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Mellin transform approach for the solution of coupled systems of fractional differential equations

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Abstract

In this paper, the solution of a multi-order, multi-degree-of-freedom fractional differential equation is addressed by using the Mellin integral transform. By taking advantage of a technique that relates the transformed function, in points of the complex plane differing in the value of their real part, the solution is found in the Mellin domain by solving a linear set of algebraic equations. The approximate solution of the differential (or integral) equation is restored, in the time domain, by using the inverse Mellin transform in its discretized form.

Keywords: fractional differential equations, Mellin transform, multi degree of freedom systems

1. Introduction

2 In the last few decades, the interest of the scientific community towards
3 the fractional calculus experienced an exceptional boost, so that its applica-
4 tions can now be found in a great variety of natural sciences. The powerful
5 of the fractional operators relies in their long memory self-structure, that

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6 makes them suitable to describe the time evolution of many physical phe-
7 nomena and, in general, to model the dynamics of complex systems. It has
8 been shown in fact, that fractional differential equations naturally arise once
9 power-type non-local interacting systems, or non-Markovian processes with
10 power-law memory, are considered [1–3]. Relevant examples can be found
11 in electrical circuits [4], in anomalous transport and diffusion processes in
12 complex media [5–8], in material sciences [9–11], in biology [12–14] and
13 biomechanics [15–17], and in many other branches of physics and engineer-
14 ing [18–20]. At the same time, the problem of the solution of these new type
15 of equations came up, and the necessity of a powerful and versatile technique
16 useful to this aim has been object of research. As a result of this effort, var-
17 ious methods are nowadays available in literature. Grünwald-Letnikov [21]
18 or other numerical algorithms [18, 22–25] and Adomian methods [26, 27] are
19 examples, but several others exists [12, 18, 28–31]. Integral transforms as the
20 Laplace and Mellin ones, have been exploited in solving particular classes of
21 fractional differential equations [18]. In [32], the authors presented a gen-
22 eral method of solution for single-degree-of-freedom (SDOF) Initial Value
23 Problems (IVP) of fractional order, that makes use of the Mellin transform.
24 The method there proposed, takes advantage of the fact that the result of
25 the inverse Mellin transform is clearly independent on the line of the com-
26 plex plane along which the integral is carried out, provided it belongs to
27 the so called fundamental strip of the transformation. Since that, exploiting
28 the property of the discretized Mellin transform according to which, in the
29 logarithm temporal scale, it can be seen as a Fourier series, a method that
30 relates the values of the transformed function in different points of the com-
31 plex plane is derived. This fundamental result allows us to find the solution
32 of the fractional integro-differential equation at hand, in the Mellin domain,
33 by solving a linear set of algebraic equations and, in the time domain, by
34 evaluating the discretized inverse Mellin transform along a proper line of the
35 fundamental strip.

36 In this paper we show that such a method, developed for a SDOF system, can
37 be straightforwardly generalized for the solution of a general linear multi-
38 order, multi-degree-of-freedom (MDOF) fractional (or integral) differential

39 equation. Again, the method is versatile and easy to implement in computer
40 routines.

41 The paper is organised as follows: in the next section, the basic con-
42 cepts of the Mellin transform and the application of the method to a SDOF
43 system is resumed, along with an illustrative application. In section 3, the
44 generalisation to a multi-order, MDOF system is presented; in section 4, the
45 application of the method to a relevant example of a structural system is
46 illustrated.

47 2. Mellin transform and SDOF systems

48 In this section, we recall the definition of the Mellin integral transform
49 and we outline the ideas underlying the method developed in [32] for a
50 SDOF. We illustrate it by giving an exemplifying application for the so-
51 lution of the fractional Kelvin-Voigt equation, that models the rheological
52 properties of a viscoelastic material, in which the classical dashpot is sub-
53 stituted by a spring-pot, characterized by a constitutive law of fractional
54 order.

