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Degree of Non-Markovianity of Quantum Evolution

Dariusz Chruściński[1] and Sabrina Maniscalco[2,3]
[1]Institute of Physics, Faculty of Physics, Astronomy, and Informatics, Nicolaus Copernicus University, Grudziądzka 5/7, 87–100 Toruń, Poland
[2]SUPA, EPS/Physics, Heriot-Watt University, Edinburgh, EH14 4AS, United Kingdom
[2]Department of Physics and Astronomy, University of Turku, 20014 Turku, Finland
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We propose a new characterization of non-Markovian quantum evolution based on the concept of non-Markovianity degree. It provides an analog of a Schmidt number in the entanglement theory and reveals the formal analogy between quantum evolution and the entanglement theory: Markovian evolution corresponds to a separable state and the non-Markovian one is further characterized by its degree. It enables one to introduce a non-Markovianity witness—an analog of an entanglement witness, and a family of measures—an analog of Schmidt coefficients, and finally to characterize maximally non-Markovian evolution being an analog of the maximally entangled state. Our approach allows us to classify the non-Markovianity measures introduced so far in a unified rigorous mathematical framework.

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Introduction.—Open quantum systems and their dynamical features are attracting increasing attention nowadays. They are of paramount importance in the study of the interaction between a quantum system and its environment, causing dissipation, decay, and decoherence [1–3]. On the other hand, the robustness of quantum coherence and entanglement against the detrimental effects of the environment is one of the major focuses in quantum-enhanced applications, as both entanglement and quantum coherence are basic resources in modern quantum technologies, such as quantum communication, cryptography, and computation [4]. Recently, much effort was devoted to the description, analysis, and classification of non-Markovian quantum evolution (see, e.g., [5–19] and the collection of papers in [20]). In particular, various concepts of non-Markovianity were introduced and several so-called non-Markovianity measures were proposed. The main approaches to the problem of (non)Markovian evolution are based on divisibility [9–12], distinguishability of states [13], quantum entanglement [10], quantum Fisher information flow [14], fidelity [15], mutual information [16,17], channel capacity [18], and geometry of the set of accessible states [19].

In this Letter we accept the definition based on divisibility [9,10]: the quantum evolution is Markovian if the corresponding dynamical map Λ_t is CP divisible (where CP stands for complete positivity), that is,

$$\Lambda_t = V_{t,s}\Lambda_s, \tag{1}$$

and $V_{t,s}$ provides a family of legitimate (completely positive and trace-preserving) propagators for all $t \geq s \geq 0$. The essential property of $V_{t,s}$ is the following composition law $V_{t,s}V_{s,u} = V_{t,u}$, for all $t \geq s \geq u$. It provides a natural generalization of a semigroup law $e^{tL}e^{sL} = e^{(t+s)L}$. Interestingly, the very property of CP divisibility is fully characterized in terms of the time-local generator L_t: if Λ_t satisfies the time-local master equation $\dot{\Lambda}_t = L_t\Lambda_t$, then Λ_t is CP divisible if and only if L_t has the standard Lindblad form for all $t \geq 0$, i.e.,

$$L_t\rho = -i[H(t),\rho] + \sum_\alpha \Big(V_\alpha(t)\rho V_\alpha^\dagger(t) - \tfrac{1}{2}\{V_\alpha^\dagger(t)V_\alpha(t),\rho\}\Big),$$

with time-dependent Lindblad (noise) operators $V_\alpha(t)$ and time-dependent effective system Hamiltonian $H(t)$ [3,21,22]. A very appealing concept of Markovianity was proposed by Breuer, Lane, and Piilo (BLP) [13]: Λ_t is Markovian if

$$\sigma(\rho_1,\rho_2;t) = \frac{d}{dt}||\Lambda_t(\rho_1-\rho_2)||_1 \leq 0, \tag{2}$$

for all pairs of initial states ρ_1 and ρ_2. BLP call $\sigma(\rho_1,\rho_2;t)$ an information flow and interpret $\sigma(\rho_1,\rho_2;t) > 0$ as a backflow of information from the environment to the system which clearly indicates the non-Markovian character of the evolution. As usual $||X||_1$ denotes the trace norm of X, i.e., $||X||_1 = \mathrm{Tr}\sqrt{XX^\dagger}$. It turns out that CP divisibility implies (2) but the converse needs not be true [23–25].

