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Knowledge of optimal quantum measurements is important for a wide range of situations, including quantum communication and quantum metrology. Quantum measurements are usually optimized with an ideal experimental realization in mind. Real devices and detectors are, however, imperfect. This has to be taken into account when optimizing quantum measurements. In this paper, we derive the optimal minimum-cost and minimum-error measurements for a general model of imperfect detection.

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## I. INTRODUCTION

Quantum measurements may be optimized with respect to a range of criteria. For example, when distinguishing between a set of quantum states $\rho_j$, occurring with prior probabilities $p_j$, we may want to minimize the average error in the result or, somewhat more generally, the average cost [1–3]. Other possibilities are to maximize the information contained in the result or to ask that measurement results should be unambiguous [2,4]. The optimal measurement is often a generalized quantum measurement. These go beyond standard projective quantum measurements and are referred to as probability operator measures (POMs) or positive operator-valued measures (POVMs) [1–3]. How to optimally perform a quantum measurement is relevant for a wide range of applications, from quantum communication, including quantum key distribution (QKD) [5], to parameter estimation and quantum metrology, such as for optimal quantum estimation of the Unruh-Hawking effect [6].

Derivations of optimal quantum measurement strategies have, however, aimed at finding the optimal *ideal* quantum measurement, that is, the best that can be done provided that an actual experimental realization is ideal. An advantage is that the theoretical derivation of the optimal ideal measurement can be separated from working out how to do the experimental realization. The same theoretically optimal measurement may be realized in very different ways, depending on the character of the quantum system to be measured. Generalized quantum measurements have been realized on photon polarization (see, e.g., [7–14]) and very recently on nitrogen-vacancy centers in diamond [15]. They could also be realized on ions or atoms with existing experimental tools [16–18].

Real experiments are of course not ideal. One might ask whether imperfections in device components change the strategy that we should aim to implement. This question has largely been overlooked. It turns out that the measurement we should aim to implement does indeed in general depend on the particular properties of the experimental devices used. An indication that this might be the case is given by the fact that certain types of pre-amplification can reduce the noise in photon-number and quadrature measurements [19] and that the use of cloning can improve estimates of states and observables for qubits when a two-outcome detector suffers from isotropic noise [20]. This is not self-evident, since an amplifier will add noise, which could degrade the performance of a measurement. Also, for unambiguous comparison of quantum states for detectors with less than unit efficiency, it is sometimes advantageous to use an amplifier in front of the detector [21]. In these works, however, specific types of amplification and cloning are evaluated for usefulness, without optimizing more generally over all possible premeasurement transforms.

In this paper, we will derive the optimal minimum-cost measurement for the case when the final detection process is not perfect. It turns out that when distinguishing between two quantum states the optimal measurement strategy we should aim for remains the same as in the ideal case, although the cost changes. For three or more states, the optimal measurement strategy in general changes, and we give an example of this. We finish with a discussion.

## II. GENERALIZED QUANTUM MEASUREMENTS

Generalized quantum measurements (POMs) can be realized in terms of a projective measurement in an extended Hilbert space [1,3]. This is used in both existing and suggested experimental realizations. The Hilbert space of the quantum system $\rho_S$ to be measured can either be extended through a direct sum, e.g., by using extra atomic levels or adding more optical paths, or through a tensor product by coupling $\rho_S$ to an auxiliary quantum system $\rho_A$. Broadly speaking, the number of dimensions in the total extended Hilbert space corresponds to the number of outcomes of the measurement. Less well known is that by realizing the measurement sequentially it is possible to limit the total number of dimensions needed at any one time to $d + 1$, where $d$ is the dimension of $\rho_S$ [22]. We can also realize the measurement by using at most $2d$ dimensions, by coupling $\rho_S$ to an auxiliary qubit, measuring the qubit, and repeating [23]. This realization is the most efficient in the sense that fewer operations are needed on average.

