



Heriot-Watt University
Research Gateway

On the Time-Inconsistent Deterministic Linear-Quadratic Control

Citation for published version:

Cai, H, Chen, D, Peng, Y & Wei, W 2022, 'On the Time-Inconsistent Deterministic Linear-Quadratic Control', *SIAM Journal on Control and Optimization*, vol. 60, no. 2, pp. 968-991. <https://doi.org/10.1137/21M1419611>

Digital Object Identifier (DOI):

[10.1137/21M1419611](https://doi.org/10.1137/21M1419611)

Link:

[Link to publication record in Heriot-Watt Research Portal](#)

Document Version:

Peer reviewed version

Published In:

SIAM Journal on Control and Optimization

Publisher Rights Statement:

© 2022 Society for Industrial and Applied Mathematics

General rights

Copyright for the publications made accessible via Heriot-Watt Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

Heriot-Watt University has made every reasonable effort to ensure that the content in Heriot-Watt Research Portal complies with UK legislation. If you believe that the public display of this file breaches copyright please contact open.access@hw.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

On the Time-Inconsistent Deterministic Linear-Quadratic Control

Hongyan Cai Danhong Chen Yunfei Peng Wei Wei *

October 13, 2021

Abstract

A fundamental theory of deterministic linear-quadratic (LQ) control is the equivalent relationship between control problems, two-point boundary value problems and Riccati equations. In this paper, we extend the equivalence to a general time-inconsistent deterministic LQ problem, where the inconsistency arises from non-exponential discount functions. By studying the solvability of the Riccati equation, we show the existence and uniqueness of the linear equilibrium for the time-inconsistent LQ problem.

Key words: time inconsistency, equilibrium control, intra-personal game, linear-quadratic control, two-point boundary value problem, Riccati equation.

*Cai: School of Mathematics and Statistics, Guizhou University, Guiyang 550025, China E-mail: hyc@gzu.edu.cn. Chen: School of Mathematics and Statistics, Guizhou University, Guiyang 550025, China. E-mail: gs.dhchen19@gzu.edu.cn. Peng: School of Mathematics and Statistics, Guizhou University, Guiyang 550025, China. Email: yfpeng@gzu.edu.cn. Wei: Department of Actuarial Mathematics and Statistics, Heriot-Watt University, Edinburgh, Scotland EH14 4AS, UK. E-mail: wei.wei@hw.ac.uk. Peng gratefully acknowledges financial support through the National Natural Science Foundation of China under grant No. 12061021. Wei gratefully acknowledges financial support through a start-up grant at Heriot-Watt University.

1 Introduction

Experimental and empirical studies suggest that decision makers often behave in a time-inconsistent manner, with individuals acting impatiently in the current moment whilst planning to act patiently in the future¹. In order to incorporate this evidence on time-inconsistency into mathematical modelling, behavioural scientists and economists suppose that the discount functions in the control models have non-exponential forms², and in consequence, the optimal control becomes inconsistent in time. As is standard in the literature on decision making, time-inconsistent problems are often considered within the self-control and intra-personal game theoretic framework and the corresponding equilibria are taken as solutions to such problems.³

This paper studies a general deterministic time-inconsistent linear-quadratic (LQ) control problem, where the inconsistency arises from non-exponential discount functions. We demonstrate two main results. First, we show an equivalence relationship between control problems, two-point boundary value problems and Riccati equations. Second, we establish the existence and uniqueness of the linear equilibrium for the time-inconsistent LQ problem. We wish these results could shed some light on the study of general time-inconsistent control problems, and justify, at least for some extent, the definition of equilibria over a set of feedback controls.

A fundamental theory of classical (time-consistent) LQ optimal control is the equivalent relationship between control problems, two-point boundary value problems and Riccati equations. In the time consistent setting, the equivalence between the three problems is a natural result of the spike variation and the linear-quadratic structure inherent to the problem⁴. However, a different picture emerges when the time consistency is lost. Due to the time inconsistency stemming from the continuously changing time preferences, a time parameter is involved in the Riccati equation that characterises the LQ problem. As a consequence, the Riccati equation turns out to be a

¹In economics, Strotz (1955) first observed that non-constant time preference rates result in time-inconsistent decisions.

²See Ekeland and Lazrak (2006), Harvey (1995), Karp (2007), Laibson (1997), Loewenstein and Prelec (1992), Bleichrodt et al. (2009) and Ebert et al. (2020) for various discount functions.

³See, for example, Phelps and Pollak (1968); Laibson (1997); O'Donoghue and Rabin (2001); Krusell and Jr. (2003) and Luttmer and Mariotti (2003).

⁴See, for example, Chapter 6 of Yong and Zhou (1999).

flow of dynamics, which switches continuously over time⁵. This feature places an obstacle in the way of using some powerful techniques in the ODE theory and thus constituting the major difficulty in the study of the time-inconsistent LQ problem. In the present paper, we extend the equivalence to the time-inconsistent LQ setting. Particularly, using a different method from the spike variation, we obtain a Riccati equation, which admits a symmetric structure and solves the time-inconsistent LQ problem. In contrast to the Riccati equation derived from the spike variation, the Riccati equation obtained in this paper does not involve the time parameter and hence characterising a single dynamics. Using the Banach fixed point theory and the extension technique in the ODE theory, we establish the unique solvability of the Riccati equation. This result, together with the equivalence, yields the existence and the uniqueness of the linear equilibrium for the time-inconsistent LQ problem.

There are studies in the literature on time-inconsistent LQ control within the intra-personal game theoretic framework. Basak and Chabakauri (2010) and Björk et al. (2014) study dynamic mean-variance asset allocation problems which can be formulated within the time-inconsistent LQ framework, where the inconsistency comes from the quadratic term of the expected state and (or) the non-constant risk aversion. In their works, the definition of an equilibrium is based on the notion of feedback control, which is formally proposed by Ekeland and Lazrak (2006) and Björk and Murgoci (2009). Particularly, this type of equilibria fit in the results in most of literature on behavioural economics, in which the equilibria are derived based on the recursive method⁶. The search of this type of equilibria is usually conducted through a verification theorem and a complicated Bellman system, of which the uniqueness of the solution remains unknown. As a result, the uniqueness of such an equilibrium is usually absent. A different type of definition of an equilibrium, which is based on the notion of open loop control, is introduced by Hu et al. (2012). Moreover, Hu et al. (2017) prove the uniqueness of the open loop equilibrium for a time-inconsistent LQ problem in the one dimensional case⁷. The third method to define (or construct)

⁵A similar phenomenon has been observed in the literature studying stochastic time-inconsistent LQ problems. See, for example, Hu et al. (2012).

⁶See the discussion in Björk and Murgoci (2009).

⁷The time inconsistency of the LQ problem discussed in Hu et al. (2012) and Hu et al. (2017) also arises from

an equilibrium is discrete approximation, which is given by Yong (2012), where the equilibrium of a continuous time-inconsistent LQ control is defined as the limit of the solutions to a sequence of discrete time problems. In the same line, Dou and Lü (2020) construct a feedback equilibrium for a time-inconsistent for a time-inconsistent LQ problem, where the underlying process takes values in a Hilbert space. Another related stream of research is in the literature combining time-inconsistent LQ control with other topics, such as mean field games. The readers could be referred to Bensoussan et al. (2013), Yong (2017) and the reference therein. Finally, it is worth noting that both non-existence and non-uniqueness results for general time-inconsistent control problems have been reported in literature. For a time-inconsistent binary control problem with non-exponential discounting, Tan et al. (2018) find that an equilibrium may not exist, while the time-consistent counterpart admits a unique optimal solution. For a behavioural portfolio management problem, Ekeland and Pirvu (2008) show that multiple linear equilibria can be found over a feedback control set, even though the value functions are obtained from the same ansatz.

The remainder of the paper is organized as follows. Section 2 introduces the formulations of the time-inconsistent LQ problem and the definition of equilibria. In particular, our definition is consistent with the definition based on the feedback control, proposed by Ekeland and Lazrak (2006) and Björk and Murgoci (2009). Section 3 demonstrates the equivalence between the time-inconsistent LQ problem, the two boundary value ODE system and the Riccati equation. Section 4 shows the unique solvability of the Riccati equation and thus providing the existence and uniqueness of the linear equilibrium for the equivalent time-inconsistent LQ problem. In section 5, concluding remarks are given.

the quadratic term of the expected state.

2 Problem setting

Let $T > 0$ be the end of a finite time horizon. Throughout the paper, we will use the following notations.

$$L^p([0, T]; \mathbb{R}^{l \times k}) = \left\{ f : [0, T] \rightarrow \mathbb{R}^{l \times k} \mid \int_0^T |f_{ij}(t)|^p dt < \infty, 1 \leq i \leq l, 1 \leq j \leq k. \right\}$$

$$L^\infty([0, T]; \mathbb{R}^{l \times k}) = \left\{ f : [0, T] \rightarrow \mathbb{R}^{l \times k} \mid \operatorname{ess\,sup}_{t \in [0, T]} |f_{ij}(t)| < \infty, 1 \leq i \leq l, 1 \leq j \leq k. \right\}$$

$$C([0, T]^m; \mathbb{R}^{l \times k}) = \left\{ f : [0, T]^m \rightarrow \mathbb{R}^{l \times k} \mid f \text{ is continuous.} \right\}$$

$$C^1([0, T]^m; \mathbb{R}^{l \times k}) = \left\{ f : [0, T]^m \rightarrow \mathbb{R}^{l \times k} \mid f \text{ and its first order derivative are continuous.} \right\}$$

For a real matrix-valued function $O(t) = (o_{ij}(t)) \in \mathbb{R}^{l \times k}$ for all $t \in [0, T]^m$ ($m = 1, 2$), we introduce the following norms,

$$\begin{cases} \|O(t)\| = \max_{1 \leq i \leq l} \sum_{j=1}^k |o_{ij}(t)|, \quad \|O\|_{L^1} = \max_{1 \leq i \leq l} \sum_{j=1}^k \|o_{ij}\|_{L^1([0, T]^m)}, \quad \|O\|_{L^\infty} = \max_{1 \leq i \leq l} \sum_{j=1}^k \|o_{ij}\|_{L^\infty([0, T]^m)}, \\ \|O\|_C = \max_{1 \leq i \leq l} \sum_{j=1}^k \|o_{ij}\|_{C([0, T]^m)}, \quad \|O\|_{C^1} = \max_{1 \leq i \leq l} \sum_{j=1}^k \left(\|o_{ij}\|_{C([0, T]^m)} + \|D o_{ij}\|_{C([0, T]^m)} \right). \end{cases}$$

We suppose the following assumptions hold throughout this paper.

(H1) $A \in L^1((0, T); \mathbb{R}^{n \times n})$, $B \in L^\infty((0, T); \mathbb{R}^{n \times m})$.

(H2) $M \in C^1([0, T] \times [0, T]; \mathbb{R}^{m \times m})$ is a positive definite symmetric matrix-valued function.

(H3) $Q \in C^1([0, T] \times [0, T]; \mathbb{R}^{n \times n})$ and $G \in C^1([0, T]; \mathbb{R}^{n \times n})$ are positive semi-definite symmetric matrix-valued functions.

(H4) $S \in C^1([0, T] \times [0, T]; \mathbb{R}^{m \times n})$.

