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Symplectic embeddings, homotopy algebras and almost Poisson gauge symmetry

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Abstract

We formulate general definitions of semi-classical gauge transformations for noncommutative gauge theories in general backgrounds of string theory, and give novel explicit constructions using techniques based on symplectic embeddings of almost Poisson structures. In the absence of fluxes the gauge symmetries close a Poisson gauge algebra and their action is governed by a P_∞ -algebra which we construct explicitly from the symplectic embedding. In curved backgrounds they close a field dependent gauge algebra governed by an L_∞ -algebra which is not a P_∞ -algebra. Our technique produces new all orders constructions which are significantly simpler compared to previous approaches, and we illustrate its applicability in several examples of interest in noncommutative field theory and gravity. We further show that our symplectic embeddings naturally define a P_∞ -structure on the exterior algebra of differential forms on a generic almost Poisson manifold, which generalizes earlier constructions of differential graded Poisson algebras, and suggests a new approach to defining noncommutative gauge theories beyond the gauge sector and the semi-classical limit based on A_∞ -algebras.

Contents

1	Introduction	3
2	Symplectic embeddings of almost Poisson structures	9
2.1	Formal deformations of cotangent bundles	9
2.2	Local symplectic embedding of Poisson manifolds	11
2.3	Local symplectic embedding of twisted Poisson manifolds	14
2.4	Local symplectic embedding of almost Poisson manifolds	14
2.5	Semi-classical limit of deformation quantization	15
3	Poisson gauge transformations	18
4	Almost Poisson gauge transformations	21
4.1	Formulation of the gauge algebra	22
4.2	Existence of Lagrangian multipliers	25
5	Homotopy algebra actions	29
5.1	L_∞ -algebras and generalized gauge symmetries	30
5.2	L_∞ -structures on Poisson gauge algebroids	32
5.3	L_∞ -structures on almost Poisson gauge algebroids	35
5.4	P_∞ -algebras of exterior differential forms	37
5.5	Comparison with the L_∞ -bootstrap	43
6	Examples	44
6.1	Linear Poisson structures	44
6.2	Poisson families	46
6.3	Magnetic Poisson structures	49
A	Cosymplectic brackets on the constraint locus	53
B	Closure of almost Poisson gauge transformations	54

1 Introduction

The construction of noncommutative gauge theories on manifolds with non-trivial tensor fields is an important problem for understanding the low-energy physics of D-branes in general backgrounds of string theory. The problem is of course not new, and has been studied for around 20 years now. But despite its relatively long history of investigation, the construction is still not completely understood in full generality. In this paper we propose a new approach to this old problem, where the main mathematical tool employed is known as a ‘symplectic embedding’.

In this section we start by providing some background and motivation from string theory, and then proceed to an informal description of our goals and main results, putting them into context with earlier physics literature on the subject. More precise statements and proofs will be given in subsequent sections, with a detailed technical analysis of the issues discussed here.

Symplectic embeddings. Consider a D-brane wrapping a submanifold M in a flat background. Then the string equations imply the vanishing of the three-form H -flux for the B -field: $H = dB = 0$; when the closed two-form B is non-degenerate its inverse defines a Poisson bivector θ on M . Quantizing an open string with its ends on the D-brane shows that, in a suitable low-energy scaling limit, its worldvolume algebra of functions undergoes a deformation in the direction of θ . The problem of constructing a noncommutative gauge theory on M then starts with a deformation quantization of the Poisson manifold (M, θ) . Conversely, the semi-classical limit of an associative noncommutative algebra of functions on a manifold M which is a flat deformation defines a Poisson bracket

$$\{x^i, x^j\}_\theta = \theta^{ij} ,$$

on local coordinate functions x^i ; in other words, the Poisson bracket is the first order semi-classical approximation to the star-product in deformation quantization. It is this semi-classical limit that we shall mostly work with in this paper, where all constructions are entirely geometric and are regarded as the classical infinitesimal data whose quantization yields the required ingredients of a noncommutative gauge theory.

In the case of a flat D-brane, the Poisson bivector θ is constant, and the worldvolume deformation is provided by the usual Moyal-Weyl star-product [62], and the construction of noncommutative gauge theories is standard and well-known [63]. For a curved D-brane in flat space, θ is not constant, and the worldvolume deformation is provided by the Kontsevich star-product [22]. The first problem one then encounters is how to define derivatives in the field theory: the usual differential is no longer a derivation of the Poisson algebra, and this obstructs the naive definition of gauge transformations to closing a Lie algebra. The problem can be formulated and solved by embedding the Poisson bracket in a ‘noncommutative phase space’

$$\begin{aligned} \{x^i, x^j\} &= \theta^{ij} , \\ \{p_i, x^j\} &= \delta_i^j - \frac{1}{2} \partial_i \theta^{jk} p_k + \dots , \\ \{p_i, p_j\} &= 0 . \end{aligned} \tag{1.1}$$

The auxiliary ‘momentum’ coordinates p_i are regarded as ‘derivatives’ when acting on functions f on M , since when θ is constant these brackets give $\{p_i, f\} = \partial_i f$; the ellipsis denotes higher

order monomials in the momenta p_k which accompany higher orders in the bivector θ and its derivatives. In general, the extra derivative terms in these brackets ensure that they fulfill the Jacobi identity order by order in p_i when θ is non-constant. Now the action of the ‘twisted’ derivatives $\{p_i, \cdot\}$ obeys the Leibniz rule, as a consequence of the Jacobi identity for the bracket. Noncommutative gauge theories have been constructed in this manner in e.g. [6]; see e.g. [66] for a review of this and other approaches, along with further references.

The Poisson brackets (1.1) also offer an alternative approach to quantization of the Poisson manifold (M, θ) . We can map the coordinates (x, p) to canonically conjugate variables (X, P) by means of a generalized Bopp shift; the existence of such a diffeomorphism is guaranteed by Darboux’s theorem, at least locally or in the case where M is covered by a single Darboux chart. The canonical coordinates (X, P) can be quantized geometrically in the usual way via a Schrödinger polarization, and mapped back to provide a polydifferential representation of the quantum version of the brackets (1.1) on the space of functions on M . By formally expanding functions of the polydifferential operators in Taylor series, this also constructs a star-product and provides a deformation quantization of the algebra of functions on M .

This construction of a noncommutative phase space is called a *symplectic embedding* of the Poisson structure θ ; in Section 6 we will see some explicit all orders examples which are given by closed analytic expressions. The brackets can be derived as the symplectic structure arising from open string quantization on the D-brane in a low-energy scaling limit [21]. In particular, it enables one to define semi-classical gauge transformations that consistently close a gauge algebra defined by the Poisson brackets: we use ‘twisted covariant derivatives’ of gauge parameters defined by the p_i and suitably restrict them by imposing constraints on the phase space that eliminate the auxiliary momentum coordinates. This is explained in detail in Section 3.

Mathematically, symplectic embeddings are a more general notion of ‘symplectic realizations’ in Poisson geometry [36,71]. We discuss the precise relation between the two notions in Section 2. The global structure which integrates a Poisson manifold is called a symplectic groupoid [70], whose source maps always define symplectic realizations. They were introduced as part of the program of quantizing Poisson manifolds. Symplectic groupoids capture the semi-classical limit of deformation quantization, as inferred by our physical arguments above, and as shown precisely in [16] where the symplectic groupoid is constructed as the classical phase space of the open string sigma-model.

Quantized differential forms. The problem of finding suitable derivative operators in a noncommutative field theory on a Poisson manifold (M, θ) can also be formulated dually as the problem of constructing a suitable noncommutative differential calculus on M . In the semi-classical limit, this amounts to finding an extension of the Poisson bracket to the exterior algebra of differential forms. The problem of constructing a differential graded Poisson algebra in this way has been addressed from several points of view, see e.g. [5, 20, 30, 37, 53, 54], while from the perspective of deformation quantization it is discussed in e.g. [2, 9, 25]. These constructions all depend on the choice of an auxiliary connection on M and typically require the Poisson bivector θ to be invertible.

Noncommutative gauge theories are treated using this formalism in [32]. Their definition of gauge transformations is similar in structure to ours in Section 3, but differs in important details. A crucial distinction is that our formulation is based on symplectic embeddings, which

always exist locally without any further data and applies to arbitrary Poisson structures, while the formulation of [32] requires the extra data of a symplectic connection on M .

