Autocorrelation functions and power spectral densities of the Stokes parameters in a polarization speckle pattern

Research Article Vol. 40, No. 1 / January 2023 / Journal of the Optical Society of America A 165

Journal of the Optical Society of America A
OPTICS, IMAGE SCIENCE, AND VISION

Autocorrelation functions and power spectral densities of the Stokes parameters in a polarization speckle pattern

WEI WANG,[1,*] STEEN G. HANSON,[2] AND MITSUO TAKEDA[3]

[1]Institute of Photonics and Quantum Sciences, School of Engineering and Physical Sciences, Heriot-Watt University, Edinburgh, EH14 4AS, UK
[2]Department of Photonics Engineering, DTU Fotonik, P.O. Box 49, DK-4000 Roskilde, Denmark
[3]Center for Optical Research and Education (CORE), Utsunomiya University, 7-1-2, Yoto, Utsunomiya, Tochigi, 321-8585, Japan
*Corresponding author: w.wang@hw.ac.uk

Received 20 September 2022; revised 14 November 2022; accepted 14 November 2022; posted 14 November 2022; published 21 December 2022

The concept of ensemble-average polarization and coherence has been applied to studying fluctuating Stokes parameters in a polarization speckle observed when coherent light is passed through a birefringent polarization scrambler. With the aid of the ensemble-average van Cittert–Zernike theorem for the propagation of ensemble-average polar-coherence, we invesitgate the autocorrelation functions and power spectra of the Stokes parameters to expose the dependence of the polarization-related scale-size distributions on the optical geometries in which the polarization speckle arises. A generalized concept of the Stokes ensemble-average coherence areas is introduced to deal with the polarization-related average areas associated with polarization speckle.

https://doi.org/10.1364/JOSAA.476171

1. INTRODUCTION

All optical fields may undergo random fluctuations, and polarization and coherence have been widely regarded as important manifestations of fluctuations [1–4]. It is common to model light from a source as a stationary and ergodic random process in time, and the usual polarization and/or coherence are then defined by infinite time averages, although statistical averages can be used whenever necessary and yield the same results as time averages. Since there exists an important class of optical phenomena called speckle phenomena, where time average and ensemble average no longer yield the same results [3,5], we must be careful to distinguish between time averages and ensemble averages. One goal of this paper is to develop the concept of ensemble-average coherence first, as explicitly introduced by Goodman [5], and apply the concept of ensemble-average polarization and coherence to the study of polarization speckle for exploring the conceptual differences in polarization and coherence associated with such random statistical processes.

Polarization speckle manifests itself as random spatial variations of the state of polarization with the Stokes parameters varying in a random way across the pattern [6,7]. Just as the intensity autocorrelation has been widely adopted to measure the coarseness of a scalar speckle pattern [8–10], so, too, will the autocorrelation functions of the Stokes parameters provide means for measuring the polarization-related spatial structure for polarization speckle. As for the correlations between the Stokes parameters, there is a great deal of previous work in the literature that is pertinent to this subject, including the covariance matrix of instantaneous Stokes parameters [11], the polarization time and length [12,13], and experimental investigation of the Stokes autocorrelations and their applications [6,14–17]. More relevant recent work is that of Kuebel and Visser [18], where the autocorrelations and cross-correlations between all the Stokes parameters have been given in terms of the cross-spectral density matrix for the study of polarization-resolved Hanbury Brown–Twiss effect.

The purpose of this paper is to apply the concept of ensemble-average polarization and coherence for investigation of the spatial structure of polarization speckle. After pointing out the physical distinctions between ensemble-averaged quantities and time-averaged ones, we apply the ensemble-average van Cittert–Zernike theorem to study the propagation of ensemble-average polar-coherence. Under the application of the complex Gaussian moment theorem, we derive the autocorrelation functions of the Stokes parameters in terms of the (generalized) Stokes parameters (rather than by a cross-spectral density matrix as in [18]) for a polarization speckle pattern. The power spectral densities of the Stokes parameters are also obtained to reveal the dependences of polarization-related scale-size distributions on two optical geometries, i.e., free-space propagation geometry and imaging geometry, where polarization speckle arises. The stressed distinctions between statistics with particular ergodicity assumptions and the presented spatial structures of polarization speckle will inspire future work of interest.

2. ENSEMBLE-AVERAGE POLARIZATION AND COHERENCE

Before giving our detailed analysis of the polarization-related spatial structure of polarization speckle, we briefly introduce the concept of ensemble-average polarization and coherence and apply this concept to deal with polarization speckle for clarification of their physical indications as compared with time averages.

Let the light from an ideally stabilized monochromatic continuous-wave (CW) laser illuminate a stationary birefringent polarization scrambler [19,20] and consider the polarization and coherence properties of light some distance beyond the depolarizing diffuser, as shown in Fig. 1(a). The optical field behind the diffuser is intricate and its intensity and state-of-polarization give an unpredictable pattern, as shown in Fig. 1(b), due to the complicated and unknown birefringent microstructure of the diffuser itself. If the diffuser were moving, the electric field, intensity, and state-of-polarization behind the depolarizing diffuser would fluctuate with time, and the usual definitions on polarization and coherence in terms of time averages would be appropriate [1–4]. However, the diffuser is not moving in the current situation of interest.

The column vector of the electric field scattered by the birefringent polarization scrambler at location $\mathbf{r}$ and at time t is given by

$$\mathbf{E}(\mathbf{r},t)=\begin{bmatrix} E_x(\mathbf{r})\exp(-j2\pi\nu t) \\ E_y(\mathbf{r})\exp(-j2\pi\nu t)\end{bmatrix}, \tag{1}$$

where ν is the frequency of the monochromatic source, and $E_x(\mathbf{r})$ and $E_y(\mathbf{r})$ are the complex polarization components. Note that these two polarization components are independent of time; therefore, the column vector $[E_x, E_y]^{\mathrm{T}}$ with its superscript T being the transpose operator represents the polarization phasor of the electric field. Using the time-average definition of the polarization matrix of such light gives the form

$$\begin{aligned}\mathcal{J}(\mathbf{r}) &= \langle \mathbf{E}(\mathbf{r},t)\otimes\mathbf{E}^{\dagger}(\mathbf{r},t)\rangle \\ &= \begin{bmatrix} E_x(\mathbf{r})E_x^*(\mathbf{r}) & E_x(\mathbf{r})E_y^*(\mathbf{r}) \\ E_y(\mathbf{r})E_x^*(\mathbf{r}) & E_y(\mathbf{r})E_y^*(\mathbf{r})\end{bmatrix},\end{aligned} \tag{2}$$

where $\langle\cdots\rangle$ stands for the time average, $\otimes$ signifies the Kronecker product, and $\dagger$ denotes a conjugate transpose operator. From the standard definition of the degree of polarization [1–4], i.e., $\mathcal{P}=\sqrt{1-4\det(\mathcal{J})/[\mathrm{tr}(\mathcal{J})]^2}$ with tr and det being the trace and determinant operations, respectively, it is not difficult to show that the light behind the diffuser is completely polarized everywhere with $\mathcal{P}=1$ due to the fact of $\det(\mathcal{J})=0$.

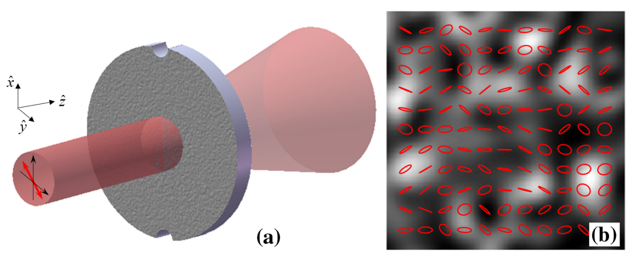

Fig. 1. (a) Generation of polarization speckle by an ideally stabilized CW laser and a stationary birefringent polarization scrambler. (b) Example of polarization speckle with the spatial variations of polarization ellipses and the fluctuations of intensity.

Similarly, the mutual coherence matrix of such light in terms of time average is given by

$$\begin{aligned}\boldsymbol{\Gamma}(\mathbf{r}_1,\mathbf{r}_2) &= \langle \mathbf{E}(\mathbf{r}_1,t)\otimes\mathbf{E}^{\dagger}(\mathbf{r}_2,t)\rangle \\ &= \begin{bmatrix} E_x(\mathbf{r}_1)E_x^*(\mathbf{r}_2) & E_x(\mathbf{r}_1)E_y^*(\mathbf{r}_2) \\ E_y(\mathbf{r}_1)E_x^*(\mathbf{r}_2) & E_y(\mathbf{r}_1)E_y^*(\mathbf{r}_2)\end{bmatrix}.\end{aligned} \tag{3}$$

With the aid of the normalized mutual coherence matrix by setting $\boldsymbol{\gamma}(\mathbf{r}_1,\mathbf{r}_2)=\boldsymbol{\Gamma}(\mathbf{r}_1,\mathbf{r}_2)/[\|\boldsymbol{\Gamma}(\mathbf{r}_1,\mathbf{r}_1)\|^{1/2}\|\boldsymbol{\Gamma}(\mathbf{r}_2,\mathbf{r}_2)\|^{1/2}]$ with $\|\cdots\|$ being the Frobenius norm of a matrix [21], it is also not difficult to show that $\|\boldsymbol{\gamma}(\mathbf{r}_1,\mathbf{r}_2)\|=1$. Therefore, by our usual definitions, based on time averages, we find that the scattered light behind the birefringent polarization scrambler is completely polarized and fully coherent everywhere, given that the diffuser is static and illuminated by entirely coherent and completely polarized light.

