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On equivalence of cyclic and dihedral zero-divisor codes having nilpotents of nilpotency degree two as generators

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Abstract
Zero-divisor codes are codes constructed using group rings where their generators are zero-divisors. Generally, zero-divisor codes can be equivalent despite their associated groups are non-isomorphic, leading to the proposed conjecture "Every dihedral zero-divisor code has an equivalent form of cyclic zero-divisor code". This paper is devoted to study equivalence of zero-divisor codes in F_2G having generators from the 2-nilradical of F_2G, consisting of all nilpotents of nilpotency degree 2 of F_2G. Essentially, algebraic structures of 2-nilradicals are first studied in general for both commutative and non-commutative F_2G before specialized into the case when G is cyclic and dihedral. Then, results are used to study the conjecture above in the cases where the codes generators are from their respective 2-nilradicals.

Keywords Group ring codes · Zero-divisor codes · Nilpotents · Code equivalence

Mathematics Subject Classification 94B99 · 20C05 · 16N40

1 Introduction

Codes defined by non-trivial ideals of group algebras are called group codes where their generators are essentially zero-divisors [1, 5]. Generally, a group code in F_2G has the form of $F_2G\{u_1, u_2, \dots, u_k\}$ where $\{u_1, u_2, \dots, u_k\} \subseteq F_2G$ is called a set of generators of the group code. In 2009, Ted and Paul Hurley introduced a new encoding method using group rings, where the resultant codes need not have ideal structures and are called group ring codes [4]. A group ring code in F_2G is defined as $Wu = \{wu | w \in W\}$, where W is a F_2-submodule

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of F_2G and $u \in F_2G$. In addition, if the generator $u \in F_2G$ is a zero-divisor or unit, then Wu is called a zero-divisor code or unit-derived code respectively. For every $u = \sum_{g \in G} a_g g \in F_2G$, define its support $supp(u) = \{g \in G \mid a_g \neq 0\}$ and weight $wt(u) = |supp(u)|$. Details about group rings can be found in [6].

It is instructive to note that zero-divisor codes are of single generator whereas group codes can have multiple generators in general. Moreover, any zero-divisor code Wu possesses module structure and is in fact a subcode of the corresponding group code F_2Gu. Note that when $W = F_2G$, the resultant zero-divisor code Wu possess left ideal structure.

Two length n binary codes are said to be equivalent if one can be obtained from another via permuting the n digits of the codewords. The main study of equivalent codes is to develop conditions which classify codes that look different as a set but perform alike as codes from both theoretical and practical points. Equivalent codes exhibit similar algebraic structures as well as always having identical parameters.

Some results on conditions for equivalence between abelian group codes are given in [3]. However, there are instances where despite two groups G_1 and G_2 are non-isomorphic, equivalence can be established between group ring codes in F_2G_1 and F_2G_2 as in [10], where $G_1 = D_{2n}$ and $G_2 = C_{2n}$ are the dihedral and cyclic groups of order $2n$ respectively. This led to the conjecture "Every zero-divisor code in F_2D_{2n} is equivalent to some zero-divisor codes in F_2C_{2n}" [10].

To answer this conjecture, a comprehensive study on equivalence between zero-divisor codes over F_2 was done by Ong and Ang in 2019 [7]. In the same paper [7], the equivalence of zero-divisor codes having idempotents as generators were studied in details. This was done by using a new classification of idempotents in F_2G which was introduced in [8, 9]. As a result, the conjecture "Every zero-divisor code in F_2D_{2n} is equivalent to some in F_2C_{2n}" was affirmed in the cases where the generators are those classified idempotents [7].

An element of a ring $u \in R$ is called a nilpotent if there exists $n \in \mathbb{Z}^+$ such that $u^n = 0$, where the smallest such n is termed the nilpotency degree of u. Consider a group algebra F_2G, where G is a finite group. In this paper, we are particularly interested with those $u \in F_2G$ with $u^2 = 0$, namely the nilpotents of nilpotency degree 2. The set of all nilpotents of nilpotency degree 2 together with 0 is called the 2-nilradical of F_2G, denoted by $Nil_2(F_2G)$.

As nilpotents are also zero-divisors, this paper is mainly devoted to study the equivalence between dihedral and cyclic zero-divisor codes having generators from their respective $Nil_2(F_2G)$. Section 2 comprises some relevant results on equivalent zero-divisor codes in [7], followed by some basic properties of nilpotents. In Sect. 3 and 4, we study $Nil_2(F_2G)$ for both commutative and non-commutative F_2G then follow up with the specific cases when G is cyclic and dihedral. The results are then applied in Sect. 5 to establish the equivalence between some dihedral and cyclic zero-divisor codes having generators from their respective 2-nilradicals. In particular, given a dihedral zero-divisor code with generator from its $Nil_2(F_2G)$, we study the sufficient conditions that ensure the code being equivalent to a cyclic zero-divisor code having generator from its respective $Nil_2(F_2G)$.

2 Preliminaries

In 1962, MacWilliams had affirmed that two linear codes of the same length are equivalent if and only if there is a weight-preserving isomorphism between them [2]. Here, we give a definition of equivalent zero-divisor codes by slightly modifying MacWilliams's into the following analogous version.

Definition 2.1 Let G_1 and G_2 be two finite groups with $|G_1| = |G_2|$. Let C_1 and C_2 be zero-divisor codes in F_2G_1 and F_2G_2 respectively. Then, C_1 and C_2 are said to be equivalent if there exists an F_2-linear isomorphism $\varphi : C_1 \to C_2$ such that for every codeword $c \in C_1$, $wt(c) = wt((c)\varphi)$.

The property $wt(c) = wt((c)\varphi)$ for every codeword $c \in C_1$ is called weight-preserving. Recall that for every F_2-submodule W of F_2G, Wu is an F_2-submodule of the corresponding group code F_2Gu. Then, the following proposition clearly holds.

