Moments of the first descending epoch for a random walk with negative drift

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Abstract

We consider the first descending ladder epoch $\tau = \min\{n \geq 1 : S_n \leq 0\}$ of a random walk $S_n = \sum_1^n \xi_i, n \geq 1$ with i.d.d. summands having a negative drift $\mathbb{E}\xi = -a < 0$. Let $\xi^+ = \max(0, \xi_1)$. It is well-known that, for any $\alpha > 1$, the finiteness of $\mathbb{E}(\xi^+)^\alpha$ implies the finiteness of $\mathbb{E}\tau^\alpha$ and, for any $\lambda > 0$, the finiteness of $\mathbb{E}\exp(\lambda\xi^+)$ implies that of $\mathbb{E}\exp(c\tau)$ where $c > 0$ is, in general, another constant that depends on the distribution of ξ_1. We consider the intermediate case, assuming that $\mathbb{E}\exp(g(\xi^+)) < \infty$ for a positive increasing function g such that $\liminf_{x\to\infty} g(x)/\log x = \infty$ and $\limsup_{x\to\infty} g(x)/x = 0$, and that $\mathbb{E}\exp(\lambda\xi^+) = \infty$, for all $\lambda > 0$. Assuming a few further technical assumptions, we show that then $\mathbb{E}\exp((1-\varepsilon)g((1-\delta)a\tau)) < \infty$, for any $\varepsilon, \delta \in (0,1)$.

Keywords: random walk, negative drift, descending ladder epoch, existence of moments, heavy tail.

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1. Introduction and the main result

Let $\xi, \xi_1, \xi_2, \ldots, \xi_n, \ldots$ be independent and identically distributed (i.i.d.) random variables (r.v.'s) with a common distribution function F having a finite negative mean $\mathbb{E}\xi = -a < 0$. Let $S_0 = 0, S_n = \sum_{k=1}^n \xi_k, n \geq 1$ be a random walk, and $\tau = \min\{n \geq 1 : S_n \leq 0\} < \infty$ a.s. its first descending ladder epoch.

The descending ladder epoch τ plays an important role in theoretical and applied probability. In particular, τ represents the length of a busy cycle in a $GI/GI/1$ queueing system. Namely, consider a FIFO single-server queue with i.i.d. interarrival times $\{t_n\}$ with a finite mean $\mathbb{E}t_1 = a$ and independent of them i.i.d. service times $\{\sigma_n\}$ with a finite mean $\mathbb{E}\sigma_1 = b < a$. Let W_n be the waiting time of customer n. Assume $W_1 = 0$, i.e. customer 1 arrives at an empty queue. The sequence $\{W_n\}$ satisfies the *Lindley recursion*

$$W_{n+1} = \max(0, W_n + \sigma_n - t_n) \quad n \geq 1. \tag{1}$$

We may let $\xi_n = \sigma_n - t_n$ and conclude that τ is the number of customers served in the queue during the *first busy cycle*, i.e. customer $\tau + 1$ is the next customer after customer 1 that finds the queue empty.

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We are interested in the existence (finiteness) of moments of τ in terms of moments of the common distribution F of the summands. In particular, the existence of a power (or an exponential) moment of τ implies corresponding convergence rates in stability and continuity theorems for various single- and multi-server queueing systems, see e.g. Theorems 2 and 11 in Chapter 4 of [1].

The following results are known (see, e.g., Theorems III.3.1 and 3.2 in [6], and also [7]). Let $\alpha > 1$ and $\lambda > 0$.

$$\text{If } \mathbb{E}(\xi^+)^\alpha < \infty, \text{ then } \mathbb{E}\tau^\alpha < \infty. \tag{2}$$

$$\text{If } \mathbb{E}\exp(\lambda\xi) < \infty, \text{ then there exists } c > 0 \text{ (that depends on } F) \text{ such that } \mathbb{E}\exp(c\tau) < \infty. \tag{3}$$

One can view (2) and (3) as two particular cases of the following implication:

$$\text{If } \mathbb{E}G(\xi^+) < \infty, \text{ then } \mathbb{E}G(C\tau) < \infty, \text{ for a certain } C > 0. \tag{4}$$

Indeed, (2) is a particular case of (4) with $G(x) = x^\alpha$, and (3) a particular case of (4) with $G(x) = \exp(\lambda x)$ (clearly, for $\lambda > 0$, exponential moments $\mathbb{E}\exp(\lambda\xi)$ and $\mathbb{E}\exp(\lambda\xi^+)$ are either finite or infinite simultaneously).

