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Motion/structure mode analysis and classification of n-RR planar parallelogram mechanisms

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Abstract

Planar parallelogram mechanisms (PPMs) have been used in many applications either as standalone pieces or as compositional joints of mechanisms, such as parallel mechanisms, remote center mechanisms and variable-DOF mechanisms. An n-RR PPM is composed of congruent base and moving platform connected with n RR links of equal link lengths. Although it is well known that an n-RR PPM has one 1-DOF circular translation mode, whether an n-RR PPM has structure/motion modes in addition to the 1-DOF motion mode and how the link parameters affect the number of extra motion/structure modes of the n-RR PPMs have not been well investigated. This paper is about the motion/structure mode analysis and classification of n-RR PPMs. Firstly, the motion/structure mode analysis of 3-RR PPMs is carried out. Then conditions for an $n(n > 3)$-RR PPM to have the same motion/structure modes as a 3-RR PPM are revealed. Finally, $n(n \geq 3)$-RR PPMs are classified into three types. In addition to one 1-DOF motion mode, Types I, II and III PPMs have two, one or zero structure mode respectively. This work provides a foundation for the construction and analysis of several variable-DOF mechanisms composed of PPMs.

Keywords: Planar parallelogram mechanism, motion mode, structure mode, parallel mechanism, variable-DOF mechanism

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1. Introduction

Planar parallelogram mechanisms (PPMs Fig. 1(a)), denoted by Pa[1, 2], Π [3, 4], or G (Circular Translation) joint [5], have been widely used in many applications either as standalone pieces or as compositional joints of mechanisms, such as parallel mechanisms [4, 2, 6, 7, 8, 9, 10], remote center mechanisms [11], and variable-DOF mechanisms [3, 5, 12, 13].

As well documented in the literature, in order to prevent a planar 4-bar PPM from entering its anti-parallelogram motion mode (Fig. 1(b)) via a singular (or toggle) position, PPMs with more than four links (Figs. 1(c) and 1(d)) have been proposed. In this paper, PPMs with more than four links are called called n($n \geq 3$)-RR PPMs, which are composed of congruent base and moving platform with n RR links of equal link lengths. A 3-RR PPM can also be regarded as singular configurations of 3-R$\underline{\text{P}}$R planar parallel mechanisms with congruent base and moving platform. Here R and $\underline{\text{P}}$ denote a revolute joint and an actuated prismatic joint respectively.

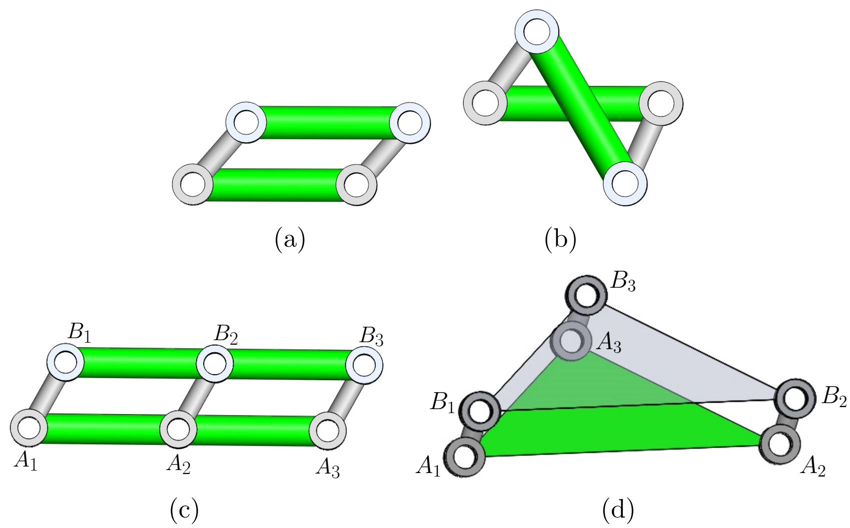


Figure 1: (a) 2-RR PPM; (b) 2-RR anti-parallelogram; (c) 3-RR PPM with collinear base and moving platform; and (d) 3-RR PPM with triangular base and moving platform.

It has been revealed in [14, 15, 16, 17, 18] that there are up to six solutions to the forward displacement analysis of 3-R$\underline{\text{P}}$R parallel mechanisms depending on the link parameters of the platform and base. In [17], a systematic degeneracy study on the forward kinematics of planar 3-R$\underline{\text{P}}$R parallel mechanisms was presented, and a new planar 3-R$\underline{\mathrm{P}}$R parallel mechanism was identified for which solutions to the forward kinematics can be obtained by solving a cubic equation. In [19], a method has been proposed to construct a 4-R$\underline{\mathrm{P}}$R parallel mechanism from a 3-R$\underline{\mathrm{P}}$R parallel mechanism by adding an extra R$\underline{\mathrm{P}}$R leg to ensure they have the same number of solutions to its forward displacement analysis. Since an n-RR PPM can be regarded as a parallel mechanism and it is well known that an n-RR PPM has one 1-DOF motion mode, two questions about n-RR PPMs arising from the multiple solutions to the the forward displacement analysis of n-R$\underline{\mathrm{P}}$R parallel mechanisms are: Does an n-RR PPM have structure/motion modes in addition to the 1-DOF motion mode? How do the link parameters affect the number of extra motion/structure modes of the n-RR PPMs? However, these questions have not been well addressed so far.

To meet the needs of mechanisms involving compositional joints and variable-DOF mechanisms, this paper is to investigate the motion/structure mode analysis and classification of n-RR PPMs.

In Section 2, the motion/structure mode analysis of 3-RR PPMs is carried out. The conditions for an n-RR PPM to have the same motion/structure modes as a 3-RR PPM are revealed in Section 3. Finally, the classification of n-RR PPMs based on their number of motion/structure modes are dealt with in Section 4.

2. Motion/Structure mode analysis of 3-RR PPMs

In this section, we will reveal the motion/structure modes of 3-RR PPMs.

2.1. Geometric description of a 3-RR PPM

A 3-RR PPM is shown in Fig. 2. It is constructed by connecting congruent base platform ($A_1A_2A_3$) and moving platform ($B_1B_2B_3$) with three RR links of equal link lengths. The link parameters of the 3-RR PPM (Fig. 2) are:

$a_1 = A_1A_2 = B_1B_2 \neq 0$, $a_2 = A_1A_3 = B_1B_3 \neq 0$, $\alpha = \angle A_2A_1A_3 = \angle B_2B_1B_3$, and $A_1B_1 = A_2B_2 = A_3B_3 = L$.