In this Letter we propose a more refined approach to non-Markovian evolution. We reveal the formal analogy with the entanglement theory: Markovian evolution corresponds to a separable state and non-Markovian evolution is characterized by a positive integer—the non-Markovianity degree—corresponding to the Schmidt number of an entangled state. The notion of non-Markovianity degree enables one to introduce a family of measures and finally to characterize maximally non-Markovian evolution being an analog of the maximally entangled state.

Schmidt number and k-positive maps.—Let us recall that a state of a composite quantum system may be uniquely characterized by its Schmidt number [26,27]: for any normalized vector $\psi \in \mathcal{H} \otimes \mathcal{H}$ let $\mathrm{SR}(\psi)$ denote the Schmidt rank of ψ, i.e., a number of nonvanishing Schmidt coefficients in the decomposition $\psi = \sum_k s_k e_k \otimes f_k$, with $s_k > 0$ and $\sum_k s_k^2 = 1$. Now, for any density operator ρ one defines its Schmidt number by

$$\mathrm{SN}(\rho) = \min_{p_k,\psi_k} \{\max_k \mathrm{SR}(\psi_k)\}, \quad (3)$$

where the minimum is performed over all decompositions $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ with $p_k > 0$ and $\sum_k p_k = 1$. Let $S_k = \{\rho | \mathrm{SN}(\rho) \leq k\}$. One has $S_1 \subset S_2 \subset \ldots \subset S_n$, where S_1 denotes a set of separable states and S_n denotes a set of all states in $\mathcal{H} \otimes \mathcal{H}$. Note that a maximally entangled state ψ satisfies $\lambda_1 = \ldots = \lambda_n$ and the corresponding projector $|\psi\rangle\langle\psi|$ defines an element of S_n. The Schmidt number does not increase under local operation, i.e., $\mathrm{SN}([\mathcal{E}_1 \otimes \mathcal{E}_2]\rho) \leq \mathrm{SN}(\rho)$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are arbitrary quantum channels. Moreover, if Φ is a k-positive map, i.e., $\mathbb{1}_k \otimes \Phi$ is positive, then for any $\rho \in S_k$ one has $[\mathbb{1}_k \otimes \Phi](\rho) \geq 0$ ($\mathbb{1}_k$ denotes an identity map acting in M_k—the space of $k \times k$ complex matrices). This simple property establishes a duality between k-positive maps and quantum bipartite states with the Schmidt number bounded by k.

Non-Markovianity degree.—The notion of k-positive maps enables one to provide a natural generalization of CP divisibility: we call a dynamical map Λ_t k divisible if and only if $V_{t,s}$ is k positive for all $t \geq s \geq 0$. Hence, n-divisible maps are CP divisible and 1 divisible are simply P divisible; i.e., $V_{t,s}$ is positive. Now, we introduce a degree of non-Markovianity which is an analog of a Schmidt number: a dynamical map Λ_t has a non-Markovianity degree $\mathrm{NMD}[\Lambda_t] = k$ if and only if Λ_t is $(n-k)$ but not $(n+1-k)$ divisible. It is clear that Λ_t is Markovian if and only if $\mathrm{NMD}[\Lambda_t] = 0$ and essentially non-Markovian if and only if $\mathrm{NMD}[\Lambda_t] = n$. Denoting by $\mathcal{N}_k = \{\Lambda_t | \mathrm{NMD}[\Lambda_t] \leq k\}$, one has a natural chain of inclusions

$$\mathcal{N}_0 \subset \mathcal{N}_1 \subset \ldots \subset \mathcal{N}_{n-1} \subset \mathcal{N}_n, \quad (4)$$

where $\mathcal{N}_0$ denotes Markovian maps and $\mathcal{N}_n$ all dynamical maps. The characterization of k-divisible maps is provided by the following.

Theorem 1.—If Λ_t is k divisible, then

$$\frac{d}{dt} ||[\mathbb{1}_k \otimes \Lambda_t](X)||_1 \leq 0, \quad (5)$$

for all operators $X \in M_k \otimes \mathcal{B}(\mathcal{H})$.