In this article, we consider the intermediate case where G is a monotone function that increases faster than any power function and slower than any exponential function. It is convenient to us to use representation $G(x) = e^{g(x)}$ and work with function g instead. Here is our main result.

Theorem 1. *Assume that $\mathbb{E}\exp(c\xi) = \infty$, for any $c > 0$. If a function g satisfies conditions $(C1) - (C3)$, introduced below, and if*

$$\mathbb{E}\exp(g(\xi)) < \infty, \tag{5}$$

then

$$\mathbb{E}\exp((1 - \varepsilon)g((a - \delta)\tau)) < \infty, \text{ for any } \varepsilon \in (0, 1) \text{ and } \delta \in (0, a). \tag{6}$$

The conditions $(C1) - (C3)$ are as follows:

- (C1) function g is positive, increasing and differentiable;

- (C2) $\lim_{x \to \infty} g'(x) = 0$;

- (C3) there exist a constant $\gamma \in (0, 1)$ such that

$$\int_1^\infty \exp(-(1 - \gamma)g(x))dx < \infty \tag{7}$$

  and positive constants x_0 and A such that, for any $x_0 < y \leq x/2$,

$$g(x) - g(x - y) \leq \gamma g(y) + A. \tag{8}$$

It follows from condition $(C2)$ that $\sup_{x \geq x_0} g'(x) \downarrow 0$ as $x_0 \to \infty$. Therefore, we may choose x_0 in condition $(C3)$ and constant $B > 0$ such that

$$g'(x) < B, \text{ for } x > x_0. \tag{9}$$

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Remark 1. Conditions $(C1) - (C3)$ are given in the form that are convenient to us, they may be weakened. For example, it is not necessary to assume differentiability, and condition $(C2)$ can be adjusted to 'dying' growth rate that also gives us inequality (9). However, inequalities (7) and (8) are more substantial since they target heavy-tailed "Weibull-type" and "lognormal-type" distributions.

Example 1. Here are examples of functions g that satisfy conditions $(C1) - (C3)$:
$g_1(x) = (\log \max(x, 1))^\alpha$, $g_2(x) = (x^+)^\beta$ and $g_3(x) = (x^+)^\beta \log(max(x, 1))$, where $\alpha > 1$ and $\beta \in (0, 1)$. More generally, the functions g_1 and g_3 continue to satisfy condition $(C1) - (C3)$ if the logarithmic function therein is replaced by a "sufficiently smooth" increasing and slowly varying function.

Remark 2. Note that one can represent (6) in an equivalent form as:

$$\mathbb{E} \exp((1 - \varepsilon)g((1 - \varepsilon)a\tau)) < \infty, \text{ for any } \varepsilon \in (0, 1).$$

On the other hand, given condition (6), the inequality in (6) also holds for function g_1 from Example 1 with $\delta = 0$ and any $\varepsilon \in (0, 1)$, and for functions g_2 and g_3 with $\varepsilon = 0$ and any $\delta \in (0, a)$. Let us show this for g_3. Indeed, for any $\delta_1 \in (0, a)$ there exist $\varepsilon_2 \in (0, 1)$ and $\delta_2 \in (0, a)$ such that

$$\lim_{x \to \infty} \frac{g_3((a - \delta_1)x)}{(1 - \varepsilon_2)g_3((a - \delta_2)x)} = \frac{(a - \delta_1)^\beta}{(1 - \varepsilon_2)(a - \delta_2)^\beta} < 1.$$

Then there exists a constant $c > 0$ such that

$$\mathbb{E} \exp(g_3((a - \delta_1)\tau)) \le c\mathbb{E} \exp((1 - \varepsilon_2)g_3((a - \delta_2)\tau)) < \infty.$$

Our proof of the theorem includes two steps. First, we show the existence of a r.v. $\widetilde{\xi} \ge_{st} \xi$ that has a strong subexponential distribution, negative mean and certain finite moments. Second, we prove that the stopping time for the random walk with new increments $\{\widetilde{\xi}_n\}$ satisfies the conditions of the theorem.

