic focusing and localized loss, we model here numerically the BEC equivalent of transverse aspects of laser resonators. We discuss possible experimental realizations in the form of periodic series of light pulses interacting with the BEC. We show that such experiments can result in trapped, structurally stable, modes analogous to Hermite-Gaussian laser modes. This outlines a new way of trapping BECs and illustrates further the analogy between BECs and laser light.

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I. INTRODUCTION

Bose-Einstein condensates (BECs) are in many ways the matter-wave equivalent of laser light [1–3]. Most importantly, all the atoms in a BEC are in the same quantum state, just like all the photons in a laser beam. While in the creation of BECs this property is usually ensured by the fundamental properties (specifically the statistics) of very cold bosons, in a laser it is usually due to a combination of the bandwidth of the gain medium (i.e., the amplification stage) and the longitudinal properties of the laser resonator [4,6]. Just like the number of photons can be coherently amplified in a laser, the number of atoms can be coherently amplified in a BEC [7–9].

Laser resonators also have an effect on the transverse structure of laser light. The origin of this transverse structure has a huge literature devoted to it (for an overview see, for example, Refs. [10,11]), and a good understanding and control has proved interesting (see, for example, the discovery of fractal modes [12]) as well as useful. For example, the standard families of laser modes, Hermite-Gaussian (HG) and Laguerre-Gaussian (LG) modes (Fig. 1), have elegant mathematical properties: they are structurally stable, i.e., they do not change shape on propagation, and form complete orthogonal sets of modes. In addition, LG modes, because of their central vortex of arbitrary topological charge, are one of the most popular choices of light beam in experiments with optical vortices. Most importantly, however, certain "stable" resonators keep the light in the proximity of the optical axis—transversally, they trap the light.

We examine here—in the limit of the Gross-Pitaevskii equation—a process in two-dimensional BECs that is analogous to the formation of transverse laser modes: free evolution (or alternatively evolution in a quadratic potential) combined with periodic focusing and spatial filtering. We believe this could be achieved experimentally with a periodic series of light pulses interacting with a BEC. We demonstrate the emergence of HG-like modes, and convert these into LG-like modes using the equivalent of the Gouy phase shift. Our main concern, however, is the question if a series of light pulses could act as a type of trap for BECs.

The paper is organized as follows: Sec. II introduces the eigenmode selection in resonators. In Sec. III we discuss the creation of specific modes in BECs and the focusing dynamics. Section IV treats the BEC equivalent of the conversion of Hermite-Gaussian modes into Laguerre-Gaussian modes. Finally the results are summarized in Sec. V. In the Appendix we show how to obtain analytically the focusing time for a BEC.

II. EIGENMODE SELECTION IN LASER RESONATORS

Figure 2(a) sketches one round trip of a light pulse (or alternatively a "slice" of a cw light beam) through a standard laser resonator. Most importantly, the light pulse is reflected back by the two mirrors facing each other, so that after one round trip, which involves reflecting off each mirror once, the light pulse is back in the same plane where it started. One round trip is much more complicated than just some changes in propagation direction [10]: as the mirrors are curved, each mirror reflection also focuses the light. In addition, localized loss occurs in one or several planes, for example, through apertures or specks of dust on mirrors, and while traveling between the mirrors the light beam diffracts. Some light beams—the resonator's eigenmodes—are unchanged (apart from a uniform change in intensity or phase) after one round trip through a resonator. In the case of geometrically stable resonators, which we consider here, the eigenmodes can be divided into families (like the HG modes and the LG modes), whereby each family forms a complete orthogonal set, i.e., every light beam can, for example, be described as a super-

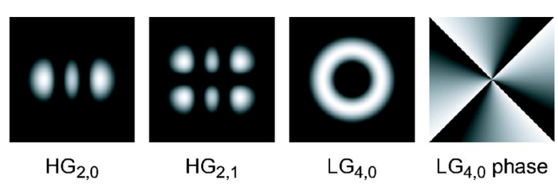

FIG. 1. Hermite-Gaussian (HG) and Laguerre-Gaussian (LG) modes. Shown here are the intensity cross sections of two HG modes, $HG_{2,0}$ and $HG_{2,1}$, and the intensity and phase (gray scale representation) cross sections of one LG mode, $LG_{4,0}$, which possesses a central charge-4 vortex.