Proposition 2.2 *Let $\varphi : F_2G_1u_1 \to F_2G_2u_2$ be a weight-preserving F_2-linear isomorphism. Restrict the domain of φ to C_1u, an F_2-submodule of $F_2G_1u_1$ by defining $\varphi_{\downarrow} : C_1u \to F_2G_2u_2$ in such a way that $(c)\varphi_{\downarrow} = (c)\varphi$ for every $c \in C_1u$. Then, C_1u is equivalent to $Im(\varphi_{\downarrow})$, an F_2-submodule of F_2G_2.*

Proposition 2.2 ensures that if $F_2G_1u_1$ and $F_2G_2u_2$ are equivalent, essentially each of the F_2-submodule of $F_2G_1u_1$ has an equivalent form in $F_2G_2u_2$. Sufficiently, the study of equivalence in this paper is done on group codes $F_2G_1u_1$ and $F_2G_2u_2$ in this paper.

For distinct $F_2G_1u_1$ and $F_2G_2u_2$, the problem of determining their equivalence has been structured into finding a weight-preserving F_2-linear isomorphism $\varphi : F_2G_1u_1 \to F_2G_2u_2$. We concentrated on φ which are expressible in terms of some bijective $\chi : G_1 \to G_2$, that is $(\sum_{g\in G_1} a_g g)\varphi = \sum_{g\in G_1} a_g\chi(g)$. Then, it is proven that the exhibition of φ being weight-preserving F_2-linear isomorphic is sufficiently dependent on the properties of χ illustrated as follows.

Definition 2.3 Let u_1 and u_2 be zero-divisors in F_2G_1 and F_2G_2 respectively. Let $\chi : A \to B$ be a bijective function for $A \subseteq G_1$ and $B \subseteq G_2$. Then:

1. χ is denoted as χ_{u_1,u_2} if $\chi(supp(u_1)) = supp(u_2)$.
2. χ is termed a u_1-homomorphism if $\chi(gh) = \chi(g)\chi(h)$ for every $g \in A$ and $h \in supp(u_1)$.

The above property of bijective map χ was first introduced in [7] and shown to be sufficient for resulting in equivalence between group codes as follows.

Theorem 2.4 *Let u_1 and u_2 be zero-divisors in F_2G_1 and F_2G_2 respectively. If there exists $\chi_{u_1,u_2} : G_1 \to G_2$ which is a u_1-homomorphism, then $F_2G_1u_1$ and $F_2G_2u_2$ are equivalent codes.*

Proof Refer to Theorem 4.4 in [7].

In the same paper [7], Theorem 4.4 was utilized to study the equivalence of zero-divisor codes having idempotents as generators. This was done by using a new classification of idempotents in F_2G which was introduced by us in [8, 9]. As a result, we are able to show that each dihedral zero-divisor codes in F_2D_{2n} with our classified idempotent generators has an equivalent cyclic form in F_2C_{2n} [7].

The remaining of this section is devoted to discuss some properties of nilpotents. Let R be a commutative ring. The set of all nilpotents in R is called the nilradical and is denoted by $Nil(R)$. It is well known that $Nil(R)$ is an ideal when R is commutative. In this paper, we use the notion of $Nil(R)$ to represent the set of all nilpotents in R, for general R. Let I be an ideal of R. If $I \subseteq Nil(R)$, then I is called a nil-ideal of R.

Let G be a finite group, then $g \in G$ is called an involution if $ord(g) = 2$. We denote the set of all involutions in G by $Inv(G)$. Let $u = 1 + g \in F_2G$ where $g \in Inv(G)$, then clearly

$u^2 = (1+g)^2 = 0$, that is $u \in Nil(F_2G)$. If $u \in Nil(F_2G) \setminus \{0\}$ with $u^2 = 0$, then u is a nilpotent of nilpotency degree 2. Recall that the set of all nilpotents of nilpotency degree 2 together with 0 is called the 2-nilradical of F_2G and is denoted by $Nil_2(F_2G)$. Clearly, $\{1+g \mid g \in Inv(G)\} \subseteq Nil_2(F_2G)$. In addition, any ideal $I \subseteq Nil_2(F_2G)$ is called a 2-nil ideal.

3 $Nil_2(F_2G)$ for commutative F_2G

Throughout this section, G is fixed to be finite abelian and thus F_2G is a commutative ring. First, we show that $Nil_2(F_2G)$ possesses ideal structure.

Proposition 3.1 *Let F_2G be a commutative ring. Then, $Nil_2(F_2G)$ is an ideal of F_2G.*

Proof Let $u_1, u_2 \in Nil_2(F_2G)$, then $u_1^2 = 0$ and $u_2^2 = 0$. Note that $(u_1+u_2)^2 = u_1^2 + u_1u_2 + u_2u_1 + u_2^2 = 0 + 2u_1u_2 + 0 = 0$ and $(u_1u_2)^2 = u_1u_2u_1u_2 = u_1^2u_2^2 = 0$. Also, for $v \in F_2G$, $vu_1 \in Nil_2(F_2G)$ since $(vu_1)^2 = vu_1vu_1 = v^2u_1^2 = v^2(0) = 0$. □

Let $u = \sum\limits_{i=1}^{k} g_i \in F_2G$ for some $k \in \mathbb{Z}^+$. As $char(F_2) = 2$, it can be easily verified that for F_2G, $u^2 = (\sum\limits_{i=1}^{k} g_i)^2 = \sum\limits_{i=1}^{k} g_i^2$. Hence $u^2 = 0$ if and only if $\sum\limits_{i=1}^{k} g_i^2 = 0$ and k must be even. This leads to Proposition 3.2.

Proposition 3.2 *Let $u \in F_2G$ for some commutative F_2G. Then, $u = \sum\limits_{i=1}^{k} g_i \in Nil_2(F_2G) \setminus \{0\}$ if and only if $supp(u)$ can be partitioned into $\frac{k}{2}$ pairs of $g_s \neq g_t$, with $s, t \in \{1, 2, \ldots, k\}$ such that $g_s^2 = g_t^2$ or equivalently $g_s + g_t \in Nil_2(F_2G) \setminus \{0\}$.*

Now, our task is to identify $Nil_2(F_2G)$. As a starting point, we show that when $|G|$ is odd, $Nil_2(F_2G)$ is trivial.