Nonassociative deformation quantization. The problem is more involved in the case of a curved background, wherein the H -flux is non-zero: $H = dB \neq 0$. In this case the Jacobi identity for the semi-classical bracket is violated, and the bivector θ defines an H -twisted Poisson structure [38, 58]. The worldvolume deformations of D-branes in this setting are still given by the Kontsevich expansion [22], which now leads to a nonassociative star-product. In this case, the symplectic embedding is described by a noncommutative (but associative) phase space of the form

$$\begin{aligned}\{x^i, x^j\} &= \theta^{ij} - \Pi^{ijk} p_k + \dots, \\ \{p_i, x^j\} &= \delta_i^j - \frac{1}{2} \partial_i \theta^{jk} p_k + \dots, \\ \{p_i, p_j\} &= 0,\end{aligned}$$

where

$$\Pi^{ijk} = \frac{1}{3} (\theta^{il} \partial_l \theta^{jk} + \text{cyclic}) = \theta^{il} \theta^{jm} \theta^{kn} H_{lmn}$$

is the Jacobiator $\{\{x^i, x^j\}_\theta, x^k\}_\theta + \text{cyclic}$, the trivector encoding the violation of the Jacobi identity for the original twisted Poisson brackets $\{x^i, x^j\}_\theta = \theta^{ij}$; its inclusion is now necessary to ensure that these extended brackets fulfill the Jacobi identity. These brackets agree precisely with the symplectic structure found in [31, 33] from quantizing an open string with its ends on a D-brane in a constant H -flux background.

Regarding again the momenta as ‘derivatives’, now the associative phase space structure is that of an algebra of pseudo-differential operators, whereby even the bracket of two functions on M is generally a differential operator. This was explained by [46], who show that the symplectic embedding in this case captures the semi-classical limit of the associative composition algebra of differential operators on M corresponding to a twisted Poisson structure [56]. In D-brane physics this poses a serious problem in the definition of gauge theories, as the gauge transformation of a gauge field will generically produce not another gauge field but a pseudo-differential operator; a resolution was proposed by [31] which involves promoting gauge parameters to functions of both coordinates x^i and momenta p_i . In the present paper we propose a different way of resolving this issue, which is similar in spirit, but involves a suitable constraint on the phase space as in the case of a Poisson structure as well as field dependent gauge transformations. The precise definitions and constructions are explained in Section 4.

This extended symplectic embedding formalism was initiated in [46] to describe the (associative) classical and quantum mechanics of an electric charge in smooth distributions of magnetic monopoles, whose phase space is described by a twisted Poisson structure. The main idea behind our construction of the “nonassociative” gauge algebra consists in starting with a given almost Poisson structure, and aiming to get the violation of the Jacobi identity under control. For this, we introduce auxiliary variables p_i and construct its symplectic embedding. In the larger space the Jacobi identity is now satisfied, but one is then faced with the problem of interpreting the auxiliary (unphysical) degrees of freedom, which cannot be removed in the nonassociative case [46]. The most non-trivial point of our construction is getting rid of the auxiliary variables

by introducing constraints in a consistent way, that is, in such a way that, in the “nonassociative” gauge algebra, the commutator of two gauge transformations is again a gauge transformation, but with a field dependent gauge parameter.

The global objects corresponding to symplectic embeddings of twisted Poisson or more generally almost Poisson structures are not presently understood. From a string theory perspective, they can be described as follows. A fundamental closed string in a constant H -flux background can be lifted to M-theory as an open membrane ending on an M5-brane in a three-form C -field background with constant flux. This gives rise to a noncommutative algebra of functions on the loop space of the M5-brane worldvolume [8], whose deformation quantization has as semi-classical limit a symplectic groupoid on the loop space obtained through transgression of the three-form H [59,60]. In this sense symplectic embeddings play a similar role in the semi-classical limit of deformation quantization of almost Poisson structures. We give a more precise account in Section 2.5.

Strong homotopy algebras. Recent advances in nonassociative deformation quantization have produced some explicit constructions of star-products which quantize twisted Poisson and even almost Poisson structures. A star-product which quantizes the phase space of a closed string in a constant R -flux background, or equivalently an electric charge moving in a uniform magnetic monopole density, was constructed in [55]; the phase space defines a twisted Poisson structure and the star-product is, in a sense, a nonassociative analog of the Moyal-Weyl star-product. In [41] a star-product was proposed which quantizes the linear almost Poisson structure of the imaginary octonion algebra, which is an example of an almost Poisson structure that is not twisted Poisson; in [45] this was applied to the quantization of the phase space of a membrane in a non-geometric R -flux background of M-theory, or equivalently an M-wave in a non-geometric Kaluza-Klein monopole density, where it was shown that the contraction of the octonionic star-product reproduces the monopole star-product. See [68] for a review of these developments and further references.

These examples have nice physical features. In particular, nonassociativity vanishes on-shell, as it should since string theory is based on a two-dimensional quantum field theory that involves strictly associative structures. Precisely, the integrated associator vanishes for these star-products, as the star-associator of three functions is a ‘total derivative’; this is also true for the nonassociative star-products of [22], provided one chooses a suitable density to integrate against (see e.g. [66]). However, this property fails in general for arbitrary almost Poisson structures, and the question arises as to what structure should be used to control the violation of associativity, or the Jacobi identity in the semi-classical limit, in a consistent way. A good candidate is Stasheff’s A_∞ -algebras [65], and also L_∞ -algebras [51] where the violation of the Jacobi identity is proportional to a higher coherent homotopy or a ‘total derivative’ of some higher bracket, together with their extension to P_∞ -algebras [69]. This suggests that strong homotopy algebras might provide the right setting for the quantization of generic twisted Poisson and almost Poisson structures; their role in deformation quantization of twisted Poisson structures was already indicated in [22,55]. In particular, the semi-classical limit of an A_∞ -algebra defines a P_∞ -algebra, and it is natural to look for P_∞ -algebras as the structure controlling the violation of the Jacobi identity. The deformation quantization of P_∞ -structures via the Kontsevich formality theorem was first considered in [52], and a bit later in [17].

As we discuss in Section 5.4, this expectation is indeed correct from the following perspective. We will show that our symplectic embeddings, which always exist locally, naturally endow the exterior algebra of differential forms on an arbitrary almost Poisson manifold (M, θ) with the structure of a P_∞ -algebra which contains the almost Poisson bracket. The P_∞ -structure controls not only the potential violation of the Jacobi identity for the almost Poisson bracket, but also the violation of the Leibniz rule for the usual differential, in a way which is compatible with the derivation properties of the original bracket with respect to the classical pointwise multiplication of functions, extended to the exterior product of differential forms. We subsequently sketch an alternative new approach to the quantization of differential forms which does not require the choice of an auxiliary connection on M and quantizes this P_∞ -algebra to an A_∞ -algebra. The details are beyond the scope of the present paper and are left for future investigation, but they illustrate another novel application of our symplectic embedding formalism, as well as its intimate relation with homotopy algebras.

The use of L_∞ -algebras as an alternative new construction of noncommutative gauge theories on arbitrary almost Poisson manifolds (M, θ) was pioneered by [11] and called the ‘ L_∞ -bootstrap’. In Section 5 we provide the dictionary between our symplectic embedding formalism and the formulation in terms of L_∞ -algebras for noncommutative gauge transformations, finding agreement with the lower degree brackets that were constructed explicitly by [11, 43]. However, our approach is much better adapted to obtaining explicit closed all orders expressions for all brackets of the gauge L_∞ -algebra. We illustrate this explicitly on several concrete non-trivial examples of relevance to physics in Section 6, wherein we obtain the complete sets of L_∞ -brackets to all orders. In particular, for the twisted Poisson structure on the phase space of an electric charge in a constant magnetic monopole density, we obtain, for the first time, the complete gauge L_∞ -structure in closed form, see Section 6.3; this should aid in understanding the symmetries of a nonassociative theory of gravity underlying the low-energy sector of closed non-geometric strings (see e.g. [68]). Technically, our approach based on symplectic embeddings is much simpler than the L_∞ -bootstrap approach of [11], particularly for explicit calculations.