It is noted that the diameter of the circumcircle of the triangular base and moving platform is

$$\rho = (a_1^2 + a_2^2 - 2a_1a_2C\alpha)^{1/2}/S\alpha \tag{1}$$

where $S*$ and $C*$ denote $\sin *$ and $\cos *$ respectively.

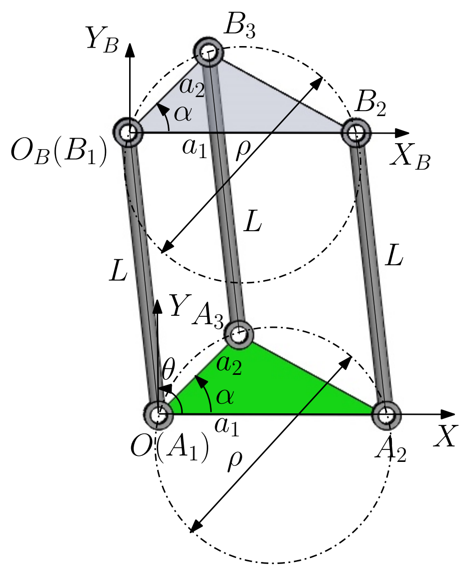


Figure 2: General 3-RR PPM.

2.2. Constraint equations

For the purpose of simplification and without loss of generality, two coordinate systems are established. The coordinate system $O - XY$ is attached to the link $A_1A_2A_3$ with O being coincident with A_1 and the X-axis passing through A_2. The coordinate system $O_B - X_BY_B$ is attached to the link $B_1B_2B_3$ with O_B being coincident with B_1 and the X_B-axis passing through B_2. Let θ and ϕ denote the angle between link A_1B_1 and the X-axis and the angle between the X_B- and X-axes respectively.

The set of constraint equations of the 3-RR PPM obtained by expressing the link lengths of links A_iB_i (2 and 3) in θ and ϕ is

$$\begin{cases} (LC\theta + a_1C\phi - a_1)^2 + (LS\theta + a_1S\phi)^2 = L^2 \\ (LC\theta + a_2C(\phi + \alpha) - a_2C\alpha)^2 + (LS\theta + a_2S(\phi + \alpha) - a_2S\alpha)^2 = L^2 \end{cases}$$

Simplification of the above equation leads to

$$\begin{cases} L(C\phi - 1)C\theta + LS\theta S\phi - (C\phi - 1)a_1 = 0 \\ LC\alpha[C\theta(C\phi - 1) + S\theta S\phi] - LS\phi C\theta S\alpha + (C\phi - 1)(LS\theta S\alpha - a_2) = 0 \end{cases} \qquad (2)$$

The motion/structure mode analysis of the 3-RR PPM can be carried out by solving Eq. (2).

2.3. Motion/structure mode analysis

Rewriting Eq.(2) as a set of linear equations in $S\theta$ and $C\theta$, we have

$$\begin{cases} RC\theta + TS\theta + Q = 0 \\ UC\theta + VS\theta + W = 0 \end{cases} \tag{3}$$

where $R = L(C\phi - 1)$, $T = LS\phi$, $Q = -a_1(C\phi - 1)$, $U = (C\phi - 1)LC\alpha - LS\phi S\alpha$, $V = S\phi LC\alpha + (C\phi - 1)LS\alpha$, $W = -a_2(C\phi - 1)$.

The determinant of the coefficient matrix of the above equation is

$$\Delta = RV - UT = -2L^2 S\alpha(C\phi - 1) \tag{4}$$

Based on the vanishing conditions of Δ, Eq. (3) can be solved into two cases: Case I: $S\alpha = 0$ and Case II: $S\alpha \neq 0$. Case II has two sub-cases: Case II-a: $S\alpha \neq 0$ and $C\phi - 1 = 0$, and Case II-b: $S\alpha \neq 0$ and $C\phi - 1 \neq 0$.

2.3.1. Case I: $S\alpha = 0$

A 3-RR PPM with $S\alpha = 0$ is called a 3-RR PPM with collinear base and moving platform (Fig. 3). For this 3-RR PPM, $a_1 \neq a_2$. Solution of Eq. (2) leads to

$$C\phi - 1 = 0 \tag{5}$$

i.e.,

$$\phi = 0 \tag{6}$$

In this case, θ can have any value (Fig. 3).

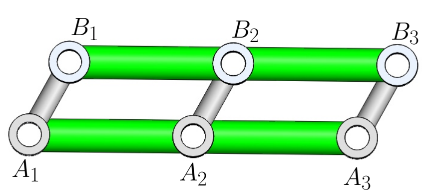


Figure 3: 3-RR PPM with collinear base and moving platform.

The above analysis shows that a 3-RR PPM with collinear base and moving platform has one 1-DOF motion mode only.

2.3.2. Case II: $S\alpha \neq 0$

A 3-RR PPM with $S\alpha \neq 0$ is called a 3-RR PPM with triangular base and moving platform (Fig. 4). For this 3-RR PPM, the structure/motion mode analysis can be carried out as follows.

Case II-a: $S\alpha \neq 0$ and $C\phi - 1 = 0$

Since $C\phi - 1 = 0$, we have

$$\phi = 0 \tag{7}$$

Substitution of Eq. (7) into Eq. (3) shows that Eq. (3) is met for any value of θ. This is the 1-DOF motion mode of the 3-RR PPMs with triangular base and moving platform (Figs. 4(a), 5(a) and 6).

Case II-b: $S\alpha \neq 0$ and $C\phi \neq 1$.