For any $(t, x) \in [0, T] \times \mathbb{R}^n$, the underlying dynamics is governed by the following controlled linear ordinary differential equation (LODE, for short)⁸

$$\begin{cases} \dot{X}(s) = A(s)X(s) + B(s)u(s), & s \in (t, T], \\ X(t) = x, \end{cases} \quad (1)$$

where the function $u \in \mathcal{U}[0, T] \equiv L^2([0, T]; \mathbb{R}^m)$ is the control and X is the state process valued in \mathbb{R}^n . It follows standard ODE theory that LODE (1) has a unique solution $X(\cdot) \equiv X_{t,x}^u(\cdot)$. At any time t with the system state $X_t = x$, the cost functional is given by

$$\begin{aligned} J(t, x; u) &= \int_t^T \left[\langle Q(t, s)X_{t,x}^u(s), X_{t,x}^u(s) \rangle + 2 \langle S(t, s)X_{t,x}^u(s), u(s) \rangle \right] ds \\ &\quad + \int_t^T \langle M(t, s)u(s), u(s) \rangle ds + \langle G(t)X_{t,x}^u(T), X_{t,x}^u(T) \rangle. \end{aligned} \quad (2)$$

As discussed earlier the non-exponential discount functions Q, S, M, G in the cost functional (2) render the underlying LQ problem generally time-inconsistent. In this paper we consider a sophisticated agent who is aware of the time-inconsistency but unable to control her future actions. In this case, she seeks to find the so-called equilibrium strategies within the intra-personal game theoretic framework, in which the individual is represented by different players at different dates. We now give the precise definition of an equilibrium control \bar{u} , which essentially entails a solution to a game in which no self at any time (or, equivalently in the current setting, at any state) is willing to deviate from \bar{u} .

Definition 1 *Let $\bar{u} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a measurable mapping and satisfies $\bar{u}(\cdot, X(\cdot)) \in \mathcal{U}[0, T]$.*

Define

$$u^{\varepsilon, v}(s, x) = \begin{cases} v, & s \in [t, t + \varepsilon], \\ \bar{u}(s, x), & s \in (t + \varepsilon, T]. \end{cases} \quad (3)$$

⁸Without any specification, any vector in this paper is a column vector.

The control \bar{u} is an equilibrium control if

$$\liminf_{\varepsilon \searrow 0} \frac{J(t, x; u^{\varepsilon, v}) - J(t, x; \bar{u})}{\varepsilon} \geq 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, v \in \mathbb{R}^m. \quad (4)$$

Furthermore, \bar{u} is called a linear equilibrium control if there is a function $\tilde{u} : [0, T] \rightarrow \mathbb{R}^{m \times n}$ such that

$$\bar{u}(t, x) = \tilde{u}(t)x \text{ for any } (t, x) \in [0, T] \times \mathbb{R}^n.$$

3 The equivalence

In this section, we focus on the equivalent relationship between control problems, two-point boundary value problems and Riccati equations.

First, we introduce the following two-point boundary value problem

$$\begin{cases} \dot{\bar{X}}(s) = [A(s) - B(s)M^{-1}(s, s)S(s, s)] \bar{X}(s) - B(s)M^{-1}(s, s)B^\top(s)\varphi(s), & s \in (t, T], \\ \dot{\varphi}(s) = -[A(s) - B(s)M^{-1}(s, s)S(s, s)]^\top \varphi(s) \\ \quad - [\hat{Q}(s, s) - S^\top(s, s)M^{-1}(s, s)S(s, s)] \bar{X}(s), & s \in [t, T), \\ \bar{X}(t) = x, \varphi(T) = G(T)\bar{X}(T), \end{cases} \quad (5)$$

where

$$\begin{aligned} \langle \hat{Q}(s, s)\bar{X}(s), \bar{X}(s) \rangle &= \langle Q(s, s)\bar{X}(s), \bar{X}(s) \rangle - \langle \dot{G}(s)\bar{X}(T), \bar{X}(T) \rangle - \int_s^T \langle Q_s(s, \tau)\bar{X}(\tau), \bar{X}(\tau) \rangle d\tau \\ &\quad - \int_s^T \left\langle M_s(s, \tau)M^{-1}(\tau, \tau) \left(B^\top(\tau)\varphi(\tau) + S(\tau, \tau)\bar{X}(\tau) \right) - 2S_s(s, \tau)\bar{X}(\tau), \right. \\ &\quad \left. M^{-1}(\tau, \tau) \left(B^\top(\tau)\varphi(\tau) + S(\tau, \tau)\bar{X}(\tau) \right) \right\rangle d\tau \end{aligned} \quad (6)$$

for any $s \in [t, T]$ and for any $y \in \mathbb{R}^n$.

Second, the Riccati equation is given by

$$\begin{cases} \dot{P}(t) + A^\top(t)P(t) + P(t)A(t) + \bar{Q}(t, t) \\ - [P(t)B(t) + S^\top(t, t)] M^{-1}(t, t) [B^\top(t)P(t) + S(t, t)] = 0, \quad 0 \leq t < T, \\ P(T) = G(T). \end{cases} \quad (7)$$

Here

$$\begin{aligned} \bar{Q}(t, t) = & Q(t, t) - \Phi^\top(T, t) \dot{G}(t) \Phi(T, t) - \int_t^T \Phi^\top(s, t) Q_t(t, s) \Phi(s, t) ds \\ & - \int_t^T \Phi^\top(s, t) \left[\Upsilon^\top(s) M_t(t, s) \Upsilon(s) - \Upsilon^\top(s) S_t(t, s) - S_t^\top(t, s) \Upsilon(s) \right] \Phi(s, t) ds, \end{aligned} \quad (8)$$

where

$$\begin{cases} \Upsilon(s) = M^{-1}(s, s) (B^\top(s)P(s) + S(s, s)), \quad \forall s \in [0, T], \\ \Phi(t, s) = \exp\left(\int_s^t (A(\tau) - B(\tau)\Upsilon(\tau)) d\tau\right), \quad \forall s, t \in [0, T]. \end{cases} \quad (9)$$

Remark 1 *There is an abuse of terminology. Compared to the conventional Riccati equations in time-consistent LQ models, the Riccati equation (7) has an integral term which contains the solution to the equation itself (See (8) and (9)). This feature highlights the difference between time-inconsistent LQ control and its time-consistent counterpart. For the reader's convenience, instead of introducing a new terminology, we still call equation (7) (equilibrium) Riccati equation.*

As is standard literature on classical time consistent LQ control and Riccati equations, we define the solution to the Riccati differential equation (7) in $C([0, T]; \mathbb{R}^{n \times n})$ as follows.

Definition 2 *A symmetric matrix-valued function $P \in C([0, T]; \mathbb{R}^{n \times n})$ is a solution to the equilibrium Riccati differential equation (7) if P satisfies the following integral equation,*

$$\begin{aligned} P(t) = & G(T) + \int_t^T (A^\top(s)P(s) + P(s)A(s) + \bar{Q}(s, s) \\ & - [P(s)B(s) + S^\top(s, s)] M^{-1}(s, s) [B^\top(s)P(s) + S(s, s)]) ds \end{aligned}$$

The result given as follows establishes the equivalence between the time-inconsistent LQ problem (4), the two-point boundary value problem (5) and the Riccati equation (7).

Theorem 1 *The following statements are equivalent.*

- (i) *A linear equilibrium control defined by Definition 1 exists.*
- (ii) *For $\forall(t, x) \in [0, T] \times \mathbb{R}^n$, the two-point equilibrium boundary value problem (5) admits a solution in $C([t, T]; \mathbb{R}^n) \times C([t, T]; \mathbb{R}^n)$.*
- (iii) *The equilibrium Riccati equation (7) admits a symmetric solution $P(\cdot) \in C([0, T]; \mathbb{R}^{n \times n})$.*

The proof of Theorem 1 is lengthy. For the ease of exposition, we divide the theorem into three propositions as follows.

Proposition 1 *Suppose that the time-inconsistent LQ problem in Definition 1 admits a linear equilibrium control, then the two-point equilibrium boundary value problem (5) admits a solution in $C([t, T]; \mathbb{R}^n) \times C([t, T]; \mathbb{R}^n)$ for any $t \in [0, T]$.*

Proof. We prove the proposition with two steps. In Step 1, we derive a representation of the equilibrium, given the existence of the equilibrium. In Step 2, using the representation obtained in step 1, we construct a solution to the two-point equilibrium boundary value problem.

Step 1. Let \bar{u} denote the linear equilibrium control defined by Definition 1, then there exists a function $\tilde{u} \in L^2((0, T); \mathbb{R}^{n \times m})$ such that

$$\bar{u}(t, x) = \tilde{u}(t)x \text{ for any } (t, x) \in [0, T] \times \mathbb{R}^n. \quad (10)$$

Consider

$$Y^{\varepsilon, t, v}(s) = \frac{X_{t,x}^{u^{\varepsilon, v}}(s) - X_{t,x}^{\bar{u}}(s)}{\varepsilon}, \quad s \in [t, T].$$

Then it follows from the linear structure of the equilibrium control (10), the definition of the

perturbation control (3) and the LODE (1) that

$$\begin{cases} \dot{Y}^{\varepsilon,t,v}(s) = \begin{cases} A(s)Y^{\varepsilon,t,v}(s) + \frac{1}{\varepsilon} [B(s)v - B(s)\tilde{u}(s)\bar{X}_{t,x}(s)], & s \in [t, t + \varepsilon], \\ (A(s) + B(s)\tilde{u}(s))Y^{\varepsilon,t,v}(s), & s \in (t + \varepsilon, T], \end{cases} \\ Y^{\varepsilon,t,v}(t) = 0. \end{cases}$$

The above ODE problem admits a unique solution $Y^{\varepsilon,t,v} \in C([0, T]; \mathbb{R}^n)$ given by

$$Y^{\varepsilon,t,v}(s) = \begin{cases} \frac{1}{\varepsilon} \int_t^s \Phi_A(s, \nu) B(\nu) [v - \tilde{u}(\nu) \tilde{\Phi}(\nu, t) x] d\nu, & s \in [t, t + \varepsilon], \\ \frac{1}{\varepsilon} \tilde{\Phi}(s, t + \varepsilon) \int_t^{t+\varepsilon} \Phi_A(t + \varepsilon, \nu) B(\nu) [v - \tilde{u}(\nu) \tilde{\Phi}(\nu, t) x] d\nu, & s \in (t + \varepsilon, T], \end{cases} \quad (11)$$

where

$$\Phi_A(s, t) = \exp\left(\int_t^s A(\nu) d\nu\right), \tilde{\Phi}(s, t) = \exp\left(\int_t^s (A(\nu) + B(\nu)\tilde{u}(\nu)) d\nu\right), \forall t, s \in [0, T]. \quad (12)$$

As $X_{t,x}^{u^{\varepsilon,v}}(s) - X_{t,x}^{\tilde{u}}(s) = \varepsilon Y^{\varepsilon,t,v}(s)$, then (11) yields that

$$\lim_{\varepsilon \searrow 0} \left\| X_{t,x}^{u^{\varepsilon,v}} - X_{t,x}^{\tilde{u}} \right\|_{C([t,T]; \mathbb{R}^n)} = 0.$$

For ease of exposition, we introduce the following functions.

$$\tilde{Q}(t, s) = Q(t, s) + \tilde{u}^\top(s) S(t, s) + S^\top(t, s) \tilde{u}(s) + \tilde{u}^\top(s) M(t, s) \tilde{u}(s), \quad s \in [t, T]. \quad (13)$$

and

$$\tilde{P}(\tau) = \tilde{\Phi}^\top(T, \tau) G(t) \tilde{\Phi}(T, t) + \int_\tau^T \tilde{\Phi}^\top(s, \tau) \tilde{Q}(t, s) \tilde{\Phi}(s, t) ds, \quad \tau \in [t, T]. \quad (14)$$

We are now calculating $\lim_{\varepsilon \searrow 0} \frac{J(t, x; u^{\varepsilon, v}) - J(t, x; \bar{u})}{\varepsilon}$.