Proposition 3.3 *Let G be an abelian group of odd order. Then, $Nil_2(F_2G) = \{0\}$.*

Proof Write $G = \mathop{\times}\limits_{i=1}^{k} C_{n_i}$ where each $C_{n_i} = \langle x_i \rangle$ has odd order n_i. Let $u = \sum\limits_{i=1}^{l} g_i \in Nil_2(F_2G)$. Suppose that $Nil_2(F_2G)$ is non-trivial, by Proposition 3.2, we can always partition $supp(u)$ into pairs of $g_s \neq g_t$ with $g_s^2 = g_t^2$. Let $g_s = \prod\limits_{i=1}^{k} x_i^{s_i}$ and $g_t = \prod\limits_{i=1}^{k} x_i^{t_i}$. Then, $g_s^2 = g_t^2$ implies that for every $i \in \{1, 2, \ldots, k\}$, $2s_i = 2t_i (\text{mod } n_i)$. Since for each i, n_i is odd, this results in $s_i = t_i (\text{mod } n_i)$, that is $g_s = g_t$. Therefore, $Nil_2(F_2G) = \{0\}$. □

Next, for the case when G is of even order, note that $Inv(G)$ is always non-empty. The following theorem identifies $Nil_2(F_2G)$ using elements in $Inv(G)$. □

Theorem 3.4 *Let G be an abelian group of even order. Then, $Nil_2(F_2G) = F_2G\{1+g \mid g \in Inv(G)\}$.*

Proof Let $G = H \times K$ where $|H|$ is odd and $|K| = 2^l$ for some $l \in \mathbb{Z}^+$. Let $K = \mathop{\times}\limits_{i=1}^{k} C_{n_i}$ where each $C_{n_i} = \langle x_i \rangle$. Let $u \in Nil_2(F_2G) \setminus \{0\}$. It follows from Proposition 3.2 that

$supp(u)$ can be partitioned into pairs of $g_s \neq g_t$, with $s, t \in \{1, 2, \ldots, k\}$ such that $g_s^2 = g_t^2$. Let $g_s = h_s \prod_{i=1}^{k} x_i^{s_i}$ and $g_t = h_t \prod_{i=1}^{k} x_i^{t_i}$ for some $h_s, h_t \in H$. Then, $g_s^2 = g_t^2$ implies that $h_s^2 \prod_{i=1}^{k} x_i^{2s_i} = h_t^2 \prod_{i=1}^{k} x_i^{2t_i}$. As $ord(h_s)$ and $ord(h_t)$ are both odd, as illustrated in the proof of Proposition 3.3, we have $h_s = h_t$. Now, from $h_s^2 \prod_{i=1}^{k} x_i^{2s_i} = h_s^2 \prod_{i=1}^{k} x_i^{2t_i}$, we have $2s_i \equiv 2t_i \pmod{n_i}$ for every $i \in \{1, 2, \ldots, k\}$. Since n_i is even, this results in $s_i \equiv t_i \pmod{\frac{n_i}{2}}$. Hence, for each $i \in \{1, 2, \ldots, k\}$, either $s_i \equiv t_i \pmod{n_i}$ or $s_i \equiv t_i + \frac{n_i}{2} \pmod{n_i}$. Then, re-enumerate $\prod_{i=1}^{k} x_i^{s_i}$ and $\prod_{i=1}^{k} x_i^{t_i}$ in the form of $\prod_{i=1}^{m} x_i^{s_i} \prod_{i=m+1}^{k} x_i^{s_i}$ and $\prod_{i=1}^{m} x_i^{t_i} \prod_{i=m+1}^{k} x_i^{t_i}$ respectively, where $s_i \equiv t_i \pmod{n_i}$ for every $i \in \{1, 2, \ldots, m\}$ and $s_i \equiv t_i + \frac{n_i}{2} \pmod{n_i}$ for every $i \in \{m+1, m+2, \ldots, k\}$. This leads to $\prod_{i=1}^{m} x_i^{t_i} = \prod_{i=1}^{m} x_i^{s_i}$ and $\prod_{i=m+1}^{k} x_i^{t_i} = \prod_{i=m+1}^{k} x_i^{s_i + \frac{n_i}{2}}$. Hence:

$$\begin{aligned}
g_s + g_t &= h_s \prod_{i=1}^{m} x_i^{s_i} \prod_{i=m+1}^{k} x_i^{s_i} + h_t \prod_{i=1}^{m} x_i^{t_i} \prod_{i=m+1}^{k} x_i^{t_i} \\
&= h_s \prod_{i=1}^{m} x_i^{s_i} \left(\prod_{i=m+1}^{k} x_i^{s_i} + \prod_{i=m+1}^{k} x_i^{s_i + \frac{n_i}{2}} \right) \\
&= h_s \prod_{i=1}^{m} x_i^{s_i} \prod_{i=m+1}^{k} x_i^{s_i} \left(1 + \prod_{i=m+1}^{k} x_i^{\frac{n_i}{2}} \right) \in F_2G \left(1 + \prod_{i=m+1}^{k} x_i^{\frac{n_i}{2}} \right)
\end{aligned}$$

Thus, $g_s + g_t \in F_2G(1 + \prod_{i=m+1}^{k} x_i^{\frac{n_i}{2}}) \subseteq F_2G\{1 + g \mid g \in Inv(G)\}$ as $\prod_{i=m+1}^{k} x_i^{\frac{n_i}{2}} \in Inv(G)$. Then, it follows from the closure property of addition that $u \in F_2G\{1 + g \mid g \in Inv(G)\}$ and $Nil_2(F_2G) \subseteq F_2G\{1 + g \mid g \in Inv(G)\}$. Conversely, let $u \in F_2G\{1 + g \mid g \in Inv(G)\}$. Express $u = \sum_{g \in Inv(G)} u_g(1 + g)$ where each $u_g \in F_2G$. Then, $u^2 = (\sum_{g \in Inv(G)} u_g(1 + g))^2 = \sum_{g \in Inv(G)} u_g^2(1 + g)^2 = \sum_{g \in Inv(G)} u_g^2(0) = 0$. Hence, $u \in F_2G\{1 + g \mid g \in Inv(G)\} \subseteq Nil_2(F_2G)$. This concludes that $Nil_2(F_2G) = F_2G\{1 + g \mid g \in Inv(G)\}$. □

We conclude this section with the case when G is cyclic. Consider $C_n = \langle x \rangle$ for some even $n \in \mathbb{Z}^+$, since $Inv(C_n) = \{x^{\frac{n}{2}}\}$, Corollary 3.5 follows directly from the theorem above.