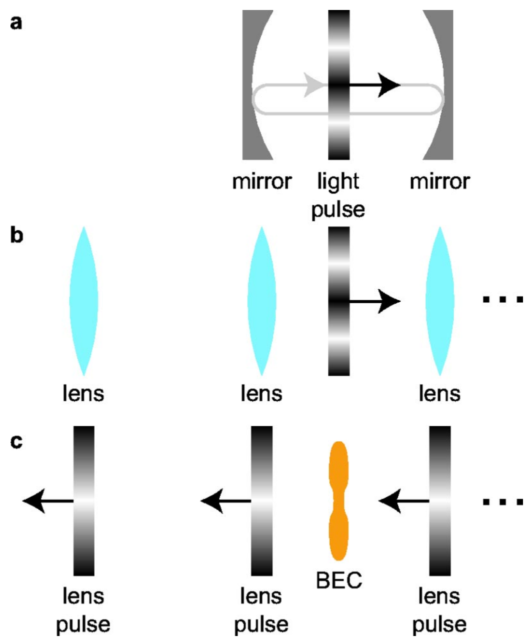

FIG. 2. Laser resonator (a) and its unfolded lens-series equivalent (b). A light beam passing through the lens-series equivalent is periodically being focused, just like light in the resonator is being focused every time it is reflected off one of the resonator's mirrors. A series of "lens pulses"—short, off-resonant light pulses with a specific intensity profile—can repeatedly focus a BEC in a similar way (c). For laser modes to form localized absorption needs to occur in the resonator; this is not shown here. Note the role reversal: in (a) and (b), a light pulse is being focused (by mirrors and lenses, respectively); in (c) light pulses focus the BEC.

position of HG modes. As already mentioned, the HG and LG modes can take on very recognizable shapes and have interesting features like high-charge vortices (see Fig. 1).

Experimentally, a pure resonator eigenmode is created in a laser by making that eigenmode the lowest-loss eigenmode, i.e., by ensuring the round-trip loss of all other eigenmodes is higher than that of the desired eigenmode. This can be done, for example, by inserting absorbing cross wires into the resonator at positions where the desired eigenmode is darker than the competing eigenmodes. For example, an absorbing wire in the central horizontal dark line of the $HG_{2,1}$ mode shown in Fig. 1 would not significantly increase that eigenmode's round-trip loss, but it would increase that of the $HG_{2,0}$ mode. The fraction of the power in the undesired eigenmodes then falls off exponentially with the round-trip number, after a few round-trips leaving only the pure lowest-loss eigenmode.

III. CREATION OF HERMITE-GAUSSIAN–LIKE MODES IN BEC

As far as transverse eigenmode selection is concerned, a resonator can be seen as a series of lenses separated by the length of the resonator [Fig. 2(b)]. If the focal lengths of the lenses match those of the resonator's mirrors, and if the important localized loss is the same, then an infinitely long lens-series equivalent of a resonator has the same eigenmodes [13]. Such an infinitely long lens-series equivalent of a resonator represents infinitely many round trips. On the other hand, eigenmode selection mostly happens during the first few round trips through the resonator, which can be represented by a few lenses. We examine here the BEC analog of such a series of lenses, interspersed with localized loss.

The BEC analog of an optical resonator's lens-series equivalent can consist of repetitions of two types of light pulses successively interacting with the BEC, whereas the two types of light pulse, respectively, act like a lens for the BEC ("lens pulse," focal time f) and cause localized loss [Fig. 2(c)]. The delay between two repetitions, L, corresponds to the length of the resonator. We refer to such a configuration as a *light-pulse resonator*. We restrict ourselves to light-pulse resonators in which the absorption events always follow immediately after the lens pulse; this allows us to model accurately the equivalent of standard laser resonators, in which the localized loss is caused by aperturing due to the mirror's final size and imperfections (like dust) on the mirror surface and therefore occurs at the mirrors.

All the pictures of BECs in light-pulse resonators shown in this paper (with the exception of two frames in Fig. 7) are snapshots taken halftime between two lens pulses, i.e., a time $L/2$ after the previous lens pulse. We refer to the sequence of events acting on the BEC between $L/2$ after the previous lens pulse and $L/2$ after the next lens pulse—time evolution over a time $L/2$, interaction with a lens pulse, absorption event, and another time evolution through $L/2$ — as one period of the resonator. We numerically model the evolution of a BEC during a number of such periods by successively modeling the effect of each individual event. This is the exact equivalent of the well-known Fox-Li method for finding laser modes [14].

We simulate time evolution according to the time-dependent Gross-Pitaevskii equation [15], which we write in the form [16]

$$i\hbar\frac{\partial\,\psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2 + V_e(\mathbf{r},t) + g|\psi(\mathbf{r},t)|^2\right]\psi(\mathbf{r},t), \tag{1}$$

where m is the mass of each atom, ω is the trap frequency, V_e is an external potential, and g is the nonlinear coefficient, which is given by

$$g = 4\pi N\frac{a}{d}\frac{\hbar^2}{m}. \tag{2}$$

d is the thickness of the BEC in the third dimension (the z direction), which needs to be very small for the BEC to behave two-dimensionally. This is usually ensured by tight z confinement by a harmonic trap that satisfies the condition $\mu < \hbar\omega_z$, where μ is the chemical potential. Note that g depends not only on m and a, the s-wave scattering length, but also on the number of atoms, N, in the BEC. As the number of particles is contained in the parameter g, which is very convenient for the purposes of this paper, the density is normalized according to