In this language we also find a significant distinction between Poisson and almost Poisson gauge symmetries. For a Poisson manifold (M, θ) the gauge L_∞ -algebra is itself a P_∞ -algebra which is obtained by a truncation of the P_∞ -structure underlying the semi-classical limit of quantization of differential forms. This suggests a path towards the construction of noncommutative gauge transformations beyond the semi-classical limit, in terms of A_∞ -algebras. However, this is not the case for the homotopy algebras which govern the closure of “nonassociative” gauge transformations: our construction of gauge transformations defines an L_∞ -algebra which is *not* a P_∞ -algebra, unless the Jacobi identity is satisfied, that is, only “associative” gauge algebras are controlled by P_∞ -algebras. The reasons for this are explained in detail in Section 5.4: the essential feature is that for an almost Poisson structure the P_∞ -structure on differential forms cannot be truncated to a subalgebra containing the gauge L_∞ -algebra.

An alternative approach to the construction of noncommutative gauge theories using homotopy algebras has been put forth recently by [24], whereby the underlying L_∞ -structure is also quantized. In contrast to our approach, the gauge theory L_∞ -algebra involves only finitely many brackets as in the classical case, and compatibility with the Leibniz rule is manifest from the outset. It would be interesting to understand better how this more algebraic approach is related to the geometric approach of the present paper.

Structure of the paper. In this paper we work solely with the kinematical sector of a complete noncommutative gauge theory, and only with the first order semi-classical approximations to proper noncommutative gauge transformations. This will elucidate, in a model independent way, the classical geometric structures underlying the full gauge algebras which quantize them. In the absence of any dynamics, we will require that our gauge transformations close to a Lie algebra. One of the main technical achievements of this paper is the ability to do this in the nonassociative case: the naive passage from Poisson to general almost Poisson gauge transformations do not immediately close, even when one modifies the requirement of closure to include field dependent gauge transformations. We show explicitly how closure can be accomplished using symplectic embeddings by the addition of terms to the gauge variations which compensate the undesired contributions to the closure formulas; we demonstrate that these extra terms are explicitly calculable. When the dynamical sector of a particular gauge theory is included, it may be possible to avoid adding these additional terms in the gauge variations and work instead with a weaker notion of closure, whereby the almost Poisson gauge algebra also involves the field equations, that is, gauge transformations only close on-shell, see Remark 5.30. Barring these model dependent extensions, it is remarkable that we are able to describe the closure of “nonassociative” gauge transformations in terms of strictly associative Lie algebras, albeit field dependent ones.

The majority of this paper can be read by taking M to be a coordinate manifold \mathbb{R}^d (or an open subset thereof), but in several places we have written tensor quantities in a more covariant form that we hope makes some of our calculations amenable to global generalizations in future work. Our main purpose here is to unravel the geometric and algebraic structures that govern the first steps to constructing noncommutative gauge theories on D-branes in general string backgrounds. This turns out to be a technically formidable task even with our restriction to local coordinate charts. Therefore, throughout we shall carefully formulate our geometric framework and then carry out all calculations via tensor calculus in local coordinates, leaving the corresponding description of the covariant tensor calculus within our framework for future study.

The outline of the remainder of this paper is as follows. In Section 2 we give a precise description of our symplectic embeddings which we motivated from physical arguments above, and compare it to the existing notions of symplectic realizations and symplectic groupoids from the mathematical literature on Poisson geometry. In Section 3 we give a precise general definition of what we mean by semi-classical gauge transformations on a Poisson manifold, and provide an explicit construction using symplectic embeddings of the Poisson structure. In Section 4 we extend these general definitions and constructions to the general case of almost Poisson manifolds. The closure condition on the semi-classical gauge algebra now requires an extension of the notion of gauge transformations using field dependent gauge parameters, which we formulate precisely in terms of 1-jets of the cotangent bundle of the underlying manifold M , and an extension of their construction by symplectic embeddings using horizontal vector fields on the jet space. In Section 5 we show how our symplectic embedding construction of gauge algebras can be cast into the standard framework of L_∞ -algebras [27, 34, 35], contrasting our approach with the approach based on the L_∞ -bootstrap. We further show how our symplectic embeddings define P_∞ -structures on differential forms and discuss their relation with the L_∞ -structures on our gauge algebras. Finally, in Section 6 we work out several explicit examples as illustrations of our formalism for both Poisson and almost Poisson gauge transformations. Two

appendices at the end of the paper collect some technical details which are used in the main text.

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2 Symplectic embeddings of almost Poisson structures

In this section we develop our notion of symplectic embeddings, which encompasses earlier constructions from the physics literature in a rigorous and precise way. We provide a detailed comparison between our symplectic embeddings and the more common concept of symplectic realizations in Poisson geometry, and discuss their relation to deformation quantization. In Section 6 we will give detailed applications to some non-trivial explicit examples.

2.1 Formal deformations of cotangent bundles

We begin with some definitions that will permeate this paper. Smooth functions and tensors on a manifold M , as well as all vector spaces, are always considered with the ground field \mathbb{R} or \mathbb{C} .

Definition 2.1. Let M be a manifold.

- (a) An *almost Poisson structure* on M is a bivector field $\theta \in \mathfrak{X}^2(M)$. The bivector field is equivalently encoded by the skew-symmetric *almost Poisson bracket* $\{\cdot, \cdot\}_\theta$ on $C^\infty(M)$ given by

$$\{f, g\}_\theta = \theta(df, dg) .$$

The Schouten-Nijenhuis bracket of the bivector θ with itself, the *Jacobiator*, is the trivector field $\Pi = [\theta, \theta] \in \mathfrak{X}^3(M)$, which satisfies

$$\frac{1}{2} \Pi(df, dg, dh) = \text{Cyc}_{f,g,h} \{f, \{g, h\}_\theta\}_\theta ,$$

where $\text{Cyc}_{f,g,h}$ indicates the cyclic sum over the functions $f, g, h \in C^\infty(M)$. The Jacobi identity $[[\theta, \theta], \theta] = 0$ for the Schouten bracket implies that the Jacobiator obeys the integrability condition

$$[\Pi, \theta] = 0 . \tag{2.2}$$

- (b) A bivector θ on M gives rise to a linear map

$$\theta^\# : \Omega^1(M) \longrightarrow \mathfrak{X}(M) , \quad \alpha \longmapsto \theta(\alpha, \cdot) . \tag{2.3}$$

If

$$\Pi = \frac{1}{2} \wedge^3 \theta^\# H$$

for a closed three-form $H \in \Omega^3(M)$, then θ is an *H -twisted Poisson structure* on M and $\{\cdot, \cdot\}_\theta$ is the corresponding *H -twisted Poisson bracket*.

- (c) The bracket $\{\cdot, \cdot\}_\theta$ obeys the Jacobi identity if and only if $\Pi = 0$. In this case θ is a *Poisson structure* and $\{\cdot, \cdot\}_\theta$ is the corresponding *Poisson bracket*.

Remark 2.4. By definition, any almost Poisson structure defines a derivation $\{f, \cdot\}_\theta$ of the commutative algebra of functions $C^\infty(M)$ for any fixed $f \in C^\infty(M)$. When θ is a Poisson bivector, this property makes $(C^\infty(M), \{\cdot, \cdot\}_\theta)$ into a *Poisson algebra*. \triangleleft

Remark 2.5. If an almost Poisson structure θ is non-degenerate, then it is automatically a twisted Poisson structure: in this case the map (2.3) is invertible, and the two-form $\theta^{-1} \in \Omega^2(M)$, defined by $\theta^{-1}(X, Y) = \theta(\theta^{\sharp-1}X, \theta^{\sharp-1}Y)$ for $X, Y \in \mathfrak{X}(M)$, is an almost symplectic structure on M and $H = d\theta^{-1}$ is the twisting three-form. In particular, if θ is a non-degenerate Poisson bivector then θ^{-1} is a symplectic structure on M . \triangleleft

We shall often work on an open subset $M = U \subset \mathbb{R}^d$ with local coordinates (x^i) . Then the bivector θ has the local coordinate expression $\theta = \frac{1}{2} \theta^{ij}(x) \partial_i \wedge \partial_j$, where $\partial_i = \partial/\partial x^i$, and the almost Poisson bracket can be expressed on U as

$$\{f, g\}_\theta = \theta^{ij} \partial_i f \partial_j g .$$

Throughout we use the Einstein convention for implicit summation over repeated upper and lower indices. Similarly, the trivector $\Pi = [\theta, \theta]$ has the local coordinate expression $\Pi = \frac{1}{3!} \Pi^{ijk}(x) \partial_i \wedge \partial_j \wedge \partial_k$, where

$$\Pi^{ijk} = \frac{1}{3} (\theta^{il} \partial_l \theta^{jk} + \theta^{kl} \partial_l \theta^{ij} + \theta^{jl} \partial_l \theta^{ki}) .$$

In this paper it will be convenient to rescale the bivector θ by a formal parameter t , and regard it as a deformation of the zero bivector $\theta_0 = 0$. If V is a real or complex vector space, we denote by $V[[t]]$ the space of formal power series in t with coefficients in V ; it can be regarded as a module over $\mathbb{R}[[t]]$ or $\mathbb{C}[[t]]$. In this paper we will work in the setting of formal Poisson structures and formal symplectic structures, see e.g. [15].