Since a statistical treatment of such light is indeed appropriate and useful, the questions then become over what statistical ensemble is the light a random process and how might we modify our definitions of polarization and coherence for such light. Suppose that, however, rather than averaging with respect to time, instead, we average over an ensemble of different birefringent polarization scramblers, each depolarizing diffuser having a different birefringent microstructure but all diffusers being statistically similar in their principal indices of refraction, the orientation angle of fast/slow axes, the mean and variance of thickness, and the lateral surface correlation length of birefringent material over the ensemble. Statistical averages are then carried out over the ensemble of these depolarizing diffusers, allowing polarization- and coherence-related concepts to be defined. Therefore, the ensemble-average polarization matrix and the ensemble-average mutual coherence matrix so defined take the forms

$$\begin{aligned}\bar{\mathcal{J}}(\mathbf{r}) &= \overline{\mathbf{E}(\mathbf{r},t)\otimes\mathbf{E}^{\dagger}(\mathbf{r},t)} \\ &= \begin{bmatrix} \overline{E_x(\mathbf{r})E_x^*(\mathbf{r})} & \overline{E_x(\mathbf{r})E_y^*(\mathbf{r})} \\ \overline{E_y(\mathbf{r})E_x^*(\mathbf{r})} & \overline{E_y(\mathbf{r})E_y^*(\mathbf{r})}\end{bmatrix},\end{aligned} \tag{4}$$

and

$$\begin{aligned}\bar{\boldsymbol{\Gamma}}(\mathbf{r}_1,\mathbf{r}_2) &= \overline{\mathbf{E}(\mathbf{r}_1,t)\otimes\mathbf{E}^{\dagger}(\mathbf{r}_2,t)} \\ &= \begin{bmatrix} \overline{E_x(\mathbf{r}_1)E_x^*(\mathbf{r}_2)} & \overline{E_x(\mathbf{r}_1)E_y^*(\mathbf{r}_2)} \\ \overline{E_y(\mathbf{r}_1)E_x^*(\mathbf{r}_2)} & \overline{E_y(\mathbf{r}_1)E_y^*(\mathbf{r}_2)}\end{bmatrix},\end{aligned} \tag{5}$$

where the overbar represents a statistical expectation. Here, a stationary polarization speckle pattern is such a process, for which the time averages of a stationary polarization speckle do not equal the averages over an ensemble of possible polarization speckle patterns. In general, $\mathcal{J}(\mathbf{r})\neq\bar{\mathcal{J}}(\mathbf{r})$ and $\boldsymbol{\Gamma}(\mathbf{r}_1,\mathbf{r}_2)\neq\bar{\boldsymbol{\Gamma}}(\mathbf{r}_1,\mathbf{r}_2)$.

Note the fact that the Stokes vector $\mathbf{S}(\mathbf{r})$ is a different representation of $\mathcal{J}(\mathbf{r})$ for characterizing light polarization, and the generalized Stokes vector $\mathbf{S}(\mathbf{r}_1, \mathbf{r}_2)$ is an alternative representation of $\boldsymbol{\Gamma}(\mathbf{r}_1, \mathbf{r}_2)$ for characterizing coherence of optical waves [1,4]. Based on time averages, they are defined by $\mathbf{S}(\mathbf{r}) = \mathbf{A}\langle \mathbf{E}(\mathbf{r}, t) \otimes \mathbf{E}^*(\mathbf{r}, t)\rangle$ and $\mathbf{S}(\mathbf{r}_1, \mathbf{r}_2) = \mathbf{A}\langle \mathbf{E}(\mathbf{r}_1, t) \otimes \mathbf{E}^*(\mathbf{r}_2, t)\rangle$ with $\mathbf{A}$ being the unitary transformation matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -j & j & 0 \end{bmatrix}. \tag{6}$$

In a similar way, we can place a bar over the symbols $\mathbf{S}(\mathbf{r})$ and $\mathbf{S}(\mathbf{r}_1, \mathbf{r}_2)$ to remind us that the averages are over the ensembles, and the definitions for the ensemble-average Stokes vector and the ensemble-average generalized Stokes vector take the forms $\bar{\mathbf{S}}(\mathbf{r}) = \mathbf{A}[\overline{\mathbf{E}(\mathbf{r}, t) \otimes \mathbf{E}^*(\mathbf{r}, t)}]$ and $\bar{\mathbf{S}}(\mathbf{r}_1, \mathbf{r}_2) = \mathbf{A}[\overline{\mathbf{E}(\mathbf{r}_1, t) \otimes \mathbf{E}^*(\mathbf{r}_2, t)}]$, respectively. More explicitly, we have

$$\begin{aligned}
\overline{S_0}(\mathbf{r}_1, \mathbf{r}_2) &= \overline{E_x(\mathbf{r}_1) E_x^*(\mathbf{r}_2)} + \overline{E_y(\mathbf{r}_1) E_y^*(\mathbf{r}_2)}, \\
\overline{S_1}(\mathbf{r}_1, \mathbf{r}_2) &= \overline{E_x(\mathbf{r}_1) E_x^*(\mathbf{r}_2)} - \overline{E_y(\mathbf{r}_1) E_y^*(\mathbf{r}_2)}, \\
\overline{S_2}(\mathbf{r}_1, \mathbf{r}_2) &= \overline{E_x(\mathbf{r}_1) E_y^*(\mathbf{r}_2)} + \overline{E_y(\mathbf{r}_1) E_x^*(\mathbf{r}_2)}, \\
\overline{S_3}(\mathbf{r}_1, \mathbf{r}_2) &= j[\overline{E_y(\mathbf{r}_1) E_x^*(\mathbf{r}_2)} - \overline{E_x(\mathbf{r}_1) E_y^*(\mathbf{r}_2)}],
\end{aligned} \tag{7}$$

and $\bar{\mathbf{S}}(\mathbf{r})$ can be found by merging $\mathbf{r}_1$ and $\mathbf{r}_2$ to a single point $\mathbf{r}$. In the situations of interest here, generally, $\bar{\mathbf{S}}(\mathbf{r}) \neq \mathbf{S}(\mathbf{r})$ and $\bar{\mathbf{S}}(\mathbf{r}_1, \mathbf{r}_2) \neq \mathbf{S}(\mathbf{r}_1, \mathbf{r}_2)$.

Although the conventional schemes of polarization and coherence based on time averages have achieved great success when applied to light from a thermal source, these time-averaged quantities of physical interests fail to characterize the inherent statistics of polarization speckle because such a stochastic field does not fluctuate with time (at least in the classical sense). The applications of the concept of ensemble-average polarization and coherence are found particularly useful in the study of polarization speckle, whose spatial properties, including intensity and state-of-polarization, are random, but no time averaging is involved. Unlike the conventional degree of polarization (which is defined on the basis of time averages and indicates the ratio of the intensity in the completely polarized wave component to the total intensity in the wave), the degree of polarization newly defined on the basis of ensemble averages may be called the *degree of ensemble-average polarization* and is related to the degree of order or disorder of the spatial distribution of polarization states for completely polarized light [6,7,22]. On the other hand, unlike the conventional concept of optical coherence providing a measure of capability for optical fields to interfere between two points for fringe generation, the concept of ensemble-average coherence indicates the statistical dependence and cross-correlation between the realizations of random processes (stochastic optical fields) at two points from fully coherent light.

Since there is a conceptual difference between ensemble averages and time averages for polarization speckle, we must be careful to distinguish between time-averaged polarization and coherence and ensemble-averaged polarization and coherence. Following Goodman [3], we shall use the ordinary symbols for polarization- and coherence-related quantities defined by time averages and identical symbols with overbars to represent the corresponding ensemble-averaged quantities. Thus, we distinguish between two degrees of polarization, $\mathcal{P}$ and $\bar{\mathcal{P}}$, and two normalized mutual coherence matrices, $\boldsymbol{\gamma}(\mathbf{r}_1, \mathbf{r}_2)$ and $\bar{\boldsymbol{\gamma}}(\mathbf{r}_1, \mathbf{r}_2)$, and so on.