Any ideal experimental realization of a generalized quantum measurement will thus employ a final projective measurement in some basis on some quantum system. The final projective measurement is preceded by a unitary transform in the extended Hilbert space. For example, for a measurement on an atom or ion, we may couple its levels to some additional atomic levels, e.g., using laser pulses or passage through a cavity [16,18], followed by a final measurement of which energy level the atom or ion occupies. If making measurements on a photon, the final measurement might be a detection of which path the photon exits from and with which polarization, in a suitable polarization basis [8–14]. This final measurement is preceded by an optical network which includes wave plates and beam splitters, effecting a unitary transform in the extended space.

Formally, any generalized measurement strategy is described by a set of measurement operators $\Pi_i$, acting in the space of $\rho_S$, the system to be measured. If the system is prepared in a state $\rho_j$, then the probability to obtain outcome $i$ is given by $p(i|j) = \mathrm{Tr}(\Pi_i \rho_j)$. The fact that $\sum_i p(i|j) = 1$ for any $\rho_j$ corresponds to the condition $\sum_i \Pi_i = \mathbf{1}$, and the fact that probabilities for all outcomes are non-negative corresponds to $\Pi_i \geqslant 0$ (and consequently $\Pi_i$ have to be Hermitian). Note that we may have more outcomes than we have dimensions in the space of the system to be measured and that $\Pi_i$ need not be orthonormal. Each measurement operator $\Pi_i$ is obtained by taking the corresponding projective measurement operators in the extended Hilbert space and projecting them onto the subspace of $\rho_S$.

## III. IMPERFECT DETECTION

Sources of error in the experimental realization can now be divided into two categories, errors in the unitary transform preceding the final measurement and errors in the final projective measurement. In this paper, we will consider the latter type of errors. Errors in the unitary transform are of course also likely to occur in any realization, and this will be the subject of future work. It is straightforward to see that also imperfections in the preparation of the state of auxiliary systems, if these are used in the realization of the measurement, will effectively result in errors in the unitary transform preceding the measurement. Suppose, for example, that we are distinguishing between qubit states, which are superpositions of $|0\rangle_S$ and $|1\rangle_S$, and that the optimal measurement can be effected by adding an extra basis state $|2\rangle_S$ and implementing the unitary transform

$$U = |a\rangle_{SS}\langle 0| + |b\rangle_{SS}\langle 1| + |c\rangle_{SS}\langle 2|,$$

where $\{|a\rangle_S, |b\rangle_S, |c\rangle_S\}$ is another orthonormal basis for the space spanned by $\{|0\rangle_S, |1\rangle_S, |2\rangle_S\}$, followed by a projective measurement in the basis $\{|0\rangle_S, |1\rangle_S, |2\rangle_S\}$. We can also implement the same measurement using an ancillary qubit instead of an additional level, by substituting, for example, $|0\rangle_S \to |0\rangle_S \otimes |0\rangle_A, |1\rangle_S \to |1\rangle_S \otimes |0\rangle_A, |2\rangle_S \to |0\rangle_S \otimes |1\rangle_A$. The ancillary qubit should then be prepared in the state $|0\rangle_A$ prior to the unitary transform analogous to $U$ taking place. If it instead is prepared in the state $|1\rangle_A$, then translated back in terms of an implementation using extra levels we are effectively implementing the wrong unitary transform:

$$U' = |c\rangle_{SS}\langle 0| + |3\rangle_{SS}\langle 1| + |a\rangle_{SS}\langle 2| + |b\rangle_{SS}\langle 3|.$$

Here the state $|3\rangle$ is another basis state, corresponding to $|1\rangle_S \otimes |1\rangle_A$ (we can add a term $|3\rangle_{SS}\langle 3|$ to $U$ above without affecting the implementation of the measurement, since the state of the system to be measured initially lies in the space spanned by $\{|0\rangle, |1\rangle\}$). Similarly, it is easy to see that preparing the ancilla in some other pure state, or in a mixed state, will also result in an operation different from the one we intended to implement.