For the proof see Supplemental Material [28]. In particular, all k-divisible maps ($k = 1, \ldots, n$) satisfy

$$\frac{d}{dt} ||\Lambda_t(X)||_1 \leq 0, \quad (6)$$

for all $X \in \mathcal{B}(\mathcal{H})$. Note that BLP condition (2) is a special case of (6) with X being traceless Hermitian operator. It is, therefore, clear that BLP condition is weaker than all conditions in the hierarchy (5) and it is satisfied for all k-divisible maps not necessarily *CP* divisible. According to our definition of Markovianity (Markovianity = CP divisibility) k-divisible maps which are not CP divisible are clearly non-Markovian. However, such non-Markovian evolutions always satisfy (6). We propose to call such dynamical maps *weakly* non-Markovian. A dynamical map which is even not P divisible will be called *essentially* non-Markovian. Hence, Λ_t is weakly non-Markovian if and only if $\Lambda_t \in \mathcal{N}_{n-1} - \mathcal{N}_0$ and it is essentially non-Markovian if and only if $\Lambda_t \in \mathcal{N}_n - \mathcal{N}_{n-1}$. Using the notion of degree of non-Markovianity Λ_t is weakly non-Markovian if and only if $0 < \mathrm{NMD}[\Lambda_t] \leq n - 1$ and it is essentially non-Markovian if and only if $\mathrm{NMD}[\Lambda_t] = n$. Note that maps which violate the BLP condition are always essentially non-Markovian. Similarly, if Λ_t is at least 2 divisible, then the relative entropy satisfies the following monotonicity property [29]

$$\frac{d}{dt} S[\Lambda_t(\rho_1)||\Lambda_t(\rho_2)] \leq 0, \quad (7)$$

for any pair ρ_1 and ρ_2. The violation of (7) means that Λ_t is at most P divisible or essentially non-Markovian. It should be stressed that there is crucial difference between CP divisibility and only k divisibility with $k < n$. CP divisibility guarantees that $V_{t,s}$ are completely positive and, hence, they may be considered as physical propagators for $s \leq t$. This is no longer true for $V_{t,s}$ which are not CP but only k positive. There was an active debate whether or not one can describe quantum evolution by maps which are more general than CP maps [30]. Usually, the departure from complete positivity is attributed to the presence of initial system-environment correlations [30]. Remarkably, in our approach the lack of complete positivity of $V_{t,s}$ corresponds to memory effects caused by the nontrivial system-environment interaction. We stress that the dynamical map Λ_t is perfectly CP; only the intermediate propagators $V_{t,s}$ are not. Note, however, that if Λ_t is k divisible then $V_{t,s}$ map a state in time s into a state in time t. One loses this property only if Λ_t is essentially non-Markovian.

Non-Markovianity witness.—Actually, if Λ_t is invertible, then it is k divisible if and only if (5) holds. Clearly, a generic map is invertible (all its eigenvalues are different from zero) and hence this result is true for a generic dynamical map (a notable exception is the Jaynes-Cummings model on resonance [1,31]). Hence, if (5) is violated for some $t > 0$, then Λ_t is not k divisible or, equivalently, $\mathrm{NMD}[\Lambda_t] > n - k$. It is, therefore, natural to call such X a non-Markovianity witness in analogy to the well-known concept of an entanglement witness. Recall, that a Hermitian operator W living in $\mathcal{H}\otimes\mathcal{H}$ is an entanglement witness [26] if and only if (i) $\langle\Psi|W|\Psi\rangle\geq 0$ for all product vectors $\Psi=\psi\otimes\phi$, and (ii) W is not a positive operator; i.e., it possesses at least one negative eigenvalue. Similarly, W is a k-Schmidt witness [32] if $\langle\Psi|W|\Psi\rangle\geq 0$ for all vectors $\Psi=\psi_1\otimes\phi_1+\cdots+\psi_k\otimes\phi_k$, that is, if $\mathrm{Tr}(\rho W)<0$, then ρ is entangled and moreover $\mathrm{SN}(\rho)>k$. Note, that if $X\geq 0$, then (5) is always satisfied due to the fact that $||[\mathbb{1}_k\otimes\Lambda_t](X)||_1=||X||_1$. Hence, similarly as W, a non-Markovianity witness X has to possess a negative eigenvalue.