55 The Mellin transform of complex order $\gamma = \rho + i\eta$, of a function $x(t)$,
56 defined in the time domain $t \geq 0$, is given as:

$$X(\gamma) = \mathcal{M}\{x(t); \gamma\} \equiv \int_0^{\infty} t^{\gamma-1} x(t) dt \quad (1)$$

57 along with its inverse transform

$$x(t) = \mathcal{M}^{-1}\{X(\gamma); t\} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\gamma) t^{-\gamma} d\eta; \quad (t > 0) \quad (2)$$

58 Eqs.(1) and (2) exist provided ρ belongs to the strip of the complex plane
59 $-p < \rho < -q$, known as fundamental strip of the Mellin transform, in which
60 the transformed function is holomorphic. The limits of the fundamental
61 strips are related to the asymptotic behaviour of the function at hand, for
62 $t \rightarrow 0$ and $t \rightarrow \infty$. In particular:

$$x(t) \sim t^p \quad (t \rightarrow 0); \quad x(t) \sim t^q \quad (t \rightarrow \infty) \quad (3)$$

63 The method is formulated by taking advantage of a discretized version of
 64 the inverse transform (2), that we introduce as:

$$x(t) \simeq \frac{\Delta\eta}{2\pi} \sum_{k=-m}^m X(\gamma_k) t^{-\gamma_k} \quad (4)$$

65 where $\gamma_k = \rho + i k \Delta\eta$ and $\bar{\eta} = m \Delta\eta$ is a properly selected cutoff value for
 66 the integral along the imaginary axes. Taking advantage of the property
 67 $X(\gamma_k) = X^*(\gamma_{-k})$, we can rewrite eq.(4) as

$$x(t) \simeq t^{-\rho} \left\{ \frac{A_0}{2b} + \frac{1}{b} \sum_{k=1}^m \left[A_k \cos\left(\frac{k\pi}{b} \ln t\right) + B_k \sin\left(\frac{k\pi}{b} \ln t\right) \right] \right\} \quad (5)$$

68 with $b = \pi/\Delta\eta$ and having indicated $A_k = \text{Re}[X(\gamma_k)]$ and $B_k = \text{Im}[X(\gamma_k)]$.
 69 From eq.(5) it is evident that the inverse Mellin transform may be seen, in
 70 the logarithm scale, as a Fourier series. Since the same function $x(t)$ is
 71 restored whichever is the value of ρ belonging to the fundamental strip,
 72 according to eq.(4), we can pose

$$\frac{\Delta\eta}{2\pi} \sum_{s=-m}^m X(\gamma_{1s}) t^{-\gamma_{1s}} \simeq \frac{\Delta\eta}{2\pi} \sum_{k=-m}^m X(\gamma_{2k}) t^{-\gamma_{2k}} \quad (6)$$

73 provided $\rho_1 = \text{Re}[\gamma_1]$ and $\rho_2 = \text{Re}[\gamma_2]$, belong to the fundamental strip.
 74 Eq.(6) allows us to relate the Mellin transform $X(\gamma)$, in points of the com-
 75 plex plane differing in their real part. To show that, let us suppose the values
 76 $X(\gamma_{2k})$ known, and that we want to evaluate the $X(\gamma_{1s})$ ($k, s = -m, \dots, m$),
 77 with $\rho_1 < \rho_2$. Indicating $\delta = \rho_2 - \rho_1$, and multiplying both sides of eq.(6)
 78 for the factor $t^{-1/2}$, we get

$$t^{-\frac{1}{2}} \sum_{s=-m}^m X(\gamma_{1s}) \exp\left(-i \frac{s\pi}{b} \ln t\right) \simeq t^{-(\delta+\frac{1}{2})} \sum_{k=-m}^m X(\gamma_{2k}) \exp\left(-i \frac{k\pi}{b} \ln t\right) \quad (7)$$