Corollary 3.5 *Let $C_n = \langle x \rangle$ with n being even. The 2-nilradical of F_2C_n is $F_2C_n(1 + x^{\frac{n}{2}})$.*

4 $Nil_2(F_2G)$ for non-commutative F_2G

Throughout this section, G is fixed to be finite non-abelian and thus F_2G is a non-commutative ring, unless otherwise stated. For $u_1, u_2 \in F_2G$, their commutator is defined as $[u_1, u_2] = u_1u_2 + u_2u_1$. Note that $u_1u_2 = u_2u_1$ if and only if $[u_1, u_2] = 0$.

For commutative F_2G, Theorem 3.4 implies that for every $g \in Inv(G)$ and $h \in G$, we have $h(1 + g) \in Nil_2(F_2G)$. However, for non-commutative F_2G, this desirable condition need not hold for every $h \in G$. In fact, the details are given by the following lemma.

Lemma 4.1 *Let $g \in Inv(G)$. Then, for each $h \in G$, $h(1+g) \in Nil_2(F_2G)$ if and only if $[g,h]=0$.*

Let us clarify some notations. The centralizer of a set $S \subseteq G$ is denoted by $C_G(S)$. For convenience, if $S=\{s\}$, we write $C_G(\{s\})$ as $C_G(s)$. Also, $Z(G)$ denotes the centre of G. Then, the next result follows from Lemma 4.1.

Proposition 4.2 *Let $g \in Inv(G) \cap Z(G)$. Then, for every $h \in G$, $h(1+g) \in Nil_2(F_2G)$. Moreover, $F_2G(1+g)$ is a 2-nil ideal.*

Recall that for commutative F_2G, $\{1+g \mid g \in Inv(G)\}$ serves as a generating set of $Nil_2(F_2G)$. Unfortunately, this property fails to hold for non-commutative F_2G in two different ways.

First, we claim that there exists $u \in Nil_2(G)$ which is not expressible as a summation of $h(1+g) \in Nil_2(F_2G)$ for $g \in Inv(G)$ and $h \in G$. Before proving this, the following proposition is necessary. Note that for a subgroup H of G, the notation $\bar{H}$ is used to denote $\sum_{h\in H} h \in F_2G$.

Proposition 4.3 *Let H be an even order subgroup of G.Then, $\bar{H} \in Nil_2(F_2G)$.*

Proof Since $|H|$ is even, this results in $\bar{H}^2 = \bar{H}\bar{H} = |H|\bar{H} = 0$. □

Consider the dihedral group $D_6 = \langle a, b \mid a^3 = b^2 = 1, ab = ba^{-1}\rangle$. Proposition 4.3 guarantees that $\bar{D}_6 \in Nil_2(F_2D_6)$. It can be easily verified that $Inv(D_6) = \{a^ib \mid i \in \{0,1,2\}\}$ where for each a^ib, $C_G(a^ib) = \{1, a^ib\}$. Note that $a^ib(1+a^ib) = 1(1+a^ib) = 1+a^ib$. This concludes that $1+b$, $1+ab$ and $1+a^2b$ are the only possible choices of $h(1+g) \in Nil_2(F_2D_{2n})$. Clearly, summation between those elements would never yield $\bar{D}_6$.

Second, note that $Nil_2(F_2G)$ need not be closed under addition, as shown by the counterexample $1+b, 1+ab \in F_2D_6$. To understand the algebraic structure of $Nil_2(F_2G)$, the conditions that are necessary or sufficient for $u_1+u_2 \in Nil_2(F_2G)$ for distinct $u_1, u_2 \in Nil_2(F_2G)$ is studied.

Proposition 4.4 *Let H and K be even order subgroups of G where H is normal. Then, $\bar{H}+\bar{K} \in Nil_2(F_2G)$.*

Proof Note that $(\bar{H}+\bar{K})^2 = \bar{H}^2 + \bar{H}\bar{K} + \bar{K}\bar{H} + \bar{K}^2$. By Proposition 4.3, we have $\bar{H}^2 = \bar{K}^2 = 0$ and this yields $(\bar{H}+\bar{K})^2 = \bar{H}\bar{K}+\bar{K}\bar{H} = \sum_{k\in K}\bar{H}k + \sum_{k\in K} k\bar{H} = \sum_{k\in K} k\bar{H} + \sum_{k\in K} k\bar{H} = 0$. □

The remaining of this section is devoted to discuss the case when G is dihedral, where the results are significantly used to prove our main result later. In this paper, $D_{2n} = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1}\rangle$ denotes a dihedral group of order $2n$ where $n > 2$. Note that $Inv(D_{2n}) = \{a^ib \mid 0 \le i \le n-1\}$ if n is odd and $Inv(D_{2n}) = \{a^ib \mid 0 \le i \le n-1\} \cup \{a^{\frac{n}{2}}\}$ if n is even.

Starting from the case of odd n, for each $g \in Inv(G)$, we first identify those corresponding $h \in G$ which results in $h(1+g) \in Nil_2(F_2D_{2n})$, then followed up by a study of the closure of addition between each pair of $h(1+g) \in Nil_2(F_2D_{2n})$. Note that D_{2n} for odd n is centerless, that is $Z(D_{2n}) = \{1\}$.