Non-Markovianity measures.—The above construction allows us to define a series of natural measures measuring departure from k divisibility,

$$\mathcal{M}_k[\Lambda_t]=\sup_X\frac{N_k^+[X]}{|N_k^-[X]|},\tag{8}$$

where

$$N_k^+[X]=\int_{\lambda_k(X;t)>0}\lambda_k(X;t)dt,$$

and, similarly for $N_k^-[X]$ (where now one integrates over time intervals such that $\lambda_k(X;t)<0$), and

$$\lambda_k(X;t)=\frac{d}{dt}||[\mathbb{1}_k\otimes\Lambda_t](X)||_1.\tag{9}$$

The supremum is taken over all Hermitian $X\in M_k\otimes\mathcal{B}(\mathcal{H})$. Note that

$$\int_0^\infty\frac{d}{dt}||[\mathbb{1}_k\otimes\Lambda_t](X)||_1dt = ||[\mathbb{1}_k\otimes\Lambda_\infty](X)||_1-||X||_1\leq 0,$$

and hence $|N_-[\Lambda_t]|\geq N_+[\Lambda_t]$, which proves that $\mathcal{M}_k[\Lambda_t]\in[0,1]$. Clearly, if $l>k$, then $\mathcal{M}_l[\Lambda_t]\geq\mathcal{M}_k[\Lambda_t]$ and, hence,

$$0\leq\mathcal{M}_1[\Lambda_t]\leq\ldots\leq\mathcal{M}_n[\Lambda_t]\leq 1,$$

which provides an analog of a similar relation among the Schmidt coefficients $s_1\geq\ldots\geq s_n$. Now, following the analogy with an entanglement theory, we may call Λ_t maximally non-Markovian if and only if $\mathcal{M}_1[\Lambda_t]=1$, which immediately implies

$$\mathcal{M}_1[\Lambda_t]=\ldots=\mathcal{M}_n[\Lambda_t]=1,\tag{10}$$

in a perfect analogy with maximally entangled state corresponding to $s_1=\ldots=s_n$.

Examples.—Let us illustrate the above introduced notions by a few simple examples.

Example 1: Consider pure decoherence of a qubit system described by the following local generator

$$L_t(\rho)=\tfrac{1}{2}\gamma(t)(\sigma_z\rho\sigma_z-\rho),\tag{11}$$

The corresponding evolution of the density matrix reads

$$\rho_t=\begin{pmatrix}\rho_{11} & \rho_{12}e^{-\Gamma(t)}\\ \rho_{12}e^{-\Gamma(t)} & \rho_{22}\end{pmatrix},\tag{12}$$

where $\Gamma(t)=\int_0^t\gamma(\tau)d\tau$. The evolution is completely positive if and only if $\Gamma(t)\geq 0$ and it is k divisible ($k=1,2$) if and only if $\gamma(t)\geq 0$. Taking $X=\sigma_x$ one finds $||\Lambda_t(X)||_1=2e^{-\Gamma(t)}$. Observe that

$$|N_-[\Lambda_t]|=N_+[\Lambda_t]+e^{-\Gamma(\infty)}-1,\tag{13}$$

and hence if $\Gamma(\infty)=0$ the evolution is maximally non-Markovian. Note, that $\Gamma(\infty)=0$ implies that $\rho_t\to\rho$, that is, asymptotically one always recovers an initial state—perfect recoherence. Actually, this example may be immediately generalized as follows: let L be a Lindblad generator and consider a time-dependent generator defined by $L_t=\gamma(t)L$. Now, L_t gives rise to a legitimate quantum dynamical map if and only if $\Gamma(t)\geq 0$ and it is k divisible ($k=1,2,\ldots,n$) if and only if $\gamma(t)\geq 0$. The corresponding dynamics is maximally non-Markovian if $\Gamma(\infty)=0$.

Example 2: Consider the qubit dynamics governed by the time-dependent generator

$$L_t(\rho)=\tfrac{1}{2}\sum_{k=1}^3\gamma_k(t)(\sigma_k\rho\sigma_k-\rho).\tag{14}$$