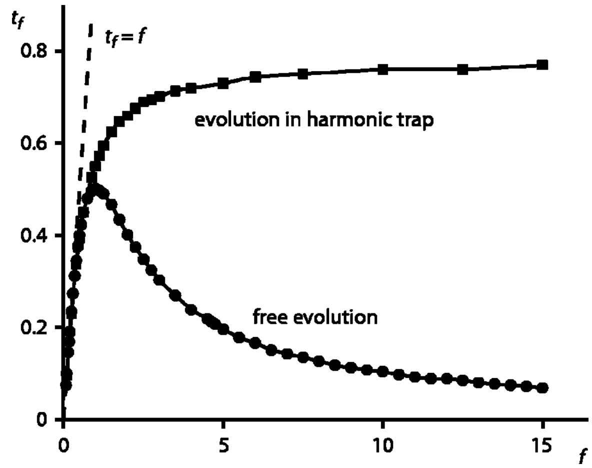

FIG. 3. Focusing of a BEC by a lens pulse. A BEC in the ground state was focused using a light pulse of focal time f, which phase shifts the BEC locally by $r^2/(2f)$. The graph shows the actual focusing time t_f, which is the time it takes the BEC to become phase flat after interaction with the lens pulse, as a function of f. Analogy with ray optics suggests that the BEC comes to a focus after a time $t_f = f$; however, to our surprise we find that this is only the case for small values of f. In the range $0 \leqslant g \leqslant 100$, over which our simulations were performed, these results are virtually independent of g.

$$\int |\psi(\mathbf{r},t)|^2 d\mathbf{r} = 1. \tag{3}$$

In our computer model we represent the BEC wave function on a square area of 16×16 dimensionless units by a 128×128 array of double-precision complex numbers. The dimensionless variables for position and time, $\mathbf{r}'$ and t', are related to the corresponding variables with dimensions, $\mathbf{r}$ and t, through the equations

$$\mathbf{r}' = \left(\frac{2m\omega}{\hbar}\right)^{1/2} \mathbf{r} \tag{4}$$

and

$$t' = \omega t. \tag{5}$$

We solve the Gross-Pitaevskii equation with a fourth-order Runge-Kutta method (a detailed description can be found in Ref. [16]).

We model the effect of a lens pulse by locally altering the phase of the wave function in the same way in which a lens (or a hologram of a lens) would alter the phase of passing light [17]. In principle (although it has not been demonstrated for the particular phase profile of a lens), this can be achieved experimentally using an off-resonant laser pulse with an intensity profile proportional to the desired local phase change, whose electric field then—through the ac Stark effect—alters the local phase of the BEC. This "phase imprinting" method [18,19] is analogous to optical phase holography. Interestingly, the time after which the BEC comes to a focus after interaction with such a *lens pulse* on its own is not what one might naively expect it to be from the optical analogy (see Fig. 3 and the Appendix, which contains analytic expressions for a focused BEC in a harmonic trap). On closer inspection, however, it turns out to occur not only for BECs but also for focused Gaussian beams [See Eq. (16.66) in Ref. [20]]. The optical effect is significant only if $\lambda f_{\mathrm{opt}}/\pi\omega_0^2 \gtrsim 1$ (λ is the optical wavelength, f_{opt} is the focal length as expected from geometrical optics, and ω_0 is the width of the beam at the focus), which is almost never the case [20,21]. On the other hand, we estimate the BEC effect to be significant whenever $f \gtrsim 1$, which is often the case. In our simulations the actual focusing time, over the range of g values we checked ($0 \leqslant g \leqslant 100$), appears to be independent of g.

At this point, it is perhaps worthwhile to briefly discuss the BEC-light analogy in slightly more detail. The time evolution of an interaction-free, untrapped, two-dimensional (2D), BEC according to the Gross-Pitaevskii equation is formally equivalent to spatial propagation of a monochromatic light beam in the paraxial approximation: during the time Δt, all Fourier (or plane-wave) components in a 2D BEC change phase by exactly the same amount as the corresponding (same transverse wave numbers, k_x and k_y) Fourier components in a monochromatic light beam on propagation through a distance $\Delta z = \Delta t(\hbar k/m)$, where $k = 2\pi/\lambda$ is the wave number of the light beam of wavelength λ [23]. This is, of course, the basis of our specific efforts to reproduce lightlike behavior in BECs. However, the analogous light beam is of subwavelength size, and the paraxial approximation describes the actual propagation of such beams very badly (it does, for example, ignore evanescent waves). It is therefore not surprising that BECs—even in the most lightlike case of no interactions and no trap—often behave quite differently from light. Increasing the strength of the (nonlinear) interactions, that is, increasing $|g|$, usually (but not always, like, for example, in the above case of focusing of 2D BECs) makes the BEC behave even more differently from light.