Definition 2.6. Let θ be an almost Poisson structure on a manifold M . Write $\pi : T^*M \rightarrow M$ for the cotangent bundle of M . A *symplectic embedding* of (M, θ) is a formal symplectic structure $\omega \in \Omega_{\text{pol}}^2(T^*M)[[t]]$ such that

- (a) ω is a deformation of the canonical symplectic structure ω_0 on T^*M , that is, $\omega|_{t=0} = \omega_0$;
- (b) The corresponding Poisson bracket $\{\cdot, \cdot\}_{\omega^{-1}}$ on the ring $C_{\text{pol}}^\infty(T^*M)[[t]]$ restricts to the zero section $M \subset T^*M$ as

$$\{\pi^* f, \pi^* g\}_{\omega^{-1}}|_M = t \{f, g\}_\theta \tag{2.7}$$

for all functions $f, g \in C^\infty(M)$; and

- (c) The zero section is a Lagrangian section of (T^*M, ω) .

Here and in the following we use the subscript pol to denote spaces of smooth tensor fields on the cotangent bundle T^*M which are polynomial on the fibres. In general, the formal symplectic structure ω is a formal power series of closed two-forms on T^*M (which are polynomial on

the fibres) given as $\omega = \omega_0 + \sum_{n=1}^{\infty} t^n \omega_n$. The inverse is a formal Poisson structure $\omega^{-1} \in \mathfrak{X}_{\text{pol}}^2(T^*M)[[t]]$ which is a formal power series of bivectors on T^*M given as

$$\omega^{-1} = \Theta_0 + \sum_{n=1}^{\infty} t^n \Theta_n ,$$

where $\Theta_0 = \omega_0^{-1}$ is the Poisson bivector field on T^*M corresponding to ω_0 . The Poisson integrability condition $[\omega^{-1}, \omega^{-1}] = 0$ implies

$$\sum_{k=0}^n [\Theta_k, \Theta_{n-k}] = 0 \tag{2.8}$$

for all $n \geq 0$, and the condition (2.7) implies $\omega^{-1}(\pi^*\alpha, \pi^*\beta)|_M = t\theta(\alpha, \beta)$ which leads to

$$\Theta_1(\pi^*\alpha, \pi^*\beta)|_M = \theta(\alpha, \beta) \quad \text{and} \quad \Theta_n(\pi^*\alpha, \pi^*\beta)|_M = 0 ,$$

for all $\alpha, \beta \in \Omega^1(M)$ and $n \geq 2$. The Lagrangian section condition further imposes $\omega(X, Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$. In this sense a symplectic embedding can be thought of as a formal deformation of an almost Poisson structure θ on M , for which $[\theta, \theta] \neq 0$, to a nondegenerate Poisson structure ω^{-1} on T^*M , for which $[\omega^{-1}, \omega^{-1}] = 0$; in particular, the canonical symplectic structure ω_0 yields a symplectic embedding for the trivial bivector $\theta_0 = 0$.

Remark 2.9. Definition 2.6 is an adapted version, to the almost Poisson case, of a symplectic realization of a Poisson structure θ on M , which more generally involves a surjective submersion $\pi : S \rightarrow M$ of a symplectic manifold S that is a Poisson morphism admitting a Lagrangian section $M \rightarrow S$. When $S = T^*M$ with π the cotangent bundle projection and $M \rightarrow S$ the zero section, a symplectic realization is a symplectic embedding, i.e. the condition (2.7) holds, but the converse is not generally true. Similarly, one defines an almost symplectic realization of a twisted Poisson structure. These can all be constructed globally in terms of integrating symplectic groupoids for (M, θ) [18, 36, 70]. In this paper we deal only with local constructions, where the existence of symplectic embeddings is always guaranteed as a formal deformation of the trivial symplectic groupoid (T^*M, ω_0) near the identity elements. \triangleleft

We shall now establish the existence of local symplectic embeddings by treating the three cases in Definition 2.1 individually, in order of increasing complexity. In the following we work on the open neighbourhood T^*U of the zero section of $M = U \subseteq \mathbb{R}^d$, and denote local coordinates thereon by (x^i, p_i) , where (p_i) are coordinates in the normal directions to $U \subset T^*U$; in particular, U is given by the equations $p_i = 0$ in T^*U . We also write $\tilde{\partial}^i = \partial/\partial p_i$.

2.2 Local symplectic embedding of Poisson manifolds

The simplest case is the original local construction of a symplectic realization of a Poisson manifold, which is due to Weinstein [71]. Let λ_0 be the Liouville one-form, i.e. the unique one-form on T^*M with the property that $s_\alpha^*\lambda_0 = \alpha$ for all one-forms $\alpha \in \Omega^1(M)$, regarded as smooth sections $s_\alpha : M \rightarrow T^*M$ of the cotangent bundle of M under the isomorphism $\Omega^1(M) \simeq \Gamma(T^*M)$. It is the tautological primitive for the canonical symplectic structure; in local coordinates, $\lambda_0 = p_i dx^i$ and $\omega_0 = d\lambda_0 = dp_i \wedge dx^i$ on T^*U . Contracting the pushforward

of the Poisson bivector field θ by the zero section with the Liouville one-form defines a vector field $X^\theta \in \mathfrak{X}(T^*U)$:

$$X^\theta = \theta(\lambda_0, \cdot) ,$$

which in local coordinates reads

$$X^\theta = \theta^{ij}(x) p_i \partial_j . \quad (2.10)$$

The vector field X^θ defines a Poisson spray, see e.g. [23]. Let $\varphi_u^\theta : T^*U \rightarrow T^*U$ be the flow of $t X^\theta$ for $u \in [0, 1]$, which is the diffeomorphism defined by

$$\frac{d\varphi_u^\theta}{du} = t X^\theta \circ \varphi_u^\theta .$$

Then the symplectic structure $\omega \in \Omega^2(T^*U)$ constructed by [71, Theorem 9.1] is given by the integrated pullback of the canonical symplectic structure ω_0 by this flow:

$$\omega := \int_0^1 (\varphi_u^\theta)^* \omega_0 du . \quad (2.11)$$

Since $\varphi_u^\theta|_{t=0}$ is the identity for all $u \in [0, 1]$, this symplectic structure is indeed a deformation of ω_0 . Note that the zero section $U \subset T^*U$ is a Lagrangian submanifold of (T^*U, ω) , and that $\omega = d\lambda$ where $\lambda = \phi^i(x, p) dp_i$ with

$$\phi^i(x, p) = \int_0^1 x^i \circ \varphi_u^\theta du . \quad (2.12)$$

The Jacobian matrix $J_\phi(x, p) = \left(\frac{\partial \phi^i}{\partial x^j}\right)$ is formally invertible (because $\varphi_u^\theta|_{t=0}$ is the identity for all $u \in [0, 1]$), and we denote its inverse by $\mathbb{1} + t\gamma(x, p)$. The corresponding cosymplectic structure then assumes the local form

$$\omega^{-1} = \frac{t}{2} \theta^{ij}(x) \partial_i \wedge \partial_j + \frac{1}{2} (\delta_j^i + t \gamma_j^i(x, p)) (\partial_i \wedge \tilde{\partial}^j + \tilde{\partial}^j \wedge \partial_i) . \quad (2.13)$$