Proposition 4.5 *Let $g \in Inv(D_{2n})$ for odd $n > 2$. Then, for every $h \in D_{2n}$, either $h(1+g) = 1+g$ or $h(1+g) \notin Nil_2(F_2D_{2n})$.*

Proof For each $g = a^i b \in Inv(D_{2n})$ with $i \in \{0, 1, 2, \cdots, n-1\}$, note that $C_{D_{2n}}(a^i b) = \{1, a^i b\}$ and $a^i b(1 + a^i b) = 1(1 + a^i b) = 1 + a^i b$. Hence, for every $h \in C_{D_{2n}}(a^i b)$, $h(1+g) = 1+g$. Suppose that $h \notin C_{D_{2n}}(a^i b)$, then $[h, g] \neq 0$. It follows from Lemma 4.1 that $h(1+g) \notin Nil_2(F_2 D_{2n})$. □

Now, for distinct $g_1, g_2 \in Inv(D_{2n})$ for odd $n > 2$, we show that the sum of $(1+g_1)$ and $(1+g_2)$ can never be in $Nil_2(F_2 D_{2n})$.

Proposition 4.6 *For distinct $g_1, g_2 \in Inv(D_{2n})$ for odd $n > 2$, $(1+g_1) + (1+g_2) \notin Nil_2(F_2 D_{2n})$.*

Proof Let $g_1 = a^i b$ and $g_2 = a^j b$ for some distinct $i, j \in \{0, 1, 2, \cdots, n-1\}$. Note that $(1 + a^i b) + (1 + a^j b) = a^i b + a^j b$. Then, $a^i b + a^j b \in Nil_2(F_2 D_{2n})$ if and only if $[a^i b, a^j b] = 0$.

Assume that $[a^i b, a^j b] = a^i b a^j b + a^j b a^i b = a^{i-j} + a^{j-i} = 0$. This results in $i - j \equiv j - i (\bmod n)$ or equivalently $2i \equiv 2j (\bmod n)$. Since n is odd, we have $i \equiv j (\bmod n)$, a contradiction. Hence, $[a^i b, a^j b] \neq 0$. □

Now, for even $n > 2$, recall that $Inv(D_{2n}) = \{a^i b \,|\, 0 \leq i \leq n-1\} \cup \{a^{\frac{n}{2}}\}$. Note that $a^{\frac{n}{2}} \in Z(D_{2n}) \cap Inv(D_{2n})$, then by Proposition 4.2, the following result holds trivially.

Proposition 4.7 *Let $n > 2$ be even. Then, $F_2 D_{2n}(1 + a^{\frac{n}{2}})$ is a 2-nil ideal.*

If $g \in Inv(D_{2n}) \setminus \{a^{\frac{n}{2}}\}$, note that we have $C_{D_{2n}}(g) = \{1, a^{\frac{n}{2}}, g, ga^{\frac{n}{2}}\}$. Then, the proposition below follows from Lemma 4.1.

Proposition 4.8 *Let $g \in Inv(D_{2n}) \setminus \{a^{\frac{n}{2}}\}$ for even $n > 2$. Then, for $h \in D_{2n}$, $h(1+g) \in Nil_2(F_2 D_{2n})$ if and only if $h \in \{1, a^{\frac{n}{2}}, g, ga^{\frac{n}{2}}\}$.*

It is instructive to note that for every $g \in Inv(D_{2n}) \setminus \{a^{\frac{n}{2}}\}$, $g(1+g) = 1+g$ and $ga^{\frac{n}{2}}(1+g) = a^{\frac{n}{2}}(1+g)$. Hence, for every $g \in Inv(D_{2n}) \setminus \{a^{\frac{n}{2}}\}$, we have $1+g, a^{\frac{n}{2}}(1+g) \in Nil_2(F_2 D_{2n})$. Then, this leads to the result below.

Proposition 4.9 *Consider distinct $a^i b, a^j b \in Inv(D_{2n})$ for even $n > 2$ and let $k_1, k_2 \in \{0, 1\}$. Then, $(a^{\frac{n}{2}})^{k_1}(1 + a^i b) + (a^{\frac{n}{2}})^{k_2}(1 + a^j b) \in Nil_2(F_2 D_{2n})$ if and only if $i \equiv j + \frac{n}{2} (\bmod n)$.*

Proof Note that $(a^{\frac{n}{2}})^{k_1}(1+a^i b) + (a^{\frac{n}{2}})^{k_2}(1+a^j b) \in Nil_2(F_2 D_{2n})$ if and only if $[(a^{\frac{n}{2}})^{k_1}(1+a^i b), (a^{\frac{n}{2}})^{k_2}(1+a^j b)] = 0$ or equivalently $[1+a^i b, 1+a^j b] = 0$ since $(a^{\frac{n}{2}})^{k_1}, (a^{\frac{n}{2}})^{k_2} \in Z(G)$. Clearly, $[1+a^i b, 1+a^j b] = 0$ if and only if $[a^i b, a^j b] = 0$. Then, it follows from the proof of Proposition 4.6 that $[a^i b, a^j b] = 0$ if and only if $2i \equiv 2j (\bmod n)$. As n is even, this leads to $i \equiv j (\bmod \frac{n}{2})$. □

In addition, each of the summations obtained from Proposition 4.9 is in fact an element in $F_2 D_{2n}(1 + a^{\frac{n}{2}})$.