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \frac{J(t, x; u^{\varepsilon, v}) - J(t, x; \bar{u})}{\varepsilon} &= \langle Q(t, t)x, x \rangle + 2 \langle S(t, t)x, v \rangle + \langle M(t, s)v, v \rangle \\ &\quad - \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \langle \tilde{Q}(t, s)X_{t,x}^{\bar{u}}(s), X_{t,x}^{\bar{u}}(s) \rangle ds \\ &\quad + \lim_{\varepsilon \searrow 0} \int_{t+\varepsilon}^T \langle \tilde{Q}(t, s) \left(X_{t,x}^{\bar{u}}(s) + X_{t,x}^{u^{\varepsilon, v}}(s) \right), Y^{\varepsilon, t, v}(s) \rangle ds \\ &\quad + \lim_{\varepsilon \searrow 0} \langle G(t) \left(X_{t,x}^{\bar{u}}(T) + X_{t,x}^{u^{\varepsilon, v}}(T) \right), Y^{\varepsilon, t, v}(T) \rangle. \end{aligned}$$

Plug the representations of $X_{t,x}^{\bar{u}}(s)$, $X_{t,x}^{u^{\varepsilon, v}}(s)$ into the above equation, we then have

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \frac{J(t, x; u^{\varepsilon, v}) - J(t, x; \bar{u})}{\varepsilon} &= \langle Q(t, t)x, x \rangle + 2 \langle S(t, t)x, v \rangle + \langle M(t, s)v, v \rangle \\ &\quad - \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \langle \tilde{\Phi}^\top(s, t)\tilde{Q}(t, s)\tilde{\Phi}(s, t)x, x \rangle ds \\ &\quad + 2 \lim_{\varepsilon \searrow 0} \int_{t+\varepsilon}^T \langle \tilde{Q}(t, s)\tilde{\Phi}(s, t)x, Y^{\varepsilon, t, v}(s) \rangle ds + 2 \lim_{\varepsilon \searrow 0} \langle G(t)\tilde{\Phi}(T, t)x, Y^{\varepsilon, t, v}(T) \rangle. \end{aligned}$$

Plug the representation of $Y^{\varepsilon, t, v}$ into the above equation, we then have

$$\begin{aligned} &\lim_{\varepsilon \searrow 0} \frac{J(t, x; u^{\varepsilon, v}) - J(t, x; \bar{u})}{\varepsilon} \\ &= \langle Q(t, t)x, x \rangle + 2 \langle S(t, t)x, v \rangle + \langle M(t, s)v, v \rangle - \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \langle \tilde{\Phi}^\top(s, t)\tilde{Q}(t, s)\tilde{\Phi}(s, t)x, x \rangle ds \\ &\quad + 2 \left\langle \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} B^\top(\nu)\tilde{\Phi}_A^\top(t+\varepsilon, \nu)\tilde{P}(t+\varepsilon)x d\nu, v \right\rangle \\ &\quad - 2 \left\langle \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \tilde{\Phi}^\top(\nu, t)\tilde{u}^\top(\nu)]B^\top(\nu)\tilde{\Phi}_A^\top(t+\varepsilon, \nu)\tilde{P}(t+\varepsilon)x d\nu, x \right\rangle. \end{aligned} \tag{15}$$

Define

$$\tilde{J}(t, x, v) \equiv \lim_{\varepsilon \searrow 0} \frac{J(t, x; u^{\varepsilon, v}) - J(t, x; \bar{u})}{\varepsilon}.$$

For any fixed $(t, x) \in [0, T) \times \mathbb{R}^n$, it follows from (15) that $\tilde{J}(t, x, v)$ is strictly convex in v .

Moreover, the definition of an equilibrium yields that $\tilde{J}(t, x, v) \geq 0$, which, together with (15), implies that $\tilde{J}(t, x, v)$ has a unique minimum point \tilde{v} given by

$$\tilde{v} = -M^{-1}(t, t) \left[S(t, t) + \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} B^\top(\nu) \Phi_A^\top(t + \varepsilon, \nu) \tilde{P}(t + \varepsilon) d\nu \right] x.$$

It follows from (12) and (14) that

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} B^\top(\nu) \Phi_A^\top(t + \varepsilon, \nu) \tilde{P}(t + \varepsilon) d\nu = B^\top(t) \tilde{P}(t), \text{ a.e. } t \in [0, T],$$

and thus

$$\tilde{v} = -M^{-1}(t, t) \left(B^\top(t) \tilde{P}(t) + S(t, t) \right) x.$$

Then the uniqueness of the minimum point \tilde{v} yields that

$$\bar{u}(t, x) = \tilde{v} = -M^{-1}(t, t) \left(B^\top(t) \tilde{P}(t) + S(t, t) \right) x \text{ for any } (t, x) \in [0, T] \times \mathbb{R}^n. \quad (16)$$

Step 2. Define

$$\varphi(s) = \tilde{P}(s) X_{t,x}^{\bar{u}}(s) \text{ for any } s \in [t, T]. \quad (17)$$

We will verify that $(X_{t,x}^{\bar{u}}, \varphi)$ is a solution to the two-point equilibrium boundary value problem (5).

Following Assumptions (H1)-(H4), and the representation of \tilde{P} (14), it is to see that \tilde{P} is absolutely continuous. Take the first order derivative on \tilde{P} , then we have

$$\begin{aligned} \dot{\tilde{P}}(t) &= -(A(t) + B(t)\tilde{u}(t))^\top \tilde{P}(t) - \tilde{P}(t)(A(t) + B(t)\tilde{u}(t)) - \tilde{Q}(t, t) \\ &\quad + \tilde{\Phi}^\top(T, t) \dot{G}(t) \tilde{\Phi}(T, t) + \int_t^T \tilde{\Phi}^\top(s, t) \tilde{Q}_t(t, s) \tilde{\Phi}(s, t) ds, \forall t \in [0, T]. \end{aligned} \quad (18)$$

Plug (16) into LODE (1), we then have

$$\dot{X}_{t,x}^{\bar{u}}(s) = \left[A(s) - B(s)M^{-1}(s,s)S(s,s) \right] X_{t,x}^{\bar{u}}(s) - B(s)M^{-1}(s,s)B^\top(s)\varphi(s). \quad (19)$$

Take the first order derivative on φ , then (16), (17), (18) and (19) yield that

$$\begin{aligned} \dot{\varphi}(s) &= - \left[A(s) - B(s)M^{-1}(s,s) \left(B^\top(s)\tilde{P}(s) + S(s,s) \right) \right]^\top \varphi(s) \\ &\quad + \tilde{P}(s) \left[A(s) - B(s)M^{-1}(s,s) \left(B^\top(s)\tilde{P}(s) + S(s,s) \right) \right] X_{t,x}^{\bar{u}}(s) - \tilde{Q}(s,s)X_{t,x}^{\bar{u}}(s) \\ &\quad + \tilde{\Phi}^\top(T,s)\dot{G}(s)\tilde{\Phi}(T,s)X_{t,x}^{\bar{u}}(s) + \int_s^T \tilde{\Phi}^\top(\tau,s)\tilde{Q}_s(s,\tau)\tilde{\Phi}(\tau,s)X_{t,x}^{\bar{u}}(s)d\tau \\ &\quad - \tilde{P}(s) \left[A(s) - B(s)M^{-1}(s,s)S(s,s) \right] X_{t,x}^{\bar{u}}(s) + \tilde{P}(s)B(s)M^{-1}(s,s)B^\top(s)\varphi(s) \\ &= - \left[A(s) - B(s)M^{-1}(s,s)S(s,s) \right]^\top \varphi(s) + \left[B(s)M^{-1}(s,s)B^\top(s)\tilde{P}(s) \right]^\top \varphi(s) \\ &\quad - \tilde{Q}(s,s)X_{t,x}^{\bar{u}}(s) + \tilde{\Phi}^\top(T,s)\dot{G}(s)\tilde{\Phi}(T,s)X_{t,x}^{\bar{u}}(s) + \int_s^T \tilde{\Phi}^\top(\tau,s)\tilde{Q}_s(s,\tau)\tilde{\Phi}(\tau,s)X_{t,x}^{\bar{u}}(s)d\tau \\ &= - \left[A(s) - B(s)M^{-1}(s,s)S(s,s) \right]^\top \varphi(s) - \left(\hat{Q}(s,s) - S^\top(s,s)M^{-1}(s,s)S(s,s) \right) X_{t,x}^{\bar{u}}(s), \end{aligned}$$

where

$$\begin{aligned} \hat{Q}(s,s) &= \tilde{Q}(s,s) - \tilde{P}^\top(s)B(s)M^{-1}(s,s)B^\top(s)\tilde{P}(s) + S^\top(s,s)M^{-1}(s,s)S(s,s) \\ &\quad - \tilde{\Phi}^\top(T,s)\dot{G}(s)\tilde{\Phi}(T,s) - \int_s^T \tilde{\Phi}^\top(\tau,s)\tilde{Q}_s(s,\tau)\tilde{\Phi}(\tau,s)d\tau. \end{aligned} \quad (20)$$

(13) and (20) lead to

$$\begin{aligned} \hat{Q}(s,s) &= Q(s,s) - \tilde{\Phi}^\top(T,s)\dot{G}(s)\tilde{\Phi}(T,s) - \int_s^T \tilde{\Phi}^\top(\tau,s)\tilde{Q}_s(s,\tau)\tilde{\Phi}(\tau,s)d\tau, \\ \tilde{Q}_s(s,\tau) &= Q_s(s,\tau) - S_s^\top(s,\tau)M^{-1}(\tau,\tau) \left(B^\top(\tau)\tilde{P}(\tau) + S(\tau,\tau) \right) \\ &\quad - \left(B^\top(\tau)\tilde{P}(\tau) + S(\tau,\tau) \right)^\top M^{-1}(\tau,\tau)S_s(s,\tau) \\ &\quad + \left(B^\top(\tau)\tilde{P}(\tau) + S(\tau,\tau) \right)^\top M^{-1}(\tau,\tau)M_s(s,\tau)M^{-1}(\tau,\tau) \left(B^\top(\tau)\tilde{P}(\tau) + S(\tau,\tau) \right). \end{aligned} \quad (21)$$

Moreover, it follows from (17) and (21) that

$$\begin{aligned}
& \langle \hat{Q}(s, s)X_{t,x}^{\bar{u}}(s), X_{t,x}^{\bar{u}}(s) \rangle \\
= & \langle Q(s, s)X_{t,x}^{\bar{u}}(s), X_{t,x}^{\bar{u}}(s) \rangle - \langle \dot{G}(s)X_{t,x}^{\bar{u}}(T), X_{t,x}^{\bar{u}}(T) \rangle - \int_s^T \langle \tilde{Q}_s(s, \tau)X_{t,x}^{\bar{u}}(\tau), X_{t,x}^{\bar{u}}(\tau) \rangle d\tau \\
= & \langle Q(s, s)X_{t,x}^{\bar{u}}(s), X_{t,x}^{\bar{u}}(s) \rangle - \langle \dot{G}(s)X_{t,x}^{\bar{u}}(T), X_{t,x}^{\bar{u}}(T) \rangle \\
& - \int_s^T \langle Q_s(s, \tau)X_{t,x}^{\bar{u}}(\tau), X_{t,x}^{\bar{u}}(\tau) \rangle d\tau \\
& - \int_s^T \left\langle M_s(s, \tau)M^{-1}(\tau, \tau) \left(B^\top(\tau)\varphi(\tau) + S(\tau, \tau)X_{t,x}^{\bar{u}}(\tau) \right) - 2S_s(s, \tau)X_{t,x}^{\bar{u}}(\tau), \right. \\
& \left. M^{-1}(\tau, \tau) \left(B^\top(\tau)\varphi(\tau) + S(\tau, \tau)X_{t,x}^{\bar{u}}(\tau) \right) \right\rangle d\tau
\end{aligned}$$

Finally, combining (17), (19) and (20) together, we have that $(X_{t,x}^{\bar{u}}, \varphi) \in C([t, T]; \mathbb{R}^n) \times C([t, T]; \mathbb{R}^n)$ solves the two-point boundary value problem (5) for any $t \in [0, T]$. This completes the proof. ■

Proposition 2 *Suppose that the two-point equilibrium boundary value problem (5) has a solution $(\varphi(\cdot), \bar{X}(\cdot)) \in C([t, T]; \mathbb{R}^n) \times C([t, T]; \mathbb{R}^n)$ for any $(t, x) \in [0, T] \times \mathbb{R}^n$, then the equilibrium Riccati equation (7) has a symmetric solution $P(\cdot) \in C([0, T]; \mathbb{R}^{n \times n})$.*

Proof. For any fixed $x \in \mathbb{R}^n$ with $x \neq 0$, denote the solution to the two point boundary problem (5) (with initial state $(0, x)$) by $(\varphi, \bar{X}) \in C([0, T]; \mathbb{R}^n) \times C([0, T]; \mathbb{R}^n)$. We define

$$\varphi(t) = P(t)\bar{X}(t) \text{ for all } t \in [0, T]. \quad (22)$$

In order to make sure the existence of such $P(t)$ in (22), we need to show $\bar{X}(t) \neq 0, \forall t \in [0, T]$. In fact, plug (22) into (5), we then have that \bar{X} satisfies the following differential equation

$$\begin{cases} \dot{\bar{X}}(t) = [A(t) - B(t)M^{-1}(t, t) (B^\top(t)P(t) + S(t, t))] \bar{X}(t), & t \in (0, T], \\ \bar{X}(0) = x. \end{cases}$$

Solving the above ordinary differential equation, we then have

$$\bar{X}(t) = \Phi(t, 0)x \text{ for all } t \in [0, T],$$

where $\Phi(t, 0)$ is given by (9).