Solving Eq. (3) for $C\theta$ and $S\theta$, we obtain

$$\begin{cases} C\theta = (TW - VQ)/\Delta \\ S\theta = -(RW - UQ)/\Delta \end{cases} \tag{8}$$

The substitution of Eq. (8) into $C^2\theta + S^2\theta = 1$ yields

$$(C\phi - 1)^2(-f_0 C\phi + f_1) = 0 \tag{9}$$

where

$$f_0 = -2a_1a_2C\alpha + a_1^2 + a_2^2$$

$$f_1 = 2L^2C^2\alpha - 2a_1a_2C\alpha - 2L^2 + a_1^2 + a_2^2$$

Eq. (9) is then reduced to a linear equation in $C\phi$

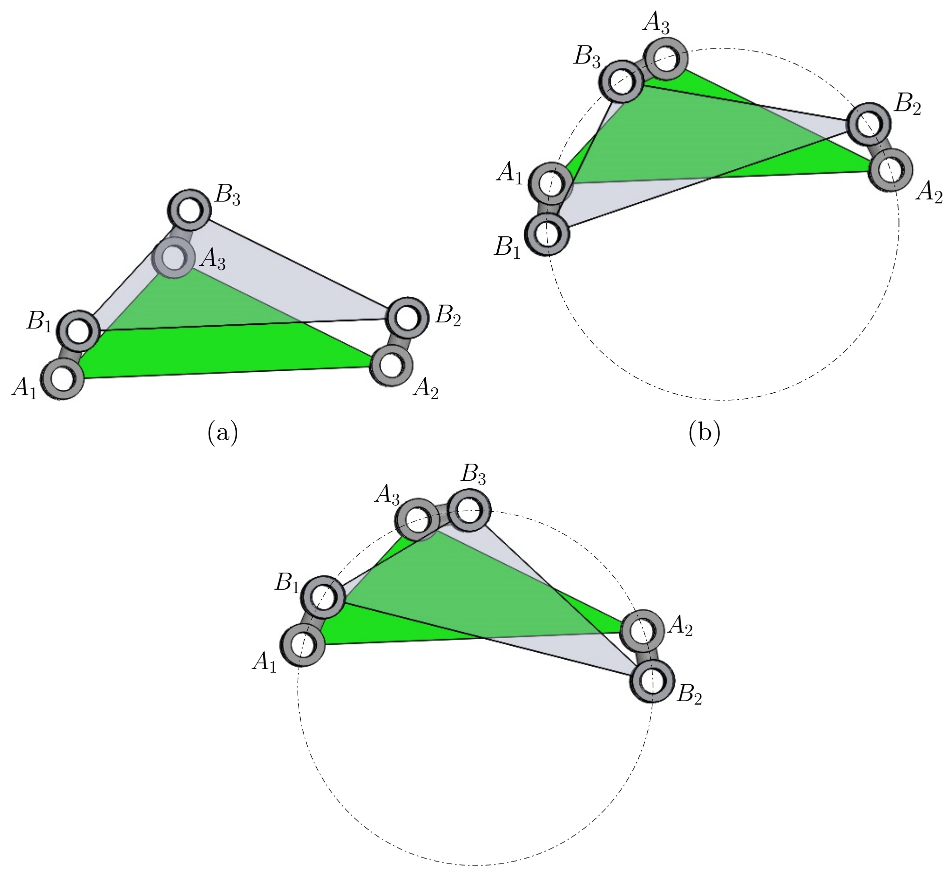
$$-f_0C\phi + f_1 = 0 \tag{10}$$

Solving Eq. (10), we obtain

$$C\phi = f_1/f_0 \tag{11}$$

Substituting $C\phi = 1 - S^2(\phi/2)$ and Eq. (1) into Eq. (10), the solution for ϕ can be further simplified as

$$S(\phi/2) = \pm L/\rho \tag{12}$$



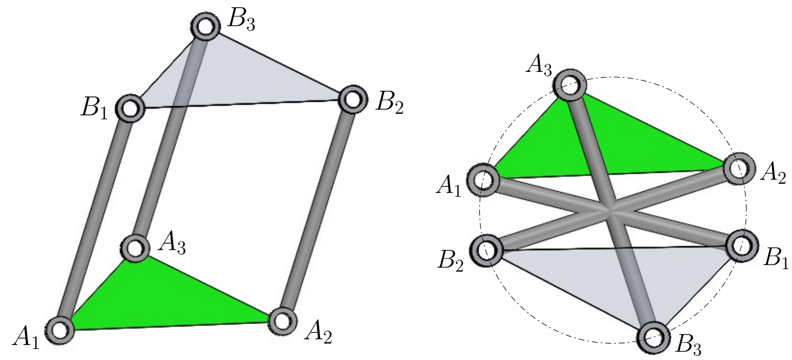
(c)

Figure 4: Example 3-RR PPM with triangular base and moving platform ($L < \rho$) in: (a) 1-DOF Motion mode I; (b) Structure mode II; and (c) Structure mode III.

Solving Eq. (12), we obtain two values for ϕ as

$$\phi = \pm 2 \arcsin(L/\rho) \tag{13}$$

For each value of ϕ obtained using Eq. (13), one unique solution to θ can be obtained using Eq. (8). These solutions are the structure modes of the 3-RR PPM (Fig. 4). The number of structure modes of a 3-RR PPM is two if $L < \rho$ (Figs. 4(b) and 4(c)), one if $L = \rho$ (Fig. 5(b)) or 0 if $L > \rho$ (Fig. 6). It can be observed from Figs. 4(b), 4(c) and 5(b) that the circumcircles of the base and moving platform coincide in a structure mode. This can be proved using an analytic approach as presented in Appendix A.



(a) (b)

Figure 5: Example 3-RR PPM with triangular base and moving platform ($L = \rho$) in: (a) 1-DOF Motion mode I; and (b) structure mode II.

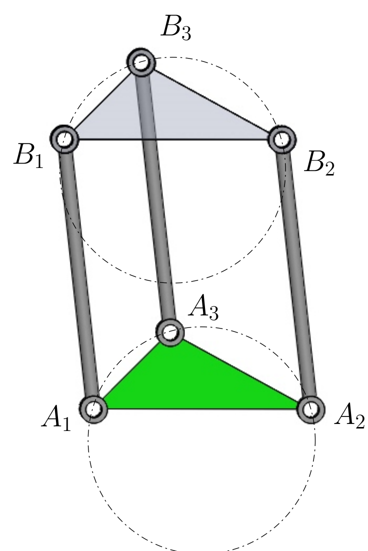


Figure 6: Example 3-RR PPM with triangular base and moving platform ($L > \rho$) in 1-DOF motion mode.