We model local absorption as a straightforward reduction in amplitude; in particular we neglect any interaction between the "absorbed" and leftover parts of the BEC. This is an idealization of a situation which, we believe, can be approximated experimentally in different ways, for example, by utilizing a two-photon Raman process to couple out (move into an untrapped state) the part of the BEC that lies in the overlap region of two laser pulses [24]. To achieve the necessary flexibility, our absorption pattern consists of up to two absorbing "cross-hairs"—two crossing thin bright lines (across the entire simulation area) with a Gaussian intensity cross section (in our simulation often a single pixel wide; for a BEC size of 300 μm, this corresponds to lines of light about 3 μm wide [25])—as well as an absorbing rectangular boundary with a Gaussian edge profile along the edge of the simulation area.

Figure 4 shows the BEC, starting with the trap ground state, after interaction with the first few light pulses of a light-pulse resonator of length $L = 0.2$ and with a focal time $f = 0.5$ (both in dimensionless time units; we use this as our standard light-pulse resonator). It will be seen that this behaves in many ways analogous to a stable optical resonator. As explained above, during each round trip a small part of the BEC is absorbed by the absorption pulse in the shape of a thin vertical line, which preferentially weakens eigenmodes

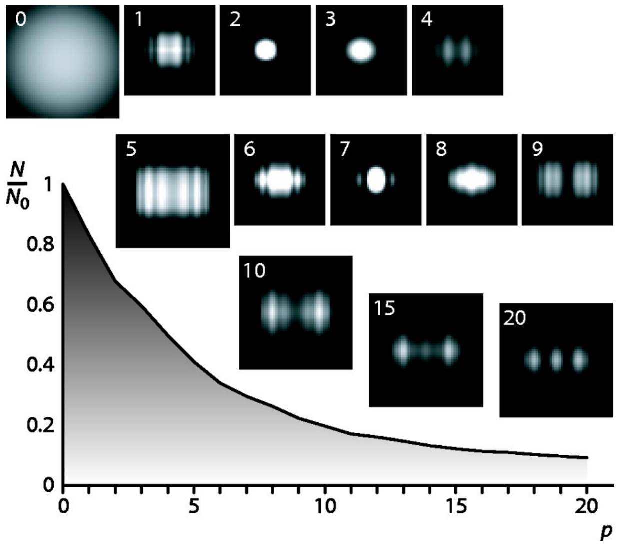

FIG. 4. BEC after interaction with p lens pulses and absorption events of a light-pulse resonator (the value of p for each frame is shown in the top-left corner), starting with the harmonic trap ground state ($p=0$), calculated for $g=100$. The pulses of focal time $f=0.5$ are separated by $L=0.2$ (in dimensionless time units); the density patterns shown here are calculated a time $L/2$ after the actual pulse, i.e., half time between the previous and the next pulse. Note that the bright central spots for $p=2, 3$, and 7 have no special significance other than that the BEC happens to be very dense in the center at the particular time during the respective resonator periods these density snapshots were taken. The curve at the bottom shows the fraction of the initial number of particles, N_0, that remains in the BEC after p periods of the light-pulse resonator. To demonstrate the loss, the density is not normalized in the frames in this figure; the full scale of grays represents in the frames for $p=0$ to 4 densities between 0 and 0.2, in the remaining frames between 0 and 0.05.

whose density is not zero there; the remaining BEC interacts with a lens pulse and diffracts. After a number of round trips only the resonator's lowest-loss eigenmode remains. Due to the losses, the number of particles changes during every period of the light-pulse resonator, which in turn changes the size of the self-interaction potential in the Gross-Pitaevskii equation, $g|\psi|^2$ (provided $g\neq 0$). As we keep the integral over the density normalized in our model, the reduction in the number of particles and any corresponding change in the interaction term size is represented as a decrease in g.

As long as the value of g—and with it the potential—changes, no stationary eigenmode can form. This means that, for $g\neq 0$, no stationary eigenmode can form before the loss has become zero. In practice, the number of particles keeps diminishing exponentially, so zero loss can never be reached with any atoms still left. However, while atoms are still left, the loss can become very small. Then the BEC becomes very similar to the eigenmode corresponding to the eigenmode for the same value of g in the hypothetical limit of no loss.

In order to calculate eigenmodes for any given value of g in the no-loss limit, we renormalize the integral over the density after each loss pulse without making a corresponding adjustment to g. This would correspond to either coherently replenishing the BEC after each loss pulse such that the number of atoms remains constant, or to tuning the value of g to "simulate" this—neither of these are easily realizable experimentally. Nevertheless, the procedure gives stationary eigenmodes that correspond to the no-loss limit.

After a number of periods of the light-pulse resonator the eigenmode has formed, that is, the BEC is not significantly changing any more over subsequent periods (apart from a uniform factor to the wave function). As a criterion for reaching this state we demand that the loss due to the aperture varied by less than 0.5%; this happened typically after about 40 iterations (light-pulse sequences), which takes several minutes to simulate on our desktop computer. Note that there appears to be an upper limit to the nonlinear coefficient, g_{max}, above which the BEC does not converge to a stable eigenmode. The exact value of g_{max} depends on the parameters of the light-pulse resonator. We determined this value for a number of different resonators and obtained values between $80\lesssim g_{\mathrm{max}}\lesssim 110$.