Proposition 4.10 *Consider distinct $a^i b, a^j b \in Inv(D_{2n})$ for even $n > 2$ and let $k_1, k_2 \in \{0, 1\}$. If $i \equiv j + \frac{n}{2} (\bmod n)$, then $(a^{\frac{n}{2}})^{k_1}(1 + a^i b) + (a^{\frac{n}{2}})^{k_2}(1 + a^j b) \in F_2 D_{2n}(1 + a^{\frac{n}{2}})$.*

Proof Suppose that $k_1 = k_2$, if $i \equiv j + \frac{n}{2} (\bmod n)$, then:

$$\begin{aligned}(a^{\frac{n}{2}})^{k_1}(1 + a^{j+\frac{n}{2}} b) + (a^{\frac{n}{2}})^{k_1}(1 + a^j b) &= (a^{\frac{n}{2}})^{k_1}(a^j b + a^{j+\frac{n}{2}} b) \\ &= (a^{\frac{n}{2}})^{k_1} a^j b(1 + a^{\frac{n}{2}}) \in F_2 D_{2n}(1 + a^{\frac{n}{2}})\end{aligned}$$

On the other hand, suppose that $k_1 \neq k_2$, if $i \equiv j + \frac{n}{2} (\mathrm{mod}\, n)$, then:

$$\begin{aligned}(a^{\frac{n}{2}})^{k_1}(1 + a^{j+\frac{n}{2}}b) + (a^{\frac{n}{2}})^{k_2}(1 + a^j b) &= (a^{\frac{n}{2}})^{k_1}(1 + a^{j+\frac{n}{2}}b + a^{\frac{n}{2}}(1 + a^j b)) \\ &= (a^{\frac{n}{2}})^{k_1}(1 + a^{\frac{n}{2}}) \in F_2 D_{2n}(1 + a^{\frac{n}{2}}).\end{aligned}$$

□

5 Main results

This section is devoted to study the equivalence of dihedral and cyclic zero-divisor codes having generators from their respective 2-nilradicals. Recall that Corollary 3.5 suggests that the 2-nilradical of F_2C_{2n} is $F_2C_{2n}(1+x^n)$ whereas Proposition 4.7 indicates that $F_2D_{2n}(1+a^{\frac{n}{2}})$ is a 2-nil ideal for even $n > 2$.

As a beginning, we start with a more general result on the equivalence between zero-divisor codes $F_2G_1(1 + g_1)$ and $F_2G_2(1 + g_2)$ for some $g_1 \in Inv(G_1)$ and $g_2 \in Inv(G_2)$. We show that the condition $|G_1| = |G_2|$ is sufficient to ensure that equivalence between $F_2G_1(1 + g_1)$ and $F_2G_2(1 + g_2)$ holds, regardless of the algebraic structures of G_1 and G_2.

Theorem 5.1 *Let G_1 and G_2 be finite groups with $|G_1| = |G_2|$ being even. If $g_1 \in Inv(G_1)$ and $g_2 \in Inv(G_2)$, then $F_2G_1(1 + g_1)$ and $F_2G_2(1 + g_2)$ are equivalent codes.*

Proof Let $|G_1| = |G_2| = 2n$. As $\{1, g_1\}$ and $\{1, g_2\}$ are order 2 subgroups of G_1 and G_2 respectively, let us consider a complete listing of the cosets of $\{1, g_1\}$ and $\{1, g_2\}$ in G_1 and G_2 respectively as follows:

$$\begin{array}{ll} H_1 = h_1\{1, g_1\} & H_1' = h_1'\{1, g_2\} \\ H_2 = h_2\{1, g_1\} & H_2' = h_2'\{1, g_2\} \\ & \vdots \\ H_n = h_n\{1, g_1\} & H_n' = h_n'\{1, g_2\} \end{array}$$

The mutually disjoint property of the cosets results in $\{\bar{H}_i \mid i \in \{1, 2, \ldots, n\}\}$ and $\{\bar{H}_i' \mid i \in \{1, 2, \ldots, n\}\}$ are bases for $F_2G_1(1+g_1)$ and $F_2G_2(1+g_2)$ respectively. Define $\varphi : F_2G_1(1+g_1) \to F_2G_2(1 + g_2)$ as a F_2-linear extension in such a way that $\varphi : \bar{H}_i \mapsto \bar{H}_i'$ for each $i \in \{1, 2, \ldots, n\}$. Clearly, φ is a F_2-linear isomorphism. In addition, the fact that φ is weight-preserving follows directly from the mutually disjoint property of cosets. □

Despite Theorem 5.1 affirms the equivalence between $F_2C_{2n}(1+x^n)$ and $F_2D_{2n}(1+a^{\frac{n}{2}})$, we give an alternative proof below by constructing a weight-preserving F_2-linear isomorphism $\varphi : F_2D_{2n}(1+a^{\frac{n}{2}}) \to F_2C_{2n}(1+x^n)$ in the form of $(\sum_{g\in G_1} a_g g)\varphi = \sum_{g\in G_1} a_g \chi(g)$ for some bijective $\chi : D_{2n} \to C_{2n}$, which is essential for further studies.

Consider the following bijection $\chi : D_{2n} \to C_{2n}$ introduced in Theorem 5.2 from [7]:

$$\chi(a^i b^j) = \begin{cases} x^{-2i-1} & j = 1 \\ x^{2i} & j = 0 \end{cases}$$

Note that $\chi(\{1, a^{\frac{n}{2}}\}) = \{1, x^n\}$. Let $g = a^i b^j \in D_{2n}$ for $i \in \{0, 1, \cdots, n-1\}$ and $j \in \{0, 1\}$, clearly $\chi(g(1)) = \chi(g)\chi(1)$. We claim that $\chi(ga^{\frac{n}{2}}) = \chi(g)\chi(a^{\frac{n}{2}})$. Suppose that $j = 0$, that is $g = a^i$, then:

$$\begin{aligned}\chi(a^i a^{\frac{n}{2}}) &= \chi(a^{i+\frac{n}{2}}) \\ &= x^{2i+n} \\ &= x^{2i}x^n \\ &= \chi(a^i)\chi(a^{\frac{n}{2}}).\end{aligned}$$

Suppose that $j = 1$, that is $g = a^i b$, then:

$$\begin{aligned}\chi(a^i b a^{\frac{n}{2}}) &= \chi(a^{i+\frac{n}{2}}b) \\ &= x^{-2i-n-1} \\ &= x^{-2i-1}x^{-n} \\ &= \chi(a^i b)\chi(a^{\frac{n}{2}}).\end{aligned}$$

As a whole, $\chi = \chi_{1+a^{\frac{n}{2}},1+x^n}$ and χ is a $(1 + a^{\frac{n}{2}})$-homomorphism. Hence by Theorem 2.4, $F_2D_{2n}(1 + a^{\frac{n}{2}})$ and $F_2C_{2n}(1 + x^n)$ are equivalent group codes.