As stated above, we will in this work consider errors in the final projective measurement. In particular, suppose that when a perfect projective measurement would have given result $j$ then our measurement will instead give result $i$ with probability $q(i|j)$, where $\sum_i q(i|j) = 1$; $q(i|j)$ are the elements of a stochastic matrix. This is a generally applicable description of errors in the final measurement. We may be able to vary the unitary transform that precedes the final measurement, but the final measurement and the $q(i|j)$ are often fixed by the nature of the detection process, including the efficiency of the detectors, and what basis states we must project on. (A final measurement in a basis slightly different than intended, for example, can also be viewed as an error in the transform preceding the final measurement.) Any nonideal measurement we are able to implement will then be described by mixed measurement operators

$$\widetilde{\Pi}_i = \sum_j q(i|j)\Pi_j, \sum_i q(i|j) = 1, \tag{1}$$

where $\Pi_j$ are the measurement operators for an unconstrained generalized measurement strategy.

For example, when trying to implement a photon-number measurement, the measurement that we are actually able to implement might be described by the measurement operators

$$\begin{aligned} \Pi_{\text{no click}} &= \sum_{n=0}^{\infty} q(0|n)|n\rangle\langle n|, \\ \Pi_{\text{click}} &= \sum_{n=0}^{\infty} q(1|n)|n\rangle\langle n|, \end{aligned} \tag{2}$$

where $|n\rangle$ is a photon-number state. This measurement thus has only two outcomes, and $q(0|n)$ and $q(1|n)$ are the probabilities to obtain "no click" and a "click," respectively, when there are $n$ photons present. A similar description results for photon-number-resolving detectors and also when a number of photodetectors are used to detect what path or with what polarization a photon exits. Yet another example is the detection of the state of a Rydberg atom by field ionization. By detecting for what field strength the atom is ionized one can infer what energy level it most likely would have been in [18,24]. The distribution functions for when the electron is released may, however, overlap for different energy levels. This also results in a measurement of the type in Eq. (1).

There is a technical point which should be mentioned in order to further justify the generality of our error model. If the measurement operators $\Pi_j$ are proportional to pure state projectors, then the ideal final measurement in the extended space can be chosen as a projection in a complete basis, with each outcome corresponding to one pure basis state $|j\rangle$. Our model for the imperfect final measurement in Eq. (1) then directly applies. The measurement operators $\Pi_j$ may, however, also be mixed. This is of course possible to realize using a final measurement comprising projectors onto more than one orthonormal state, together with coarse graining in deciding what the final result is. However, the index $j$ in $q(i|j)$ best refers to pure state projectors $|j\rangle\langle j|$ in the ideal final measurement, rather than to projectors onto more than one orthonormal state. This is because the "subprojectors" in a mixed $\Pi_j$ will in general have different misidentification probabilities and hence would need different $q(i|j)$.

Nevertheless, also for mixed $\Pi_j$, we may always arrange for the corresponding final measurement operator to be a pure state projector in an extended space. This means that the description of the imperfect final measurement in Eq. (1) again applies. For example, suppose that the ideal final measurement in the extended Hilbert space is described by the two projectors $|1\rangle_{SS}\langle 1| + |2\rangle_{SS}\langle 2|$ and $|3\rangle_{SS}\langle 3|$. We may then further couple the system to a qubit, initially in the state $|0\rangle_q$, using, e.g., the unitary operation

$$\begin{aligned} U = &|1\rangle_{SS}\langle 1| \otimes |0\rangle_{qq}\langle 0| + |2\rangle_{SS}\langle 2| \otimes |0\rangle_{qq}\langle 0| \\ &+ |1\rangle_{SS}\langle 1| \otimes |1\rangle_{qq}\langle 1| + |2\rangle_{SS}\langle 2| \otimes |1\rangle_{qq}\langle 1| \\ &+ |3\rangle_{SS}\langle 3| \otimes |1\rangle_{qq}\langle 0| + |3\rangle_{SS}\langle 3| \otimes |0\rangle_{qq}\langle 1|. \end{aligned} \tag{3}$$