Once the eigenmode has formed, the loss due to each absorption event can be very low; we have achieved losses below 0.2% per absorption event. However, before the eigenmode has formed the losses can be very high; just how high depends on the starting conditions and the particular choice of shape of the loss pulse. Note that there is a trade-off between round-trip loss and number of round trips it takes to form eigenmode: thinner cross wires mean lower loss per round trip, but require more round trips. For well-chosen parameters, of the order of 10% of the BEC are left by the time the BEC begins to resemble the stationary eigenmode. For the formation of eigenmodes with an odd number of horizontal or vertical nodal lines, a break in the symmetry is required as opposite peaks are π out of phase. This symmetry breaking can be facilitated in a number of ways, for example, by imprinting a phase of π onto one half of the condensate [26].

The location of the absorbing pulses decides which eigenmode is selected and affects the round-trip loss, whereby a number of different locations might result in the same eigenmode. Any absorption should be located where the desired eigenmode is dark to minimize the relative loss to that mode. The fastest method is to place an absorbing line at every one of the $n+m$ dark lines crossing the mode horizontally and vertically; however, this results in large initial losses and links back to the above-mentioned trade-off. Most of the lower modes ($m,n\leqslant 5$) can be generated using only one absorption line at any one of the mode's dark lines (this is what we used to calculate Fig. 4). As the shape of the resulting eigenmode changes only very little over time (in the optical case it does not change at all: the eigenmode is structurally stable), the absorbing pulse can occur at any time during one period of the light-pulse resonator. However, the size of the eigenmode does change, so absorbing pulses fired at different times might have to be at different positions to coincide with a dark line of the same desired mode.

Figure 5 shows some of the stable patterns that emerge for a modest nonlinearity ($g=30$, typically corresponding to approximately 10^3 atoms of ^{87}Rb in a 20π-Hz trap) by varying the localized absorption pattern—a 100% absorptive cross hair of 1 pixel width placed in crossing node lines (see Fig. 5). Like all other pictures in this paper (unless otherwise stated), this figure shows only the central 6.25×6.25 units of

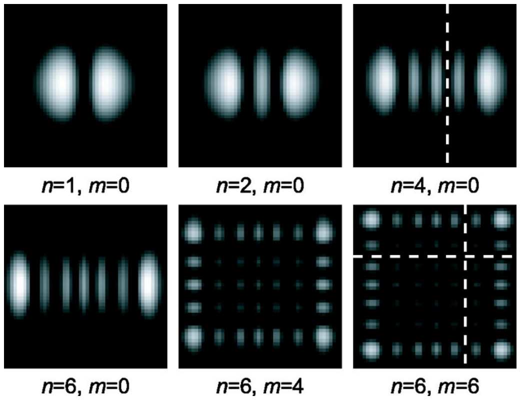

FIG. 5. Density patterns of some eigenmodes of the standard light-pulse resonator with a nonlinearity of $g=30$. By varying the cross-hair position, different patterns—characterized here by their number of nodal lines in the horizontal and vertical direction, n and m, respectively—emerge, which strongly (but not exactly) resemble optical Hermite-Gaussian modes. These pictures show the BEC halfway between two lens pulses. A possible position of the cross hair (white dashed lines) is shown in the two rightmost frames. The pictures show only the central 6.25×6.25 units of the modeled area.

the modeled area for clarity, and the gray values represent linear densities, ranging from 0 (black) to 0.2 (white). The patterns look very similar to Hermite-Gaussian modes, more so for small values of the nonlinear parameter g, less so for large values (Fig. 6). In all cases, the eigenmodes have the telltale π phase difference between adjacent peaks, just like the optical HG modes.

As an alternative to the absorption events, far off-resonant pulses could locally disturb the BEC in places where the local density of the desired mode is zero. This would effectively transfer a small part of the BEC in an undesired mode into a variety of other modes, amongst them the desired mode. It would not directly introduce loss; however, loss would still occur as some of the modes populated by the disturbance would expand outwards until they hit an absorbing element (in computer simulations an absorbing boundary specifically put into the system, in experiments, for example, the wall of the vacuum chamber). No simulations of this scheme have been performed to date, so the magnitude of any loss saving is unknown.

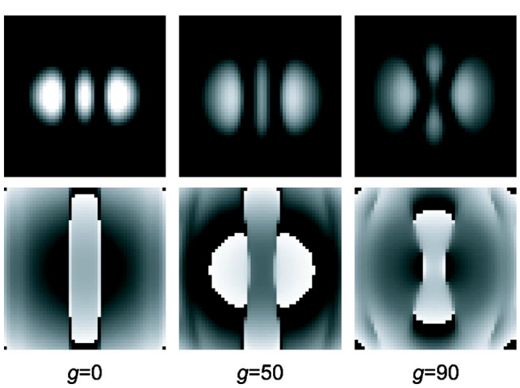

FIG. 6. Density (top) and corresponding phase patterns (below) for various values of the nonlinear coefficient, g. In all cases, the shape of the absorption region is a narrow vertical straight line.