The concept of ensemble average has long been known and utilized widely as a common mathematical tool in statistical optics [1–4]. Unlike the conventional time-average polarization and coherence (which are practical quantities observable by experiment), the ensemble-average quantities are conceptual quantities defined only mathematically and are not observable by experiment unless an infinite number of realizations (events) are physically generated. Obviously, polarization speckle is just a single realization (event) of a spatial random process represented by spatial variations of polarization ellipses and spatial fluctuations of intensity. Nonetheless, if the polarization speckle field has spatial ergodicity, we can replace the ensemble averaging with the space averaging that is experimentally observable, which gives a practical value to the concept of ensemble-average polarization and coherence. In the context of polarization speckle, which is a fully coherent and completely polarized random optical wave, the *ensemble-average incoherence* with $||\bar{\boldsymbol{\gamma}}(\mathbf{r}_1, \mathbf{r}_2)|| = 0$ indicates that intensities and states of polarization at the two positions $\mathbf{r}_1$ and $\mathbf{r}_2$ fluctuate independently without any spatial correlation. While the *full ensemble-average coherence* with $||\bar{\boldsymbol{\gamma}}(\mathbf{r}_1, \mathbf{r}_2)|| = 1$ gives an opposite situation where the intensities and the states of polarization are perfectly correlated and fluctuate in a synchronized manner if the two measurement points approach each other arbitrarily closely. As far as ensemble-average polarization is concerned, the polarization speckle with $\bar{\mathcal{P}} = 0$ can be referred to as the *isotropic polarization speckle* since the randomness for varying polarization ellipses does not change when measured along any spatial directions. On the other hand, the polarization speckle with $\bar{\mathcal{P}} = 1$ can be called the *uniform polarization speckle* since the state-of-polarization is uniformly distributed with identical polarization ellipses across the whole observation area, although its intensity fluctuates in space. The conventional scalar speckle can be understood as the uniform polarization speckle phenomenon, whose statistical properties and applications are comprehensively discussed in a monograph by Goodman [8].

3. AUTOCORRELATION FUNCTIONS AND POWER SPECTRA OF THE STOKES PARAMETERS

Due to the fact of random spatial variations of the state-of-polarization in a polarization speckle pattern, our interest here is in the coarseness of polarization-related spatial structures and the distributions of scale sizes in their random spatial fluctuations of the Stokes parameters. In this section, we will consider some important aspects of the spatial structure of polarization speckle, namely, the autocorrelation functions and the power spectral densities of the Stokes parameters of polarization speckle in two optical geometries: free-space scattering geometry and imaging geometry.

A. Propagation of Ensemble-Average Polar-Coherence

In statistical optics, both the mutual coherence matrix and the generalized Stokes vector are regarded as representing the light coherence properties by taking the vector nature of the stochastic electric fields into account; therefore, we can term the *polar-coherence* for the polarization-related coherence properties as contrasting with the conventional scalar coherence of light. Note the fact that the wave equation governing the propagation of light is the same, no matter if we are ultimately interested in time- or ensemble-average properties of light. The laws governing the propagation of polar-coherence are identical for time- and ensemble-averaged quantities [3]. Therefore, we are allowed to apply all previously acquired knowledge on the propagation of time-average polar-coherence [23–26] to problems involving propagation of ensemble-average polar-coherence in the context of polarization speckle.

We consider the case when a coherent source illuminates a birefringent polarization scrambler, and the scattered light is observed some distance z from that surface, as illustrated in Fig. 2. The birefringent polarization scrambler is assumed to be stationary in time without specifying its rough surface or birefringent microstructure. Over an ensemble of ideally rough surfaces with a very short lateral correlation width of surface height fluctuations, there is little relationship between the phase differences of two polarization components of the light scattered from two closely spaced surface elements, at least until the spacing becomes close to a wavelength of the illuminating light. From an ensemble-averaging point of view, the ensemble-average generalized Stokes vector of the light scattered by a depolarizing diffuser, and observed very close to that surface, is Dirac delta-correlated with essentially the same correlation extent as the generalized Stokes vector of an incoherent source. Mathematically, we can represent the ensemble-average generalized Stokes vector of light just leaving the surface by

$$\overline{\mathbf{S}^{\mathrm{Sca}}}(\alpha_1,\beta_1;\alpha_2,\beta_2)=\kappa\overline{\mathbf{S}^{\mathrm{Sca}}}(\alpha_1,\beta_1)\delta(\alpha_1-\alpha_2,\beta_1-\beta_2), \tag{8}$$

where κ is a constant with dimensions meters squared, $\overline{\mathbf{S}^{\mathrm{Sca}}}$ with its superscript Sca indicates the ensemble-average Stokes vector of the scattered light just leaving the rough surface of the depolarizing diffuser and δ is the 2D delta function.

In analogy to the theoretical analysis and experimental demonstration of the propagation of time-averaged polar-coherence [23–27], we can write the ensemble-average van Cittert–Zernike theorem as

$$\overline{\mathbf{S}^{\mathrm{Obs}}}(x_1,y_1;x_2,y_2)=\frac{\kappa e^{-j\psi}}{(\lambda z)^2}\int\int_{-\infty}^{\infty}\overline{\mathbf{S}^{\mathrm{Sca}}}(\alpha,\beta)\times\exp\left[\frac{j2\pi}{\lambda z}(\Delta x\alpha+\Delta y\beta)\right]\mathrm{d}\alpha\mathrm{d}\beta, \tag{9}$$

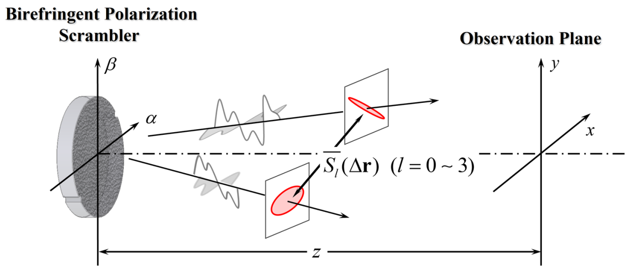

Fig. 2. Polarization speckle arises in free-space propagation geometry.

where the superscript Obs indicates the observation plane, $\psi=\pi[(x_2^2+y_2^2)-(x_1^2+y_1^2)]/(\lambda z)$ and $\Delta x=x_2-x_1$, $\Delta y=y_2-y_1$, and λ is the wavelength of the incident radiation. Similarly, Eq. (9) can be rewritten in terms of $\bar{\mathcal{J}}$ and $\bar{\mathbf{\Gamma}}$. That is

$$\overline{\mathbf{\Gamma}^{\mathrm{Obs}}}(x_1,y_1;x_2,y_2)=\frac{\kappa e^{-j\psi}}{(\lambda z)^2}\int\int_{-\infty}^{\infty}\overline{\mathcal{J}^{\mathrm{Sca}}}(\alpha,\beta)\times\exp\left[\frac{j2\pi}{\lambda z}(\Delta x\alpha+\Delta y\beta)\right]\mathrm{d}\alpha\mathrm{d}\beta. \tag{10}$$

As expected, up to scaling constants, the ensemble-average generalized Stokes vector and the ensemble-average mutual coherence matrix are given by Fourier transforms of the distributions of the ensemble-average Stokes vector and the ensemble-average polarization matrix leaving the surface of the birefringent polarization scrambler, respectively. When the ensemble average is replaced with the spatial average, Eqs. (9) and (10) can be regarded as a natural generalization of the spatial-average version of the van Cittert–Zernike theorem for vector fields [22,28–30].

B. Free-Space Propagation Geometry

Just as the autocorrelation function of intensity has been widely adopted to characterize the average scale-size of a scalar speckle, a suitable description for the fluctuating state-of-polarization in polarization speckle will be the autocorrelation functions of the Stokes parameters at two points, ensemble-average quantities, which are defined by

$$\overline{\Gamma_{S_l}}(\Delta x,\Delta y)=\overline{S_l(x_1,y_1)S_l(x_2,y_2)}, \tag{11}$$

for $l=0\sim3$. When Eq. (11) is written, we have made use of an assumption that the autocorrelation functions of the Stokes parameters depend only on the difference of observation coordinates. Note that two polarization components E_x and E_y of a polarization speckle are complex Gaussian random processes. From the complex Gaussian moment theorem [3], we have

$$\overline{\Gamma_{S_0}}(\Delta x,\Delta y)=\overline{S_0}^2+\tfrac{1}{2}[|\overline{S_0}(\Delta x,\Delta y)|^2+|\overline{S_1}(\Delta x,\Delta y)|^2+|\overline{S_2}(\Delta x,\Delta y)|^2+|\overline{S_3}(\Delta x,\Delta y)|^2]$$

$$\overline{\Gamma_{S_1}}(\Delta x,\Delta y)=\overline{S_1}^2+\tfrac{1}{2}[|\overline{S_0}(\Delta x,\Delta y)|^2+|\overline{S_1}(\Delta x,\Delta y)|^2-|\overline{S_2}(\Delta x,\Delta y)|^2-|\overline{S_3}(\Delta x,\Delta y)|^2]$$

$$\overline{\Gamma_{S_2}}(\Delta x,\Delta y)=\overline{S_2}^2+\tfrac{1}{2}[|\overline{S_0}(\Delta x,\Delta y)|^2-|\overline{S_1}(\Delta x,\Delta y)|^2+|\overline{S_2}(\Delta x,\Delta y)|^2-|\overline{S_3}(\Delta x,\Delta y)|^2]$$

$$\overline{\Gamma_{S_3}}(\Delta x,\Delta y)=\overline{S_3}^2+\tfrac{1}{2}[|\overline{S_0}(\Delta x,\Delta y)|^2-|\overline{S_1}(\Delta x,\Delta y)|^2-|\overline{S_2}(\Delta x,\Delta y)|^2+|\overline{S_3}(\Delta x,\Delta y)|^2], \quad \textbf{(12)}$$

where $\overline{S_l}$ and $\overline{S_l}(\Delta x,\Delta y)$ for $l=0\sim 3$ are the ensemble-average Stokes parameters and the ensemble-average generalized Stokes parameters of light in the observation plane, respectively. Rather than by using a cross-spectral density matrix [18], the above Stokes autocorrelations expressed in terms of the (generalized) Stokes parameters have revealed the complicated relationship between the local polarization (provided by the Stokes parameters at a certain point) and all the possible polar-coherence (provided by the generalized Stokes parameters at two points). The first term in each expression stems from the contribution of the corresponding ensemble-averaged Stokes parameter. While the second term, consisting of a combination of the generalized Stokes parameters, indicates that each Stokes autocorrelation depends not only on its corresponding generalized Stokes parameter but also on other generalized Stokes parameters. Detailed proof of Eq. (12) can be found in Appendix A.