For later use, and in particular for comparison with the generic cases of almost Poisson structures, it is useful to cast the symplectic embedding described by (2.11) into the setting of Definition 2.6 by developing its asymptotic series in t . For this, we expand the bivector γ as a formal power series

$$\gamma_i^j(x, p) = \sum_{n=1}^{\infty} t^{n-1} \gamma_i^{j|i_1 \dots i_n}(x) p_{i_1} \dots p_{i_n} \quad (2.14)$$

using (2.12), where the local functions $\gamma_i^{j|i_1 \dots i_n}(x)$ are proportional to the components of the bivector θ and their derivatives. Alternatively, they can be found by solving the Poisson integrability condition $[\omega^{-1}, \omega^{-1}] = 0$, which using $[\theta, \theta] = 0$ yields local first order differential equations

$$\tilde{\partial}^l \gamma_i^k - \tilde{\partial}^k \gamma_i^l + t (\gamma_j^l \tilde{\partial}^j \gamma_i^k - \gamma_j^k \tilde{\partial}^j \gamma_i^l) = \partial_i \theta^{lk} + t (\gamma_i^j \partial_j \theta^{lk} + \theta^{kj} \partial_j \gamma_i^l - \theta^{lj} \partial_j \gamma_i^k) \quad (2.15)$$

for the bivector γ in terms of the given bivector θ . Substituting the formal power series (2.14) in (2.15) then yields an infinite system of recursive differential equations given by

$$\gamma_i^{k|l} - \gamma_i^{l|k} = \partial_i \theta^{lk} , \quad (2.16)$$

and

$$\begin{aligned}
& (n+1) \left(\gamma_i^{k|i_2 \dots i_{n+1}} - \gamma_i^{l|ki_2 \dots i_{n+1}} \right) \\
& + \sum_{m=1}^n (n-m+1) \left(\gamma_j^{l|i_1 \dots i_m} \gamma_i^{k|j i_{m+1} \dots i_{n+1}} - \gamma_j^{k|i_1 \dots i_m} \gamma_i^{l|j i_{m+1} \dots i_{n+1}} \right) \\
& = \gamma_i^{j|i_2 \dots i_{n+1}} \partial_j \theta^{lk} + \theta^{kj} \partial_j \gamma_i^{l|i_2 \dots i_{n+1}} - \theta^{lj} \partial_j \gamma_i^{k|i_2 \dots i_{n+1}} , \quad (2.17)
\end{aligned}$$

for $n \geq 1$. A formal power series solution of (2.15) was constructed in this way by [40], with the first two leading orders given by

$$\begin{aligned}
\gamma_i^{j|k} &= -\frac{1}{2} \partial_i \theta^{jk} \\
\gamma_i^{j|kl} &= -\frac{1}{12} \left(2 \theta^{lm} \partial_i \partial_m \theta^{kj} + \partial_i \theta^{km} \partial_m \theta^{jl} \right) . \quad (2.18)
\end{aligned}$$

Remark 2.19. The solution (2.18) is not unique. There is no general notion of equivalence of symplectic embeddings for a general Poisson manifold. If (M, θ) is integrable, there is a notion of Morita self-equivalence, see e.g. [15]. We will return to the question of uniqueness, as well as the meaning of the symplectic structure on T^*M away from the zero section, later on where we will find that they have natural interpretations in terms of gauge algebras (see Remark 5.13 and Proposition 3.7, respectively). \triangleleft

Example 2.20. Let $M = \mathbb{R}^d$ with a constant Poisson structure θ . This is the only case in which the solution $\gamma_i^j = 0$ to (2.15) is possible; this is also the solution that follows from (2.12) for constant θ . With this choice, the symplectic embedding of (\mathbb{R}^d, θ) is given by the strict deformation $(T^*\mathbb{R}^d, \omega_0 + t\theta^*)$ of the cotangent symplectic groupoid for \mathbb{R}^d , where θ^* is the vertical two-form on $T^*\mathbb{R}^d$ induced by the linear dual of the bivector θ on the vector space \mathbb{R}^d , that is, $\theta^*(\tilde{\partial}^i, \tilde{\partial}^j) = \theta^{ij}$ and $\theta^*(\partial_i, \partial_j) = 0 = \theta^*(\partial_i, \tilde{\partial}^j)$. The integrating symplectic groupoid is the direct product of the pair groupoid $\mathbb{R}^r \times \mathbb{R}^r$ and the cotangent groupoid $T^*\mathbb{R}^{d-r}$ for \mathbb{R}^{d-r} where r is the rank of θ , see e.g. [16]. However, there are also non-zero solutions of (2.15) in this case. \triangleleft

Remark 2.21. An explicit global extension of the local symplectic embedding (2.11) to an open neighbourhood $N \subset T^*M$ of the zero section is given in [12, 23]. The construction depends on the choice of an affine connection ∇ on M . It amounts to replacing the vector field (2.10) with

$$X^{\theta, \nabla} = \theta^{ij}(x) p_i \partial_j + p_k p_l \theta^{ki}(x) \Gamma_{ij}^l(x) \tilde{\partial}^j ,$$

where $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^l(x) \partial_l$, and replacing φ_u^θ with the corresponding flow $\varphi_u^{\theta, \nabla}$ of $tX^{\theta, \nabla}$ in (2.11). The connection ∇ induces a Lie algebroid connection on the cotangent Lie algebroid $(T^*M, \theta^\sharp, [\cdot, \cdot]_\theta)$ associated to an integrable Poisson manifold (M, θ) , where the bracket of this Lie algebroid extends the natural Lie bracket on exact one-forms, given by $[df, dg]_\theta = d\{f, g\}_\theta$, in a unique way to the Koszul bracket

$$[\alpha, \beta]_\theta := L_{\theta^\sharp \alpha} \beta - L_{\theta^\sharp \beta} \alpha - d\theta(\alpha, \beta)$$

where L denotes the Lie derivative. This construction makes N into a local symplectic groupoid $N \rightrightarrows M$ (cf. [36]), with source map π and target map $\pi \circ \varphi_1^{\theta, \nabla}$, which integrates the cotangent Lie algebroid. \triangleleft

2.3 Local symplectic embedding of twisted Poisson manifolds

The definition (2.11) makes sense for any bivector θ and defines a symplectic structure, but it yields a symplectic embedding only when θ is a Poisson structure. However, when θ is an H -twisted Poisson structure, it is possible to modify ω accordingly [18] and turn it into an almost symplectic embedding of (M, θ) : the two-form

$$\omega_H = \int_0^1 (\varphi_u^\theta)^* (\omega_0 + t \pi^* H(X^\theta, \cdot, \cdot)) du \quad (2.22)$$

is an almost symplectic structure with $d\omega_H = t \pi^* H$ which satisfies (2.7).

Remark 2.23. Similarly to the untwisted case (see Remark 2.21), a twisted Poisson structure makes the cotangent bundle into a Lie algebroid $(T^*M, \theta^\sharp, [\cdot, \cdot]_{\theta, H})$ with Lie bracket

$$[\alpha, \beta]_{\theta, H} := [\alpha, \beta]_\theta + H(\theta^\sharp \alpha, \theta^\sharp \beta, \cdot) .$$

In particular, for $f, g \in C^\infty(M)$ this gives

$$[df, dg]_{\theta, H} = d\{f, g\}_\theta + H(X_f, X_g, \cdot)$$

with $X_f = \theta^\sharp df$. If θ is integrable, the corresponding integrating Lie groupoid is called a twisted symplectic groupoid in [18]; it is provided by extending this local almost symplectic embedding construction by replacing X^θ with $X^{\theta, \nabla}$ and φ_u^θ with $\varphi_u^{\theta, \nabla}$ in (2.22) [23], as explained in Remark 2.21. \triangleleft

In this local picture, it is possible to locally ‘untwist’ the almost symplectic embedding to a bonafide symplectic embedding by interpreting the closed three-form $H \in \Omega^3(M)$ as the curvature of a gerbe connection and following the approach of [4, 55]. For this, we choose a suitable covering of the manifold M by good open subsets U_a . We can write H in terms of local two-forms $B_a \in \Omega^2(U_a)$ as $H = dB_a$. On intersections $U_{ab} := U_a \cap U_b$, the two-form $F_{ab} := B_b - B_a$ is closed and hence exact, so it can be expressed in terms of one-form fields $A_{ab} \in \Omega^1(U_{ab})$ as $F_{ab} = dA_{ab}$; triple intersections involve local gauge transformations by functions $f_{abc} \in C^\infty(U_a \cap U_b \cap U_c)$ satisfying a suitable integrability condition. Then the two-form $\omega_a \in \Omega^2(U_a)$ defined by

$$\omega_a = \omega_H - t \pi^* B_a$$

is a symplectic structure on T^*U_a , i.e. $d\omega_a = 0$. The local symplectic two-forms ω_a are related by the flows φ_u^{ab} at $u = 1$ generated by the vector fields $t\theta(A_{ab}, \cdot) \in \mathfrak{X}(U_{ab})$, such that

$$(\varphi_1^{ab})^* \{F, G\}_{\omega_b^{-1}} = \{(\varphi_1^{ab})^* F, (\varphi_1^{ab})^* G\}_{\omega_a^{-1}}$$

for $F, G \in C^\infty(T^*U_{ab})$. This defines a twisted sheaf (or stack) of Poisson algebras on T^*M [64].