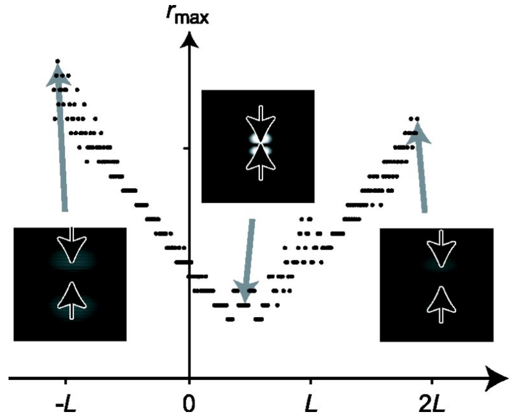

FIG. 7. Free time evolution of an eigenmode, calculated for $g=0$. If—after the eigenmode has formed—the series of light pulses is switched off at $t=0$ (the last few pulses arrive at $t=-2L,-L,0$), the BEC approximately retains its shape but changes its size. The graph shows the distance from the center of the maximum of the density, which indicates the relative size; the points do not lie on a smooth curve due to numerical errors. In analogy with optical higher-order Hermite-Gaussian modes, the BEC passes through a focus, where it is smallest. On either side of the focus, its size changes approximately symmetrically. The data points for $t<0$ were calculated using "backward propagation" in time. The graph is calculated for $L=0.2$ (in adimensional time units); the vertical range then represents three adimensional length units. To avoid numerical artifacts in obtaining this graph, the BEC was simulated over an area larger than the standard 16×16 nondimensional units.

Repeating these simulations for a switched-off harmonic trap shows that the results are virtually indistinguishable from those obtained with the trap switched on. This is not surprising as the time scale of effects due to the trap (one dimensionless unit) is several times larger than the time scale of effects due to the lens pulses (the focal time). It suggests that a series of lens pulses—a light-pulse resonator without absorption events—could act as a type of trap. Indeed, in the absence of absorption events, and to within the accuracy of the Gross-Pitaevskii equation, a series of light pulses acts as a lossless trap. It is worth noting that, much like a stable laser resonator "traps" not only light *waves* but also light *rays*, a light-pulse resonator could perhaps not only trap atoms in a BEC but also hotter or fermionic atoms.

If the series of light pulses, as well as any other potential (in particular the harmonic trap), is switched off after a stationary eigenmode has formed, the eigenmode evolves freely. Figure 7 investigates the size change of the eigenmode during this free evolution; the shape remains almost unchanged. This property is in analogy to structural stability of the eigenmodes of stable optical resonators, which can be understood in terms of imaging of every plane into every other plane [27].

All the results shown so far were calculated for our standard resonator with a time between pulses—corresponding to the resonator length—of $L=0.2$, and a lens-pulse focal time of $f=0.5$. Figure 8 shows $n=1$, $m=0$ eigenmodes from different light-pulse resonators. In the optical analogy, altering

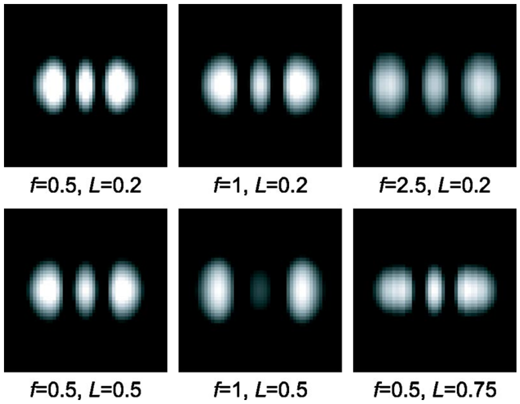

FIG. 8. Density patterns for the output of different configurations of resonator, calculated for $g=0$. f represents the focal time of the lens pulse, L the time between subsequent pulses (both in dimensionless units). For the particular choice of loss-pulse shape, the loss per period of the light-pulse resonator for these eigenmodes is, from top left to bottom right, 1.8%, 1.3%, 5.9%, 4.4%, 10.2%, and 9.7%.

resonator parameters while keeping the resonator geometrically stable (that is, satisfying $L<4f$) results in eigenmodes of a different size; the analogous effect in BEC light-pulse resonators can be seen in Fig. 8

IV. CONVERSION INTO LAGUERRE-GAUSSIAN–LIKE MODES

To examine the similarities between the eigenmodes of light-pulse resonators and optical HG modes beyond the level of appearance, we investigate here the analogy of a conversion of optical HG modes, which have a rectangular symmetry, into circularly symmetric LG modes under cylindrical focusing [28]. This conversion is due to a fairly subtle effect: the Gouy phase shift, which occurs when HG modes pass through a focus. Like HG modes, LG modes are structurally stable. They are interesting because they can contain higher-charge vortices at their center.