Note from Eq. (12) that there is a high redundancy among four terms with similar expressions. To simplify the expressions and highlight the underlying physics of the Stokes autocorrelations, we define four linear transforms as follows:

$$\mathcal{L}_0\{g_0,g_1,g_2,g_3\}=g_0+g_1+g_2+g_3,$$
$$\mathcal{L}_1\{g_0,g_1,g_2,g_3\}=g_0+g_1-g_2-g_3,$$
$$\mathcal{L}_2\{g_0,g_1,g_2,g_3\}=g_0-g_1+g_2-g_3,$$
$$\mathcal{L}_3\{g_0,g_1,g_2,g_3\}=g_0-g_1-g_2+g_3, \quad \textbf{(13)}$$

with g_l (for $l=0\sim 3$) being an expression. Let us also introduce formally the normalized ensemble-average generalized Stokes vector by the Frobenius norm of the ensemble-average Stokes vector

$$\overline{\boldsymbol{\gamma}_S}(\Delta x,\Delta y)=\bar{\mathbf{S}}(\Delta x,\Delta y)/\left\|\bar{\mathbf{S}}(0,0)\right\|$$
$$=\bar{\mathbf{S}}(\Delta x,\Delta y)/(\overline{S_0}\sqrt{1+\bar{\mathcal{P}}^2}), \quad \textbf{(14)}$$

where the denominator $||\bar{\mathbf{S}}(0,0)||=||\bar{\mathbf{S}}||$ is the Frobenius norm of the ensemble-average Stokes vector. When Eq. (14) is derived, we have made use of the definition for the degree of ensemble-average polarization: $\bar{\mathcal{P}}=\sqrt{\overline{S_1}^2+\overline{S_2}^2+\overline{S_3}^2}/\overline{S_0}$.

With the aid of the normalized ensemble-average Stokes vector, $\widehat{\bar{\mathbf{S}}}=\bar{\mathbf{S}}/||\bar{\mathbf{S}}||$, we are now ready to present the autocorrelation functions of the Stokes parameters of the polarization speckle in the observation plane:

$$\overline{\Gamma_{S_l}}(\Delta x,\Delta y)=(1+\overline{\mathcal{P}}^2)\overline{S_0}^2[\widehat{\overline{S_l}}^2+2^{-1}\mathcal{L}_l\{|\overline{\gamma_{S_0}}(\Delta x,\Delta y)|^2,|\overline{\gamma_{S_1}}(\Delta x,\Delta y)|^2,|\overline{\gamma_{S_2}}(\Delta x,\Delta y)|^2,|\overline{\gamma_{S_3}}(\Delta x,\Delta y)|^2\}], \quad \textbf{(15)}$$

for $l=0\sim 3$. With reference to Eq. (10), the constant ensemble-average quantities: $\overline{S_l}$, $\widehat{\overline{S_l}}$, $\overline{\gamma_{S_l}}$ and $\bar{\mathcal{P}}$ in the observation plane are related to the distributions of the ensemble-average Stokes parameters at the scattering spot through

$$\overline{S_0}=\frac{\kappa}{(\lambda z)^2}\int\int_{-\infty}^{\infty}\overline{S_0^{\rm Sca}}(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta$$

$$\widehat{\overline{S_l}}=\frac{\iint_{-\infty}^{\infty}\overline{S_l^{\rm Sca}}(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta}{\sqrt{\sum_{m=0}^{3}[\int\int_{-\infty}^{\infty}\overline{S_m^{\rm Sca}}(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta]^2}}$$

$$\left|\overline{\gamma_{S_l}}(\Delta x,\Delta y)\right|=\frac{\left|\iint_{-\infty}^{\infty}\overline{S_l^{\rm Sca}}(\alpha,\beta)\exp\left[\frac{j2\pi}{\lambda z}(\Delta x\alpha+\Delta y\beta)\right]\mathrm{d}\alpha\mathrm{d}\beta\right|}{\sqrt{\sum_{m=0}^{3}\left[\iint_{-\infty}^{\infty}\overline{S_m^{\rm Sca}}(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta\right]^2}}$$

$$\bar{\mathcal{P}}=\frac{\sqrt{\sum_{k=1}^{3}\left[\iint_{-\infty}^{\infty}\overline{S_k^{\rm Sca}}(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta\right]^2}}{\iint_{-\infty}^{\infty}\overline{S_0^{\rm Sca}}(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta}. \quad \textbf{(16)}$$

The power spectral densities of the Stokes parameters $\overline{\mathcal{G}_{S_l}}(f_x,f_y)$ for $l=0\sim 3$ in a polarization speckle pattern are also ensemble-average quantities, representing the spatial power distributions of fluctuating Stokes parameters over the 2D frequency plane and are given by the Fourier transforms of the corresponding autocorrelation functions:

$$\overline{\mathcal{G}_{S_l}}(f_x,f_y)=\iint_{-\infty}^{\infty}\overline{\Gamma_{S_l}}(\Delta x,\Delta y)\exp[j2\pi(\Delta x f_x+\Delta y f_y)]\mathrm{d}\Delta x\mathrm{d}\Delta y. \quad \textbf{(17)}$$

With the help of the autocorrelation theorem [31], the power spectral densities of the Stokes parameters $\overline{\mathcal{G}_{S_l}}(f_x,f_y)$ can be reduced to

$$\overline{\mathcal{G}_{S_l}}(f_x,f_y)={}_{\mathrm{I}}\overline{\mathcal{G}_{S_l}}(f_x,f_y)+{}_{\mathrm{II}}\overline{\mathcal{G}_{S_l}}(f_x,f_y)$$
$$=(1+\overline{\mathcal{P}}^2)\overline{S_0}^2[\widehat{\overline{S_l}}^2\delta(f_x,f_y)+(\lambda z)^2/(2\mathcal{Q})\times\mathcal{L}_l\{\overline{\mathcal{R}_{S_0^{\rm Sca}}}(f_x,f_y),\overline{\mathcal{R}_{S_1^{\rm Sca}}}(f_x,f_y),\overline{\mathcal{R}_{S_2^{\rm Sca}}}(f_x,f_y),\overline{\mathcal{R}_{S_3^{\rm Sca}}}(f_x,f_y)\}], \quad \textbf{(18)}$$

where $\overline{\mathcal{R}_{S_l^{\mathrm{Sca}}}}(f_x, f_y) = \int\int_{-\infty}^{\infty} \overline{S_l^{\mathrm{Sca}}}(\alpha, \beta)\overline{S_l^{\mathrm{Sca}}}(\alpha - \lambda z f_x, \beta - \lambda z f_y)\mathrm{d}\alpha\mathrm{d}\beta$ and $\mathcal{Q} = \sum_{m=0}^{3}\left[\iint_{-\infty}^{\infty} \overline{S_m^{\mathrm{Sca}}}(\alpha, \beta)\mathrm{d}\alpha\mathrm{d}\beta\right]^2$. The first terms in Eq. (18) (denoted by the symbols $_{\mathrm{I}}\overline{\mathcal{G}_{S_l}}$) are the Dirac delta functions corresponding to the zero-frequency discrete powers contributed by the ensemble-averaged Stokes parameters $\overline{S_l^{\mathrm{Sca}}}$ of the scattering spot. The second terms (denoted by the symbols $_{\mathrm{II}}\overline{\mathcal{G}_{S_l}}$), consisting of different combinations of the normalized and scaled autocorrelation functions of the Stokes parameters $S_l^{\mathrm{Sca}}(\alpha, \beta)$ of the scattering spot, represent the distributions of varying powers over spatial frequency for the fluctuating parts of the Stokes parameters in the polarization speckle pattern.