Hence, $\bar{X}(t) \neq 0, \forall t \in [0, T]$ follows from $x \neq 0$.

Next, we derive the differential equation that $P(t)$ satisfies. As φ and \bar{X} are continuous and weakly differentiable, we have that $P \in W^{1,1}((0, T); \mathbb{R}^{n \times n})$. Furthermore, taking the first order derivative on the both sides of (22), we then have

$$\dot{\varphi}(s) = \dot{P}(s)\bar{X}(s) + P(s)\dot{\bar{X}}(s) \text{ for all } s \in [0, T].$$

Moreover, it is from (5), (6) and (22) to see

$$\begin{cases} \dot{P}(s) + P(s)A(s) + A^\top(s)P(s) + \hat{Q}(s, s) \\ - [P(s)B(s) + S^\top(s, s)] M^{-1}(s, s) [B^\top(s)P(s) + S(s, s)] = 0, \quad s \in [0, T), \\ P(T) = G(T). \end{cases} \quad (23)$$

Next, we verify that $\bar{Q}(t, t) = \hat{Q}(t, t)$. It follows from (6) that

$$\begin{aligned} & \langle \hat{Q}(s, s)\Phi(s, 0)x, \Phi(s, 0)x \rangle \\ = & \langle Q(s, s)\Phi(s, 0)x, \Phi(s, 0)x \rangle - \langle \dot{G}(s)\Phi(T, 0)x, \Phi(T, 0)x \rangle \\ & - \int_s^T \langle Q_s(s, \tau)\Phi(\tau, 0)x, \Phi(\tau, 0)x \rangle d\tau \\ & - \int_s^T \left\langle M_s(s, \tau)M^{-1}(\tau, \tau) \left(B^\top(\tau)P(\tau) + S(\tau, \tau) \right) \Phi(\tau, 0)x \right. \\ & \left. - 2S_s(s, \tau)\Phi(\tau, 0)x, M^{-1}(\tau, \tau) \left(B^\top(\tau)P(\tau) + S(\tau, \tau) \right) \Phi(\tau, 0)x \right\rangle d\tau \\ = & \langle Q(s, s)\Phi(s, 0)x, \Phi(s, 0)x \rangle - \langle \Phi^\top(T, s)\dot{G}(s)\Phi(T, s)\Phi(s, 0)x, \Phi(s, 0)x \rangle \\ & - \int_s^T \langle \Phi^\top(\tau, s)Q_s(s, \tau)\Phi(\tau, s)\Phi(s, 0)x, \Phi(s, 0)x \rangle d\tau \\ & - \int_s^T \left\langle \Phi^\top(\tau, s)\Upsilon^\top(\tau) \left[M_s(s, \tau)\Upsilon(\tau)\Phi(\tau, s) - 2S_s(s, \tau)\Phi(\tau, s) \right] \Phi(s, 0)x, \Phi(s, 0)x \right\rangle d\tau. \end{aligned}$$

Then the representation $\bar{Q}(t, t)$ (see (8)) yields that

$$\langle \hat{Q}(s, s)\Phi(s, 0)x, \Phi(s, 0)x \rangle = \langle \bar{Q}(s, s)\Phi(s, 0)x, \Phi(s, 0)x \rangle,$$

which suggests that $\bar{Q}(t, t) = \hat{Q}(t, t)$.

Therefore, comparing (23) and (7), we have P solves the Riccati equation (7).

It now suffices to prove the symmetry of P . It is easy to see from (8) that $\bar{Q}^\top(t, t) = \bar{Q}(t, t)$ for all $t \in [0, T]$. Moreover, the Riccati equation (7) yields that

$$\begin{cases} \dot{P}^\top(s) + P^\top(s)A(s) + A^\top(s)P^\top(s) + \bar{Q}(s, s) \\ - [B^\top(s)P(s) + S(s, s)]^\top M^{-1}(s, s) [B^\top(s)P^\top(s) + S(s, s)] = 0, \quad s \in [0, T], \\ P^\top(T) = G(T). \end{cases} \quad (24)$$

Define

$$\begin{cases} \mathbb{P}(s) = P^\top(s) - P(s), & s \in [0, T], \\ \mathbb{A}(s) = A(s) - B(s)M^{-1}(s, s) [B^\top(s)P^\top(s) + S(s, s)], & s \in [0, T]. \end{cases} \quad (25)$$

One can see from (23) and (24) that

$$\begin{aligned} & \frac{d}{ds} (P^\top(s) - P(s)) + (P^\top(s) - P(s))A(s) + A^\top(s)(P^\top(s) - P(s)) \\ &= (P^\top(s) - P(s))B(s)M^{-1}(s, s) [B^\top(s)P^\top(s) + S(s, s)] \\ & \quad + [B^\top(s)P^\top(s) + S(s, s)]^\top M^{-1}(s, s)B^\top(s)(P^\top(s) - P(s)). \end{aligned} \quad (26)$$

Furthermore, it follows from (23), (24), (25) and (26) that

$$\begin{cases} \dot{\mathbb{P}}(s) + \mathbb{P}(s)\mathbb{A}(s) + \mathbb{A}^\top(s)\mathbb{P}(s) = 0, \quad s \in [0, T], \\ \mathbb{P}(T) = 0. \end{cases}$$

It is easy to see that (26) admits a unique solution $\mathbb{P}(s) = 0$ for all $s \in [0, T]$, which is equivalent

to $P^\top(s) = P(s)$ for all $s \in [0, T]$.

This completes the proof. ■

Proposition 3 *If the equilibrium Riccati equation (7) has a symmetric solution $P(\cdot) \in C([0, T]; \mathbb{R}^{n \times n})$, then the problem has an equilibrium control \bar{u} given by*

$$\bar{u}(t, x) = -M^{-1}(t, t) \left(B^\top P(t) + S(t, t) \right) x, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n. \quad (27)$$

Proof. Let P solves the equilibrium Riccati equation (7), then

$$\begin{aligned} & \dot{P}(s) + P(s) \left[A(s) - B(s)M^{-1}(s, s) \left(B^\top(s)P(s) + S(s, s) \right) \right] \\ & + \left[A(s) - B(s)M^{-1}(s, s) \left(B^\top(s)P(s) + S(s, s) \right) \right]^\top P(s) + \bar{Q}(s, s) \\ & = S^\top(s, s)M^{-1}(s, s)S(s, s) - P(s)B(s)M^{-1}(s, s)B^\top(s)P(s), \text{ for any } s \in [0, T]. \end{aligned} \quad (28)$$

Plug the feedback control \bar{u} given by (27) into the LODE (1), we then have

$$\begin{cases} \dot{\bar{X}}(s) = A(s)\bar{X}(s) - B(s)M^{-1}(s, s) \left(B^\top(s)P(s) + S(s, s) \right) \bar{X}(s), & s \in (t, T], \\ \bar{X}(t) = x, \end{cases}$$

which admits the unique solution \bar{X} given by

$$\bar{X}(s) = X_{t,x}^{\bar{u}}(s) = \Phi(s, t)x, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, s \in [t, T].$$

It follows from (28) and the representation of \bar{X} that

$$\begin{aligned} & \frac{d}{ds} \langle P(s)\bar{X}(s), \bar{X}(s) \rangle + \langle P(s)B(s)M^{-1}(s, s)B^\top(s)P(s)\bar{X}(s), \bar{X}(s) \rangle \\ & = \langle S^\top(s, s)M^{-1}(s, s)S(s, s)\bar{X}(s), \bar{X}(s) \rangle - \langle \bar{Q}(s, s)\bar{X}(s), \bar{X}(s) \rangle, \text{ for any } s \in [t, T]. \end{aligned} \quad (29)$$

Integrate (29) from t to T and plug the representations (7), (8) and (27) into the integral, we

then have

$$\begin{aligned}
\langle P(t)x, x \rangle &= \langle G(T)\bar{X}(T), \bar{X}(T) \rangle + \int_t^T \left\langle M^{-1}(s, s)B^\top(s)P(s)\bar{X}(s), B^\top(s)P(s)\bar{X}(s) \right\rangle ds \\
&\quad - \int_t^T \left\langle M^{-1}(s, s)S(s, s)\bar{X}(s), S(s, s)\bar{X}(s) \right\rangle ds + \int_t^T \langle \bar{Q}(s, s)\bar{X}(s), \bar{X}(s) \rangle ds \\
&= \int_t^T \langle Q(s, s)\bar{X}(s), \bar{X}(s) \rangle ds - \int_t^T \left\langle M^{-1}(s, s)S(s, s)\bar{X}(s), S(s, s)\bar{X}(s) \right\rangle ds \\
&\quad + \langle G(T)\bar{X}(T), \bar{X}(T) \rangle + \int_t^T \left\langle M^{-1}(s, s)B^\top(s)P(s)\bar{X}(s), B^\top(s)P(s)\bar{X}(s) \right\rangle ds \\
&\quad - \int_t^T \left\langle \dot{G}(s)\bar{X}(T), \bar{X}(T) \right\rangle ds - \int_t^T \int_s^T \left\langle Q_s(s, \tau)\bar{X}(\tau), \bar{X}(\tau) \right\rangle d\tau ds \\
&\quad - \int_t^T \int_s^T \left\langle [M_s(s, \tau)\Upsilon(\tau) - 2S_s(s, \tau)]\bar{X}(\tau), \Upsilon(\tau)\bar{X}(\tau) \right\rangle d\tau ds \\
&= \int_t^T \langle Q(t, s)\bar{X}(s), \bar{X}(s) \rangle ds - \int_t^T \left\langle M^{-1}(s, s)S(s, s)\bar{X}(s), S(s, s)\bar{X}(s) \right\rangle ds \\
&\quad + \langle G(t)\bar{X}(T), \bar{X}(T) \rangle + \int_t^T \left\langle M^{-1}(s, s)B^\top(s)P(s)\bar{X}(s), B^\top(s)P(s)\bar{X}(s) \right\rangle ds \\
&\quad - \int_t^T \left\langle M^{-1}(s, s) \left(B^\top(s)P(s) + S(s, s) \right) \bar{X}(s), \left(B^\top(s)P(s) - S(s, s) \right) \bar{X}(s) \right\rangle ds \\
&\quad + \int_t^T \left\langle [M(t, s)\bar{u}(s, \bar{X}(s)) + 2S(t, s)\bar{X}(s)], \bar{u}(s, \bar{X}(s)) \right\rangle ds \\
&= \langle G(t)\bar{X}(T), \bar{X}(T) \rangle + \int_t^T \langle Q(t, s)\bar{X}(s), \bar{X}(s) \rangle ds \\
&\quad + \int_t^T \left\langle [M(t, s)\bar{u}(s, \bar{X}(s)) + 2S(t, s)\bar{X}(s)], \bar{u}(s, \bar{X}(s)) \right\rangle ds
\end{aligned}$$

which yields that

$$J(t, x; \bar{u}) = \langle P(t)x, x \rangle \text{ for any } (t, x) \in [0, T] \times \mathbb{R}^n. \quad (30)$$

In order to verify \hat{u} given by (27) is the equilibrium control, we consider the perturbation control $u^{\varepsilon, v}(t, x)$ given by (3). Solving the control system (1) with $u^{\varepsilon, v}(t, x)$, we have the control systems (1) with $u^{\varepsilon, v}$ has a unique solution $X_{t, x}^{u^{\varepsilon, v}}$ given by

$$X_{t, x}^{u^{\varepsilon, v}}(s) = \begin{cases} X_{t, x}^v(s), & s \in [t, t + \varepsilon], \\ \Phi(s, t + \varepsilon)X_{t, x}^v(t + \varepsilon), & s \in (t + \varepsilon, T]. \end{cases} \quad (31)$$