We then minimize, respect to $X^*(\gamma_{1k})$, the squared modulus of the difference
 between the two sides of eq.(7), integrated over a proper time interval, that

is

$$\int_{t_1}^{t_2} \frac{1}{t} \left[\sum_{s=-m}^m X(\gamma_{1s}) \exp\left(-i \frac{s\pi}{b} \ln t\right) - t^{-\delta} \sum_{k=-m}^m X(\gamma_{2k}) \exp\left(-i \frac{k\pi}{b} \ln t\right) \right] \times \left[\text{c.c.} \right] dt = \min_{X^*(\gamma_{1s})} \quad (8)$$

79 where [c.c.] stands for complex conjugate. In order overcome the singularity
 80 in $t = 0$, we choose the lower limit of the integral $t_1 = e^{-b}$ and we consider
 81 $t_2 = e^b$ as upper limit. In this way, the overlapping of the response, evaluated
 82 along the two different lines $\rho = \rho_1$ and $\rho = \rho_2$ of the fundamental strip, is
 83 guaranteed in a wide time interval. Making now the change of variable

$$\ln t = \xi; \quad \frac{dt}{t} = d\xi; \quad \ln t_1 = -b; \quad \ln t_2 = b \quad (9)$$

84 performing variations of eq.(8) respect to $X^*(\gamma_{1s})$, and taking advantage of
 85 the orthogonality of the exponentials $\exp\left(-i \frac{s\pi}{b} \xi\right)$ on the interval $\xi = [-b, b]$,
 86 we get the following relation between $X(\gamma_{1s})$ and $X(\gamma_{2k})$:

$$X(\gamma_{1s}) = \frac{1}{2b} \sum_{k=-m}^m X(\gamma_{2k}) a_{sk}(\delta) \quad (10)$$

87 where

$$a_{sk}(\delta) = \int_{-b}^b \exp\left(-\left(\delta - i\pi \frac{s-k}{b}\right) \xi\right) d\xi = 2b \frac{\sin((s-k)\pi + ib\delta)}{(s-k)\pi + ib\delta} \quad (11)$$

88 Thanks to eq.(11), we are able to solve, in the Mellin domain, whichever
 89 multi-order linear fractional integro-differential equation, by simply solving
 90 a linear set of algebraic equations.

91 Before addressing the more general case of a MDOF problem, which is
 92 the object of the following sections, we briefly outline here the application of
 93 the method for the solution of a relevant SDOF physical problem, that is the
 94 response, in term of strain, of a sample of viscoelastic material enforced by
 95 a given stress history. We model the material through a fractional Kelvin-

96 Voigt element, whose constitutive law is given by

$$f(t) = c_0 x(t) + c_\alpha ({}^C \mathbf{D}_{0+}^\alpha x)(t) \quad (12)$$

where ${}^C \mathbf{D}_{0+}^\alpha$ is the Caputo's fractional differential operator of order $0 < \alpha < 1$ and $f(t)$ is the stress history applied to the sample. Assuming the system quiescent for $t < 0$, the Cauchy problem that is to be solved, reads as

$$\begin{cases} f(t) = c_0 x(t) + c_\alpha ({}^C \mathbf{D}_{0+}^\alpha x)(t) & (13a) \\ x(0) = 0 & (13b) \end{cases}$$

97 By Mellin transforming eq.(13a), we get the corresponding equation in the
98 complex plane, that reads as:

$$c_\alpha \sum_{k=0}^{n-1} \frac{\Gamma(\alpha - \gamma)}{\Gamma(1 - \gamma)} [x(t) t^{\gamma - \alpha}]_0^\infty + c_\alpha \frac{\Gamma(1 - \gamma + \alpha)}{\Gamma(1 - \gamma)} X(\gamma - \alpha) + c_0 X(\gamma) = F(\gamma) \quad (14)$$