The ideal final measurement may then be realized as a projective measurement on the qubit, with $|0\rangle_{qq}\langle 0|$ corresponding to $|1\rangle_{SS}\langle 1| + |2\rangle_{SS}\langle 2|$ and $|1\rangle_{qq}\langle 1|$ corresponding to $|3\rangle_{SS}\langle 3|$. This procedure is easily generalized so that any mixed measurement operators $\Pi_j$ will correspond to a pure state projector.

Related to this, one realizes that it may well be advantageous to arrange to use as few final measurement states as possible, since this reduces the complexity of the final detection process, and we may be able to select the few measurement basis states with the most favorable $q(i|j)$. Any extra unitary transforms prior to the final measurement, including ones analogous to the one in Eq. (3), are of course also likely to introduce experimental errors. In this initial work, however, we want to optimize with respect to imperfections in the final projective measurement, in order to see what the optimal unitary transform preceding it would be.

## IV. MINIMUM-COST MEASUREMENTS FOR IMPERFECT DETECTION

We will now derive the optimal minimum-cost strategy when the final measurement we can realize is restricted, so that the measurement operators are given by Eq. (1). The measurement we are actually realizing is described by the measurement operators $\widetilde{\Pi}_i$, whereas the measurement we are aiming to realize is described by the measurement operators $\Pi_j$.

To briefly review results related to optimal minimum-cost measurements [1], suppose that quantum state $\rho_j$ occurs with prior probability $p_j$. We choose a measurement with measurement operators $\Pi_i$, and obtaining result $i$ when the state prepared was actually $\rho_j$ carries a cost of $C_{ij}$. The average cost will then be given by

$$\bar{C} = \mathrm{Tr}\sum_{ij} C_{ij}p_j\Pi_i\rho_j = \mathrm{Tr}\sum_i W_i\Pi_i = \mathrm{Tr}\Gamma, \tag{4}$$

where

$$W_i = \sum_j C_{ij}p_j\rho_j \text{ and } \Gamma = \sum_i W_i\Pi_i = \sum_i \Pi_i W_i \tag{5}$$

are called the risk operator corresponding to result $i$ and the Lagrange operator, respectively. $\Gamma$ takes care of the constraint $\sum_i \Pi_i = \mathbf{1}$ and is Hermitian. For a minimum-error measurement we may choose $C_{ij} = -\delta_{ij}$ and $W_i = -p_i\rho_i$. The minimum-cost measurement operators satisfy the conditions

$$(W_i - \Gamma)\Pi_i = \Pi_i(W_i - \Gamma) = 0\forall i, \tag{6}$$

$$W_i - \Gamma \geqslant 0\forall i. \tag{7}$$

Consider now a mixed measurement strategy with measurement operators given by Eq. (1). The average cost for this measurement strategy will be

$$\begin{aligned} \widetilde{C} &= \mathrm{Tr}\sum_i W_i\widetilde{\Pi}_i = \mathrm{Tr}\sum_{ijk} C_{ik}p_k\rho_k q(i|j)\Pi_j \\ &= \mathrm{Tr}\sum_{jk}\widetilde{C}_{jk}p_k\rho_k\Pi_j = \mathrm{Tr}\sum_j \widetilde{W}_j\Pi_j, \end{aligned} \tag{8}$$

where

$$\widetilde{C}_{jk} = \sum_i C_{ik}q(i|j) \quad \text{and} \quad \widetilde{W}_j = \sum_k \widetilde{C}_{jk}p_k\rho_k. \tag{9}$$

It immediately follows that the ideal strategy we should aim to perform, if the final measurement has the misidentification probabilities $q(i|j)$, is optimal for the modified costs $\widetilde{C}_{jk}$ and modified risk operators $\widetilde{W}_j$. It is also clear that if the states $\rho_j$ are orthogonal, corresponding to "perfectly distinguishable" classical states, then the measurement in the presence of misidentification probabilities does not change; it remains a projective measurement on (the subspaces of) the different $\rho_j$. The fact that the measurement strategy changes is in this sense a *quantum* feature.