In the optical case, the cylindrical focusing that converts HG modes into LG modes is usually performed with a pair of cylindrical lenses. In analogy, we use here a pair of cylindrical-lens pulses, i.e., far-off-resonant laser pulses with an intensity profile that varies quadratically in one direction and that is constant in the perpendicular direction. Figure 9 shows some results that were calculated for $g=50$. The density is certainly not shaped like a perfect ring; instead, most of the density is contained in a distorted ring, and instead of containing a charge-l vortex at the center, the wave function contains l vortices of charge 1. The resulting BEC is also not structurally stable. The conversion is therefore not perfect, but the emergence of vortices with a combined charge of l and (to a lesser degree) the conversion of the intensity structure into a distorted ring are indications that it is happening at least in part. In other simulations (not shown) we confirmed that this is also the case for different choices of g (including $g=0$), and that, in the appropriate cases, the shape of the BEC also shows signs of the multiple concentric rings

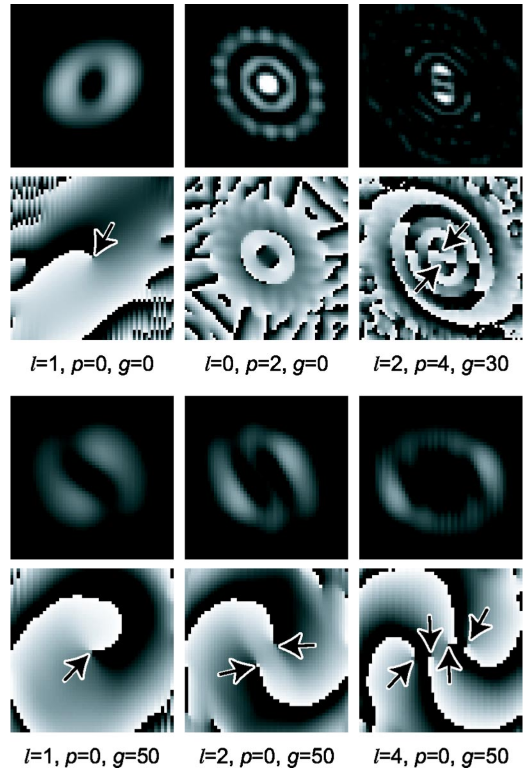

FIG. 9. Modeled density (first and third picture row) and phase (second and fourth picture row; gray-level representation) immediately after a BEC initially in an eigenmode with n horizontal and m vertical nodal lines has interacted with a pair of cylindrical-lens pulses. The intensity patterns show strong signs of the ring structure characteristic of LG modes (p bright rings and a central bright spot if $l=0$, $p+1$ bright rings if $l\neq 0$), although this ring structure is not perfect and becomes less clear for larger values of g (see $g=50$, bottom row). The phase patterns show the presence of, respectively, l vortices within the innermost intensity ring, situated at positions where all possible phases meet (marked by arrows). From top left to bottom right, the parameters of the initial eigenmode and lens pair (f, focal time of each lens; s, separation between lenses; both were optimized in each case to give visually the best ring structure) are as follows: $n=1$, $m=0$, $f=0.46$, $s=0.35$; $n=2$, $m=2$, $f=0.25$, $s=0.18$; $n=6$, $m=4$, $f=0.25$, $s=0.18$; $n=1$, $m=0$, $f=0.23$, $s=0.35$; $n=2$, $m=0$, $f=0.23$, $s=0.3$; $n=4$, $m=0$, $f=0.1$, $s=0.11$.

characteristic of LG modes with a radial mode index $p>0$.

In the optical case, the focal lengths of the cylindrical lenses and their positions need to be carefully matched to the parameters of the incoming HG mode to achieve good conversion into a LG mode. We tried to similarly optimize the parameters of the cylindrical-lens pulses. However, the similarity to a pure LG mode of the resulting density was always limited, which is not surprising as the BEC was initially not in a pure HG mode and as the evolution of the BEC is different from that of light. Finally, optimization of the pulse parameters was not made easier by the complications in the focusing characteristics of a BEC mentioned earlier (Fig. 3).

V. CONCLUSIONS

The emergence of Hermite-Gaussian–like modes in a BEC is a very graphical further illustration of the similarity between BECs and laser light. However, any experimental verification of our scheme would have to take into account the high losses associated with the absorption events.

Our results indicate that a series of lens pulses acts as a trap. Although analogy with optical resonators, in agreement with our approximate description with the Gross-Pitaevskii equation, would suggest that such a trap is lossless, a more detailed microscopic description of the BEC predicts finite losses during the interaction with the lens pulses and when the density gets high due to mechanisms such as three-body recombination [29,30].