99 being $F(\gamma) = \mathcal{M}\{f(t); \gamma\}$ the Mellin transform of the forcing action. In
100 the hypothesis of stable system $c_0, c_\alpha > 0$, and assuming: i) $f(t) \equiv 0$ from
101 a time instant $t = \bar{t}$, condition that doesn't represent a limitation since,
102 because of the causal properties of the system at hand, the response at $t = \bar{t}$
103 is only determined by the past stress history, ii) $\rho < \alpha$, then eq.(14) reduces
104 to

$$c_\alpha C(\gamma, \alpha) X(\gamma - \alpha) + c_0 X(\gamma) = F(\gamma) \quad (15)$$

105 having defined $C(\gamma, \alpha) = \frac{\Gamma(1 - \gamma + \alpha)}{\Gamma(1 - \gamma)}$. It is clear that the solution can't
106 be sought directly from eq.(15) because values of the transformed function
107 in different points of the complex plane are involved. However, by taking
108 advantage of eq.(10), we are in the position to rewrite eq.(15), in terms of
109 only the values that the transformed function assumes along the same line
110 of the fundamental strip. Evaluating the resulting equation in the points
111 $\gamma_k = \rho + i k \Delta \eta$ ($k = -m, \dots, m$), we obtain a system of $2m + 1$ equations in
112 the $2m + 1$ unknown values $X(\gamma_k)$:

$$\mathbf{MX} = \mathbf{F} \quad (16)$$

113 in which we have defined

$$M_{kj} = \frac{1}{2b} (c_\alpha C_k^\alpha a_{k-m-1, j-m-1}(\alpha) + c_0 \delta_{kj}) \quad (17a)$$

$$114 \quad X_k = X(\rho + i(k-m-1)\Delta\eta) \quad (17b)$$

$$115 \quad F_k = F(\rho + i(k-m-1)\Delta\eta) \quad (17c)$$

116 and $k, j = 1 \div 2m+1$, $C_k^\alpha = C(\rho + i(k-m-1)\Delta\eta, \alpha)$. Once solved eq.(16),
 117 the approximate solution, in the time domain, can be restored by using the
 118 discretized inverse Mellin integral (4). Such integral is meaningful provided
 119 ρ belongs to the fundamental strip of the transformed function $X(\gamma)$, that
 120 a priori is not known. However, because of the initial condition (13b), the
 121 lower limit is at least equal to -1 while, by comparison with other similar
 122 situations, we can guess the upper limit to be infinite [32].

123 3. MDOF systems

124 In this section we show that the method outlined above for the solution
 125 of a multi-order SDOF fractional differential equation, can be straightfor-
 126 wardly generalized to the more general case of a MDOF system. An attempt
 127 to address the problem can be found in [33] where, defining in a proper way
 128 the state vector of the system at hand, the problem can be reduced, in the
 129 modal space, to a mutually independent set of fractional differential equa-
 130 tions. Aim of this section is to show that this step can be overcome, and
 131 that the problem can be directly addressed in the relative physical space.
 132 We first illustrate the method by dealing with the most general linear multi-
 133 order, MDOF fractional differential equation with constant coefficients. Then,
 134 in the next section, we give a relevant application for the solution of the dy-
 135 namics of a structural system, enforced by a time dependent action, in which
 136 the presence of viscoelastic dampers cause the appearance of differential op-
 137 erators of not integer order in the equation of motion.

Let us assume that we are involved in solving the following M-degree of

freedom Cauchy problem of order $n - 1 < \alpha_N < n$, with $n \in \mathbb{N}$:

$$\begin{cases} \mathbf{A}_1 ({}^C\mathbf{D}_{0^+}^{\alpha_1} \mathbf{x})(t) + \mathbf{A}_2 ({}^C\mathbf{D}_{0^+}^{\alpha_2} \mathbf{x})(t) + \dots + \mathbf{A}_N ({}^C\mathbf{D}_{0^+}^{\alpha_N} \mathbf{x})(t) = \mathbf{f}(t) & (18a) \\ \mathbf{x}(0) = \mathbf{0}; \quad \dots \quad \mathbf{x}^{(n-1)}(0) = \mathbf{0} & (18b) \end{cases}$$