It follows from (2) and (31), (27), (30) that

$$\begin{aligned}
& J(t, x; u^{\varepsilon, v}) \\
= & \int_t^{t+\varepsilon} \left[\langle Q(t, s) X_{t,x}^v(s), X_{t,x}^v(s) \rangle + 2 \langle S(t, s) X_{t,x}^v(s), v \rangle + \langle M(t, s) v, v \rangle \right] ds \\
& + \int_{t+\varepsilon}^T \left\langle (Q(t, s) - Q(t + \varepsilon, s)) \Phi(s, t + \varepsilon) X_{t,x}^v(t + \varepsilon), \Phi(s, t + \varepsilon) X_{t,x}^v(t + \varepsilon) \right\rangle ds \\
& - 2 \int_{t+\varepsilon}^T \left\langle (S(t, s) - S(t + \varepsilon, s)) \Phi(s, t + \varepsilon) X_{t,x}^v(t + \varepsilon), \right. \\
& \left. M^{-1}(s, s) \left(B^\top(s) P(s) + S(s, s) \right) \Phi(s, t + \varepsilon) X_{t,x}^v(t + \varepsilon) \right\rangle ds \\
& + \int_{t+\varepsilon}^T \left\langle (M(t, s) - M(t + \varepsilon, s)) M^{-1}(s, s) \left(B^\top(s) P(s) + S(s, s) \right) \Phi(s, t + \varepsilon) X_{t,x}^v(t + \varepsilon), \right. \\
& \left. M^{-1}(s, s) \left(B^\top(s) P(s) + S(s, s) \right) \Phi(s, t + \varepsilon) X_{t,x}^v(t + \varepsilon) \right\rangle ds \\
& + \left\langle (G(t) - G(t + \varepsilon)) \Phi(T, t + \varepsilon) X_{t,x}^v(t + \varepsilon), \Phi(T, t + \varepsilon) X_{t,x}^v(t + \varepsilon) \right\rangle \\
& + \left\langle P(t + \varepsilon) X_{t,x}^v(t + \varepsilon), X_{t,x}^v(t + \varepsilon) \right\rangle.
\end{aligned}$$

Moreover, it follows from (31) that $\lim_{\varepsilon \rightarrow 0} X_{t,x}^{u^{\varepsilon, v}}(\cdot) = \Phi(\cdot, t)x$ in $C([t, T]; \mathbb{R}^n)$, then we have

$$\begin{aligned}
& \lim_{\varepsilon \searrow 0} \frac{J(t, x; u^{\varepsilon, v}) - J(t, x; \bar{u})}{\varepsilon} \\
= & \langle Q(t, t)x, x \rangle + 2 \langle S(t, t)x, v \rangle + \langle M(t, t)v, v \rangle - \int_t^T \langle Q_t(t, s) \Phi(s, t)x, \Phi(s, t)x \rangle ds \\
& + 2 \int_t^T \left\langle S_t(t, s) \Phi(s, t)x, M^{-1}(s, s) \left(B^\top(s) P(s) + S(s, s) \right) \Phi(s, t)x \right\rangle ds \\
& - \int_t^T \left\langle M_t(t, s) M^{-1}(s, s) \left(B^\top P(s) + S(s, s) \right) \Phi(s, t)x, \right. \\
& \left. M^{-1}(s, s) \left(B^\top(s) P(s) + S(s, s) \right) \Phi(s, t)x \right\rangle ds + 2 \langle B^\top(t) P(t)x, v \rangle \\
& - \langle \dot{G}(t) \Phi(T, t)x, \Phi(T, t)x \rangle + \left\langle \left(\dot{P}(t) + P(t)A(t) + A^\top(t)P(t) \right) x, x \right\rangle.
\end{aligned}$$

which, together with (7), yields that

$$\begin{aligned}
& \lim_{\varepsilon \searrow 0} \frac{J(t, x; u^{\varepsilon, v}) - J(t, x; \bar{u})}{\varepsilon} \\
&= \left\langle \left(\dot{P}(t) + P(t)A(t) + A^\top(t)P(t) + \bar{Q}(t, t) \right) x, x \right\rangle + 2 \left\langle \left(B^\top(t)P(t) + S(t, t) \right) x, v \right\rangle + \langle M(t, t)v, v \rangle \\
&= \left\langle M^{-1}(t, t) \left[B^\top(t)P(t) + S(t, t) \right] x, \left[B^\top(t)P(t) + S(t, t) \right] x \right\rangle \\
&\quad + 2 \left\langle \left[B^\top(t)P(t) + S(t, t) \right] x, v \right\rangle + \langle M(t, t)v, v \rangle \\
&= \left\langle M(t, t) \left[M^{-1}(t, t) \left(B^\top(t)P(t) + S(t, t) \right) x + v \right], M^{-1}(t, t) \left(B^\top(t)P(t) + S(t, t) \right) x + v \right\rangle \\
&\geq 0, \quad \forall (t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m.
\end{aligned}$$

This suggests that \bar{u} is an equilibrium control and thus completing the proof. ■

4 Solvability of the time inconsistent LQ problem

4.1 Solvability of the equilibrium Riccati equation

In this section, we will study the solvability of the equilibrium Riccati equation (7). Throughout this section, besides Assumptions (H1)-(H4), we work with the following assumption

(H5) $Q_t(t, s)$, $M_t(t, s)$, $\dot{G}(t)$, $Q(t, s) - S^\top(t, s)M^{-1}(t, s)S(t, s)$ and $Q_t(t, s) - S_t^\top(t, s)M_t^{-1}(t, s)S_t(t, s)$ are positive semi-definite, $\forall 0 \leq t \leq s \leq T$.

Remark 2 *In the one dimensional case with $S \equiv 0$, let us suppose that $Q(t, s) = Q(s-t)$, $M(t, s) = M(s-t)$, $G(T-t) = G(t)$ and view Q, M, G as discount functions. Then Assumption (H5) states the decreasing property of discount functions, which is commonly assumed in economics.*

We define a map \mathbb{F} as follows:

$$\begin{aligned}
\mathbb{F}(t; s, P) &= \Phi^\top(T, t)\dot{G}(s)\Phi(T, t) + \int_t^T \Phi^\top(\tau, t)Q_s(s, \tau)\Phi(\tau, t)d\tau \\
&\quad + \int_t^T \Phi^\top(\tau, t) \left[\Upsilon^\top(\tau)M_s(s, \tau)\Upsilon(\tau) - \Upsilon^\top(\tau)S_s(s, \tau) - S_s^\top(s, \tau)\Upsilon(\tau) \right] \Phi(\tau, t)d\tau, \quad \forall s, t \in [0, T],
\end{aligned} \tag{32}$$

It is easy to see that $\mathbb{F}(t; s, P)$ is symmetric and

$$\bar{Q}(t, t) = Q(t, t) - \mathbb{F}(t; t, P) \text{ for all } t \in [0, T].$$

Lemma 1 *Suppose $P_1, P_2 \in C([0, T]; \mathbb{R}^{n \times n})$ are symmetric matrix-value functions, then*

$$\|\mathbb{F}(s; t, P_2) - \mathbb{F}(s; t, P_1)\| \leq 4T\gamma e^{4\omega} \|P_1 - P_2\|_C \text{ for any } s, t \in [0, T], \quad (33)$$

where

$$\begin{cases} \omega = \|A\|_{L^1} + T\|B\|_{L^\infty} \|M^{-1}\|_C (\|B\|_{L^\infty} (\|P_1\|_C + \|P_2\|_C) + \|S\|_C), \\ \alpha = \|M^{-1}\|_C (\|B\|_{L^\infty} (\|P_1\|_C + \|P_2\|_C) + \|S\|_C), \\ \gamma = \|M^{-1}\|_C (1 + \|B\|_{L^\infty})^2 \left[\|G\|_{C^1} + T\|Q\|_{C^1} + (1 + T\alpha) (\alpha\|M\|_{C^1} + \|S\|_{C^1}) \right]. \end{cases} \quad (34)$$

Proof. Define

$$\Psi_i(t, s) = \exp \left(\int_s^t \left(A(\tau) - B(\tau)M^{-1}(\tau, \tau) \left(B^\top(\tau)P_i(\tau) + S(\tau, \tau) \right) \right) d\tau \right), \quad s, t \in [0, T], i = 1, 2. \quad (35)$$

Then we have

$$\|\Psi_i(t, s)\| \leq \exp(\|A\|_{L^1} + T\alpha\|B\|_{L^\infty}), \quad s, t \in [0, T], \quad i = 1, 2. \quad (36)$$

Thus, it follows from (32) and (34) that

$$\begin{aligned} \|\mathbb{F}(t; s, P_1)\| &\leq \|\Psi_1(T, t)\|^2 \|\dot{G}(s)\| + \int_t^T \|\Psi_1(\tau, t)\|^2 \left[\|Q_s(s, \tau)\| + \left\| \left(B^\top(\tau)P_1(\tau) \right. \right. \right. \\ &\quad \left. \left. \left. + S(\tau, \tau) \right)^\top M^{-1}(\tau, \tau) \left(M_s(s, \tau)M^{-1}(\tau, \tau) \left(B^\top(\tau)P_1(\tau) + S(\tau, \tau) \right) - 2S_s(s, \tau) \right) \right\| \right] d\tau \\ &\leq [\|G\|_{C^1} + T\|Q\|_{C^1} + T\alpha(\alpha\|M\|_{C^1} + 2\|S\|_{C^1})] \exp(2\|A\|_{L^1} + 2T\alpha\|B\|_{L^\infty}). \end{aligned} \quad (37)$$

For any fixed $s \in [0, T]$, we define

$$\mathbb{E}(t) = \mathbb{F}(t; s, P_2) - \mathbb{F}(t; s, P_1) \text{ for all } t \in [0, T].$$

Then, some algebra yields that $\mathbb{E} \in C([0, T]; \mathbb{R}^{n \times n}) \cap W^{1,1}((0, T); \mathbb{R}^{n \times n})$ satisfies the following lyapunov equation

$$\begin{cases} \dot{\mathbb{E}}(t) + (A(t) - B(t)\Upsilon_2(t))^\top \mathbb{E}(t) + \mathbb{E}(t) (A(t) - B(t)\Upsilon_2(t)) \\ \quad + (P_1(t) - P_2(t))^\top B(t)M^{-1}(t, t) [B^\top(t)\mathbb{F}(t; s, P_1) - M_s(s, t)\Upsilon_1(t) + S_s(s, t)] \\ \quad + [\mathbb{F}(t; s, P_1)B(t) - \Upsilon_2^\top(t)M_s(s, t) + S_s^\top(s, t)] M^{-1}(t, t)B^\top(t) (P_1(t) - P_2(t)) = 0, \quad t \in [0, T), \\ \mathbb{E}(T) = 0, \end{cases}$$

where $\Upsilon_i(t) = M^{-1}(t, t) (B^\top(t)P_i(t) + S(t, t))$, $i = 1, 2$.