We use the following notation and conventions. For a distribution function F on the real line, $\overline{F}(x) = 1 - F(x)$ is its *tail distribution* function. For two strictly positive functions h_1 and h_2, equivalence $h_1(x) \sim h_2(x)$ means that $\lim_{x \to \infty} h_1(x)/h_2(x) = 1$. For two r.v.s η_1 and η_2, stochastic inequality $\eta_1 \le_{st} \eta_2$ means that $\mathbb{P}(\eta_1 > x) \le \mathbb{P}(\eta_2 > x)$, for all x. For an increasing function g, its (generalised) *inverse function* g^{-1} is defined as $g^{-1}(t) = \inf\{x : g(x) > t\}$. Then the sets $\{g(x) > t\}$ and $\{x > g^{-1}(t)\}$ do coincide. A function f is *slowly varying* if $f(\lambda x)/x \to 1$, as $x \to \infty$, for $\lambda > 0$, and *regularly varying with exponent* α if $f(\lambda x)/x \to \lambda^\alpha$.

2. Proof of the theorem

Recall the following definitions. Let F be a distribution on the real line with right-unbounded support. We say that F is *long-tailed* if $\lim_{x \to \infty} \overline{F}(x - 1)/\overline{F}(x) = 1$. Since the tail function $\overline{F}$ is monotone non-increasing, its long-tailedness implies that $\lim_{x \to \infty} \overline{F}(x - y)/\overline{F}(x) = 1$, for any $y > 0$.

Further, let a distribution F have right-unbounded support and finite mean $m = \int_0^\infty \overline{F}(y)dy$ on the positive half line. We say that F is *strong subexponential* and write $F \in \mathcal{S}^*$ if $\int\limits_0^x \overline{F}(x - y)\overline{F}(y)dy \sim 2m\overline{F}(x)$, as $x \to \infty$.

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The strong subexponentiality is a *tail property*: if a distribution function F is strong subexponential and if G is another distribution function such that $\overline{F}(x) \sim \overline{G}(x)$, then G is also strong subexponential (see, e.g., [4], Theorem 3.11).

2.1. Step one: an upper-bound random variable having a strong subexponential distribution

Let $\zeta = \exp(g(\xi))$. Since $\mathbb{E}\zeta < \infty$, $\mathbb{P}\{\zeta > x\} = o(x^{-1})$, as $x \to \infty$. Then, in particular, one can choose $K \geq \exp(g(x_0))$ (where constant x_0 is from condition $(C3)$) such that $\mathbb{P}\{\zeta > x\} \leq Kx^{-1}$, for all $x > 0$, and introduce a new non-negative r.v. $\widehat{\zeta}$ with the tail distribution

$$\mathbb{P}\{\widehat{\zeta} > x\} = \min(1, Kx^{-1}), \ x \geq 0. \tag{10}$$

Clearly, $\mathbb{P}\{\zeta > x\} \leq \mathbb{P}\{\widehat{\zeta} > x\}$, for all x, and $\mathbb{E}\widehat{\zeta}^{1-\varepsilon} < \infty$, for $\varepsilon \in (0,1)$.

Lemma 1. *Under the assumptions* $(C1) - (C3)$, *the r.v.* $\widehat{\xi} = g^{-1}(\ln(\widehat{\zeta}))$ *has a strong subexponential distribution* $\widehat{F}$.

Proof. We use the following result.

Proposition 1. *(This is a part of Theorem 3.30 from [4]). Let F be a long-tailed distribution on the real line. Let $R(x) = -\ln\overline{F}(x)$. Suppose that there exist $\gamma < 1$ and $A' < \infty$ such that*

$$R(x) - R(x - y) \leq \gamma R(y) + A', \tag{11}$$

for all $x > 0$ and $y \in [0, x/2]$. If, in addition,

$$\text{the function } \exp(-(1-\gamma)R(x)) \text{ is integrable over } [0,\infty), \tag{12}$$

then $F \in \mathcal{S}^*$.