138 with $\alpha_1 < \alpha_2 < \dots < \alpha_N$, $\mathbf{x}(t)^T = [x_1(t), x_2(t), \dots, x_M(t)]$ and $\mathbf{f}(t)^T =$
 139 $[f_1(t), f_2(t), \dots, f_M(t)]$. In the hypothesis of stable systems (\mathbf{A}_j positive defi-
 140 nite for $j = 1, \dots, N$), assuming (because of causality and so without affecting
 141 generality) the forcing action $\mathbf{f}(t)$ different from zero in the finite time inter-
 142 val $[0, \bar{t}]$ and zero otherwise, and posing $\rho < \{\alpha\}$ (being $\{\alpha\}$ the not integer
 143 part of α), the Mellin transform of eq.(18), reads as:

$$C(\gamma, \alpha_1) \mathbf{A}_1 \mathbf{X}(\gamma - \alpha_1) + \dots + C(\gamma, \alpha_N) \mathbf{A}_N \mathbf{X}(\gamma - \alpha_N) = \mathbf{F}(\gamma) \quad (19)$$

144 with $\mathbf{X}(\gamma)^T = [X_1(\gamma), X_2(\gamma), \dots, X_M(\gamma)]$, $\mathbf{F}(\gamma)^T = [F_1(\gamma), F_2(\gamma), \dots, F_M(\gamma)]$,
 145 and $C(\gamma, \alpha_j)$ as defined in the previous section. In a more compact form we
 146 can also write:

$$\sum_{j=1}^N C(\gamma, \alpha_j) \mathbf{A}_j \mathbf{X}(\gamma - \alpha_j) = \mathbf{F}(\gamma) \quad (20)$$

147 As for the SDOF case, the solution of the problem cannot be pursued di-
 148 rectly from eq.(20), because the values $\mathbf{X}(\gamma - \alpha_j)$ ($j = 1, \dots, N$) are involved
 149 in the same equation. However, we know from eq.(10), that the values of the
 150 transformed function, in points belonging to different line of the fundamental
 151 strip, are related as

$$\mathbf{X}(\gamma_s - \alpha_j) = \frac{1}{2b} \sum_{k=-m}^m \mathbf{X}(\gamma_k) a_{sk}(\alpha_j) \quad (21)$$

152 with $s = -m \div m$ and remembering that m is related to the cut-off value $\bar{\eta}$
 153 of the inverse Mellin integral as $m\Delta\eta = \bar{\eta}$. Evaluating eq.(20) in the point
 154 γ_s , and exploiting eq.(21), we end up with the following equation:

$$\sum_{k=-m}^m \mathbf{M}(s, k) \mathbf{X}(\gamma_k) = \mathbf{F}(\gamma_s) \quad (22)$$

155 where we defined

$$\mathbf{M}(s, k) = \frac{1}{2b} \sum_{j=1}^N C(\gamma_s, \alpha_j) a_{sk}(\alpha_j) \mathbf{A}_j \quad (23)$$

156 By evaluating eq.(22) for $s = -m, \dots, m$, we obtain a linear set of $2m + 1$
 157 algebraic equations in the $2m + 1$ vectorial unknown $\mathbf{X}(\gamma_s)$. By defining
 158 the $(2m + 1) \cdot M$ dimensional super-vectors $\boldsymbol{\chi}^T = [\mathbf{X}_{-m}^T, \dots, \mathbf{X}_m^T]$, $\boldsymbol{\phi}^T =$
 159 $[\mathbf{F}_{-m}^T, \dots, \mathbf{F}_m^T]$ and the $(2m + 1) \cdot M \times (2m + 1) \cdot M$ block matrix $\boldsymbol{\mu}$, composing
 160 the $\mathbf{M}(s, k)$ sub-matrices as follows:

$$\boldsymbol{\mu} = \begin{pmatrix} \mathbf{M}(-m, -m) & \mathbf{M}(-m, -m + 1) & \cdots & \mathbf{M}(-m, m) \\ \mathbf{M}(-m + 1, -m) & & & \vdots \\ \vdots & & & \vdots \\ \mathbf{M}(m, -m) & \cdots & \cdots & \mathbf{M}(m, m) \end{pmatrix}$$