The second terms of the Stokes power spectra $_{\mathrm{II}}\overline{\mathcal{G}_{S_l}}$ yield the continuous portions of the spectra and have values at $f_x = f_y = 0$ of

$$_{\mathrm{II}}\overline{\mathcal{G}_{S_l}}(0, 0) = \frac{(1+\bar{\mathcal{P}}^2)(\lambda z \overline{S_0})^2}{2\mathcal{Q}} \iint_{-\infty}^{\infty} \mathcal{L}_l\{\overline{S_0^{\mathrm{Sca}}}^2(\alpha, \beta), \overline{S_1^{\mathrm{Sca}}}^2(\alpha, \beta), \overline{S_2^{\mathrm{Sca}}}^2(\alpha, \beta), \overline{S_3^{\mathrm{Sca}}}^2(\alpha, \beta)\}\mathrm{d}\alpha\mathrm{d}\beta. \quad \textbf{(19)}$$

To illustrate the results above, we use an example of circular uniform distributions of the Stokes parameters at the scattering spot with its diameter D:

$$\overline{S_l^{\mathrm{Sca}}}(\alpha, \beta) = {}_0S_l \mathrm{circ}(2\sqrt{\alpha^2 + \beta^2}/D), \quad \textbf{(20)}$$

where $_0S_l$ (for $l = 0 \sim 3$) are constants and $\mathrm{circ}(r) = 1$ for $0 \le r \le 1$, and 0 otherwise. After substituting Eq. (20) into Eqs. (9) and (14), we have the modulus for each component of the normalized ensemble-average generalized Stokes vector

$$|\overline{\gamma_{S_l}}(\Delta x, \Delta y)| = \left| \frac{_0S_l}{\sqrt{\sum_{m=0}^{3}(_0S_m)^2}} \frac{2J_1\left(\pi D\sqrt{\Delta x^2 + \Delta y^2}/(\lambda z)\right)}{\pi D\sqrt{\Delta x^2 + \Delta y^2}/(\lambda z)} \right|, \quad \textbf{(21)}$$

where $J_1(\cdots)$ is a Bessel function of the first kind, order one. With proper normalization, the autocorrelation functions of the Stokes parameters of the polarization speckle in the free-space scattering geometry become

$$\overline{\Gamma_{S_l}}(\Delta x, \Delta y) = (1+\bar{\mathcal{P}}^2)\overline{S_0}^2 \left\{ \overset{\frown}{S_l}^2 + 2^{-1}\mathcal{L}_l\left\{\overset{\frown}{S_0}^2, \overset{\frown}{S_1}^2, \overset{\frown}{S_2}^2, \overset{\frown}{S_3}^2\right\} \times \left|2J_1\left(\pi D\sqrt{\Delta x^2 + \Delta y^2}/(\lambda z)\right) / \left[\pi D\sqrt{\Delta x^2 + \Delta y^2}/(\lambda z)\right]\right|^2 \right\}, \quad \textbf{(22)}$$

where $\overline{S_0} = \kappa\pi D^2 {}_0S_0/(2\lambda z)^2$, $\overset{\frown}{S_l} = {}_0S_l/\sqrt{\sum_{m=0}^{3}(_0S_m)^2}$, and $\bar{\mathcal{P}} = \sqrt{(_0S_1)^2 + (_0S_2)^2 + (_0S_3)^2}/{}_0S_0$. It is interesting to note that *the polarization speckle maintains the degree of ensemble-average polarization during its free-space propagation*. The corresponding power spectral densities of the Stokes parameters of polarization speckle can then be shown to be

$$\overline{\mathcal{G}_{S_l}}(f_x, f_y) = (1+\bar{\mathcal{P}}^2)\overline{S_0}^2 \left\{ \overset{\frown}{S_l}^2 \delta(f_x, f_y) + \mathcal{L}_l\{\overset{\frown}{S_0}^2, \overset{\frown}{S_1}^2, \overset{\frown}{S_2}^2, \overset{\frown}{S_3}^2\} \frac{4(\lambda z)^2}{\pi^2 D^2}\left[\arccos(\lambda z|\vec{f}|/D) - (\lambda z|\vec{f}|/D)\sqrt{1 - (\lambda z|\vec{f}|/D)^2}\right] \right\} \quad \textbf{(23)}$$

with $|\vec{f}| = \sqrt{f_x^2 + f_y^2}$.

As a special case for $\bar{\mathcal{P}} = 0$ with $_0S_1 = {}_0S_2 = {}_0S_3 = 0$, the Stokes autocorrelations $\overline{\Gamma_{S_l}}(\Delta x, \Delta y)$ are reduced to

$$\overline{\Gamma_{S_0}}(\Delta x, \Delta y) = \overline{S_0}^2 \left\{ 1 + 2\left| \frac{J_1\left(\pi D\sqrt{\Delta x^2 + \Delta y^2}/(\lambda z)\right)}{\pi D\sqrt{\Delta x^2 + \Delta y^2}/(\lambda z)} \right|^2 \right\}$$

$$\overline{\Gamma_{S_1}}(\Delta x, \Delta y) = \overline{\Gamma_{S_2}}(\Delta x, \Delta y) = \overline{\Gamma_{S_3}}(\Delta x, \Delta y) = 2\overline{S_0}^2 \left| \frac{J_1\left(\pi D\sqrt{\Delta x^2 + \Delta y^2}/(\lambda z)\right)}{\pi D\sqrt{\Delta x^2 + \Delta y^2}/(\lambda z)} \right|^2. \quad \textbf{(24)}$$

Similarly, the power spectral densities of the Stokes parameters when $\bar{\mathcal{P}} = 0$ can be reduced to

$$\overline{\mathcal{G}_{S_0}}(f_x, f_y) = \overline{S_0}^2 \left\{ \delta(f_x, f_y) + \frac{4(\lambda z)^2}{\pi^2 D^2}\left[\arccos(\lambda z|\vec{f}|/D) - (\lambda z|\vec{f}|/D)\sqrt{1 - (\lambda z|\vec{f}|/D)^2}\right] \right\},$$

$$\overline{\mathcal{G}_{S_1}}(f_x, f_y) = \overline{\mathcal{G}_{S_2}}(f_x, f_y) = \overline{\mathcal{G}_{S_3}}(f_x, f_y) = \frac{4(\lambda z)^2\overline{S_0}^2}{\pi^2 D^2}\left[\arccos(\lambda z|\vec{f}|/D) - (\lambda z|\vec{f}|/D) \times \sqrt{1 - (\lambda z|\vec{f}|/D)^2}\right]. \quad \textbf{(25)}$$

Figure 3 shows cross-sections of the autocorrelations and power spectral densities of the Stokes parameters for uniform polarization speckle with $\bar{\mathcal{P}} = 1$. In this example, a set of $\overset{\frown}{S_l}$ (i.e., $\overset{\frown}{S_0} = \sqrt{0.5}$, $\overset{\frown}{S_1} = -\sqrt{0.25}$, $\overset{\frown}{S_2} = \sqrt{0.15}$, $\overset{\frown}{S_3} = -\sqrt{0.1}$) has been chosen. The autocorrelation functions and the power spectral densities of the Stokes parameters for isotropic polarization speckle with $\bar{\mathcal{P}} = 0$ are illustrated in Fig. 4. We conclude that, in any polarization speckle, large-scale (low frequency) sizes are the most populous, and no scale sizes greater than a certain cutoff frequency are present. Due to the different means

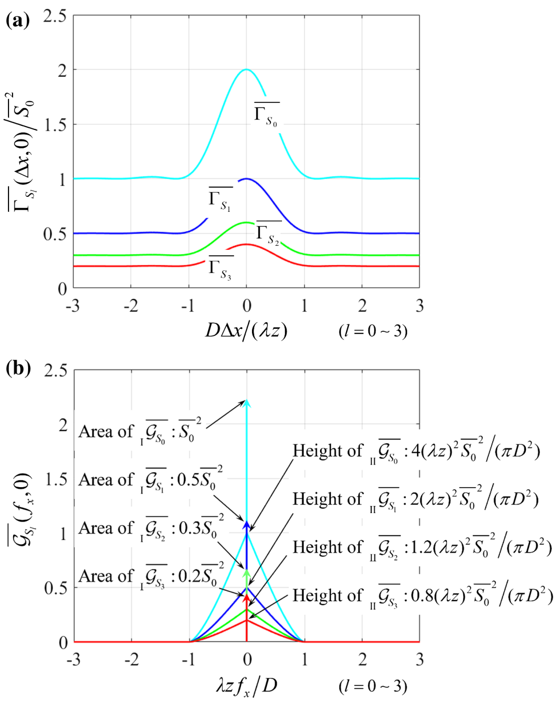

Fig. 3. Cross sections of (a) autocorrelation functions and (b) power spectral densities of the Stokes parameters of uniform polarization speckle with $\bar{\mathcal{P}} = 1$ for free-space propagation and a uniform circular scattering spot.