In summary, a 3-RR PPM with triangular base and moving platform has one 1-DOF motion mode and at most two structure modes. Figures 4, 5 and 6 show the structure/motion modes of an example 3-RR PPM ($L < \rho$) that has one 1-DOF motion mode and two structure modes, an example 3-RR PPM ($L = \rho$) that has one 1-DOF motion mode and one structure mode, and an example 3-RR PPM ($L > \rho$) that has one 1-DOF motion mode and no structure mode respectively.

3. Conditions for an $n(n>3)$-RR PPM to have the same motion/structure modes as a 3-RR PPM

An $n(n>3)$-RR PPM (Fig. 7) can be obtained from a 3-RR PPM (Fig. 2) by adding $(n-3)$ RR links, A_iB_i (i=4,5,$\cdots$) with link length L. In addition to the link parameters of the 3-RR PPM, the remaining link parameters of the $n(n>3)$-RR PPM include the polar coordinates of A_i (or B_i) (i=4,5,$\cdots$) in its local coordinate systems: a_{i-1} and α_{i-1}.

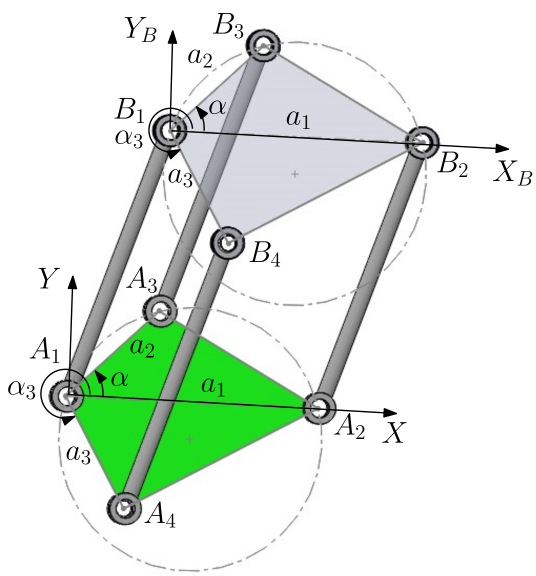


Figure 7: n-RR PPM.

Following the procedure in Section 2.2, a set of constraint equations of the $n(n>3)$-RR PPM is obtained as

$$\begin{cases} L(C\phi-1)C\theta+LS\theta S\phi-(C\phi-1)a_1=0 \\ LC\alpha[C\theta(C\phi-1)+S\theta S\phi]-LS\phi C\theta S\alpha+(C\phi-1)(LS\theta S\alpha-a_2)=0 \\ LC\alpha_{i-1}[(C\phi-1)C\theta+S\theta S\phi] \\ -LS\phi C\theta S\alpha_{i-1}+(C\phi-1)(LS\theta S\alpha_{i-1}-a_{i-1})=0 \quad i\text{=4, 5, } \cdots \end{cases} \tag{14}$$

The first two equations in Eq. (14) is in fact the constraint equation (Eq. (2)) of its associated 3-RR PPM. The remaining equations are obtained by calculating the link length of link A_iB_i (i=4, 5, $\cdots$).

It is apparent $\phi = 0$ is a solution to Eq. (14) and corresponds to the 1-DOF motion mode of the $n(n > 3)$-RR PPM.

An $n(n > 3)$-RR PPM has one 1-DOF motion mode. The number of its structure modes is not greater than that of its associated 3-RR PPM. In the following, we will reveal the conditions for an $n(n > 3)$-RR PPM to have the same number of structure modes as its associated 3-RR PPM.

3.1. $n(n > 3)$-RR PPMs with no structure mode

Section 2 shows that 3-RR PPMs have no structure mode if the base and moving platform are collinear or if the planar base and moving platform are triangular and $L > \rho$. Any $n(n > 3)$-RR PPMs obtained by adding extra RR legs to the above 3-RR PPMs have no structure mode (Fig. 8).

3.2. $n(n > 3)$-RR PPMs with a structure mode

Section 2 shows that only 3-RR PPMs with triangular base and moving platform and $L \leq \rho$ have 1 or 2 structure modes. The conditions for $n(n > 3)$-RR PPMs obtained by adding extra RR legs to the above 3-RR PPMs to have the same number of structure modes as their associated 3-RR PPM can be obtained in the following two steps.

Step 1 Solve the constraint equations of the 3-RR PPM to obtain one set of θ and ϕ associated with a structure mode.

Step 2 Substitute θ and ϕ obtained in Step 1 into the constraint equation associated with link A_iB_i (i=4, 5, $\cdots$) to obtain the conditions in a_{i-1} and α_{i-1}.

For example, for a 3-RR PPM with link parameters: $a_1 = 20$, $a_2 = 10$, $\alpha = \pi/4$, and $L = 3$. Using Step 1, we can obtain ϕ=0.2889086253 *rad* and $\theta = 4.571919167$ *rad*. Following Step 2, $n(n > 3)$-RR PPMs that have the same number of motion/structure modes as the 3-RR PPM must meet the following conditions

$$-0.8288922843C\alpha_{i-1} + 0.2427769286S\alpha_{i-1} + 0.0414446142a_{i-1} = 0 \quad i=4,5,\cdots \tag{15}$$

Multiplying Eq. (15) with a_{i-1}, we can rewrite the equation in Cartesian coordinates as

$$-0.8288922843x_{i-1} + 0.2427769286y_{i-1} + 0.0414446142(x_{i-1}^2 + y_{i-1}^2) = 0 \quad i=4,5,\cdots \tag{16}$$