Theorem 5.2 *Let $n > 2$ be even. Then, $F_2D_{2n}(1 + a^{\frac{n}{2}})$ and $F_2C_{2n}(1 + x^n)$ are equivalent codes.*

For every $u \in F_2D_{2n}(1 + a^{\frac{n}{2}})$, note that $F_2D_{2n}u$ is a subideal of $F_2D_{2n}(1 + a^{\frac{n}{2}})$. Restrict the domain of φ to $F_2D_{2n}u$ by defining $\varphi_\downarrow : F_2D_{2n}u \to F_2C_{2n}(1 + x^n)$ in such a way that $(c)\varphi_\downarrow = (c)\varphi$ for every $c \in F_2D_{2n}u$. By Proposition 2.2, we have $F_2D_{2n}u$ is equivalent to $Im(\varphi_\downarrow)$ with respect to the weight-preserving F_2 linear isomorphism $\varphi_\downarrow$.

While identifying $Im(\varphi_\downarrow)$, one might conjecture that $Im(\varphi_\downarrow) = F_2C_{2n}(u)\varphi_\downarrow$ is a subideal of $F_2C_{2n}(1 + x^n)$. Here, we construct a counterexample for this conjecture by first showing that although $\chi(supp(u)) = supp((u)\varphi_\downarrow)$, χ need not be a u-homomorphism.

Let $u = \sum_{i=1}^{k} v_i(1 + a^{\frac{n}{2}})$ for some $k \in \mathbb{Z}^+$ and each $v_i \in D_{2n}$. Note that $(u)\varphi_\downarrow = \sum_{i=1}^{k} \chi(v_i)(1 + x^n)$. For each codeword $c \in F_2D_{2n}u$, write $c = wu = (\sum_{j=1}^{l} w_j)u$ for some $l \in \mathbb{Z}^+$ and each $w_i \in D_{2n}$. This leads to $c = (\sum_{j=1}^{l} w_j) \sum_{i=1}^{k} v_i(1+a^{\frac{n}{2}}) = \sum_{j=1}^{l}\sum_{i=1}^{k} w_j v_i(1+a^{\frac{n}{2}})$. Denote each $w_j v_i = u_{ij}$, then $c = \sum_{j=1}^{l}\sum_{i=1}^{k} u_{ij}(1 + a^{\frac{n}{2}})$ and the $(1 + a^{\frac{n}{2}})$-homomorphic property ensures that for every u_{ij}, $\chi(u_{ij}h) = \chi(u_{ij})\chi(h)$ for each $h \in \{1, a^{\frac{n}{2}}\}$.

To further show that χ is not necessarily a u-homomorphism, it is sufficient to prove that not all $\chi(u_{ij})$ are expressible as $\chi(w_j)\chi(v_i)$. We divide the proof into four cases:

Case 1 When $w_j = a^s$ and $v_i = a^t$ for some $s, t \in \{0, 1, 2, \ldots n - 1\}$, then it can be easily shown that $\chi(w_j v_i) = \chi(w_j)\chi(v_i)$.

Case 2 When $w_j = a^s$ and $v_i = a^t b$ for some $s, t \in \{0, 1, 2, \ldots n - 1\}$, then:

$$\begin{aligned}\chi(w_j v_i) = \chi(a^s a^t b) &= \chi(a^{s+t}b) \\ &= x^{-2(s+t)-1} \\ &= x^{-2s}x^{-2t-1} \\ &= \chi(a^{-s})\chi(a^t b).\end{aligned}$$

Note that $\chi(a^{-s})\chi(a^t b) \neq \chi(a^s)\chi(a^t b)$ whenever $s \notin \{0, \frac{n}{2}\}$.
Case 3 When $w_j = a^s b$ and $v_i = a^t$ for some $s, t \in \{0, 1, 2, \ldots n-1\}$, then it can be easily shown that $\chi(w_j v_i) = \chi(w_j)\chi(v_i)$.
Case 4 When $w_j = a^s b$ and $v_i = a^t b$ for some $s, t \in \{0, 1, 2, \ldots n-1\}$, then:

$$\begin{aligned}\chi(w_j v_i) = \chi(a^s b a^t b) &= \chi(a^{s-t}) \\ &= x^{2(s-t)} \\ &= x^{-2(-s-1)-1)}x^{-2t-1} \\ &= \chi(a^{-s-1}b)\chi(a^t b).\end{aligned}$$

Note that $\chi(w_j v_i) = \chi(a^{-s-1}b)\chi(a^t b) \neq \chi(a^s b)\chi(a^t b)$.

Now, suppose that $u = \sum_{i=1}^{k} v_i(1 + a^{\frac{n}{2}})$ for some $k \in \mathbb{Z}^+$ and each $v_i = a^t \in D_{2n}$ for some $t \in \{0, 1, 2, \ldots, n-1\}$, we have $\chi(w_j v_i) = \chi(w_j)\chi(v_i)$, then:

$$\begin{aligned}(c)\varphi_{\downarrow} &= \left(\sum_{j=1}^{l}\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}} u_{ij}h\right)\varphi_{\downarrow} \\ &= \left(\sum_{j=1}^{l}\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}} w_j v_i h\right)\varphi_{\downarrow} \\ &= \sum_{j=1}^{l}\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}} \chi(w_j v_i h) \\ &= \sum_{j=1}^{l}\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}} \chi(w_j)\chi(v_i)\chi(h) \\ &= \left(\sum_{j=1}^{l}\chi(w_j)\right)\left(\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}}\chi(v_i)\chi(h)\right) \\ &= (w)\varphi_{\downarrow}(u)\varphi_{\downarrow} \in F_2C_n(u)\varphi_{\downarrow}.\end{aligned}$$

Hence, by Theorem 2.4, the u-homomorphic property of χ with $\chi = \chi_{u,(u)\varphi_{\downarrow}}$ leads to the following proposition.