2.4 Local symplectic embedding of almost Poisson manifolds

The existence of local symplectic embeddings for arbitrary almost Poisson structures is established in [42]. The construction of Section 2.2 is spoiled for a non-zero Jacobiator $\Pi \neq 0$, as then $(C^\infty(M), t\{\cdot, \cdot\}_\theta)$ cannot be embedded as a Poisson subalgebra of the cosymplectic algebra

$(C_{\text{pol}}^\infty(T^*M)[[t]], \{\cdot, \cdot\}_{\omega^{-1}})$. Nevertheless, we can gleam off the general form of the symplectic embedding from the two special cases (2.13) and (2.22).

For this, we take the zero section $U \subset T^*U$ to be a Lagrangian submanifold of (T^*U, ω) and account for the additional terms involving $[\theta, \theta] \neq 0$. This means that the formal Poisson bivector ω^{-1} can be written as

$$\omega^{-1} = \frac{t}{2} \underline{\theta}^{ij}(x, p) \partial_i \wedge \partial_j + \frac{1}{2} (\delta_j^i + t \gamma_j^i(x, p)) (\partial_i \wedge \tilde{\partial}^j + \tilde{\partial}^j \wedge \partial_i), \quad (2.24)$$

where the bivector γ has a formal power series expansion as in (2.14), with leading order term given in (2.18), while

$$\underline{\theta}^{ij}(x, p) = \theta^{ij}(x) - t \Pi^{ijk}(x) p_k + \sum_{n=2}^{\infty} t^n \theta^{ij|i_1 \dots i_n}(x) p_{i_1} \dots p_{i_n} \quad (2.25)$$

and the local functions $\theta^{ij|i_1 \dots i_n}(x)$ for $n \geq 2$ are proportional to the components of the trivector Π and their derivatives. The various functions satisfy local first order differential equations determined from the Poisson integrability condition (2.8) with

$$\begin{aligned} \Theta_0 &= \frac{1}{2} (\partial_i \wedge \tilde{\partial}^i + \tilde{\partial}^i \wedge \partial_i), \\ \Theta_1 &= \frac{1}{2} \theta^{ij}(x) \partial_i \wedge \partial_j + \frac{1}{2} \gamma_j^{i|k}(x) p_k (\partial_i \wedge \tilde{\partial}^j + \tilde{\partial}^j \wedge \partial_i), \\ \Theta_2 &= -\frac{1}{2} \Pi^{ijk}(x) p_k \partial_i \wedge \partial_j + \frac{1}{2} \gamma_j^{i|kl}(x) p_k p_l (\partial_i \wedge \tilde{\partial}^j + \tilde{\partial}^j \wedge \partial_i), \\ \Theta_n &= \frac{1}{2} \theta^{ij|i_1 \dots i_{n-1}}(x) p_{i_1} \dots p_{i_{n-1}} \partial_i \wedge \partial_j + \frac{1}{2} \gamma_j^{i|i_1 \dots i_n}(x) p_{i_1} \dots p_{i_n} (\partial_i \wedge \tilde{\partial}^j + \tilde{\partial}^j \wedge \partial_i), \end{aligned} \quad (2.26)$$

for $n \geq 3$. This cosymplectic structure indeed satisfies the two requisite requirements: (a) it coincides with the original almost Poisson structure θ along the zero section of T^*U , as in (2.7); and (b) in the case of a Poisson bivector θ , the symplectic embedding (2.24) restores the symplectic embedding (2.13) of a Poisson structure. For example, an explicit form for the order t^2 term according to [42] reads

$$\begin{aligned} \theta^{ij|kl} &= \frac{3}{16} (\Pi^{jlm} \partial_m \theta^{ki} + \Pi^{jkm} \partial_m \theta^{li} - \Pi^{ilm} \partial_m \theta^{kj} - \Pi^{ikm} \partial_m \theta^{lj}) \\ &\quad - \frac{1}{8} (\theta^{km} \partial_m \Pi^{ijl} + \theta^{lm} \partial_m \Pi^{ijk}). \end{aligned} \quad (2.27)$$

Remark 2.28. The formalism here covers as well the special case of twisted Poisson structures from Section 2.3, and it explicitly realises the local ‘untwisting’ of the almost symplectic realization (2.22) to a symplectic embedding ω . For example, consider the case of a topologically trivial closed three-form $H \in \Omega^3(M)$, that is, $H = dB$ for a globally defined two-form $B \in \Omega^2(M)$, with associated linear map $B^\flat : \mathfrak{X}(M) \rightarrow \Omega^1(M)$. In this case we can choose the trivial one-form $A = 0$ (up to gauge equivalence), for which the associated flows are identity maps of M and the corresponding map

$$(\omega^{-1})^\sharp = (\omega_{dB}^{-1})^\sharp (\mathbb{1} - t(\pi^* B)^\flat (\omega_{dB}^{-1})^\sharp)^{-1} : \Omega_{\text{pol}}^1(T^*M)[[t]] \longrightarrow \mathfrak{X}_{\text{pol}}(T^*M)[[t]]$$

defines a B -transformation of the almost cosymplectic structure ω_{dB}^{-1} . ◁

2.5 Semi-classical limit of deformation quantization

In the context of deformation quantization, a Poisson bracket may be regarded as the semi-classical limit of the commutator bracket of an associative noncommutative star-product which

quantizes a Poisson manifold (M, θ) ; the existence of such star-products is provided by the famous Kontsevich formality theorem [39]. More precisely, a star-product on M is a product on $C^\infty(M)[[\hbar]]$ (regarded as a $\mathbb{C}[[\hbar]]$ -module for a formal deformation parameter \hbar) of the form

$$f \star g = fg + \sum_{n=1}^{\infty} \hbar^n B_n(f, g)$$

for smooth functions $f, g \in C^\infty(M)$, where each $B_n : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ for $n \geq 1$ is a bidifferential operator, and the Poisson structure is recovered through $\{f, g\}_\theta = \frac{1}{\hbar} [f, g]_\star \Big|_{\hbar=0}$ where $[f, g]_\star = f \star g - g \star f$.

The relation between symplectic embeddings and the semi-classical limit of deformation quantization of a Poisson structure θ is well-known (at least implicitly) in both the physics and mathematics literature; after all, symplectic realizations were originally introduced with the quantization problem for Poisson manifolds in mind. On $M = U \subseteq \mathbb{R}^d$, the Fourier integral representation of the Kontsevich star-product on $(U, t\theta)$ can be brought to the form

$$f \star g(x) = \int_{(\mathbb{R}^d)^* \times (\mathbb{R}^d)^*} \hat{f}(p) \hat{g}(p') a_\hbar(p, p', x) e^{\frac{i}{\hbar} \Sigma(p, p', x)} \frac{dp dp'}{(2\pi \hbar)^d}, \quad (2.29)$$

where \hat{f} and \hat{g} are the asymptotic Fourier transforms of $f, g \in C^\infty(U)$, and the function $a_\hbar(p, p', x)$ is regular at $\hbar = 0$. In the semi-classical limit $\hbar \rightarrow 0$, the leading contribution is given by the oscillatory phase $\Sigma(p, p', x)$, which is a formal power series

$$\Sigma(p, p', x) = \langle p + p', x \rangle + \sum_{n=1}^{\infty} t^n \Sigma_n(p, p', x),$$

where $\langle \cdot, \cdot \rangle$ is the dual pairing between \mathbb{R}^d and $(\mathbb{R}^d)^*$, and each $\Sigma_n(p, p', x)$ for $n \geq 1$ is a homogeneous polynomial in $p, p' \in (\mathbb{R}^d)^*$ of degree $n + 1$, with $\Sigma_n(p, 0, x) = \Sigma_n(0, p, x)$, whose homogeneous part $\tilde{\Sigma}_n(p, p', x)$ in p satisfies $\tilde{\Sigma}_n(p, p, x) = 0$. It formally generates a local symplectic groupoid structure on (T^*U, ω_0) [19], and in this manner the formal symplectic groupoid can be regarded as a semi-classical version of the full noncommutative algebra of functions $(C^\infty(U)[[\hbar]], \star)$.