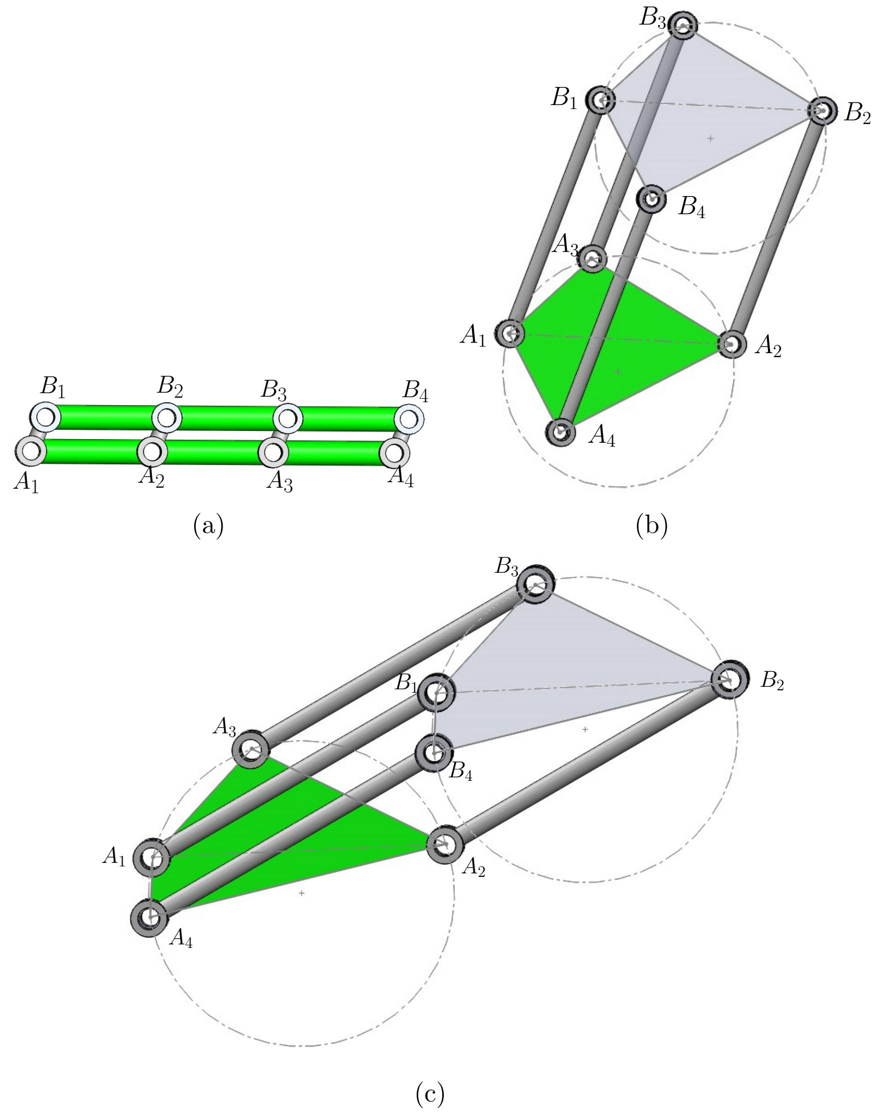


Figure 8: Example 4-RR PPMs with no structure mode: (a) Case with collinear base and moving platform; (b) Case with a non-cyclic base and moving platform; and (c) Case with a cyclic base and moving platform ($L > \rho$).

where $x_{i-1} = a_{i-1}C\alpha_{i-1}$ and $y_{i-1} = a_{i-1}S\alpha_{i-1}$.

Equation (16) represents a circle (Fig.9) on which A_i (or B_i) (i=4,5,$\cdots$) should be located. One can verify numerically that this circle is in fact defined by R joint centers A_1, A_2 and A_3 in $O - XY$ (or B_1, B_2 and B_3 in

$O_B - X_BY_B$). This means that the base and moving platform are cyclic. In Appendix B, it will be proved using an algebraic approach that an n-RR parallelogram has the same structure modes as a 3-RR PPM with a structure mode if and only if the base and moving platform of n-RR PPM are cyclic. Therefore, an $n(n > 3)$-RR PPM has a structure mode if and only if the base and moving platform are cyclic and $L \leq \rho$. The number of structure modes is two if $L < \rho$ (Fig. 10) or one if $L = \rho$ (Fig. 11). Otherwise, the $n(n > 3)$-RR PPM have no structure mode (Fig. 12).

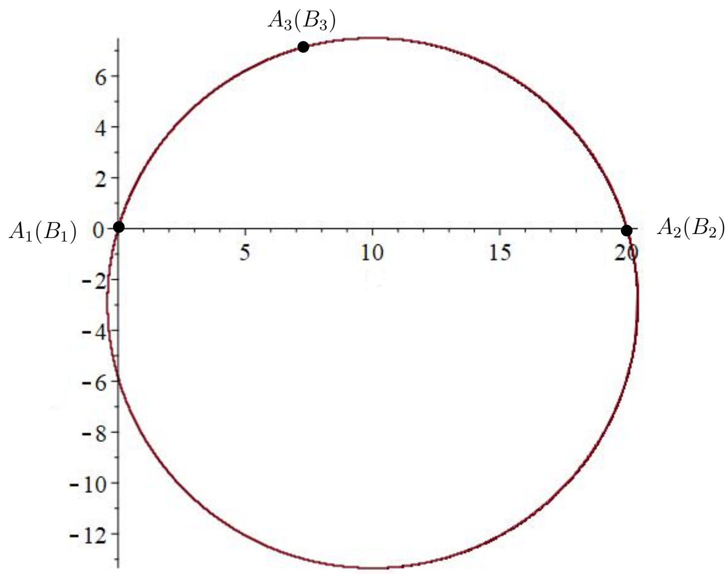


Figure 9: Circle on which joint centers of the extra legs of an $n(n > 3)$-RR PPM should be located

4. Classification of n-RR PPMs

Sections 2 and 3 show that an $n(n \geq 3)$-RR PPM has one 1-DOF motion mode and up to two structure modes. In the 1-DOF motion mode, the moving platform of the $n(n \geq 3)$-RR PPM undergoes circular translation. Based on these results, $n(n \geq 3)$-RR PPMs can be classified into the following three types:

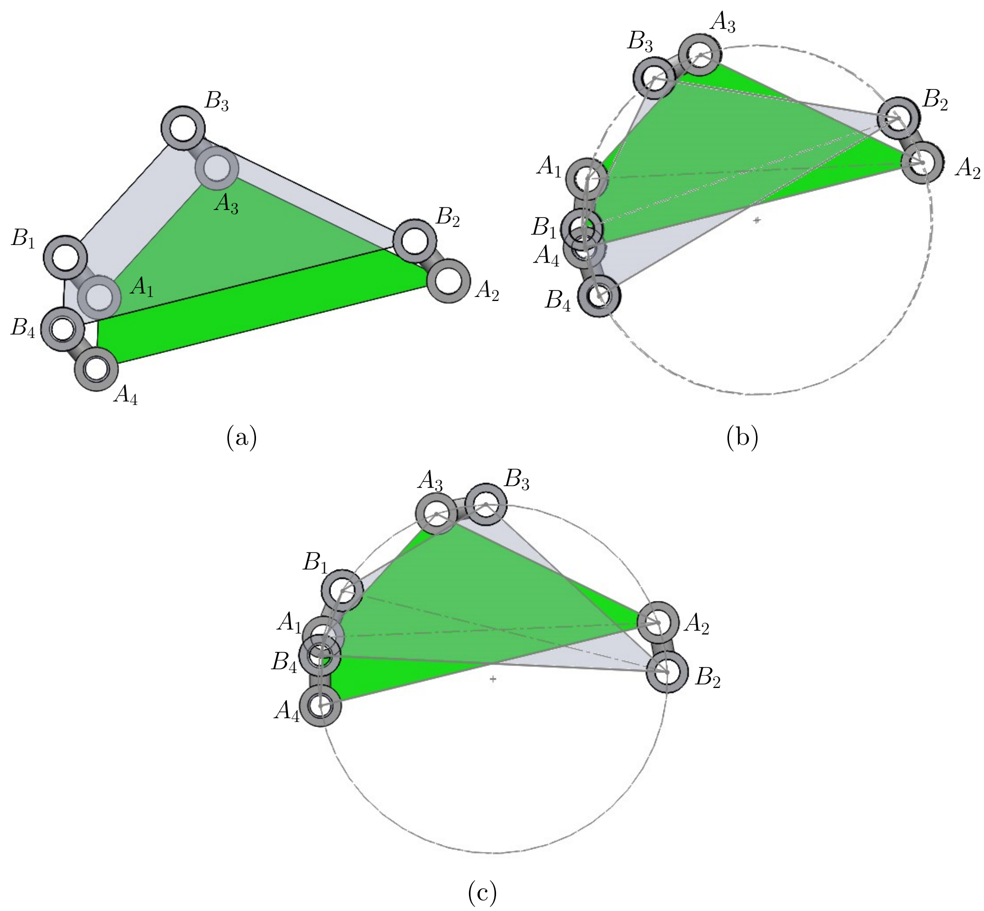
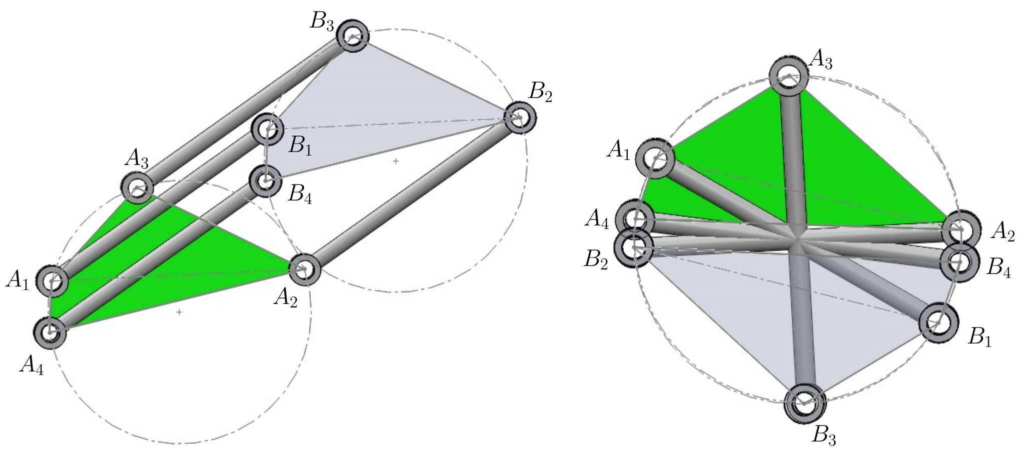


Figure 10: Example 4-RR PPM with cyclic base and moving platform ($L < \rho$) in: (a) 1-DOF Motion mode I; (b) Structure mode II; and (c) Structure mode III.

1. Type 1 PPMs: $n(n \geq 3)$-RR PPMs with cyclic base and moving platform and $L < \rho$ (Figs. 4 and 10). A Type 1 PPM has one 1-DOF motion mode and two structure modes.
   In each structure mode, circumcircles of the base and moving platform coincide. One can obtain the poses of the moving platform by a rotation about the center of the circumcircles by $\pm 2arcsin(L/\rho)$.
2. Type 2 PPMs: $n(n \geq 3)$-RR PPMs with cyclic base and moving platform and $L = \rho$ (Figs. 5 and 11). A Type 2 PPM has one 1-DOF motion mode and one structure mode.



(a) (b)

Figure 11: Example 4-RR PPM with cyclic base and moving platform ($L = \rho$) in: (a) 1-DOF Motion mode I; (b) Structure mode II.

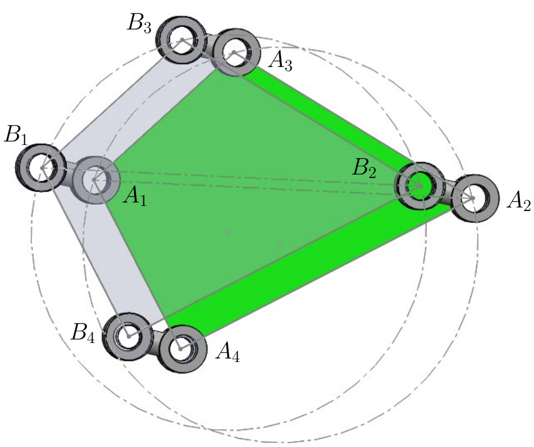


Figure 12: Example 4-RR PPM with non-cyclic base and moving platform and $L < \rho$ in 1-DOF motion mode.

In the structure mode, circumcircles of the base and moving platform coincide, and all the RR links intersect at the center of the circumcircles. Therefore, the moving platform undergoes 1-DOF infinitesimally rotation about the center of the circumcircles.