We can freely choose which final measurement basis states are assigned to which initial state $\rho_i$, and should choose so that the obtained cost is optimal. This can be done by checking what the optimal measurement is for each possible assignment and picking the best one; there are $m!$ assignments if there are $m$ basis states. Roughly speaking, the most probable initial states should be associated with those basis states which we can identify most accurately. Related to this, if the final detection process is defective enough, then for some assignments it may happen that when a final outcome $i$ is obtained this is more likely to have occurred as a result of another initial state $\rho_j$ than the initial state $\rho_i$ itself. As a particularly simple example, consider distinguishing with minimum error between the orthogonal states $|0\rangle$ and $|1\rangle$, occurring with probabilities $p_0$ and $p_1$. We make an imperfect projection in the $\{|0\rangle,|1\rangle\}$ basis, with $\widetilde{\Pi}_0 = q(0|0)|0\rangle\langle 0| + q(0|1)|1\rangle\langle 1|, \widetilde{\Pi}_1 = q(1|0)|0\rangle\langle 0| + q(1|1)|1\rangle\langle 1|$. If, for example, $q(0|1)p_1 > q(0|0)p_0$, then the final result "0" is more likely to have occurred because the state was $|1\rangle$ rather than $|0\rangle$, and we should guess "$|1\rangle$" even if we obtain result "0." We should therefore also check what the optimal cost is for different reassignments of final outcomes to other states.

Nevertheless, checking all possible different assignments of outcomes, in order to obtain the overall optimal detection strategy for imperfect detection, is straightforward if there is a finite number of outcomes. We will proceed to look at examples.

### A. Distinguishing between two nonorthogonal states

Suppose that we want to distinguish between $\rho_0$ and $\rho_1$, occurring with prior probabilities $p_0$ and $p_1$, with minimum cost, with misidentification probabilities $q(0|0),q(0|1),q(1|0),q(1|1)$. The cost of obtaining result $i$ when the prepared state was $j$ is $C_{ij}$, for $i,j = 0,1$. The ideal measurement strategy we should try to implement is optimal for the modified risk operators

$$\widetilde{W}_0 = \widetilde{C}_{00} p_0 \rho_0 + \widetilde{C}_{01} p_1 \rho_1,$$
$$\widetilde{W}_1 = \widetilde{C}_{10} p_0 \rho_0 + \widetilde{C}_{11} p_1 \rho_1. \tag{10}$$

The optimal measurement for distinguishing between two nonorthogonal states with minimum cost was given by Helstrom [1]. It is a projection in the eigenbasis of the operator $\widetilde{O} = \widetilde{W}_0 - \widetilde{W}_1$. Using the first equation in Eq. (9) and the second equation in Eq. (1), we find that

$$\widetilde{O} = [q(0|0) + q(1|1) - 1](W_0 - W_1). \tag{11}$$

This means that unless $q(0|0) + q(1|1) = 1$, $\widetilde{O}$ is proportional to $W_0 - W_1$, and therefore the measurement strategy we should aim to implement does not change when detection is imperfect, no matter how asymmetric the misidentification probabilities are. If $q(0|0) + q(1|1) > 1$, then the optimal $\Pi_0$ is a projector onto the eigenstates of $\widetilde{O}$ with negative eigenvalues, and $\Pi_1$ is a projector onto the eigenstates with positive eigenvalues. If there are any zero eigenvalues, then the corresponding eigenstates may be assigned to either result without changing the average cost. If the results are sufficiently "scrambled," more precisely, when $q(0|0) + q(1|1) < 1$, we should still perform the same measurement, but with $\Pi_0$ and $\Pi_1$ swapped. If $q(0|0) + q(1|1) = 1$, then this implies $q(1|1) = q(1|0)$ and $q(0|0) = q(0|1)$, i.e., that the measurement results are completely random. There is then no point in making a measurement at all. We should just pick the $\rho_i$ which gives the least average cost based on the prior probabilities and costs $C_{ij}$.