This determines a formal Poisson submersion $\pi_\theta : (T^*U, \omega_0^{-1}) \rightarrow (U, t\theta)$ with Lagrangian zero section which in components is defined by

$$\pi_\theta(x, p)^i = \tilde{\partial}'^i \Sigma(p, p', x) \Big|_{p'=0} = x^i + \sum_{n=1}^{\infty} t^n \Sigma^{i|i_1 \dots i_n}(x) p_{i_1} \dots p_{i_n}, \quad (2.30)$$

where $\tilde{\partial}'^i = \partial / \partial p'_i$ and $\Sigma^{i|i_1 \dots i_n}(x) p_{i_1} \dots p_{i_n} = \tilde{\partial}'^i \Sigma_n(p, p', x) \Big|_{p'=0}$. This map is sometimes called a ‘generalized Bopp shift’ in the physics literature, see e.g. [40, 47]. In [42] it was shown how to construct the (not unique) local functions $\Sigma^{i|i_1 \dots i_n}(x)$ by solving the (formal) Poisson map equation

$$\{\pi_\theta^* f, \pi_\theta^* g\}_{\omega_0^{-1}} = t \pi_\theta^* \{f, g\}_\theta$$

order by order in t , with the first three leading orders which are compatible with quantization

given by

$$\begin{aligned}\Sigma^{i|j} &= -\frac{1}{2}\theta^{ij} , \\ \Sigma^{i|jk} &= \frac{1}{24}\theta^{kl}\partial_l\theta^{ij} + \frac{1}{24}\theta^{jl}\partial_l\theta^{ik} , \\ \Sigma^{i|jmn} &= -\frac{1}{12}\left(2\Sigma^{l|mn}\partial_l\theta^{ij} + 2\Sigma^{l|jn}\partial_l\theta^{im} + 2\Sigma^{l|jm}\partial_l\theta^{in}\right. \\ &\quad \left.+ \frac{1}{12}\theta^{lm}\theta^{kn}\partial_l\partial_k\theta^{ij} + \frac{1}{12}\theta^{lj}\theta^{kn}\partial_l\partial_k\theta^{im} + \frac{1}{12}\theta^{lj}\theta^{km}\partial_l\partial_n\theta^{in}\right) .\end{aligned}$$

Writing $\tilde{\pi}(x, p) = p$ for the projection to the normal directions to $U \subset T^*U$, the inverse of the map $\pi_\theta \times \tilde{\pi} : T^*U \rightarrow T^*U$ then replaces (2.30) with the canonical cotangent bundle projection $\pi(x, p) = x$ and yields the local symplectic embedding constructed in Section 2.2; this is the sense in which the symplectic embedding ‘integrates’ the Poisson manifold $(M, t\theta)$. The existence of invertible generalized Bopp shifts is always guaranteed, at least locally, by virtue of Darboux’s theorem.

Example 2.31. If $\theta = \theta_0 = 0$, then $\Sigma_n = 0$ for all $n \geq 1$ and $\pi_0(x, p) = x$. More generally, if $M = \mathbb{R}^d$ with a constant Poisson structure θ , then

$$\Sigma(p, p', x) = \langle p + p', x \rangle + \frac{t}{2}\theta^{ij}p_i p'_j ,$$

and the function a_{\hbar} in (2.29) is identically equal to 1. In this case the generalized Bopp shift

$$\pi_\theta(x, p)^i = x^i - \frac{t}{2}\theta^{ij}p_j$$

reproduces the symplectic embedding of Example 2.20. ◁

In the general case, we may again formulate deformation quantization in the direction of an almost Poisson bracket through a suitable nonassociative star-product. For twisted Poisson manifolds these star-products can be constructed through Kontsevich’s formalism (see e.g. [4, 22, 55, 64]). For generic almost Poisson manifolds the existence and uniqueness of nonassociative Weyl star-products is established by [48].

The relation between symplectic embeddings and the semi-classical limit of deformation quantization of a generic almost Poisson structure θ is much more involved. For twisted Poisson manifolds, the twisted symplectic groupoids of [18] capture the semi-classical limit of the nonassociative algebra of functions, but these are not directly induced by our symplectic embedding formalism, which deals with strictly associative structures. Instead, in [46, Section 9] it was shown that symplectic embeddings capture the semi-classical limit of the associative composition algebra of differential operators on M induced by the star-product [56], which coincides with the noncommutative algebra of functions precisely when the Jacobiator \mathcal{J} vanishes. In this setting, the semi-classical limit of a nonassociative star-product quantizing a generic almost Poisson structure θ was formulated as a generalized Bopp shift by [42] as the submersion $\pi_\theta : T^*U \rightarrow U$ satisfying

$$\{\pi_\theta^*f, \pi_\theta^*g\}_{\omega_0^{-1}} = t(\pi_\theta \times \tilde{\pi})^*\{\pi^*f, \pi^*g\}_{\underline{\theta}} ,$$

where $\underline{\theta} = \frac{1}{2}\theta^{ij}(x, p)\partial_i \wedge \partial_j$ is the bivector introduced in Section 2.4 with the formal power series expansion (2.25). The map π_θ has precisely the same expansion (2.30) as in the case of a Poisson structure, and inverting $\pi_\theta \times \tilde{\pi}$ then gives a local symplectic embedding in the weaker sense of Definition 2.6, as constructed explicitly in Section 2.4.

3 Poisson gauge transformations

In this section we focus on the case of Poisson structures, i.e. $\Pi = 0$. Recalling our discussion of deformation quantization from Section 2.5, a noncommutative gauge theory is a formal deformation of an ordinary gauge theory, obtained by replacing pointwise products of gauge fields with star-products. A *Poisson gauge theory* is a limit of a noncommutative gauge theory whose gauge algebra is the semi-classical limit of the noncommutative gauge algebra with closure defined by the commutators $[f, g]_\star$, in the sense defined below; we return to this point of view in Section 5.4 where we will make this notion somewhat more precise in the context of homotopy Poisson algebras. In this paper we deal only with the kinematical data of a Poisson gauge theory. We also discuss only the case of gauge theories with structure group $U(1)$ for simplicity.

A conventional local $U(1)$ gauge transformation is specified by a pair

$$(f, A) \in C^\infty(U) \times \Omega^1(U)$$

on an open subset $U \subseteq M$ consisting of a gauge parameter f and a gauge field A . The commutative ring of functions $C^\infty(U)$ is an enveloping algebra for an abelian Lie algebra which acts on $\Omega^1(U)$ as $(f, A) \mapsto A + \delta_f^0 A$ via the gauge variation

$$\delta_f^0 A = df .$$

The closure condition for the gauge variations defines an abelian Lie algebra:

$$[\mathbb{1} + \delta_f^0, \mathbb{1} + \delta_g^0]A := ((\mathbb{1} + \delta_f^0) \circ (\mathbb{1} + \delta_g^0) - (\mathbb{1} + \delta_g^0) \circ (\mathbb{1} + \delta_f^0))A = 0$$

for $f, g \in C^\infty(U)$.

In a Poisson gauge theory, this construction is deformed in the following way. The idea is then to mimick the representation of the Poisson algebra as infinitesimal diffeomorphisms of M : The map $f \mapsto X_f := \theta^\sharp df$ is a Lie algebra homomorphism from $(C^\infty(M), \{\cdot, \cdot\}_\theta)$ to the Lie algebra of vector fields $(\mathfrak{X}(M), [\cdot, \cdot])$:

$$[X_f, X_g] = X_{\{f, g\}_\theta} . \tag{3.1}$$

For later reference, we note that this construction makes the trivial line bundle $M \times \mathbb{R}$ into a Lie algebroid over M .