Solving the above lyapunov equation with (35), we have that

$$\begin{aligned} \mathbb{E}(t) &= \int_t^T \Psi_2^\top(\tau, t) \left[[\mathbb{F}(\tau; s, P_1)B(\tau) - \Upsilon_2^\top(\tau)M_s(s, \tau) + S_s^\top(s, \tau)] M^{-1}(\tau, \tau)B^\top(\tau) (P_1(\tau) - P_2(\tau)) \right. \\ &\quad \left. + (P_1(\tau) - P_2(\tau))^\top B(\tau)M^{-1}(\tau, \tau) (B^\top(\tau)\mathbb{F}(\tau; s, P_1) - M_s(s, \tau)\Upsilon_1(t) + S_s(s, \tau)) \right] \Psi_2(\tau, t) d\tau. \end{aligned}$$

As $\mathbb{E}(t) = \mathbb{F}(t; s, P_2) - \mathbb{F}(t; s, P_1)$, it follows from (34), (36) and (37) that

$$\begin{aligned} \|\mathbb{F}(t; s, P_2) - \mathbb{F}(t; s, P_1)\| &\leq 2Te^{2\omega} \|B\|_{L^\infty} \|M^{-1}\|_C \left[\|\mathbb{F}(\cdot; s, P_1)\|_C \|B\|_{L^\infty} + \alpha \|M\|_{C^1} + \|S\|_{C^1} \right] \|P_2 - P_1\|_C \\ &\leq 4Te^{4\omega} \|M^{-1}\|_C (1 + \|B\|_{L^\infty})^2 \left[\|G\|_{C^1} + T\|Q\|_{C^1} \right. \\ &\quad \left. + (1 + T\alpha) (\alpha \|M\|_{C^1} + \|S\|_{C^1}) \right] \|P_2 - P_1\|_C \\ &\leq 4\gamma Te^{4\omega} \|P_2 - P_1\|_C \end{aligned}$$

This completes the proof. \blacksquare

The following lemma, which demonstrates different types of solutions to the Riccati equation (7), will be used in the proof of the main result of this section.

Lemma 2 Suppose $P(\cdot) \in C([0, T]; \mathbb{R}^{n \times n})$ is symmetric. Then the following statements are equivalent.

(i) P solves the Riccati differential equation (7).

(ii) P solves the following Riccati equation

$$P(t) = \Psi^\top(T, t)G(T)\Psi(T, t) + \int_t^T \Psi^\top(s, t) \left[Q(s, s) - \mathbb{F}(s; s, P) - \left(B^\top(s)P(s) + S(s, s) \right)^\top M^{-1}(s, s) \left(B^\top(s)P(s) + S(s, s) \right) \right] \Psi(s, t) ds, \quad t \in [0, T]$$

where

$$\Psi(s, t) = \exp \left(\int_t^s A(\tau) d\tau \right) \text{ for all } s, t \in [0, T].$$

(iii) P solves the following Riccati integral equation

$$P(t) = \Phi^\top(T, t)G(T)\Phi(T, t) + \int_t^T \Phi^\top(s, t) \left[P(s)B(s)M^{-1}(s, s)B^\top(s)P(s) - \mathbb{F}(s; s, P) + Q(s, s) - S^\top(s, s)M^{-1}(s, s)S(s, s) \right] \Phi(s, t) ds, \quad t \in [0, T].$$

Proof. The result is an immediate consequence of direct calculation. Indeed, let

$Y(t) := (\Psi^{-1})^\top(T, t)P(t)\Psi^{-1}(T, t)$. Then the the equivalence between (i) and (ii) follows from taking the first order derivative on Y and the Riccati equation (7). Similarly, let $Z(t) := (\Phi^{-1})^\top(T, t)P(t)\Phi^{-1}(T, t)$. Then the the equivalence between (i) and (iii) follows from taking the first order derivative on Z and the Riccati equation (7). This completes the proof. ■

The following Lemma provides a prior estimate for the solution to the Riccati equation.

Lemma 3 Suppose $P(\cdot) \in C([0, T]; \mathbb{R}^{n \times n})$ is symmetric and solves the Riccati equation (7), then $\mathbb{F}(\cdot; s, P)$ and P are positive semi-definite and

$$\|P\|_C \leq e^{2\|A\|_{L^1}} (\|G(T)\| + T\|Q\|_C). \quad (38)$$

Proof. It follows from the assumption (H5) and (32) that

$$\begin{aligned}
& \langle \mathbb{F}(t; s, P)x, x \rangle \\
&= \langle \dot{G}(s)\Phi(T, t)x, \Phi(T, t)x \rangle + \int_t^T \left\langle \left[Q_s(s, \tau) - 2\Upsilon^\top(\tau)S_s(s, \tau) + \Upsilon^\top(\tau)M_s(s, \tau)\Upsilon(\tau) \right] \Phi(\tau, t)x, \Phi(\tau, t)x \right\rangle d\tau \\
&= \langle \dot{G}(s)\Phi(T, t)x, \Phi(T, t)x \rangle + \int_t^T \left\langle \left(Q_s(s, \tau) - 2S_s^\top(s, \tau)M_s^{-1}(s, \tau)S_s(s, \tau) \right) \Phi(\tau, t)x, \Phi(\tau, t)x \right\rangle d\tau \\
&\quad + \int_t^T \left\langle M_s(s, \tau) \left[\Upsilon(\tau) - M_s^{-1}(s, \tau)S_s(s, \tau) \right] \Phi(\tau, t)x, \left[\Upsilon(\tau) - M_s^{-1}(s, \tau)S_s(s, \tau) \right] \Phi(\tau, t)x \right\rangle d\tau \\
&\geq 0 \text{ for all } (s, t, x) \in [0, T] \times [0, T] \times \mathbb{R}^n. \tag{39}
\end{aligned}$$

Moreover, for any $x \in \mathbb{R}^n$, using Lemma 2, we have

$$\begin{aligned}
& \langle P(t)x, x \rangle \\
&= \langle G(T)\Phi(T, t)x, \Phi(T, t)x \rangle - \int_t^T \left\langle \dot{G}(s)\Phi(T, t)x, \Phi(T, t)x \right\rangle ds \\
&\quad + \int_t^T \left\langle \left[P(s)B(s)M^{-1}(s, s)B^\top(s)P(s) + Q(s, s) - S^\top(s, s)M^{-1}(s, s)S(s, s) \right] \Phi(s, t)x, \Phi(s, t)x \right\rangle ds \\
&\quad - \int_t^T \int_s^T \left\langle \left[Q_s(s, \tau) + \Upsilon^\top(\tau)M_s(s, \tau)\Upsilon(\tau) - 2\Upsilon^\top(\tau)S_s(s, \tau) \right] \Phi(\tau, t)x, \Phi(\tau, t)x \right\rangle d\tau ds \\
&= \langle G(t)\Phi(T, t)x, \Phi(T, t)x \rangle + \int_t^T \left\langle \left[Q(t, s) - 2\Upsilon^\top(s)S(t, s) + \Upsilon^\top(s)M(t, s)\Upsilon(\tau) \right] \Phi(s, t)x, \Phi(s, t)x \right\rangle ds \\
&= \langle G(t)\Phi(T, t)x, \Phi(T, t)x \rangle + \int_t^T \left\langle \left[Q(t, s) - S^\top(t, s)M^{-1}(t, s)S(t, s) \right] \Phi(s, t)x, \Phi(s, t)x \right\rangle ds \\
&\quad + \int_t^T \left\langle M(t, s) \left(\Upsilon(s) - M^{-1}(t, s)S(t, s) \right) \Phi(s, t)x, \left(\Upsilon(s) - M^{-1}(t, s)S(t, s) \right) \Phi(s, t)x \right\rangle ds.
\end{aligned}$$

Thanks to (H2), (H3) and (H5), we thus have that

$$\langle P(t)x, x \rangle \geq 0 \text{ for all } (t, x) \in [t_{12}, T] \times \mathbb{R}^n. \tag{40}$$

Moreover, it follows from (39) that

$$\begin{aligned}
\langle P(t)x, x \rangle &= \langle G(T)\Psi(T, t)x, \Psi(T, t)x \rangle + \int_t^T \left\langle \left[Q(s, s) - \mathbb{F}(s; s, P) \right. \right. \\
&\quad \left. \left. - \left(B^\top(s)P(s) + S(s, s) \right)^\top M^{-1}(s, s) \left(B^\top(s)P(s) + S(s, s) \right) \right] \Psi(s, t)x, \Psi(s, t)x \right\rangle ds \\
&\leq \langle G(T)\Psi(T, t)x, \Psi(T, t)x \rangle + \int_t^T \langle Q(s, s)\Psi(s, t)x, \Psi(s, t)x \rangle ds, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n
\end{aligned}$$

which, together with (40), yields (38) and thus completing the proof. ■

The main theorem in this section demonstrates the unique solvability of the Riccati equation.

Theorem 2 *The equilibrium Riccati equation (7) admits a unique symmetric solution $P(\cdot) \in C([0, T]; \mathbb{R}^{n \times n})$. Moreover, P is positive semi-definite.*

Proof. The proof can be divided into two steps. In Step 1, we prove the local existence and uniqueness of the solution to the Riccati equation (7), while we extend the result to the global case in Step 2.

For the ease of exposition, we introduce the following notations, which will be used in the proof without further clarifications.

Denote by $\mathbb{B}(\eta, r_0)$ the closed ball (in $\mathbb{R}^{n \times n}$) centered at η and of radius $r_0 > 0$. Define

$$\left\{ \begin{array}{l} r = e^{2\|A\|_{L^1}} (\|G\|_C + T\|Q\|_C), \\ \bar{\rho} = \|M^{-1}\|_C (4r\|B\|_{L^\infty} + \|S\|_C), \\ \bar{\beta} = \|A\|_{L^1} + T\bar{\rho}\|B\|_{L^\infty}, \\ \bar{\omega} = \|A\|_{L^1} + 2T\bar{\rho}\|B\|_{L^\infty}, \\ \bar{\gamma} = \|M^{-1}\|_C (1 + \|B\|_{L^\infty})^2 \left[\|G\|_{C^1} + T\|Q\|_{C^1} + (1 + 2T\bar{\rho})(2\bar{\rho}\|M\|_{C^1} + \|S\|_{C^1}) \right]. \end{array} \right. \quad (41)$$

For Ψ , we can find a constant $\tau_1 > 0$ such that

$$\|\Psi(t, s) - I\| \leq \frac{1}{2(1 + e^{2\bar{\beta}})} \text{ for any } s, t \in [0, T] \text{ with } |s - t| \leq \tau_1, \quad (42)$$

where the existence of τ_1 follows from the uniform continuity of Ψ .

Moreover, we define

$$\begin{cases} \tau_2 = \frac{r}{2e^{4\bar{\beta}}(\bar{\rho}^2\|M\|_C + \|G\|_{C^1} + T\|Q\|_{C^1} + T\bar{\rho}(\bar{\rho}\|M\|_{C^1} + 2\|S\|_{C^1}) + \|Q\|_C)}, \\ \tau_3 = \frac{1}{4e^{2\|A\|_{L^1}}(\bar{\rho}\|B\|_{L^\infty} + 2T\bar{\gamma}e^{4\bar{\omega}})} \end{cases} \quad (43)$$

and

$$\tau = \min\{\tau_1, \tau_2, \tau_3\}. \quad (44)$$

It is easy to see that $\tau > 0$.

Step 1. Define a map \mathbb{H}_1 on $C([T - \tau, T]; \mathbb{B}(G(T), r))$ given by

$$\begin{aligned} (\mathbb{H}_1 P)(t) &= \Psi^\top(T, t)G(T)\Psi(T, t) + \int_t^T \Psi^\top(s, t) \left[Q(s, s) - \mathbb{F}(s; s, P) \right. \\ &\quad \left. - \left(B^\top(s)P(s) + S(s, s) \right)^\top M^{-1}(s, s) \left(B^\top(s)P(s) + S(s, s) \right) \right] \Psi(s, t) ds. \end{aligned} \quad (45)$$

Following (37) and using the notations defined by (41)- (44), we have

$$\|\mathbb{F}(t; s, P)\| \leq e^{2\bar{\beta}} [\|G\|_{C^1} + T\|Q\|_{C^1} + T\bar{\rho}(\bar{\rho}\|M\|_{C^1} + 2\|S\|_{C^1})] \text{ for any } P \in C([0, T]; \mathbb{B}(0, 2r)).$$

Thus,

$$\begin{aligned}
& \|(\mathbb{H}_1 P)(t) - G(T)\| \\
= & \left\| (\Psi(T, t) - I)^\top G(T) \Psi(T, t) + G(T) (\Psi(T, t) - I) + \int_t^T \Psi^\top(s, t) \left[Q(s, s) - \mathbb{F}(s; s, P) \right. \right. \\
& \left. \left. - \left(B^\top(s) P(s) + S(s, s) \right)^\top M^{-1}(s, s) \left(B^\top(s) P(s) + S(s, s) \right) \right] \Psi(s, t) ds \right\| \\
\leq & \left(1 + e^{2\bar{\beta}} \right) \|G(T)\| \|\Psi(T, t) - I\| + \int_t^T \|\Psi\|_C^2 \left[\|Q\|_C + \|\mathbb{F}(s; s, P)\| \right. \\
& \left. + \left\| B^\top(s) P(s) + S(s, s) \right\|^2 \left\| M^{-1}(s, s) \right\| \right] ds \\
\leq & e^{4\bar{\beta}} \left(\bar{\rho}^2 \|M\|_C + \|G\|_{C^1} + T \|Q\|_{C^1} + T \bar{\rho} (\bar{\rho} \|M\|_{C^1} + 2 \|S\|_{C^1}) + \|Q\|_C \right) (T - t) \\
& + \left(1 + e^{2\bar{\beta}} \right) \|G(T)\| \|\Psi(T, t) - I\| \\
\leq & r
\end{aligned} \tag{46}$$

for all $P \in C([T - \tau, T]; \mathbb{B}(G(T), r))$.