Even though the measurement strategy we should aim for does not change, the minimum cost, given in Eq. (8), does of course increase as compared with the cost for the ideal measurement. These results also hold for the special case of distinguishing between $\rho_0$ and $\rho_1$ with minimum error. The strategy we should try to implement stays the same, but the error probability increases. That this holds even when the misidentification probabilities $q(i|j)$ are not symmetric is not entirely intuitive.

### B. Distinguishing between three symmetric states

We will now see that when distinguishing between three pure quantum states the measurement strategy we should aim for may change when the detection is imperfect. Consider the three equiprobable states

$$|\psi_1\rangle = -|0\rangle, |\psi_2\rangle = \frac{1}{2}(|0\rangle + \sqrt{3})|1\rangle, |\psi_3\rangle = \frac{1}{2}(|0\rangle - \sqrt{3})|1\rangle. \tag{12}$$

The ideal measurement that distinguishes between these states with minimum error has the measurement operators $\Pi_i = 2/3|\psi_i\rangle\langle\psi_i| = 2/3\rho_i$ for $i = 1,2,3$ [1,2,9].

Suppose now that in the final detection outcome 1 is sometimes misidentified as outcome 2 or 3 with equal probability $q$, but that otherwise the detection is perfect. That is, we have $q(1|1) = 1 - 2q$ and $q(2|1) = q(3|1) = q$, with $0 \leqslant q \leqslant 1/2$. Also, $q(2|2) = q(3|3) = 1$ and $q(1|2) = q(3|2) = q(1|3) = q(2|3) = 0$. We find $\widetilde{C}_{11} = 2q - 1, \widetilde{C}_{12} = \widetilde{C}_{13} = -q, \widetilde{C}_{22} = \widetilde{C}_{33} = -1$, and all other $\widetilde{C}_{ij} = 0$. Furthermore,

$$\widetilde{W}_1 = \frac{1}{3}[(2q-1)\rho_1 - q(\rho_2 + \rho_3)],$$
$$\widetilde{W}_2 = -\frac{1}{3}\rho_2, \quad \widetilde{W}_3 = -\frac{1}{3}\rho_3. \tag{13}$$

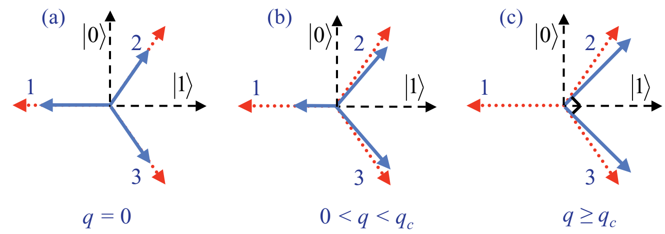


FIG. 1. (Color online) The optimal measurement operators we should aim for when distinguishing between the three symmetric quantum states in Eq. (12), when the final detection is imperfect so that outcome 1 is misidentified as outcome 2 or 3 with probability $q$. The states to be distinguished are shown using dotted arrows (red), and the measurement operators are (un-normalized) projectors onto the states indicated by solid arrows (blue). The basis states $|0\rangle$ and $|1\rangle$ are indicated with dashed arrows (black). In (a), the detection is perfect, and the measurement operators are proportional to projectors onto the states themselves. In (b), $0 < q < q_c$, and, in (c), $q \geqslant q_c$. It pays to concentrate more on the states we can identify more reliably.