To apply Proposition 1, we need to verify the long-tailedness of $\widehat{F}$ and conditions (11) and (12). First, we show the long-tailedness of $\widehat{F}$. For a fixed $y > 0$ and large x, we have $\mathbb{P}\{\widehat{\xi} > x + y\} = K\exp(-g(x+y))$.

From the first-order Taylor expansion $g(x + y) = g(x) + yg'(z)$, for some $z \in (x, x + y)$, and from condition $(C2)$ we get

$$1 \geq \frac{\mathbb{P}\{\widehat{\xi} > x + y\}}{\mathbb{P}\{\widehat{\xi} > x\}} \geq \frac{\exp(-g(x) - yg'(z))}{\exp(-g(x))} = \exp(-yg'(z)) = \exp(o(1)) = (1 + o(1)),$$

as $x \to \infty$. Thus, the distribution of $\widehat{\xi}$ is long-tailed.

Second, we verify condition (11). It is equivalent to

$$\frac{\overline{H}(x - y)}{\overline{H}(x)} \leq \frac{\exp(A')}{\overline{H}^\gamma(y)}, \tag{13}$$

where $\overline{H}(x) = \mathbb{P}\{\widehat{\zeta} > \exp(g(x))\}$. We take γ from condition $(C3)$. Next we show the existence of an appropriate constant A'.

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Let $x_1 = \inf\{x : \overline{H}(x) < 1\}$. Since we have chosen $K > \exp(g(x_0))$, we get $x_1 \geq x_0$. We consider four cases depending on whether $\overline{H}(x) = 1$ or $\overline{H}(x) = K \exp(-g(x))$.

Assume $x \leq x_1$. Then inequality (13) holds if we take $A' \geq 0$.

Assume $x - y \leq x_1 < x$. Then (13) is equivalent to $K^{-1} \exp(g(x)) \leq \exp(A')$. Since $x/2 \leq x - y \leq x_1$, inequality $A' \geq g(2x_1) - \ln K$ is a sufficient condition on A' to satisfy (13).

Assume $y \leq x_1 < x - y$. Then (13) is equivalent to $\exp(g(x) - g(x - y)) \leq \exp(A')$. Since $g(x - y) = g(x) - yg'(z)$, for $z \in (x - y, x)$, we have $g(x) - g(x - y) = yg'(z) < Bx_1$. Therefore, it is sufficient to assume $A' \geq Bx_1$.

Next, assume $y > x_1$. Then (13) is equivalent to $\exp(g(x) - g(x - y)) \leq K^{-\gamma} \exp(\gamma g(y) + A')$. From condition $(C3)$ it is sufficient to assume $A' \geq A + \gamma \ln K$ for the Proposition 2 to hold.

Finally, condition (12) follows directly from $(C3)$. ∎

By construction, $\mathbb{E} \exp((1 - \varepsilon)g(\widehat{\xi})) = \mathbb{E}\widehat{\zeta}^{1-\varepsilon} < \infty$. However, we need our upper-bound to have sufficiently close mean to the original. Thus, we need the following lemma.

Lemma 2. *Assume that conditions* $(C1) - (C3)$ *hold. For any* $\delta \in (0, a)$*, we can introduce a r.v.* $\widetilde{\zeta}$ *such that* $\widetilde{\xi} = g^{-1}(\ln(\widetilde{\zeta}))$ *has a strong subexponential distribution,* $\widetilde{\xi} \geq_{st} \xi$ *and, in addition,* $\mathbb{E}\widetilde{\xi} < \mathbb{E}\xi + \delta = -a + \delta < 0$.