This shows that $\mathbb{H}_1 C([T - \tau, T]; \mathbb{B}(G(T), r)) \subseteq C([T - \tau, T]; \mathbb{B}(G(T), r))$.

Furthermore, for any $P_1, P_2 \in C([T - \tau, T]; \mathbb{B}(G(T), r))$, it is easy to see from (45) that

$$\begin{aligned}
(\mathbb{H}_1 P_1)(t) - (\mathbb{H}_1 P_2)(t) &= \int_t^T \Psi^\top(s, t) \left[(P_2(s) - P_1(s)) B(s) M^{-1}(s, s) \left(B^\top(s) P_2(s) + S(s, s) \right) \right. \\
& \left. + \left(B^\top(s) P_1(s) + S(s, s) \right)^\top M^{-1}(s, s) B^\top(s) (P_2(s) - P_1(s)) \right. \\
& \left. + \mathbb{F}(s; s, P_2) - \mathbb{F}(s; s, P_1) \right] \Psi(s, t) ds.
\end{aligned}$$

As a result,

$$\begin{aligned}
\|(\mathbb{H}_1 P_1)(t) - (\mathbb{H}_1 P_2)(t)\| &\leq \int_t^T \|\Psi(s, t)\|^2 \left[2 \left\| B^\top(s) P_1(s) + S(s, s) \right\| \left\| M^{-1}(s, s) \right\| \|B(s)\| \|P_2(s) - P_1(s)\| \right. \\
& \left. + \|\mathbb{F}(s; s, P_2) - \mathbb{F}(s; s, P_1)\| \right] ds.
\end{aligned} \tag{47}$$

It follows from (33) that

$$\|\mathbb{F}(s; t, P_2) - \mathbb{F}(s; t, P_1)\| \leq 4T\bar{\gamma}e^{4\bar{\omega}} \|P_1 - P_2\|_C \text{ for any } s, t \in [0, T], P_1, P_2 \in C([0, T]; \mathbb{B}(0, 2r)) \tag{48}$$

Combining (47) and (48) and the notations defined in (43), (44), we obtain

$$\|(\mathbb{H}_1 P_1)(t) - (\mathbb{H}_1 P_2)(t)\| \leq \frac{1}{2} \|P_2 - P_1\|_C \text{ for any } P_1, P_2 \in C([T - \tau, T]; \mathbb{B}(G(T), r)).$$

The above inequality and (46) shows that \mathbb{H}_1 is a contraction mapping from $C([T - \tau, T]; \mathbb{B}(G(T), r))$ to itself. Therefore, it follows from the Banach fixed point theorem that, there is a unique $\mathbb{P}_1 \in C([T - \tau, T]; \mathbb{B}(G(T), r))$ such that

$$\begin{aligned} \mathbb{P}_1(t) &= \Psi^\top(T, t)G(T)\Psi(T, t) + \int_t^T \Psi^\top(s, t) \left[Q(s, s) - \mathbb{F}(s, s, \mathbb{P}_1) \right. \\ &\quad \left. - \left(B^\top(s)\mathbb{P}_1(s) + S(s, s) \right)^\top M^{-1}(s, s) \left(B^\top(s)\mathbb{P}_1(s) + S(s, s) \right) \right] \Psi(s, t) ds, \quad \forall t \in [T - \tau, T]. \end{aligned}$$

Hence, according to Lemma 2, we have that \mathbb{P}_1 solves the Riccati equation (7) on $[T - \tau, T]$, and the positive semi-definiteness of \mathbb{P}_1 follows from Lemma 3.

Step 2. For any fixed $P \in C([T - 2\tau, T - \tau]; \mathbb{R}^{n \times n})$, define

$$\begin{cases} \hat{P}_1(t) = \begin{cases} P(t), & t \in [T - 2\tau, T - \tau], \\ \mathbb{P}_1(t), & t \in [T - \tau, T], \end{cases} \\ \hat{Y}_1(t) = M^{-1}(t, t) \left(B^\top(t)\hat{P}_1(t) + S(t, t) \right), \\ \hat{\Phi}_1(t, s) = \exp \left(\int_s^t \left(A(\tau) - B(\tau)M^{-1}(\tau, \tau) \left(B^\top(\tau)\hat{P}_1(\tau) + S(\tau, \tau) \right) \right) d\tau \right), \quad \forall s, t \in [T - 2\tau, T], \end{cases} \quad (49)$$

and

$$\begin{aligned} \mathbb{F}_1(t; s, \hat{P}_1) &= \hat{\Phi}_1^\top(T, t) \dot{G}(s) \hat{\Phi}_1(T, t) + \int_t^T \hat{\Phi}_1^\top(\tau, t) Q_s(s, \tau) \hat{\Phi}_1(\tau, t) d\tau \\ &\quad + \int_t^T \hat{\Phi}_1^\top(\tau, t) \left[\hat{Y}_1^\top(\tau) M_s(s, \tau) \hat{Y}_1(\tau) - \hat{Y}_1^\top(\tau) S_s(s, \tau) - S_s^\top(s, \tau) \hat{Y}_1(\tau) \right] \hat{\Phi}_1(\tau, t) d\tau \end{aligned} \quad (50)$$

for $s, t \in [T - 2\tau, T]$.

We introduce a map \mathbb{H}_2 on $C([T - 2\tau, T - \tau]; \mathbb{B}(\mathbb{P}_1(T - \tau), r))$ given by

$$\begin{aligned} (\mathbb{H}_2 P)(t) &= \Psi^\top(T - \tau, t) \mathbb{P}_1(T - \tau) \Psi(T - \tau, t) + \int_t^{T - \tau} \Psi^\top(s, t) \left[Q(s, s) - \mathbb{F}_1(s; s, \hat{P}_1) \right. \\ &\quad \left. - \left(B^\top(s) P(s) + S(s, s) \right)^\top M^{-1}(s, s) \left(B^\top(s) P(s) + S(s, s) \right) \right] \Psi(s, t) ds. \end{aligned}$$

Moreover, combining (41), (49) and (50), we have

$$\begin{aligned} & \|(\mathbb{H}_2 P)(t) - \mathbb{P}_1(T - \tau)\| \\ & \leq \|(\Psi(T - \tau, t) - I)^\top \mathbb{P}_1(T - \tau) \Psi(T - \tau, t) + \mathbb{P}_1(T - \tau) (\Psi(T - \tau, t) - I)\| \\ & \quad + \int_t^{T - \tau} \|\Psi(s, t)\|^2 \left[\|Q(s, s)\| + \|\mathbb{F}_1(s; s, \hat{P}_1)\| + \|B^\top(s) P(s) + S(s, s)\|^2 \|M^{-1}(s, s)\| \right] ds \\ & \leq e^{4\bar{\beta}} \left(\bar{\rho}^2 \|M\|_C + \|G\|_{C^1} + \|Q\|_C + T \|Q\|_{C^1} + T \bar{\rho} (\bar{\rho} \|M\|_{C^1} + 2 \|S\|_{C^1}) \right) (T - \tau - t) \\ & \quad + (1 + e^{2\bar{\beta}}) \|\mathbb{P}_1(T - \tau)\| \|\Psi(T - \tau, t) - I\| \end{aligned}$$

and

$$\begin{aligned} (\mathbb{H}_2 P_1)(t) - (\mathbb{H}_2 P_2)(t) &= \int_t^T \Psi^\top(s, t) \left[(P_2(s) - P_1(s)) B(s) M^{-1}(s, s) \left(B^\top(s) P_2(s) + S(s, s) \right) \right. \\ &\quad + \left(B^\top(s) P_1(s) + S(s, s) \right)^\top M^{-1}(s, s) B^\top(s) (P_2(s) - P_1(s)) \\ &\quad \left. + \mathbb{F}(s; s, \hat{P}_2) - \mathbb{F}(s; s, \hat{P}_1) \right] \Psi(s, t) ds \end{aligned}$$

for all $P_1, P_2 \in C([T - 2\tau, T - \tau]; \mathbb{B}(\mathbb{P}_1(T - \tau), r))$.

It follows from (41), (44), (49) and (50) that $\mathbb{H}_2 C([T - 2\tau, T - \tau]; \mathbb{B}(\mathbb{P}_1(T - \tau), r)) \subseteq C([T - 2\tau, T - \tau]; \mathbb{B}(\mathbb{P}_1(T - \tau), r))$ and

$$\|(\mathbb{H}_2 P_1)(t) - (\mathbb{H}_2 P_2)(t)\| \leq \frac{1}{2} \|P_2 - P_1\|_C \text{ for any } P_1, P_2 \in C([T - 2\tau, T - \tau]; \mathbb{B}(\mathbb{P}_1(T - \tau), r)).$$

Thus, it follows from the Banach fixed point theorem that, there is a unique $\mathbb{P}_2 \in C([T - 2\tau, T -$

$\tau]; \mathbb{B}(\mathbb{P}_1(T - \tau), r)$) such that

$$\begin{aligned} \mathbb{P}_2(t) &= \Psi^\top(T - \tau, t) \mathbb{P}_1(T - \tau) \Psi(T - \tau, t) + \int_t^{T - \tau} \Psi^\top(s, t) \left[Q(s, s) - \mathbb{F}(s; s, \hat{\mathbb{P}}_2) \right. \\ &\quad \left. - \left(B^\top(s) \mathbb{P}_2(s) + S(s, s) \right)^\top M^{-1}(s, s) \left(B^\top(s) \mathbb{P}_2(s) + S(s, s) \right) \right] \Psi(s, t) ds, \quad \forall t \in [T - 2\tau, T - \tau], \end{aligned} \quad (51)$$

It is easy to see from (51) that \mathbb{P}_2 is symmetric. Also, Lemma 2 yields that \mathbb{P}_2 solves the Riccati equation (7) and Lemma 3 shows that \mathbb{P}_2 is positive semi-definite.