The optimal measurements for such mirror-symmetric situations are known [25,26]. For small $q$, the optimal strategy has three measurement operators. When $q$ increases, it pays less and less try to identify $|\psi_1\rangle$, and the trace $a$ of $\Pi_1 = a\rho_1$ decreases, starting from 2/3. At the same time, $\mathrm{Tr}\Pi_2 = \mathrm{Tr}\Pi_3$ increases. $\Pi_{2,3}$ are un-normalized projectors onto pure states that become closer and closer to $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$. When $q \geqslant q_c = (1 + 1/\sqrt{3})/2 \approx 0.211$, $a = 0$ and the optimal measurement has only two nonzero measurement operators. Then $\Pi_1 = 0$, and $\Pi_{2,3}$ are projectors onto the states $|\pm\rangle$. This remains optimal for all $q_c \leqslant q \leqslant 1/2$. Thus, for any nonzero value of $q$, the measurement we should aim for is different from the optimal minimum-error measurement for $q = 0$. This is illustrated in Fig. 1.

The case we have considered, distinguishing between three or more symmetric states, is relevant for quantum key distribution (QKD) (see, e.g., [27] and references therein). As argued above, our detection error model directly applies to photodetection. In a realization of similar QKD protocols, it is therefore likely that optimal operation would require similar modifications of the measurements performed.

## V. CONCLUSIONS

Knowledge of optimal quantum measurements for realistic experimental components is important in order to be able to select the best possible measurements for a given situation. In this paper, we derived the optimal minimum-cost measurement one should aim to implement for a general model of imperfect detection. When a perfect measurement would have given result $j$, then the detection gives result $i$ with probability $q(i|j)$. This leads to a modification of the costs of different outcomes and hence the measurement strategy we should aim to implement in general changes. In the special case of distinguishing between two quantum states, the measurement stays the same, and only the cost changes. For three states we gave a simple example, relevant, e.g., for quantum key distribution, where the optimal measurement strategy changes if the detection is imperfect.

It should be noted that for any kind of state discrimination, not just minimum-error discrimination, the case where several copies of the states to be distinguished are available is naturally accounted for. If we, for example, are given $N$ copies of a state $\rho_j$, then the optimal measurement can be found (both in the ideal case and when the final detection is imperfect) by distinguishing instead between the states $\{\rho_j^{\otimes N}\}$. The optimal measurement strategy then is not restricted to separate measurements on the individual copies.

One should also note that, somewhat related to the results in [20], we may use cloning in the final measurement basis in order to "purify" an imperfect measurement. That is, if the final ideal measurement would be a projective measurement in the basis $\{|j\rangle\}$, then we may use cloning in this basis, $C(\rho) = \sum_j \langle j|\rho|j\rangle |j\rangle\langle j|^{\otimes M}$, to improve the effective misidentification probabilities. Following the cloning transform, we use the imperfect measurement $M$ times, once on each cloned copy, and then pick the most advantageous final outcome based on the relative frequencies of different results. In particular, in the limit of $M \to \infty$, we can effectively implement a perfect projection in the basis $\{|j\rangle\}$, since then we can tell from the frequencies of the different results $i$, which are determined by $q(i|j)$, what the correct result $j$ would have been. Nevertheless, such a cloning procedure is likely to introduce further errors in the measurement due to its complexity, in case it is at all possible to implement. Therefore it may well be advantageous, when errors in the transform prior to the final measurement are also taken into account, to keep the measurement procedure as simple as possible. What is provided in this paper is a way to optimize a minimum-error measurement once we have settled on a particular final imperfect detection process, characterized by some fixed misidentification probabilities $q(i|j)$.

It would be interesting to investigate how imperfect detection affects other types of measurements, such as unambiguous or error-free measurements [2]. If the detection is imperfect, then it may not be possible to distinguish some states unambiguously anymore. We should then instead consider a maximum confidence measurement [28]. Also, errors in an experimental realization will not only come from imperfections in the final detection but necessarily and importantly also from errors in operations on the system to be measured prior to the final detection. Finding optimal measurements for such imperfections will be the subject of further work.

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