161 we can rewrite eq.(22) in the more compact form:

$$\boldsymbol{\mu} \boldsymbol{\chi} = \boldsymbol{\phi} \quad (24)$$

162 Once found the approximate solution of the problem in the Mellin domain
 163 by solving eq.(24), we can restore the sought solution in the time domain,
 164 by evaluating the inverse transform (4).

165 4. Applications

166 In order to show the versatility and the powerful of the method, we
 167 present here its application for the solution of the dynamics of a structural
 168 system, equipped with viscoelastic dampers, subject to the action of a time-
 169 dependent load. In order to validate the method, we choose the structural
 170 parameters in such a way that the problem is diagonal in its modal space,
 171 and so easily solvable by applying the method to the resulting uncoupled
 172 SDOF fractional equations, as in the previous section.

173 The equation of motion of a M -degree-of-freedom structure, equipped

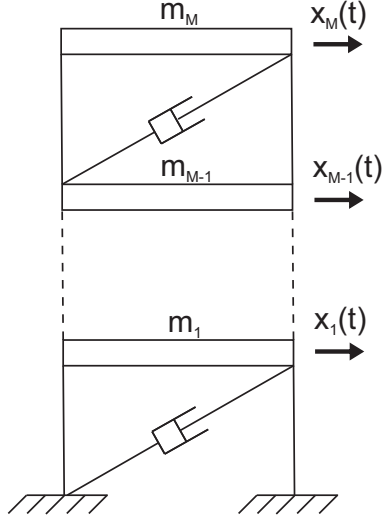


Figure 1: Schematic model of a multi-degree-of-freedom structural system, passively protected with viscoelastic dampers.

174 with viscoelastic dampers (see fig.1), reads as:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{C}_\alpha ({}^C\mathbf{D}_{0+}^\alpha \mathbf{x})(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \quad (25)$$

in which \mathbf{M} is the mass matrix, \mathbf{C} and \mathbf{C}_α are respectively the damping and fractional damping matrices, and \mathbf{K} represent the stiffness matrix of the structure. Assuming the system quiescent for $t < 0$, in the same hypothesis as above, the corresponding equation in the Mellin domains reads:

$$C(\gamma, 2)\mathbf{M}\mathbf{X}(\gamma - 2) + C(\gamma, 1)\mathbf{C}\mathbf{X}(\gamma - 1) + C(\gamma, \alpha)\mathbf{C}_\alpha\mathbf{X}(\gamma - \alpha) + \mathbf{K}\mathbf{X}(\gamma) = \mathbf{F}(\gamma) \quad (26)$$

175 Evaluating eq.(26) in the point $\gamma_s = \rho + is\Delta\eta$ of the fundamental strip, we
 176 can write it in terms of the only values $X(\gamma_k)$ of the transformed function,
 177 by making use of eq.(21), as follows:

$$\sum_{k=-m}^m \mathbf{M}(s, k) \mathbf{X}(\gamma_k) = \mathbf{F}(\gamma_s) \quad (27)$$

with

$$\mathbf{M}(s, k) = \frac{1}{2b} [C(\gamma_s, 2) a_{sk}(2) \mathbf{M} + C(\gamma_s, 1) a_{sk}(1) \mathbf{C} + C(\gamma_s, \alpha) a_{sk}(\alpha) \mathbf{C}_\alpha] + \delta_{sk} \mathbf{K} \quad (28)$$