Continue the above process, we then obtain the symmetric matrix-valued function $\{\mathbb{P}_k | k = 1, 2, \dots, \lceil \frac{T}{\tau} \rceil, \lceil \frac{T}{\tau} \rceil + 1\}$ given by

$$\begin{aligned} \mathbb{P}_k(t) &= \Psi^\top(T - (k - 1)\tau, t) \mathbb{P}_{k-1}(T - (k - 1)\tau) \Psi(T - (k - 1)\tau, t) + \int_t^{T - (k-1)\tau} \Psi^\top(s, t) \left[Q(s, s) - \mathbb{F}(s; s, \hat{\mathbb{P}}_k) \right. \\ &\quad \left. - \left(B^\top(s) \mathbb{P}_k(s) + S(s, s) \right)^\top M^{-1}(s, s) \left(B^\top(s) \mathbb{P}_k(s) + S(s, s) \right) \right] \Psi(s, t) ds \end{aligned}$$

for all $t \in [T - k\tau, T - (k - 1)\tau]$ and

$$\begin{aligned} \mathbb{P}_j(t) &= \Psi^\top(T - (j - 1)\tau, t) \mathbb{P}_{j-1}(T - (j - 1)\tau) \Psi(T - (j - 1)\tau, t) + \int_t^{T - (j-1)\tau} \Psi^\top(s, t) \left[Q(s, s) - \mathbb{F}(s; s, \hat{\mathbb{P}}_j) \right. \\ &\quad \left. - \left(B^\top(s) \mathbb{P}_j(s) + S(s, s) \right)^\top M^{-1}(s, s) \left(B^\top(s) \mathbb{P}_j(s) + S(s, s) \right) \right] \Psi(s, t) ds \end{aligned}$$

for all $t \in [0, T - \lceil \frac{T}{\tau} \rceil \tau]$, where $j = \lceil \frac{T}{\tau} \rceil + 1$. Here

$$\begin{cases} \hat{\mathbb{P}}_k(t) = \begin{cases} \mathbb{P}_k(t), & t \in [T - k\tau, T - (k - 1)\tau), \\ \vdots \\ \mathbb{P}_1(t), & t \in [T - \tau, T], \end{cases} \\ \hat{\Upsilon}_k(t) = M^{-1}(t, t) \left(B^\top(t) \hat{\mathbb{P}}_k(t) + S(t, t) \right), \\ \hat{\Phi}_k(t, s) = \exp \left(\int_s^t \left(A(\tau) - B(\tau) M^{-1}(\tau, \tau) \left(B^\top(\tau) \hat{\mathbb{P}}_k(\tau) + S(\tau, \tau) \right) \right) d\tau \right), \quad \forall s, t \in [T - k\tau, T], \end{cases}$$

and

$$\begin{aligned} \mathbb{F}(t; s, \hat{\mathbb{P}}_k) &= \hat{\Phi}_k^\top(T, t) \dot{G}(s) \hat{\Phi}_k(T, t) + \int_t^T \hat{\Phi}_k^\top(\tau, t) Q_s(s, \tau) \hat{\Phi}_k(\tau, t) d\tau \\ &+ \int_t^T \hat{\Phi}_k^\top(\tau, t) \left[\hat{\Upsilon}_k^\top(\tau) M_s(s, \tau) \hat{\Upsilon}_k(\tau) - \hat{\Upsilon}_k^\top(\tau) S_s(s, \tau) - \hat{\Upsilon}_k(\tau) S_s^\top(s, \tau) \right] \hat{\Phi}_k(\tau, t) d\tau, \quad s, t \in [0, T]. \end{aligned}$$

We define

$$P(t) = \begin{cases} \mathbb{P}_1(t), & t \in [T - \tau, T], \\ \vdots \\ \mathbb{P}_k(t), & t \in [T - k\tau, T - (k-1)\tau), \\ \vdots \\ \mathbb{P}_j(t), & t \in [0, T - \lfloor \frac{T}{\tau} \rfloor \tau). \end{cases}$$

Moreover, $P \in C([0, T]; \mathbb{R}^{n \times n})$ satisfies the Riccati integral equation (37) and P is symmetric positive semi-definite. This implies that the Riccati differential equation (7) has a unique symmetric solution $P \in C([0, T]; \mathbb{R}^{n \times n})$. This completes the proof. ■

4.2 Existence and Uniqueness of the linear equilibrium

Having obtained the equivalence result and the solvability of the equilibrium Riccati equation, we are poised to show the existence and uniqueness of the linear equilibrium for the time inconsistent LQ problem.

Theorem 3 *The time-inconsistent LQ problem in Definition 1 admits a unique linear equilibrium.*

Proof. The existence of the linear equilibrium is an immediate result of Proposition 3.

To prove the uniqueness, we let $\bar{u}(t, x) = \tilde{u}(t)x$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$ denote a linear equilibrium of the LQ problem.

Then (16) yields that

$$\tilde{u}(t) = -M^{-1}(t, t) \left(B^\top(t) \tilde{P}(t) + S(t, t) \right) \text{ for all } t \in [0, T]. \quad (52)$$

It now suffices to prove the uniqueness of \tilde{P} .

It follows from (12) and (13) that

$$\left\{ \begin{array}{l} \tilde{\Phi}(s, t) = \exp \left(\int_t^s \left(A(\nu) - B(\nu) M^{-1}(t, t) \left(B^\top(\nu) \tilde{P}(\nu) + S(\nu, \nu) \right) \right) d\nu \right), \forall t, s \in [0, T], \\ \tilde{Q}(t, s) = \left(B^\top(s) \tilde{P}(s) + S(s, s) \right)^\top M^{-1}(s, s) M(t, s) M^{-1}(s, s) \left(B^\top(s) \tilde{P}(s) + S(s, s) \right) \\ \quad + Q(t, s) - \left(B^\top(s) \tilde{P}(s) + S(s, s) \right)^\top M^{-1}(s, s) S(t, s) \\ \quad - S^\top(t, s) M^{-1}(s, s) \left(B^\top(s) \tilde{P}(s) + S(s, s) \right), \forall s \in [t, T]. \end{array} \right. \quad (53)$$

Let

$$\tilde{\Upsilon}(s) = M^{-1}(s, s) \left(B^\top(s) P(s) + S(s, s) \right), \forall s \in [0, T].$$

Then one can see from (53) that

$$\begin{aligned} & Q(t, t) - \tilde{\Phi}^\top(T, t) \dot{G}(t) \tilde{\Phi}(T, t) - \int_t^T \tilde{\Phi}^\top(s, t) \tilde{Q}_t(t, s) \tilde{\Phi}(s, t) ds \\ = & Q(t, t) - \tilde{\Phi}^\top(T, t) \dot{G}(t) \tilde{\Phi}(T, t) - \int_t^T \tilde{\Phi}^\top(s, t) \left[\tilde{\Upsilon}^\top(s) M_t(t, s) \tilde{\Upsilon}(s) \right. \\ & \left. + Q_t(t, s) - \tilde{\Upsilon}^\top(s) S_t(t, s) - S_t^\top(t, s) \tilde{\Upsilon}(s) \right] \tilde{\Phi}(s, t) ds, \forall t \in [0, T]. \end{aligned} \quad (54)$$

Using (14), we have $\tilde{P}(T) = G(T)$ and $\tilde{P}^\top(t) = \tilde{P}(t)$ for all $t \in [0, T]$. Moreover, plug (52) and (53), (54) into (18), we then have

$$\left\{ \begin{array}{l} \dot{\tilde{P}}(t) + A^\top(t) \tilde{P}(t) + \tilde{P}(t) A(t) - \left(B^\top(t) \tilde{P}(t) + S(t, t) \right)^\top M^{-1}(t, t) \left(B^\top(t) \tilde{P}(t) + S(t, t) \right) \\ \quad + Q(t, t) - \tilde{\Phi}^\top(T, t) \dot{G}(t) \tilde{\Phi}(T, t) - \int_t^T \tilde{\Phi}^\top(s, t) \left[Q_t(t, s) + \tilde{\Upsilon}^\top(s) M_t(t, s) \tilde{\Upsilon}(s) \right. \\ \quad \left. - \tilde{\Upsilon}^\top(s) S_t(t, s) - S_t^\top(t, s) \tilde{\Upsilon}(s) \right] \tilde{\Phi}(s, t) ds = 0, \forall t \in [0, T], \\ \tilde{P}(T) = G(T), \end{array} \right.$$

which shows $P \in C([0, T]; \mathbb{R}^{n \times n})$ is a symmetric solution of the equilibrium Riccati equation (7). Then the uniqueness of the linear equilibrium follows from Theorem 2. This completes the proof.

■

5 Concluding remarks

The equivalence between LQ problems, two-point boundary value problems and Riccati equations plays a significant role in the study of time-consistent LQ problems. We have extended this equivalence result to the time-inconsistent setting. In contrast to the time-inconsistent Riccati equation obtained from the spike variation, the Riccati equation does not involve the time parameter and thus characterising a single dynamics. Thanks to the unique solvability of the Riccati equation and the equivalence, we have obtained the existence and uniqueness of the linear equilibrium for the time-inconsistent LQ problem.

We have to point out that the uniqueness result is only for linear equilibria and deterministic LQ problems. For general time-inconsistent LQ problems, such as stochastic LQ problems, and general equilibria, the uniqueness result remains open.

References

- BASAK, S. AND G. CHABAKAURI (2010): “Dynamic mean-variance asset allocation,” *Review of financial Studies*, 23, 2970–3016.
- BENSOUSSAN, A., K. SUNG, AND S. C. P. YAM (2013): “Linear–quadratic time-inconsistent mean field games,” *Dynamic Games and Applications*, 3, 537–552.
- BJÖRK, T. AND A. MURGOCI (2009): “A general theory of Markovian time inconsistent stochastic control problems,” *Working Paper*.
- BJÖRK, T., A. MURGOCI, AND X. ZHOU (2014): “Mean-variance Portfolio Optimization with State Dependent Risk Aversion,” *Mathematical Finance*, 24, 1–24.

- BLEICHRODT, H., K. I. ROHDE, AND P. P. WAKKER (2009): “Non-hyperbolic time inconsistency,” *Games and Economic Behavior*, 66, 27–38.
- DOU, F. AND Q. LÜ (2020): “Time-inconsistent linear quadratic optimal control problems for stochastic evolution equations,” *SIAM Journal on Control and Optimization*, 58, 485–509.
- EBERT, S., W. WEI, AND X. Y. ZHOU (2020): “Weighted discounting—On group diversity, time-inconsistency, and consequences for investment,” *Journal of Economic Theory*, 189, 105089.
- EKELAND, I. AND A. LAZRAK (2006): “Being Serious About Non-Commitment: Subgame Perfect Equilibrium in Continuous Time,” Working Paper.
- EKELAND, I. AND T. PIRVU (2008): “Investment and consumption without commitment,” *Mathematics and Financial Economics*, 2, 57–86.
- HARVEY, C. (1995): “Proportional Discounting of Future Costs and Benefits,” *Mathematics of Operations Research*, 20, 381–399.
- HU, Y., H. JIN, AND X. Y. ZHOU (2012): “Time-inconsistent stochastic linear–quadratic control,” *SIAM journal on Control and Optimization*, 50, 1548–1572.
- (2017): “Time-inconsistent stochastic linear-quadratic control: characterization and uniqueness of equilibrium,” *SIAM Journal on Control and Optimization*, 55, 1261–1279.
- KARP, L. (2007): “Non-constant discounting in continuous time,” *Journal of Economic Theory*, 132, 557–568.
- KRUSSELL, P. AND A. S. JR. (2003): “Consumption-Savings Decision with Quasi-Geometric Discounting,” *Econometrica*, 71, 365–375.
- LAIBSON, D. (1997): “Golden Eggs and Hyperbolic Discounting,” *Quarterly Journal of Economics*, 112, 443–378.

- LOEWENSTEIN, G. AND D. PRELEC (1992): “Anomalies in intertemporal choice: Evidence and an interpretation,” *The Quarterly Journal of Economics*, 573–597.
- LUTTMER, E. AND T. MARIOTTI (2003): “Subjective Discounting in an Exchange Economy,” *Journal of Politic Economy*, 11, 959–989.
- O’DONOGHUE, T. AND M. RABIN (2001): “Choice and Procrastination,” *Quarterly Journal of Economics*, 116, 112–160.
- PHELPS, E. AND R. POLLAK (1968): “On Second-Best National Saving and Game-Equilibrium Growth,” *Review of Economic Studies*, 35, 185–199.
- STROTZ, R. (1955): “Myopia and Inconsistency in Dynamic Utility Maximization,” *Review of Economic Studies*, 23, 165–180.
- TAN, K. S., W. WEI, AND X. Y. ZHOU (2018): “Failure of Smooth Pasting Principle and Nonexistence of Equilibrium Stopping Rules under Time-Inconsistency,” *arXiv preprint arXiv:1807.01785*.
- YONG, J. (2012): “A deterministic linear quadratic time-inconsistent optimal control problem,” *arXiv preprint arXiv:1204.1856*.
- (2017): “Linear-quadratic optimal control problems for mean-field stochastic differential equations-time-consistent solutions,” *Transactions of the American Mathematical Society*, 369, 5467–5523.
- YONG, J. AND X. Y. ZHOU (1999): *Stochastic controls: Hamiltonian systems and HJB equations*, vol. 43, Springer Science & Business Media.