As described in this paper, a light-pulse resonator confines a BEC in two dimensions. It should be possible to achieve confinement in all three dimensions by a combination of two orthogonal light-pulse resonators or similar arrangements. Alternatively, the rapid falloff of the intensity away from the focal plane due to extremely tight focusing of the lens pulses, which leads to an approximately quadratic Stark potential along the propagation axis, could perhaps be used to extend the confinement to allthree dimensions.

We are currently investigating the BEC analogy to holographic light shaping. In this way, a BEC can be shaped in two and three dimensions. This could, for example, be useful in quantum computation with arrays of trapped atoms [31]. The analogy can even be extended to the extremely versatile volume holograms, whereby the BEC analog of a controllable volume phase hologram—a far off-resonant light pulse with a controllable time-varying intensity—might even be easier to realize than the original.

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APPENDIX: ANALYTICAL FOCUSING RESULTS

Let us briefly discuss the focusing dynamics in two dimensions. For low densities or small scattering lengths we can assume a noninteracting gas, i.e., $g=0$. Using the propagator for the now linear Schrödinger equation with a harmonic external potential we obtain the full dynamics of the condensate,

$$\Psi(r,t) = \exp\left(-i\frac{r^2}{2}\cot(\omega t)\right)\frac{1+i\cot(\omega t)}{1+i[\cot(\omega t)-1/f]} \times \exp\left(\frac{r^2}{2\sin^2(\omega t)\{1+i[\cot(\omega t)-1/f]\}}\right), \tag{A1}$$

where we have used the initial state

$$\Psi(r,t=0) = \exp\left[-\frac{1}{2}r^2 - i\frac{1}{2f}r^2\right]. \tag{A2}$$

This initial state is a ground state whose phase has been altered by a lens pulse of focusing time f. All lengths are in units of $(\hbar/m\omega)^{1/2}$ and the energy is in units of $\hbar\omega$. From the minimum of the width of the Gaussian $|\Psi(r,t)|^2$ we obtain the actual focusing time

$$t_f = \frac{1}{2}\arctan(2f). \tag{A3}$$

If $f \gg 1$ we get $t_f = \pi/4$. In the opposite limit, $f \ll 1$, $t_f \approx f$, in accordance with Fig. 3.

We can also calculate an analytical expression for the actual focusing time when the interactions between the atoms are not negligible and the inequality $\mu/\hbar\omega \gg 1$ (where μ is the chemical potential) is satisfied—the Thomas-Fermi regime. During the focusing the condensate is structurally stable, i.e., the shape of the cloud is preserved and only the time and length scales change during the evolution. We rewrite the Gross-Pitaevskii equation in the hydrodynamic form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0, \tag{A4}$$

$$m\frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla}\left(\frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 r^2 + g\rho - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right), \tag{A5}$$

where $\rho(r,t)$ is the density and the velocity $\mathbf{v}$ is defined as $\mathbf{v}=(\hbar/m)\boldsymbol{\nabla}\Theta(\mathbf{r},t)$, where $\nabla\Theta(\mathbf{r},t)$ is the gradient of the condensate phase. In the Thomas-Fermi approximation we neglect the last term on the right-hand side of Eq. (A5). The density will now take the form

$$\rho(r,t) = \frac{\mu}{g}\left\{1 - \left[\frac{r}{b(t)}\right]^2\right\}\frac{1}{b(t)^2}, \tag{A6}$$

where r is in units of $(2\mu/m\omega^2)^{1/2}$, t is in units of $\hbar/\mu$, and μ is the chemical potential at $t=0$. If we choose the phase to be quadratic in r, $\Theta(r,t)=(1/2)r^2 q(t)$, and insert this into Eqs. (A4) and (A5) together with the parabolic density $\rho(r,t)$, we see that the dynamics are completely described by the parameter $b(t)$ if and only if $b(t)$ fulfills

$$\ddot{b} = \frac{1}{b^3} - b \tag{A7}$$

with $q(t)=\dot{b}/b$ and the initial conditions $b(0)=1$ and $\dot{b}(0)=-1/f$. Equation (A7) can be solved exactly to give

$$b(t) = \sqrt{1 + \frac{1}{2f^2} - \frac{1}{2}\sqrt{\frac{1}{f^4} + \frac{4}{f^2}}\sin\left(2t + \arctan\left[\frac{1}{2f}\right]\right)}. \tag{A8}$$

The actual focusing time, defined by $\dot{b}=0$, is consequently

$$t_f = \frac{1}{2}\arctan(2f). \tag{A9}$$

As can be seen from Eqs. (A3) and (A9) the focusing time is the same both for the noninteracting case and in the Thomas-Fermi limit. This is a coincidence and holds only in two dimensions since the radial excitation frequencies are given by $\Omega=2n\omega$ for the ideal gas and $\Omega=\omega\sqrt{2n^2+2nm+2n+m}$ [32] for the Thomas-Fermi cloud. For $n=1$, $m=0$, corresponding to the lowest breathing mode, i.e., focusing, the frequencies are the same.

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