Proof. Since the distributions of ξ and $\widehat{\xi}$ have right-unbounded support, for all $V > 0$ we can find $V' > V$ such that there exists r.v. $\widetilde{\xi}$ with right tail

$$
\mathbb{P}\{\widetilde{\xi} > t\} = \begin{cases} \mathbb{P}\{\xi > t\}, & t < V, \\ \mathbb{P}\{\xi > V\}, & V \leq t < V', \\ \mathbb{P}\{\widehat{\xi} > t\}, & t \geq V'. \end{cases}
$$

Clearly, $\xi \leq_{st} \widetilde{\xi} \leq_{st} \widehat{\xi}$. Since $\widetilde{\xi}$ and $\widehat{\xi}$ have the same right tail, $\widetilde{\xi}$ has a strong subexponential distribution. By choosing sufficiently large V we can make $\mathbb{E}\widetilde{\xi} = \mathbb{E}\left(\widetilde{\xi}; \widetilde{\xi} \leq V\right) + \int_V^\infty \mathbb{P}\{\widetilde{\xi} > t\}dt = \mathbb{E}\left(\xi; \xi \leq V\right) + \int_V^\infty \mathbb{P}\{\widetilde{\xi} > t\}dt < -a + \delta$. ∎

2.2. Step two: existence of moments of the first descending epoch for strong subexponential distributions

We have introduced a r.v. $\widetilde{\xi}$ with negative drift $\mathbb{E}\widetilde{\xi} = -\widetilde{a} < -a + \delta < 0$ and a finite moment $\mathbb{E} \exp((1 - \varepsilon)g(\widetilde{\xi})) < \infty$, such that $\xi \leq_{st} \widetilde{\xi}$. Now we want to show that the stopping time $\widetilde{\tau}$ satisfies $\mathbb{E} \exp((1 - \varepsilon)g((a - \delta)\widetilde{\tau})) < \infty$.

Without loss of generality, we may assume that the distribution $\widetilde{F}$ of the r.v.'s $\widetilde{\xi}_k$ is bounded below, i.e. $\widetilde{\xi}_k \geq -L$ a.s., for some $L \in (0, \infty)$. Indeed, let us choose an arbitrary $L > 0$ and take $\xi_i' = \max(\widetilde{\xi}_i, -L)$, $i \geq 1$. Then the random walk $S_0' = 0$, $S_n' = \sum_{k=1}^n \xi_k'$ satisfies $S_n' \geq \widetilde{S}_n$ a.s., for all n and, therefore, $\tau' = \inf\{n \geq 1 : S_n' \leq 0\} \geq \widetilde{\tau}$ a.s.

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By taking L large enough, we can make $\mathbb{E}\xi' = \mathbb{E}\widetilde{\xi} - \mathbf{E}(\widetilde{\xi} + L; \widetilde{\xi} \leq -L)$ as close to $\mathbb{E}\widetilde{\xi}$ as one wishes and, in particular, smaller than zero. Since $\sup_{x \leq 0} g(x) < \infty$, condition (5) implies the finiteness of $\mathbb{E}\exp(g(\xi'))$ too. If we prove the statement of Theorem 1 for the random walk with increments ξ'_n, then we prove it for the initial random walk, too.

We write $h(\cdot) = (1 - \varepsilon)g(\cdot)$ for short. We prove now that $\mathbb{E}\exp(h((a - \delta)\widetilde{\tau})) < \infty$. Let $\chi = S_{\widetilde{\tau}}$, $\chi \in [-L, 0]$. We have

$$(a - \delta)\widetilde{\tau} = (a - \delta)\widetilde{\tau} + \chi - \chi \leq ((a - \delta)\widetilde{\tau} + \chi) + L = \sum_{i=1}^{\widetilde{\tau}}(\widetilde{\xi}_i + a - \delta) + L.$$

Let $\psi_i = \widetilde{\xi}_i + a - \delta$. Thus, $\mathbb{E}\psi_1 < 0$ and, since $\mathbb{P}\{\psi_1 > x\} \sim \mathbb{P}\{\widetilde{\xi}_1 > x\}$, r.v. ψ_1 has a strong subexponential distribution. From inequality (9) and the first-order Taylor expansion for h we get $h(x + y) \leq h(x) + (1 - \varepsilon)By$, for $x > x_0$, and thus,

$$\mathbb{E}\exp(h((a-\delta)\widetilde{\tau})) \leq \mathbb{E}\exp\left(h\left(\sum_{i=1}^{\widetilde{\tau}}\psi_i + L\right)\right) \leq \exp\left(h\left(x_0 + L\right)\right) + \exp\left((1 - \varepsilon)BL\right)\mathbb{E}\exp\left(h\left(\sum_{i=1}^{\widetilde{\tau}}\psi_i\right)\right).$$