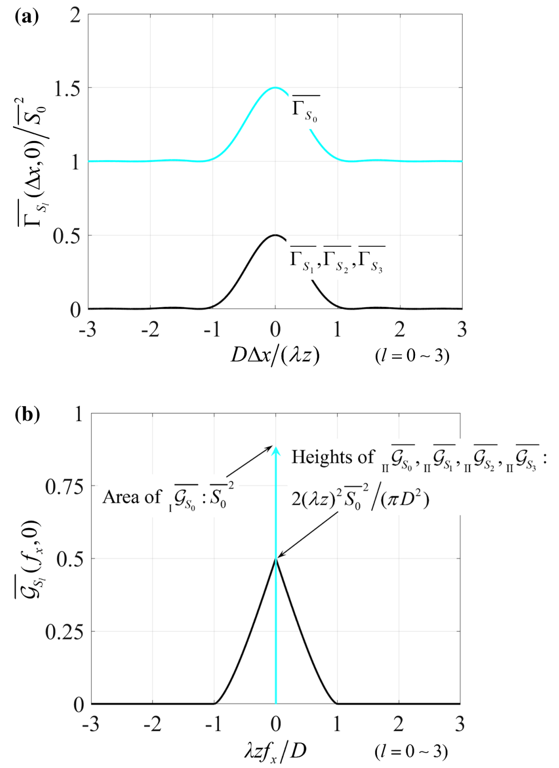

Fig. 4. Cross sections of (a) autocorrelation functions and (b) power spectral densities of the Stokes parameters of isotropic polarization speckle with $\bar{\mathcal{P}} = 0$ for free-space propagation and a uniform circular scattering spot.

of the Stokes parameters in a polarization speckle, the areas under the Dirac delta functions at zero spatial frequency are different. Meanwhile, the exact power spectral densities of the Stokes parameters in a polarization speckle pattern also depend on the geometric shape of the scattering spot.

Similar to the case of conventional scalar speckle, where the normalized correlation function of speckle intensity has been adopted as the averaged speckle size, here we would like to develop a similar concept for measures of the average "sizes" for the Stokes parameters in a polarization speckle from the above example of calculation of the autocorrelation functions and power spectral densities. Since the normalized covariance function of the Stokes parameter S_0 (intensity of polarization speckle) is related to the autocorrelation function of the Stokes parameter S_0 by $\overline{c_{S_0}}(\Delta x, \Delta y) = [\overline{\Gamma_{S_0}}(\Delta x, \Delta y) - \overline{S_0}^2]/[\overline{\Gamma_{S_0}}(0, 0) - \overline{S_0}^2]$, the equivalent area of S_0 detection in a polarization speckle is given by

$$\mathcal{A}^{\mathrm{C}}_{S_0} = \iint_{-\infty}^{\infty} \overline{c_{S_0}}(\Delta x, \Delta x)\mathrm{d}\Delta x \mathrm{d}\Delta y$$
$$= \iint_{-\infty}^{\infty} [|\overline{\gamma_{S_0}}(\Delta x, \Delta x)|^2 + |\overline{\gamma_{S_1}}(\Delta x, \Delta x)|^2$$
$$+ |\overline{\gamma_{S_2}}(\Delta x, \Delta x)|^2 + |\overline{\gamma_{S_3}}(\Delta x, \Delta x)|^2]\mathrm{d}\Delta x \mathrm{d}\Delta y. \quad \textbf{(26)}$$

Here, we call $\mathcal{A}^{\mathrm{C}}_{S_0}$ the *Stokes S_0 correlation area* or the *ensemble-average coherence area of the Stokes parameter S_0* for a polarization speckle. Similarly, we can also introduce ensemble-average coherence areas for other Stokes parameters in a polarization speckle. Therefore, all the Stokes correlation areas $\mathcal{A}^{\mathrm{C}}_{S_l}$ for $l = 0 \sim 3$ can be defined in a unified manner. They are

$$\mathcal{A}^{\mathrm{C}}_{S_l} = \iint_{-\infty}^{\infty} \mathcal{L}_l \{ |\overline{\gamma_{S_0}}(\Delta x, \Delta y)|^2, |\overline{\gamma_{S_1}}(\Delta x, \Delta y)|^2,$$
$$|\overline{\gamma_{S_2}}(\Delta x, \Delta y)|^2, |\overline{\gamma_{S_3}}(\Delta x, \Delta y)|^2 \} \mathrm{d}\Delta x \mathrm{d}\Delta y. \quad \textbf{(27)}$$

As expected, we may have different correlation areas for detection of fluctuating Stokes parameters in a polarization speckle. More generally, for a scattering spot with nonuniformly distributed Stokes parameters, these correlation areas $\mathcal{A}^{\mathrm{C}}_{S_l}$ for $(l = 0 \sim 3)$ can be defined by

$$\mathcal{A}^{\mathrm{C}}_{S_l} = (\lambda z)^2 \left\{ \sum_{m=0}^{3} \left[\iint_{-\infty}^{\infty} \overline{S^{\mathrm{Sca}}_m}(\alpha, \beta)\mathrm{d}\alpha \mathrm{d}\beta \right]^2 \right\}^{-1} \iint_{-\infty}^{\infty} \mathcal{L}_l \{$$
$$\overline{S^{\mathrm{Sca}}_0}^2(\alpha, \beta), \overline{S^{\mathrm{Sca}}_1}^2(\alpha, \beta), \overline{S^{\mathrm{Sca}}_2}^2(\alpha, \beta), \overline{S^{\mathrm{Sca}}_3}^2(\alpha, \beta) \} \mathrm{d}\alpha \mathrm{d}\beta. \quad \textbf{(28)}$$

When the above expression is derived, we have substituted Eqs. (16)–(27) and made use of Parseval's theorem [31].

C. Imaging Geometry

In the previous section, we considered fundamental properties related to the spatial structure of polarization speckle in a free-space propagation geometry. We now turn our attention

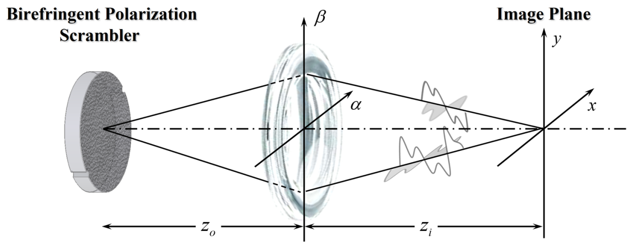

Fig. 5. Imaging geometry for polarization speckle formation.

to an imaging geometry, as shown in Fig. 5, to derive the autocorrelation functions and power spectral densities of the Stokes parameters for the Stokes images with polarization speckle. A birefringent polarization scrambler with rough surface is illuminated with coherent light. A portion of the scattered light is collected by the simple positive lens and is then brought to a focus in the image plane. The distance between the depolarizing diffuser and the lens is z_o and the image distance from the lens is z_i.

To simplify our analysis, we use an approximation first proposed by Zernike [32] and explained in detail by Goodman [3] for scalar coherence theory. Here, two assumptions have been used, i.e., the illumination area on the depolarizing diffuser is large as compared with the correlation area of the scattered light, and the illumination region is broad as compared with the extend of the resolution cell given by the lens. Under the condition that the scattering grains generated by the depolarizing diffuser in the object plane are not resolved by the imaging system, it is possible to treat the lens pupil itself as a δ-correlated effective secondary source with uniform distributions of the Stokes parameters. Therefore, we can apply the ensemble-average van Cittert–Zernike theorem to this effective source to obtain the ensemble-average generalized Stokes vector in the image plane:

$$\overline{\mathbf{S}^{\mathrm{Img}}}(x_1, y_1; x_2, y_2) = \frac{\kappa e^{-j\psi}\overline{\mathbf{S}^{\mathrm{Pup}}}}{(\lambda z_i)^2} \iint_{-\infty}^{\infty} |P(\alpha, \beta)|^2 \times \exp\left[\frac{j2\pi}{\lambda z_i}(\Delta x\alpha + \Delta y\beta)\right] \mathrm{d}\alpha \mathrm{d}\beta, \tag{29}$$

where $\overline{\mathbf{S}^{\mathrm{Pup}}}$, assumed to be constant, is the ensemble-average Stokes vector across the exit pupil, and $P(\alpha, \beta)$, defining the shape of the pupil, is the exit pupil function, and z_i is the distance from the exit pupil to the image plane.

Similar to the free-space propagation geometry, the ensemble-average generalized Stokes vector in the image plane can also be found by means of scalar multiplication of a suitably scaled Fourier transform of squared modulus of the pupil function with the constant ensemble-average Stokes vector at the exit pupil. Therefore, the desired autocorrelation functions of the Stokes parameters in imaged polarization speckle can also be found with the aid of the ensemble-average van Cittert–Zernike theorem, where $|\gamma_{S_l}(\Delta x, \Delta y)|$ is given by

$$|\gamma_{S_l}(\Delta x, \Delta y)| = \left|\overline{S_l^{\mathrm{Pup}}}\right| \Big/ \sqrt{\sum_{m=0}^{3} \overline{S_m^{\mathrm{Pup}}}^2} \left| \iint_{-\infty}^{\infty} \left|\widehat{P}(\alpha, \beta)\right|^2 \times \exp\left[j\frac{2\pi}{\lambda z_i}(\Delta x\alpha + \Delta y\beta)\right] \mathrm{d}\alpha \mathrm{d}\beta \right|, \tag{30}$$

with $\widehat{P}$ representing a normalized pupil function defined through [3]

$$\left|\widehat{P}(\alpha, \beta)\right|^2 = |P(\alpha, \beta)|^2 \Big/ \iint_{-\infty}^{\infty} |P(\alpha, \beta)|^2 \mathrm{d}\alpha \mathrm{d}\beta. \tag{31}$$