It is clear that (14) provides a simple generalization of (11) by introducing two additional decoherence channels. The corresponding dynamical map reads

$$\Lambda_t(\rho)=\sum_{\alpha=0}^3p_\alpha(t)\sigma_\alpha\rho\sigma_\alpha,\tag{15}$$

where $\sigma_0=\mathbb{1}$, and the probability distribution $p_\alpha(t)$ may be easily calculated in terms of $\gamma_k(t)$ (see Ref. [33]). Interestingly, in this example there is an essential difference between CP divisibility (= Markovianity) and only P divisibility: CP divisibility is equivalent to

$$\gamma_1(t)\geq 0,\qquad\gamma_2(t)\geq 0,\qquad\gamma_3(t)\geq 0,\tag{16}$$

whereas P divisibility is equivalent to much weaker conditions [33]

$$\begin{aligned}\gamma_1(t)+\gamma_2(t)&\geq 0,\\ \gamma_1(t)+\gamma_3(t)&\geq 0,\\ \gamma_2(t)+\gamma_3(t)&\geq 0,\end{aligned}\tag{17}$$

for all $t\geq 0$. Actually, the BLP condition reproduces (17). Now, violation of at least one inequality from (17) implies essential non-Markovianity. Suppose for example that $\gamma_2(t)+\gamma_3(t)\ngeq 0$. Assuming that $\Gamma_2(\infty)=\Gamma_3(\infty)=0$

one finds that $\mathcal{M}_1[\Lambda_t] = 1$, that is, Λ_t is maximally non-Markovian. Interestingly, if there are at most two decoherence channels, then there is no difference between CP and P divisibility. Note, that random unitary dynamics (15) is unital; i.e., $\Lambda_t(\mathbb{I}) = \mathbb{I}$ and, hence, during the evolution the entropy never decreases $S[\Lambda_t(\rho)] \geq S(\rho)$ for any initial qubit state ρ. One easily shows that P divisibility is equivalent to

$$\frac{d}{dt} S[\Lambda_t(\rho)] \geq 0, \tag{18}$$

for any qubit state ρ. Hence, for any weakly non-Markovian random unitary dynamics the von Neumann entropy monotonically increases. Violation of (18) proves that Λ_t is essentially non-Markovian.

Example 3: Consider a qubit dynamics governed by the following local generator

$$L_t = \gamma_+(t) L_+ + \gamma_-(t) L_-, \tag{19}$$

where $L_+(\rho) = \frac{1}{2}([\sigma_+, \rho\sigma_-] + [\sigma_+\rho, \sigma_-])$ and $L_-(\rho) = \frac{1}{2}([\sigma_-, \rho\sigma_+] + [\sigma_-\rho, \sigma_+])$, with $\sigma_+ = |2\rangle\langle 1|$ and $\sigma_- = |1\rangle\langle 2|$. L_+ generates pumping from the ground state $|1\rangle$ to an excited state $|2\rangle$ and L_- generates a decay from $|2\rangle$ to $|1\rangle$. One shows that L_t generates legitimate dynamical map if and only if

$$0 \leq \int_0^t \gamma_\pm(s) e^{\Gamma(s)} ds \leq e^{\Gamma(t)} - 1, \tag{20}$$

where $\Gamma(t) = \int_0^t [\gamma_-(\tau) + \gamma_+(\tau)] d\tau$. In particular, it follows from (20) that $\Gamma(t) \geq 0$. Now, Λ_t is CP divisible if and only if

$$\gamma_-(t) \geq 0, \qquad \gamma_+(t) \geq 0, \tag{21}$$

and it is P divisible if and only if

$$\gamma_-(t) + \gamma_+(t) \geq 0. \tag{22}$$

Note, that (21) implies (20). However, it is not true for (22): i.e., P divisibility requires both (20)—it guarantees that Λ_t is completely positive—and (22).

Bloch equations and P divisibility.—The above examples illustrating qubit dynamics may be easily rewritten in terms of the Bloch vector $x_k(t) = \mathrm{Tr}[\sigma_k \Lambda_t(\rho)]$. Example 2 gives rise to