178 Evaluating eq.(27) for $s = -m, \dots, m$, we get the solving algebraic set of
179 equations, as obtained in (24).

180 In order to validate the method, we consider the simpler case in which
181 $\mathbf{C} = \lambda_1 \mathbf{K}$, $\mathbf{C}_\alpha = \lambda_\alpha \mathbf{K}$, condition necessary for the system to be diagonal
182 in its modal space. Under these conditions, defining the dynamical matrix
183 $\mathbf{D} = \mathbf{M}^{-1} \mathbf{K}$, the equation of motion of the system takes the form:

$$\ddot{\mathbf{x}}(t) + \mathbf{D} (\lambda_1 \dot{\mathbf{x}}(t) + \lambda_\alpha ({}^C \mathbf{D}_{0+}^\alpha \mathbf{x})(t) + \mathbf{x}(t)) = \mathbf{g}(t) \quad (29)$$

184 with $\mathbf{g}(t) = \mathbf{M}^{-1} \mathbf{f}(t)$. Let us rewrite now the physical position vector
185 $\mathbf{x}(t)$, in the basis composed by the normalized eigenvector ϕ_i ($i = 1, \dots, N$)
186 of the dynamical matrix \mathbf{D} . Labelling $\mathbf{y}(t)$ the modal vector and Φ the
187 transformation matrix between the physical and the modal space, whose
188 column are the eigenvectors ϕ_i , we can write:

$$\mathbf{x}(t) = \Phi \mathbf{y}(t) \quad (30)$$

189 By inserting eq.(30) into eq.(29), and pre-multiplying both sides by Φ^T , we
190 get

$$\ddot{\mathbf{y}}(t) + \mathbf{U}_D (\lambda_1 \dot{\mathbf{y}}(t) + \lambda_\alpha ({}^C \mathbf{D}_{0+}^\alpha \mathbf{y})(t) + \mathbf{y}(t)) = \mathbf{h}(t) \quad (31)$$

191 having defined $\mathbf{U}_D = \Phi^T \mathbf{D} \Phi = \text{diag}\{\epsilon_1, \dots, \epsilon_M\}$ (with ϵ_j the j -th eigenvalue
192 of \mathbf{D}) and $\mathbf{h}(t) = \Phi^T \mathbf{g}(t)$ the forcing action in the modal space. The eq.(30)
193 is a set of independent fractional differential equations, having the form:

$$\ddot{y}_j(t) + \epsilon_j (\lambda_1 \dot{y}_j(t) + \lambda_\alpha ({}^C \mathbf{D}_{0+}^\alpha y_j)(t) + y_j(t)) = h_j(t) \quad (32)$$

194 Once the solution of eq.(32) is found for $j = 1, \dots, M$, the response in the
195 physical space is restored by eq.(30).

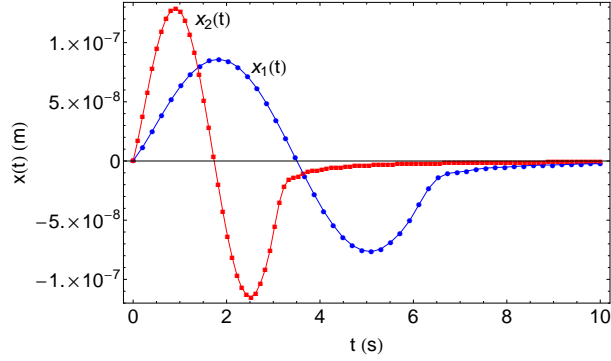
196 We consider as first example a 2-DOF structure, with the value $\alpha = 0.2$
 197 for the order of the fractional derivative. For simplicity we assume significant
 198 only the stiffness due to the main structure, disregarding the stiffness of the
 199 viscoelastic dampers. Indicating with $K = 6EI/\ell^2$ the stiffness of each
 200 ℓ long beam, where E is the Young's modulus of the material and I the
 201 moment of inertia of the beam cross section, and with m_1 and m_2 the values
 202 of the two masses, then the stiffness matrix \mathbf{K} and the mass matrix \mathbf{M} , read
 203 as:

$$\mathbf{K} = \begin{pmatrix} 2K & -K \\ -K & K \end{pmatrix}; \quad \mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad (33)$$