After substituting into Eqs. (15) and (18), we have the autocorrelation functions and the power spectral densities of the Stokes parameters of polarization speckle in the image plane:

$$\begin{aligned} &\overline{\Gamma_{S_l}}(\Delta x, \Delta y) \\ &= (1 + \overline{\mathcal{P}}^2)\overline{S_0}^2 \left\{ \widehat{\overline{S_l}}^2 + 2^{-1}\mathcal{L}_l \left\{ \widehat{\overline{S_0}}^2, \widehat{\overline{S_1}}^2, \widehat{\overline{S_2}}^2, \widehat{\overline{S_3}}^2 \right\} \right. \\ &\left. \times \left| \iint_{-\infty}^{\infty} \left|\widehat{P}(\alpha, \beta)\right|^2 \exp\left[j2\pi(\Delta x\alpha + \Delta y\beta)/(\lambda z_i)\right] \mathrm{d}\alpha \mathrm{d}\beta \right|^2 \right\} \end{aligned} \tag{32}$$

and

$$\begin{aligned} \overline{\mathcal{G}_{S_l}}(f_x, f_y) = (1 + \overline{\mathcal{P}}^2)\overline{S_0}^2 \Bigg[&\widehat{\overline{S_l}}^2 \delta(f_x, f_y) \\ &+ \frac{(\lambda z_i)^2}{2}\mathcal{L}_l \left\{ \widehat{\overline{S_0}}^2, \widehat{\overline{S_1}}^2, \widehat{\overline{S_2}}^2, \widehat{\overline{S_3}}^2 \right\} \mathcal{R}_{|\widehat{P}|^2}(f_x, f_y) \Bigg], \end{aligned} \tag{33}$$

where $\overline{S_0} = \kappa S_0^{\mathrm{Pup}} A/(\lambda z)^2$ with A being the area of the exit pupil, $\widehat{\overline{S_l}} = S_l^{\mathrm{Pup}} \Big/ \sqrt{\sum_{m=0}^{3} (S_m^{\mathrm{Pup}})^2}$, $\bar{\mathcal{P}} = \sqrt{(S_1^{\mathrm{Pup}})^2 + (S_2^{\mathrm{Pup}})^2 + (S_3^{\mathrm{Pup}})^2} / S_0^{\mathrm{Pup}}$, and $\mathcal{R}_{|\widehat{P}|^2}(f_x, f_y) = \iint_{-\infty}^{\infty} |\widehat{P}(\alpha, \beta)|^2 |\widehat{P}(\alpha - \lambda z_i f_x, \beta - \lambda z_i f_y)|^2 \mathrm{d}\alpha \mathrm{d}\beta$.

Aside from scaling factors, the polarization speckle in an imaging geometry has similar expressions for the autocorrelation functions and the power spectral densities of the Stokes parameters as those for free-space propagation geometry. When the lens is not apodized (i.e., when $P = 0$ or 1), then $|\widehat{P}|^2 = |\widehat{P}|$, and the autocorrelation function of $|\widehat{P}|^2$ is (within a normalizing constant) equivalent to the autocorrelation function of the P itself. Thus, these parts of the Stokes power spectral densities have the same shape as a normalized optical transfer function of the unaberrated system, up to normalizing factors. For the usual case of a circular lens pupil, the corresponding autocorrelation functions and the power spectral densities of the Stokes parameters of imaged polarization speckle have the same results, as shown in Figs. 3 and 4.

Two additional comments should be made here. First, since light just behind lens pupil behaves essentially as a delta-correlated source, any lens aberrations usually represented as phase errors across the lens pupil have no effect on the correlation properties of the polarization speckle observed in the image plane. Second, although we have considered only a simple thin lens in the analysis above, identical results will be obtained for any imaging system, and the distance z_i now will be interpretated as the distance from the exit pupil to the image plane.

Before closing the discussion about the autocorrelation functions and the power spectral densities of the Stokes parameters, the point should be made that we have already applied the concepts of ensemble-average polarization and coherence in the study of polarization speckle. Note the fact that the main mathematical relationship and/or function composition between the polarization and coherence-related quantities is independent of which kind of averaging is used [3]. The main equations have the same mathematical forms for ensemble average and time average because these averaging operations have the common mathematical nature of being a linear operation based on integration (with the only difference in the integration being performed with respect to time or the ensemble). Therefore, all our previously acquired knowledge on the Stokes autocorrelations and power spectral densities derived on the basis of ensemble average can be applied directly to problems involving time average, including but not limited to the study of the polarization-resolved Hanbury Brown–Twiss effect and polarization-sensitive ghost imaging.

4. CONCLUSION

The concept of ensemble-average polarization and coherence has been applied to the study of spatially fluctuating Stokes parameters in a stationary polarization speckle, which is a nonergodic statistical process since time and ensemble averages do not yield the same results. With the aid of the ensemble-average van Cittert–Zernike theorem for propagation of the ensemble-average polar-coherence, the polarization-related coarseness of polarization speckle has been investigated through the autocorrelation functions and the power spectral densities of the Stokes parameters with four average sizes introduced for the Stokes measures. In terms of the (generalized) Stokes parameters, both free-space geometry and imaging geometry are considered to explore the dependence of these second-order polar-coherence effects on the optical geometries where the polarization speckle arises. Further, the formalisms of the Stokes autocorrelations and the Stokes power spectral densities obtained in the context of polarization speckle will be beneficial to understanding the fourth-order statistical properties of the stochastic optical field and therefore open up new opportunities with wide applications involving other higher-order statistics of random electromagnetic wave no matter the statistical averages are carried out with respect to time or the ensemble.

APPENDIX A

Here, we present details of the derivation that leads to the expressions in Eq. (12).

Let us use the autocorrelation of the Stokes parameter S_0 at two positions $\mathbf{r}_1$ and $\mathbf{r}_2$ defined by $\overline{\Gamma_{S_0}}(\mathbf{r}_1, \mathbf{r}_2) \triangleq \overline{S_0(\mathbf{r}_1) S_0(\mathbf{r}_2)}$ as an example for illustration. After expressing the instantaneous Stokes parameters in terms of the fluctuating electric fields, i.e., $S_0(\mathbf{r}) = E_x(\mathbf{r}) E_x^*(\mathbf{r}) + E_y(\mathbf{r}) E_y^*(\mathbf{r})$, we find that

$$\begin{aligned}\overline{\Gamma_{S_0}}(\mathbf{r}_1, \mathbf{r}_2) = {} & \overline{E_x(\mathbf{r}_1) E_x^*(\mathbf{r}_1) E_x(\mathbf{r}_2) E_x^*(\mathbf{r}_2)} \\ & + \overline{E_x(\mathbf{r}_1) E_x^*(\mathbf{r}_1) E_y(\mathbf{r}_2) E_y^*(\mathbf{r}_2)} \\ & + \overline{E_y(\mathbf{r}_1) E_y^*(\mathbf{r}_1) E_x(\mathbf{r}_2) E_x^*(\mathbf{r}_2)} \\ & + \overline{E_y(\mathbf{r}_1) E_y^*(\mathbf{r}_1) E_y(\mathbf{r}_2) E_y^*(\mathbf{r}_2)}. \end{aligned} \quad \textbf{(A1)}$$

It is reasonable to assume that each polarization component of the electric field E_k (for $k = x, y$) reaching the observation plane arises from the superposition of many independent contributors from the scattering spot and, consequently, can be considered as a realization of a complex Gaussian random process. With the aid of the complex Gaussian moment theorem, i.e., $\overline{E_1^* E_2^* E_3 E_4} = \overline{E_1^* E_3}\,\overline{E_2^* E_4} + \overline{E_1^* E_4}\,\overline{E_2^* E_3}$ and the definitions of $\overline{\mathcal{J}}$ and $\overline{\boldsymbol{\Gamma}}$ in Eqs. (4) and (5), we can rewrite Eq. (A1) as

$$\begin{aligned}\overline{\Gamma_{S_0}} = {} & \overline{E_x(\mathbf{r}_1) E_x^*(\mathbf{r}_1)}\,\overline{E_x(\mathbf{r}_2) E_x^*(\mathbf{r}_2)} + \overline{E_x^*(\mathbf{r}_1) E_x(\mathbf{r}_2)}\,\overline{E_x(\mathbf{r}_1) E_x^*(\mathbf{r}_2)} \\ & + \overline{E_x(\mathbf{r}_1) E_x^*(\mathbf{r}_1)}\,\overline{E_y(\mathbf{r}_2) E_y^*(\mathbf{r}_2)} + \overline{E_x^*(\mathbf{r}_1) E_y(\mathbf{r}_2)}\,\overline{E_x(\mathbf{r}_1) E_y^*(\mathbf{r}_2)} \\ & + \overline{E_y(\mathbf{r}_1) E_y^*(\mathbf{r}_1)}\,\overline{E_x(\mathbf{r}_2) E_x^*(\mathbf{r}_2)} + \overline{E_y^*(\mathbf{r}_1) E_x(\mathbf{r}_2)}\,\overline{E_y(\mathbf{r}_1) E_x^*(\mathbf{r}_2)} \\ & + \overline{E_y(\mathbf{r}_1) E_y^*(\mathbf{r}_1)}\,\overline{E_y(\mathbf{r}_2) E_y^*(\mathbf{r}_2)} + \overline{E_y^*(\mathbf{r}_1) E_y(\mathbf{r}_2)}\,\overline{E_y(\mathbf{r}_1) E_y^*(\mathbf{r}_2)} \\ = {} & [\overline{\mathcal{J}_{xx}}(\mathbf{r}_1) + \overline{\mathcal{J}_{yy}}(\mathbf{r}_1)][\overline{\mathcal{J}_{xx}}(\mathbf{r}_2) + \overline{\mathcal{J}_{yy}}(\mathbf{r}_2)] + |\overline{\Gamma_{xx}}(\mathbf{r}_1, \mathbf{r}_2)|^2 \\ & + |\overline{\Gamma_{xy}}(\mathbf{r}_1, \mathbf{r}_2)|^2 + |\overline{\Gamma_{yx}}(\mathbf{r}_1, \mathbf{r}_2)|^2 + |\overline{\Gamma_{yy}}(\mathbf{r}_1, \mathbf{r}_2)|^2. \end{aligned} \quad \textbf{(A2)}$$