$$\frac{d}{dt} x_k(t) = -\frac{1}{T_k(t)} x_k(t), \qquad k = 1, 2, 3, \tag{23}$$

where $T_1(t) = [\gamma_2(t) + \gamma_3(t)]^{-1}$, and similarly for $T_2(t)$ and $T_3(t)$. Quantities $T_k(t)$ correspond to local relaxation times. It is therefore clear that P divisibility is equivalent to $T_k(t) \geq 0$ for $k = 1, 2, 3$. This proves the essential difference between CP divisibility and P divisibility. CP divisibility requires that all local decoherence rates satisfy $\gamma_k(t) \geq 0$, whereas P divisibility requires only $T_k(t) \geq 0$. Hence, one may have temporarily negative decoherence rates but always positive relaxation times. From a physical point of view this shows that the two main non-Markovianity measures used in the literature describe very different ways in which memory effects manifest themselves. Violation of CP divisibility [10] reflects the presence of reverse quantum jumps [6], restoring previously lost coherence and occurring when one of the decay rates becomes negative. The BLP non-Markovianity [13], which in this case corresponds to the violation of P divisibility, instead occurs when at least one of the relaxation times becomes temporarily negative; i.e., instead of relaxation one of the components $x_k(t)$ temporarily grows. This in turn stems from a temporary and partial increase of information on the open system, as measured by trace distance. Note, that CP divisibility is equivalent to P divisibility plus three extra conditions

$$\frac{1}{T_1} + \frac{1}{T_2} \geq \frac{1}{T_3}, \qquad \frac{1}{T_1} + \frac{1}{T_3} \geq \frac{1}{T_2}, \qquad \frac{1}{T_2} + \frac{1}{T_3} \geq \frac{1}{T_1}.$$

Finally, let us observe that the initial volume of the Bloch ball shrinks during the evolution according to

$$V(t) = e^{-[\Gamma_1(t) + \Gamma_2(t) + \Gamma_3(t)]} V(0),$$

where $V(t)$ denotes a volume of the set of accessible states at time t. Authors of [19] characterized non-Markovian evolution as a departure from $(d/dt)V(t) \leq 0$. One has $(d/dt)V(t) = -[\gamma_1(t) + \gamma_2(t) + \gamma_3(t)]V(t)$ and, hence, $(d/dt)V(t) \leq 0$ if and only if

$$\gamma_1(t) + \gamma_2(t) + \gamma_3(t) \geq 0. \tag{24}$$

This condition is much weaker than (17). To violate (24) the evolution has to be essentially non-Markovian (i.e., Λ_t cannot be even P divisible). Actually, the geometric condition of [19] is always weaker than P divisibility (and hence k divisibility). Any k-divisible dynamics necessarily satisfies $(d/dt)V(t) \leq 0$.

A similar conclusion may be drawn from Example 3: the corresponding Bloch equations read

$$\begin{aligned} \frac{d}{dt} x_k(t) &= -\frac{1}{T_\perp(t)} x_k(t), \qquad k = 1, 2, \\ \frac{d}{dt} x_3(t) &= -\frac{1}{T_{||}(t)} x_3(t) + \Delta(t), \end{aligned} \tag{25}$$

where $\Delta(t) = [\gamma_+(t) - \gamma_-(t)]$, and $T_\perp(t) = 2/[\gamma_-(t) + \gamma_+(t)]$ and $T_{||}(t) = T_\perp(t)/2$ are transverse and longitudinal local relaxation times, respectively. Again, P divisibility is equivalent to $T_\perp, T_{||}(t) \geq 0$, provided that the Bloch vector stays within a Bloch ball.

Conclusions.—In this Letter we provided further characterization of non-Markovian evolution in terms of the non-Markovianity degree. This simple concept, being an analog of the Schmidt number in the entanglement theory, enables one to compare quantum evolutions. We say that $\Lambda_t^{(1)}$ is more non-Markovian than $\Lambda_t^{(2)}$ if $\mathrm{NMD}[\Lambda_t^{(1)}] > \mathrm{NMD}[\Lambda_t^{(2)}]$. Similarities and differences between the existing non-Markovianity measures in specific open system models have been discussed in several papers [18,19,23–25,31,34,35]. However, their general connection was still an open problem. Here we have solved this problem in full generality by defining a hierarchy of non-Markovianity measures. They interpolate between the well-known RHP [10] and BLP [13] measures and, hence, they provide refinement of non-Markovianity measures usually used in the literature. This way our approach allows us to classify the non-Markovianity measures introduced so far in a unified framework. Finally, we define a notion of maximally non-Markovian evolution which is an analog of a maximally entangled state. Maximally, non-Markovian evolution may be of crucial importance if non-Markovianity can be shown to be a resource for quantum technologies, as recent results suggest [18]. Finally, if the evolution is only k divisible with $k < n$ one may ask about additional properties of the dynamical map. In particular an interesting issue might be the optimality of the family of "propagators" [36] $V_{t,s}$.

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