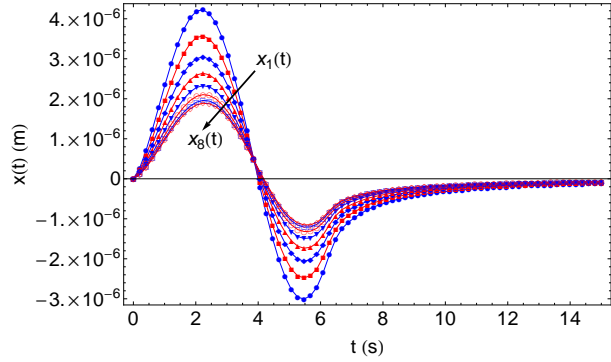
204 In fig.2(a) is reported the solution obtained from eq.(27), compared to
 205 the one obtained by first solving the system in its modal space, when the
 206 following forcing function acts on every degree of freedom of the system:

$$\mathbf{f}(t) = \begin{cases} \sin t & 0 \leq t \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \quad (34)$$

207 The values of the parameters $\lambda_1 = 1$, $\lambda_\alpha = 3$, and the values $m_1 = m_2 =$
 208 10^4 Kg and $E = 2 \times 10^{10}$ N/m² for the physical quantities defined above
 209 have been used. We considered a length $\ell = 4$ m for the beams, and a
 210 square 0.3×0.3 m cross section. Both the solutions have been calculated by
 211 evaluating the discretized inverse Mellin transform on the line $\rho = 0.5$ of the
 212 complex plane, considering a cut-off $\bar{\eta} = 50$ and a sampling step $\Delta\eta = 0.5$.



(a) 2-DOF



(b) 8-DOF

Figure 2: Response of a structural system, equipped with viscoelastic dampers and enforced by the time dependent load given in eq.(34), applied to the whole structure in the 2-DOF case and only to the first mass (m_1) in the 8-DOF case. Comparison between the solution obtained from eq.(27) (dotted line) and from the modal analysis eq.(31) (continuous line).

213 In fig.2(b) is reported the solution for a 8-DOF system, enforced by a
 214 load with a time dependence as in eq.(34), and acting only on the first mass
 215 of the structure. We used the value $\alpha = 0.5$ for the order of the fractional
 216 derivative, $E = 2 \times 10^8 \text{N/m}^2$ for the Young's modulus of the material and
 217 the same values for the λ_1 , λ_α parameters and for the others physical and
 218 geometrical quantities of the system, as for the 2-DOF case.

219 **5. Conclusions**

220 We generalized the method of solution for single-degree-of freedom (SDOF)
221 fractional differential equations, presented by the authors in [32], to the more
222 general case of multi-degree-of-freedom (MDOF) systems. By taking advan-
223 tage of the theory of the Mellin transform in the complex plane, the method
224 allows us to solve the most general multi-order, MDOF fractional (integro-
225)differential equation with constant coefficients, regardless is the number and
226 the order of the differential operators. An approximate solution is found in
227 the Mellin domain by solving a linear set of algebraic equations, and the cor-
228 responding solution in the physical domain is restored by using a discretized
229 version of the inverse Mellin transform. The powerful and the versatility of
230 the method have been proved by mean of an illustrative application for the
231 solution of the dynamics of two examples of passively protected structural
232 systems enforced by a time dependent load, in which the presence of vis-
233 coelastic dampers causes the appearance of non integer order operators in
234 the equations of motion. Such solutions have been verified by a compari-
235 son with the ones obtained by analysing the system in its modal space, in
236 which the relative dynamics reduce to a set of decouples fractional differen-
237 tial equations.

238 The method revealed robust, computationally efficient, and easily imple-
239 mentable for any order and number of the fractional operators present in
240 the MDOF fractional differential equations.

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