It follows from Eqs. (4), (5), and (7) that the generalized Stokes parameters and the elements of the mutual coherence matrix are related by the formulas

$$\begin{aligned} \overline{\Gamma_{xx}}(\mathbf{r}_1, \mathbf{r}_2) &= [\overline{S_0}(\mathbf{r}_1, \mathbf{r}_2) + \overline{S_1}(\mathbf{r}_1, \mathbf{r}_2)]/2, \\ \overline{\Gamma_{yy}}(\mathbf{r}_1, \mathbf{r}_2) &= [\overline{S_0}(\mathbf{r}_1, \mathbf{r}_2) - \overline{S_1}(\mathbf{r}_1, \mathbf{r}_2)]/2, \\ \overline{\Gamma_{xy}}(\mathbf{r}_1, \mathbf{r}_2) &= [\overline{S_2}(\mathbf{r}_1, \mathbf{r}_2) + j\overline{S_3}(\mathbf{r}_1, \mathbf{r}_2)]/2, \\ \overline{\Gamma_{yx}}(\mathbf{r}_1, \mathbf{r}_2) &= [\overline{S_2}(\mathbf{r}_1, \mathbf{r}_2) - j\overline{S_3}(\mathbf{r}_1, \mathbf{r}_2)]/2, \end{aligned} \quad \textbf{(A3)}$$

and the similar relationships also hold between the Stokes parameters and the elements of the polarization matrix. After substitution of Eq. (A3), we arrive at

$$\begin{aligned}\overline{\Gamma_{S_0}}(\Delta x, \Delta y) = {} & \overline{S_0}^2 + \tfrac{1}{2}[|\overline{S_0}(\Delta x, \Delta y)|^2 + |\overline{S_1}(\Delta x, \Delta y)|^2 \\ & + |\overline{S_2}(\Delta x, \Delta y)|^2 + |\overline{S_3}(\Delta x, \Delta y)|^2]. \end{aligned} \quad \textbf{(A4)}$$

When the above expression is derived, we have made use of a condition of spatial stationarity for polarization speckle so that the corresponding ensemble-average Stokes parameters are spatially independent. As asserted in Eq. (12), the autocorrelation functions of the remaining Stokes parameters $\overline{\Gamma_{S_1}}$, $\overline{\Gamma_{S_2}}$, and $\overline{\Gamma_{S_3}}$ can be derived in a similar way.

Funding. Scottish Univeristies Physics Alliance (SSG040).

Acknowledgment. The authors thank Mr. J. Du for his help in preparing figures for this work.

Disclosures. The authors declare no conflicts of interest.

Data availability. No data were generated or analyzed in the presented research.

REFERENCES

1. E. L. O'Neill, *Introduction to Statistical Optics* (Addison-Wesley, 1963).
2. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University, 1995).
3. J. W. Goodman, *Statistical Optics*, 2nd ed. (Wiley-Blackwell, 2015).
4. E. Wolf, *Introduction to the Theory of Coherence and Polarization of Light* (Cambridge University, 2007).
5. J. W. Goodman, "Role of coherence concepts in the study of speckle," Proc. SPIE **194**, 86–94 (1979).
6. W. Wang, S. G. Hanson, and M. Takeda, "Statistics of polarization speckle: theory versus experiment," Proc. SPIE **7288**, 738803 (2009).
7. M. Takeda, W. Wang, and S. G. Hanson, "Polarization speckles and generalized Stokes vector wave: a review," Proc. SPIE **7387**, 73870V (2010).
8. J. W. Goodman, *Speckle Phenomena in Optics: Theory and Applications*, 2nd ed. (SPIE, 2020).
9. L. I. Goldfischer, "Autocorrelation function and power spectral density of laser-produced speckle patterns," J. Opt. Soc. Am. **55**, 247–252 (1965).
10. J. W. Goodman, "Some effects of target-induced scintillation on optical radar performance," Proc. IEEE **53**, 1688–1700 (1965).
11. C. Brosseau, *Fundamentals of Polarized Light* (Wiley, 1998).
12. T. Setälä, A. Shevchenko, M. Kaivola, and A. T. Friberg, "Polarization time and length for random optical beams," Phys. Rev. A **78**, 033817 (2008).
13. T. Voipio, T. Setälä, A. Shevchenko, and A. T. Friberg, "Polarization dynamics and polarization time of random three-dimensional electromagnetic fields," Phys. Rev. A **82**, 063807 (2010).
14. W. Wang, R. Ishijima, A. Matsuda, S. G. Hanson, and M. Takeda, "Pseudo-Stokes vector correlation and its applications to micro-displacement measurement," Strain **46**, 12–18 (2010).
15. R. K. Singh, D. N. Naik, H. Itou, Y. Miyamoto, and M. Takeda, "Characterization of spatial polarization fluctuations in scattered field," J. Opt. **16**, 105010 (2014).
16. S. Zhang, Y. Yang, S. G. Hanson, M. Takeda, and W. Wang, "Statistics of the derivatives of complex signal derived from Riesz transform and its application to pseudo-Stokes vector correlation for speckle displacement measurement," Appl. Opt. **54**, 8561–8565 (2015).
17. T. Sarkar, R. Parvin, M. M. Brundavanam, and R. K. Singh, "Higher-order Stokes-parameter correlation to restore the twisted wave front propagating through a scattering medium," Phys. Rev. A **104**, 013525 (2021).
18. D. Kuebel and T. D. Visser, "Generalized Hanbury Brown-Twiss effect for Stokes parameters," J. Opt. Soc. Am. A **36**, 362–367 (2019).
19. N. Ma, S. G. Hanson, M. Takeda, and W. Wang, "Coherence and polarization of polarization speckle generated by a rough-surfaced retardation plate depolarizer," J. Opt. Soc. Am. A **32**, 2346–2352 (2015).
20. J. Ritter, N. Ma, W. Osten, M. Takeda, and W. Wang, "Depolarizing surface scattering by a birefringent material with rough surface," Opt. Commun. **430**, 456–460 (2019).
21. R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University, 1990).
22. M. Takeda, W. Wang, D. N. Naik, and R. K. Singh, "Spatial statistical optics and spatial correlation holography: a review," Opt. Rev. **21**, 849–861 (2014).
23. F. Gori, M. Santarsiero, R. Borghi, and G. Piquero, "Use of the van Cittert–Zernike theorem for partially polarized sources," Opt. Lett. **25**, 1291–1293 (2000).
24. O. Korotkova and E. Wolf, "Generalized Stokes parameters of random electromagnetic beams," Opt. Lett. **30**, 198–199 (2005).
25. M. A. Alonso, O. Korotkova, and E. Wolf, "Propagation of the electric correlation matrix and the van Cittert–Zernike theorem for random electromagnetic fields," J. Mod. Opt. **53**, 969–978 (2006).
26. A. S. Ostrovsky, G. Martínez-Niconoff, P. Martínez-Vara, and M. A. Olvera-Santamaría, "The van Cittert–Zernike theorem for electromagnetic fields," Opt. Express **17**, 1746–1752 (2009).
27. H. C. Kandpal, B. Kanseri, and S. Rath, "Direct determination of the generalized Stokes parameters from the usual Stokes parameters," Opt. Lett. **34**, 719–721 (2009).
28. R. K. Singh, D. N. Naik, H. Itou, M. M. Brundabanam, Y. Miyamoto, and M. Takeda, "Vectorial van Cittert–Zernike theorem based on spatial averaging: experimental demonstrations," Opt. Lett. **38**, 4809–4812 (2013).
29. R. V. Vinu and R. K. Singh, "Experimental determination of generalized Stokes parameters," Opt. Lett. **40**, 1227–1230 (2015).
30. M. Takeda, "Spatial stationarity of statistical optical fields for coherence holography and photon correlation holography," Opt. Lett. **38**, 3452–3455 (2013).
31. J. W. Goodman, *Introduction to Fourier Optics*, 4th ed. (W. H. Freeman, 2017).
32. F. Zernike, "The concept of degree of coherence and its application to optical problems," Physica **5**, 785–795 (1938).