Proposition 5.3 *Let $n > 2$ be even and $u \in F_2D_{2n}(1 + a^{\frac{n}{2}})$ such that $a^i b \notin supp(u)$ for every $i \in \{0, 1, \ldots, n-1\}$. Then, $F_2D_{2n}u$ and $F_2C_{2n}(u)\varphi_{\downarrow}$ are equivalent codes.*

Next, suppose that $u = \sum_{i=1}^{k} v_i(1 + a^{\frac{n}{2}})$ for some $k \in \mathbb{Z}^+$ and each $v_i = a^t b \in D_{2n}$ for some $t \in \{0, 1, 2, \ldots, n-1\}$, we have $\chi(w_j v_i) = \chi(w_j^{-1})\chi(v_i)$ when $w_j = a^s$ for some $s \in \{0, 1, 2, \ldots, n-1\}$ and $\chi(w_j v_i) = \chi(w_j^*)\chi(v_i)$ when $w_j = a^s b$ for some $s \in \{0, 1, 2, \ldots, n-1\}$, where $w_j^* = a^{-s-1}b$. Without loss of generality, we write $w = \sum_{j=1}^{\alpha} w_j + \sum_{j=\alpha+1}^{l} w_j$ for some $0 \leq \alpha \leq l$ such that $w_j = a^s$ when $0 \leq j \leq \alpha$ and $w_j = a^s b$ when $\alpha + 1 \leq j \leq l$. Then:

$$
\begin{aligned}
(c)\varphi_{\downarrow} &= \left(\sum_{j=1}^{l}\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}} u_{ij}h\right)\varphi_{\downarrow}\\
&= \left(\sum_{j=1}^{l}\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}} w_j v_i h\right)\varphi_{\downarrow}\\
&= \sum_{j=1}^{l}\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}} \chi(w_j v_i h)\\
&= \sum_{j=1}^{\alpha}\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}} \chi(w_j^{-1})\chi(v_i)\chi(h) + \sum_{j=\alpha+1}^{l}\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}} \chi(w_j^{*})\chi(v_i)\chi(h)\\
&= \left(\sum_{j=1}^{\alpha}\chi(w_j^{-1}) + \sum_{j=\alpha+1}^{l}\chi(w_j^{*})\right)\left(\sum_{i=1}^{k}\sum_{h\in\{1,a^{\frac{n}{2}}\}}\chi(v_i)\chi(h)\right)\\
&= \left(\sum_{j=1}^{\alpha}\chi(w_j^{-1}) + \sum_{j=\alpha+1}^{l}\chi(w_j^{*})\right)(u)\varphi_{\downarrow} \in F_2C_n(u)\varphi_{\downarrow}.
\end{aligned}
$$

Define a map $f : D_{2n} \to D_{2n}$ in such a way that $f : a^s \mapsto a^{-s}$ and $f : a^s b \mapsto a^{-s-1}b$ for each $s \in \{0, 1, \ldots, n-1\}$. Note that f is bijective, this implies that $Im(\varphi_{\downarrow}) = F_2C_{2n}((u)\varphi_{\downarrow})$.

Proposition 5.4 *Let $n > 2$ be even and $u \in F_2D_{2n}(1 + a^{\frac{n}{2}})$ such that $a^i \notin supp(u)$ for every $i \in \{0, 1, \ldots, n-1\}$. Then, $F_2D_{2n}u$ and $F_2C_{2n}(u)\varphi_{\downarrow}$ are equivalent codes.*

The following example shows that $Im(\varphi_{\downarrow})$ need not possess ideal structure in general, thus need not be $F_2C_{2n}(u)\varphi_{\downarrow}$.

Example 5.5 Consider the case when $n = 4$ and let $u = 1 + a^2 + b + a^2b = 1(1 + a^2) + b(1 + a^2) \in F_2D_8(1 + a^2)$. Note that $(u)\varphi_{\downarrow} = 1 + x^3 + x^4 + x^7 = 1(1 + x^4) + x^3(1 + x^4) \in F_2C_8(1 + x^4)$. Note that F_2D_8u and $F_2C_8(u)\varphi_{\downarrow}$ are not equivalent since it can be verified that $F_2D_8u = L_{F_2}(u, au)$ and $F_2D_8u = L_{F_2}((u)\varphi_{\downarrow}, x(u)\varphi_{\downarrow}, x^2(u)\varphi_{\downarrow})$, having dimension 2 and 3 respectively. In fact, $F_2D_8u = L_{F_2}(u, au)$ is equivalent to $W_2(u)\varphi_{\downarrow} = L_{F_2}((u)\varphi_{\downarrow}, (au)\varphi_{\downarrow})$. Note that $W_2(u)\varphi_{\downarrow} \subset F_2C_8(u)\varphi_{\downarrow}$.

The main results of this section are summarized as follows. We proved that for $u \in \{1 + g | g \in Inv(D_{2n})\} \subseteq Nil_2(F_2D_{2n})$, $F_2D_{2n}u$ has an equivalent form of cyclic zero-divisor code, that is $F_2C_{2n}(1 + x^n)$ by Proposition 5.1. In addition, for even $n > 2$, partition D_{2n} into $A_1 = \{1, a, a^2, \ldots, a^{n-1}\}$ and $A_2 = \{b, ab, a^2b, \ldots, a^{n-1}b\}$. Let $u \in F_2D_{2n}(1 + a^{\frac{n}{2}}) \subseteq Nil_2(F_2D_{2n})$. If either $supp(u) \subseteq A_1$ or $supp(u) \subseteq A_2$, then Proposition 5.3 and Proposition 5.4 ensure that $F_2D_{2n}u$ has an equivalent form of cyclic zero-divisor code. The remaining studies on whether $F_2D_{2n}u$ has an equivalent form of cyclic zero-divisor code for the case when $supp(u) \nsubseteq A_1$ and $supp(u) \nsubseteq A_2$ are left as a future direction.

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