Further,

$$\mathbb{E}\exp\left(h\left(\sum_{i=1}^{\widetilde{\tau}}\psi_i\right)\right) = \int_0^\infty \mathbb{P}\left\{\exp\left(h\left(\sum_{i=1}^{\widetilde{\tau}}\psi_i\right)\right) > t\right\}dt \leq \int_0^\infty \mathbb{P}\left\{\sum_{i=1}^{\widetilde{\tau}}\psi_i > h^{-1}(\ln t)\right\}dt. \qquad (14)$$

Next, we need the following result:

Proposition 2. *(Theorem 1 in [5]). Let* $\mathbb{E}\psi < 0$ *and let* τ *be a stopping time for* $\{\psi_n\}$. *Denote* $M_\tau = \max_{0 \leq j \leq \tau} \sum_{i=1}^{j} \psi_i$ *and let* $F_\psi(x)$ *be the distribution function of* ψ. *Under condition* $F_\psi \in \mathcal{S}^*$ *we have*

$$\lim_{x \to \infty} \frac{P\{M_\tau > x\}}{\overline{F}_\psi(x)} = \mathbb{E}\tau.$$

Clearly, we can apply Proposition 2: we have $\mathbb{E}\psi < 0$, and ψ has a strong subexponential distribution. Also, the r.v. $\widetilde{\tau}$ is a stopping time w.r.t. $\{\widetilde{\xi}_n\}$ and, therefore, w.r.t. $\{\psi_n\}$. Thus, the conditions of Proposition 2 hold. Now, combining this with (14), we get that, for every $\Delta > 0$, there exists a constant N such that

$$\mathbb{E}\exp\left(h\left(\sum_{i=1}^{\widetilde{\tau}}\psi_i\right)\right) \leq N + \int_N^\infty \mathbb{P}\{M_{\widetilde{\tau}} > h^{-1}(\ln t)\}dt \leq N + (\mathbb{E}\widetilde{\tau} + \Delta)\int_N^\infty \mathbb{P}\{\widetilde{\xi} + a - \delta > h^{-1}(\ln t)\}dt,$$

and the integral on the right-hand side of the latter inequality is finite. This concludes the proof of the theorem.

3. Further comments

In our theorem, the coefficients $(1 - \varepsilon)$ and $(1 - \delta)$ appear because the first moment of the upper-bound distribution in (10) is infinite. The following nice result may help to eliminate the coefficients under certain assumptions discussed below.

Proposition 3. *(Corollary 1 in [2]) Let* ζ *be a nonnegative r.v. and* $\mathbb{E}\zeta^\alpha < \infty$ *for some* $\alpha > 0$. *Then there exists a r.v.* $\widehat{\zeta}$ *such that* $\mathbb{E}\widehat{\zeta}^\alpha < \infty$, $\mathbb{P}\{\widehat{\zeta} > t\}$ *is a function of regular variation with exponent* $-\alpha$, *and* $\zeta \leq_{st} \widehat{\zeta}$.

We can apply Proposition 3 with $\alpha = 1$, $\zeta = \exp(g(\xi))$, and then the upper bound $\widehat{\zeta}$ has the tail distribution $\mathbb{P}\{\widehat{\zeta} \geq x\} \sim l(x)/x$, which is integrable. Here $l(x)$ is a slowly varying function. If in addition $l(x)$ is sufficiently smooth (to be justified), there is a chance to show that $\widehat{\xi} = g^{-1}(\ln \widehat{\zeta})$ has a strong subexponential distribution and $\mathbb{E} \exp(g(\widehat{\xi})) < \infty$. Then the statement of the theorem holds with $\varepsilon = \delta = 0$.

Another way to apply Proposition 3 is to provide an alternative proof of Theorem III.3.1 in [6]. Indeed, in this case the distribution of $(\xi^+)^\alpha$ possesses an integrable majorant having a regularly varying distribution. Since any power of a regularly varying function is also a regularly varying function, the distribution of ξ^+ possesses a majorant having a regularly varying distribution with finite moment of order α. And it is known that any regularly varying distribution with finite mean is strong subexponential.

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