3. Type 3 PPMs: $n(n \geq 3)$-RR PPMs other than Types 1 and 2, including mechanisms with cyclic base and moving platform and $L > \rho$

(Figs. 6 and 8(c)), mechanisms with collinear base and moving platform (Figs. 8(a)), mechanisms with non-cyclic base and moving platform (Figs. 8(a), 8(b) and 12). A Type 3 PPM has one 1-DOF motion mode but has no structure mode.

5. Conclusions

An approach has been proposed to the motion/structure mode analysis of n-RR planar parallelogram mechanisms. The analysis has shown that an $n(n \geq 3)$-RR planar parallelogram mechanism may have zero, one or two structure modes in addition to the apparent 1-DOF motion mode. $n(n \geq 3)$-RR planar parallelogram mechanisms have been classified into three types with zero, one, and two structure modes respectively.

This work is being extended to the construction and analysis of several variable-DOF mechanisms constructed using planar parallelogram mechanisms and the design and analysis of parallel mechanisms that can generate certain types of trajectories, such as circular and spherical trajectories, under simplified control.

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Appendix A. Alternative approach to the structure/motion mode analysis of a 3-RR PPM with triangular base and platform

In section 2.3.2, an approach to the structure/motion mode analysis was presented and the geometric characteristics of the 3-RR PPM with a triangular base and platform (Fig. 4) in its structure mode were revealed numerically. In order to reveal the geometric characteristics of the 3-RR PPM in its structure mode analytically, an alternative approach to the structure/motion mode analysis will be presented below. Coordinate systems $O - XY$ and $O_B - X_B Y_B$ (Fig. A.1) are attached to the base and moving platform in such a way that O and O_B are located at the center of the base and moving platform respectively [16] and that the coordinates of A_i in $O - XYZ$ and the coordinates of B_i in $O_B - X_B Y_B$ are both (A_{ix}, A_{iy}). Let $\{x \;\; y\}^T$ and ϕ denote the position vector of O_B in the coordinate system $O - XY$ and the orientation of the coordinate system of $O_B - X_B Y_B$ with respect to the coordinate system $O - XY$.

The set of constraint equations of the 3-RR PPM obtained by expressing the link lengths of links $A_i B_i$ (i=1, 2 and 3) in x, y, and ϕ is

$$\begin{cases} x^2 + y^2 + 2A_{1x}z_1 + 2A_{1y}z_2 - 2r^2C\phi + 2r^2 = L^2 \\ x^2 + y^2 + 2A_{2x}z_1 + 2A_{2y}z_2 - 2r^2C\phi + 2r^2 = L^2 \\ x^2 + y^2 + 2A_{3x}z_1 + 2A_{3y}z_2 - 2r^2C\phi + 2r^2 = L^2 \end{cases} \tag{A.1}$$

where $r = (A_{1x}^2 + A_{1y}^2)^{1/2} = (A_{2x}^2 + A_{2y}^2)^{1/2} = (A_{3x}^2 + A_{3y}^2)^{1/2}$ is the radius of

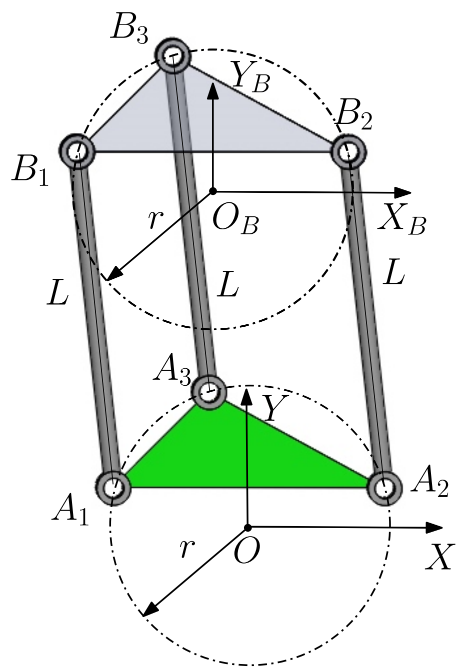

Figure A.1: General 3-RR PPM with origins of coordinate frames at the centre of the base and moving platform.

the circumcircle defined by A_1, A_2 and A_3 (or B_1, B_2 and B_3), and

$$\begin{cases} z_1 = xC\phi + yS\phi - x \\ z_2 = -xS\phi + yC\phi - y \end{cases} \tag{A.2}$$

Equation (A.1) can be re-written as

$$\begin{cases} x^2 + y^2 + 2A_{1x}z_1 + 2A_{1y}z_2 = L^2 + 2r^2(C\phi - 1) \\ x^2 + y^2 + 2A_{2x}z_1 + 2A_{2y}z_2 = L^2 + 2r^2(C\phi - 1) \\ x^2 + y^2 + 2A_{3x}z_1 + 2A_{3y}z_2 = L^2 + 2r^2(C\phi - 1) \end{cases} \tag{A.3}$$

Subtracting the first equation of Eq. (A.3) from the second and third equations of Eq. (A.3), we have

$$\begin{cases} (A_{2x} - A_{1x})z_1 + (A_{2y} - A_{1y})z_2 = 0 \\ (A_{3x} - A_{1x})z_1 + (A_{3y} - A_{1y})z_2 = 0 \end{cases} \tag{A.4}$$

i.e.,

$$\begin{bmatrix} A_{2x} - A_{1x} & A_{2y} - A_{1y} \\ A_{3x} - A_{1x} & A_{3y} - A_{1y} \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \mathbf{0} \tag{A.5}$$

The determinant of the above coefficient matrix is not equation to 0 since A_1, A_2 and A_3 (or B_1, B_2 and B_3) are not collinear. Solving Eq. (A.5), we obtain

$$\begin{cases} z_1 = 0 \\ z_2 = 0 \end{cases} \tag{A.6}$$

Substituting Eq. (A.6) into Eqs. (A.3) and (A.2), we obtain

$$x^2 + y^2 = L^2 + 2r^2(C\phi - 1) \tag{A.7}$$

and

$$\begin{bmatrix} C\phi - 1 & S\phi \\ -S\phi & C\phi - 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = 0 \tag{A.8}$$

Case II-a: $C\phi - 1 = 0$

In this case, we have

$$\phi = 0 \tag{A.9}$$

Eq. (A.8) is always true since the determinant of the coefficient matrix is equal to 0. Substitution of Eq. (A.9) into Eq. (A.7), we obtain

$$x^2 + y^2 = L^2 \tag{A.10}$$

This is the 1-DOF motion mode of the 3-RR PPM with triangular base and moving platform (Figs. 4(a), 5(a) and 6).

Case II-b: $C\phi \neq 1$. In this case, the determinant of the coefficient matrix in Eq. (A.8) is not equal to 0. Solving Eq. (A.8) for x and y, we obtain

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \tag{A.11}$$

Substituting Eq. (A.11) into Eq. (A.7), we have

$$L^2 + 2r^2(C\phi - 1) = 0 \tag{A.12}$$

Substituting $C\phi = 1 - 2S^2(\phi/2)$ into Eq. (A.12), the solution for ϕ can be further simplified as

$$S(\phi/2) = \pm L/(2r) = \pm L/\rho \tag{A.13}$$

Solving Eq. (A.13), we obtain two values for ϕ as

$$\phi = \pm 2 \arcsin(L/\rho) \tag{A.14}$$

These solutions (Eqs. (A.11)) and (A.14)) are the structure modes of the 3-RR PPM (Fig. 4). The number of structure modes of a 3-RR PPM is two if $L < \rho$ (Figs. 4(b) and 4(c)), one if $L = \rho$ (Fig. 5(b)) or 0 if $L > \rho$ (Fig. 6). Equation (A.11) shows that in a structure mode (Figs. 4(b), 4(c) and 5(b)), the circumcircles of the base and moving platform of the 3-RR PPM coincide.

Appendix B. Conditions for adding extra RR legs to a 3-RR PPM with a structure mode without changing its structure/motion modes

In order to derive the analytical conditions for adding extra RR legs to a 3-RR PPM with a structure mode without changing its structure/motion modes, coordinate systems $O-XY$ and $O_B-X_BY_B$ are attached to the base and moving platform of the n-RR PPM in such a way that O and O_B are located at the centers of the circumcicles defined by A_1, A_2 and A_3 on the base and B_1, B_2 and B_3 on the moving platform respectively and that the coordinates of A_i in $O-XYZ$ and the coordinates of B_i in $O_B-X_BY_B$ are both (A_{ix}, A_{iy}). Let $\{x \;\; y\}^T$ and ϕ denote the position vector of O_B in the coordinate system $O-XY$ and the orientation of the coordinate system of $O_B-X_BY_B$ with respect to the coordinate system $O-XY$.

Following the procedure in Appendix A, we can obtain the set of constraint equations of the n-RR PPM as

$$\left\{ \begin{array}{l} x^2 + y^2 + 2A_{1x}z_1 + 2A_{1y}z_2 + 2r^2(1 - C\phi) = L^2 \\ x^2 + y^2 + 2A_{2x}z_1 + 2A_{2y}z_2 + 2r^2(1 - C\phi) = L^2 \\ x^2 + y^2 + 2A_{3x}z_1 + 2A_{3y}z_2 + 2r^2(1 - C\phi) = L^2 \\ x^2 + y^2 + 2A_{ix}z_1 + 2A_{iy}z_2 + 2r_i^2(1 - C\phi) = L^2 \end{array} \right. \tag{B.1}$$

where $r = (A_{1x}^2 + A_{1y}^2)^{1/2} = (A_{2x}^2 + A_{2y}^2)^{1/2} = (A_{3x}^2 + A_{3y}^2)^{1/2}$ is the radius of the circumcircle defined by A_1, A_2 and A_3 (or B_1, B_2 and B_3), $r_i = (A_{ix}^2 + A_{iy}^2)^{1/2}$ represents the distance between A_i and O (or between B_i and O_B), and

$$\left\{ \begin{array}{l} z_1 = xC\phi + yS\phi - x \\ z_2 = -xS\phi + yC\phi - y \end{array} \right. \tag{B.2}$$

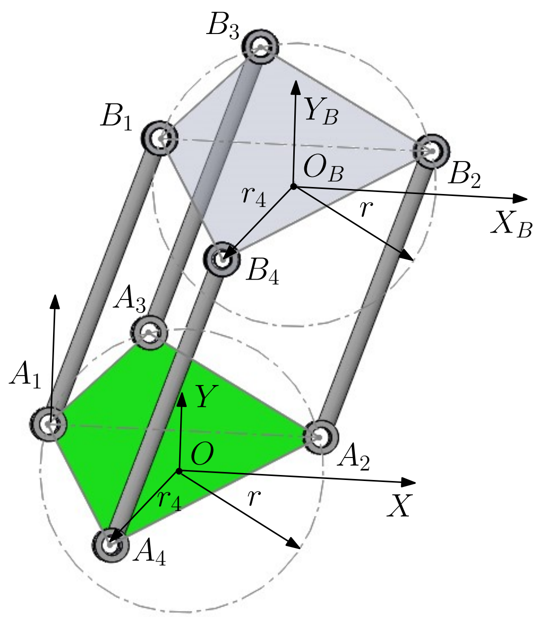

Figure B.1: n-RR PPM with origins of coordinate frames at the centre of the base and moving platform.

As presented in Appendix A, there are two solutions to the first three equations in Eq.(B.1): 1) Eqs. (A.9) and (A.10), and 2) Eqs. (A.11) and (A.14).

It can be readily verified that fourth equation in Eq.(B.1) is satisfied if Eqs. (A.9) and (A.10) are met. In the following, we will derive the condition for the fourth equation in Eq.(B.1) to be satisfied under Eqs. (A.11) and (A.14).

Substituting Eqs. (A.11) and (A.14) into the fourth equation in Eq.(B.1), we have

$$2r_i^2 \times [2(L/(2r)^2] - L^2 = 0 \tag{B.3}$$

Simplification of Eq. (B.3) leads to

$$(r_i/r)^2 = 1 \tag{B.4}$$

i.e.

$$r_i = r \tag{B.5}$$

Equation (B.5) shows A_i (or B_i) is on the circumcircle defined by A_1, A_2 and A_3 (or B_1, B_2 and B_3). Thus it has been proved that the conditions for adding extra RR legs to a 3-RR PPM with a structure mode without affecting its structure/motion modes is that both the base and moving platform are cyclic (Figs. 10 and 11).