Definition 3.2. Let (M, θ) be a Poisson manifold. A (*local*) *Poisson gauge transformation* on an open subset $U \subseteq M$ is an action of the Lie algebra $(C^\infty(U), t\{\cdot, \cdot\}_\theta)$ on the affine space $\Omega^1(U)[[t]]$ of the form

$$(f, A) \mapsto A + \delta_f^\theta A$$

for $(f, A) \in C^\infty(U) \times \Omega^1(U)$, which is a deformation of an abelian gauge transformation:

$$\delta_f^\theta A|_{t=0} = \delta_f^0 A = df , \tag{3.3}$$

and satisfies the derivation property

$$\delta_{f, g}^\theta A = g \delta_f^\theta A + f \delta_g^\theta A$$

over the algebra $C^\infty(U)$. The closure condition for the Lie algebra of gauge variations is the *Poisson gauge algebra*

$$[\mathbb{1} + \delta_f^\theta, \mathbb{1} + \delta_g^\theta]A = \delta_{t\{f, g\}_\theta}^\theta A . \tag{3.4}$$

Remark 3.5. In noncommutative gauge theory, the space $\Omega^1(U)[[\hbar]]$ is naturally acted upon by the vector space $C^\infty(U)[[\hbar]]$ under the left and right actions by multiplication with the star-product extended over one-forms by replacing the bidifferential operators B_n with Lie bidifferential operators. In the semi-classical limit, this naturally defines a skew-symmetric $\Omega^1(U)$ -valued bracket between gauge parameters and gauge fields that we denote by $\{f, A\}_\theta = -\{A, f\}_\theta$. Defining $DA \in \Omega^1(U) \otimes \Omega^1(U)$ in local coordinates by $DA := L_{\partial_i} A \otimes dx^i$, this bracket reads as

$$\{f, A\}_\theta = (\theta \otimes \mathbb{1})(df, DA) ,$$

which can be expressed as

$$\{f, A\}_\theta = \theta^{ij} \partial_i f L_{\partial_j} A$$

in local coordinates. In the following we will use the abbreviation $\theta^\otimes := \theta \otimes \mathbb{1}$. This bracket obeys the usual derivation property over the algebra $C^\infty(U)$ in its first entry, as well as with respect to the $C^\infty(U)$ -bimodule $\Omega^1(U)$ in its second entry since

$$D(g A) = dg \otimes A + g DA$$

implies

$$\{f, g A\}_\theta = \{f, g\}_\theta A + g \{f, A\}_\theta ,$$

for all $g \in C^\infty(U)$.

In later sections we will also use an extension of this bracket to differential forms of arbitrary degree. For $\alpha, \beta \in \Omega^1(U)$, we define their symmetric bracket $\{\alpha, \beta\}_\theta \in \Omega^2(U)$ by

$$\{\alpha, \beta\}_\theta := \theta^\otimes(D\alpha, D\beta) .$$

This is then extended to a graded skew-symmetric bracket $\{\cdot, \cdot\}_\theta$ on the entire exterior algebra $\Omega^\bullet(U)$ as a graded biderivation of degree 0. \triangleleft

Example 3.6. The simplest example (apart from the zero bivector) of a Poisson gauge transformation comes from taking $M = \mathbb{R}^d$ and a constant skew-symmetric $d \times d$ matrix $\theta = (\theta^{ij})$, regarded as a bivector on \mathbb{R}^d . Then the gauge variations

$$\delta_f^\theta A = df + t \{A, f\}_\theta$$

fulfill the requirements of Definition 3.2. In the following we extend this result to generic Poisson manifolds (M, θ) , which will generally require the use of formal power series. \triangleleft

By the discussion of Section 2.5, it is natural to expect that symplectic embeddings should play a role in determining the semi-classical limit of a noncommutative gauge theory, that is, in Poisson gauge theories. The purpose of this section is to demonstrate that this is indeed the case, focusing in detail on the realization of gauge symmetries. Our main observation here is that a Poisson gauge algebra is canonically defined by the restriction of a symplectic embedding (T^*M, ω) of (M, θ) to a constraint submanifold defined by the gauge fields. For this, we regard a gauge field $A \in \Omega^1(U)$ as a section $s_A : U \rightarrow T^*U$, whose image is a submanifold $\text{im}(s_A) \subset T^*U$; in local coordinates where $A = A_i(x) dx^i$, $s_A(x) = (x, A(x)) \in T^*U$ for $x \in U$. Define the local one-form $\Phi_A \in \Omega^1(T^*U)$ by

$$\Phi_A = \lambda_0 - \pi^* A$$

where we recall that λ_0 is the Liouville one-form on T^*U . The one-form Φ_A vanishes precisely on the submanifold $\text{im}(s_A) \subset T^*U$. In local coordinates where $\Phi_A = (\Phi_A)_i dx^i$ with

$$(\Phi_A)_i = p_i - A_i(x) ,$$

this is the submanifold of T^*U defined by the constraints $(\Phi_A)_i = 0$. Since $s_A^* \lambda_0 = A$, this is equivalently presented as $s_A^* \Phi_A = 0$ in $\Omega^1(U)$.

Note that s_A is a Lagrangian section of (T^*U, ω_0) if and only if A is a flat connection, i.e. $dA = 0$. Thus in general, $\text{im}(s_A)$ is *not* a Lagrangian submanifold of (T^*U, ω) . In physics parlance, the local equations $(\Phi_A)_i = 0$ do not define first class constraints, as expected since otherwise they would eliminate all local degrees of freedom. In the present context, the consistent elimination of the ‘auxiliary’ variables p_i which are adjoined in symplectic embeddings of Poisson manifolds [46] acquires a natural meaning through

Proposition 3.7. *Let (T^*M, ω) be a local symplectic embedding of a Poisson manifold (M, θ) , and let $U \subseteq M$ be an open subset. For $(f, A) \in C^\infty(U) \times \Omega^1(U)$, the gauge variation*

$$\delta_f^\theta A := s_A^* \{ \pi^* f, \Phi_A \}_{\omega^{-1}} \quad (3.8)$$

is a Poisson gauge transformation.

Proof. Using (2.13) the gauge variations (3.8) read

$$\delta_f^\theta A = df + t s_A^* \gamma(df, \cdot) + t \{A, f\}_\theta , \quad (3.9)$$

where the one-form $s_A^* \gamma(df, \cdot) \in \Omega^1(U)[[t]]$ is given by

$$s_A^* \gamma(df, \cdot) = \gamma_i^j(x, A(x)) \partial_j f(x) dx^i$$

in local coordinates on U . We need to check the closure condition (3.4) for this definition of gauge transformations.

We begin with some preliminary definitions. For a one-form which is a functional $\mathcal{F}(A) \in \Omega^1(U)$ of the gauge field A , we define the gauge variation

$$\delta_f^\theta \mathcal{F}(A) := \mathcal{F}(A + \delta_f^\theta A) - \mathcal{F}(A) . \quad (3.10)$$

In particular,

$$\delta_g^\theta \{A, f\}_\theta = \{ \delta_g^\theta A, f \}_\theta .$$

If $\underline{\mathcal{F}} \in \Omega_{\text{pol}}^1(T^*U)$, then we define

$$\delta_f^\theta (s_A^* \underline{\mathcal{F}}) := s_A^* (\tilde{\partial}^j \underline{\mathcal{F}}) \delta_f^\theta A_j ,$$

where $\delta_f^\theta A := \delta_f^\theta A_i dx^i$.

We are now ready to calculate the composition of two gauge variations. We obtain

$$\begin{aligned} \delta_f^\theta (\delta_g^\theta A) &= s_A^* (\tilde{\partial}^j \{ \pi^* g, \Phi_A \}_{\omega^{-1}}) \delta_f^\theta A_j + t \{ \delta_f^\theta A, g \}_\theta \\ &= s_A^* (\tilde{\partial}^j \{ \pi^* g, \Phi_A \}_{\omega^{-1}}) s_A^* \{ \pi^* f, (\Phi_A)_j \}_{\omega^{-1}} + t \{ s_A^* \{ \pi^* f, \Phi_A \}_{\omega^{-1}}, g \}_\theta . \end{aligned}$$

Thus for the left-hand side of the closure condition (3.4) one finds

$$\begin{aligned} & \delta_f^\theta(\delta_g^\theta A) - \delta_g^\theta(\delta_f^\theta A) \\ &= s_A^*(\tilde{\partial}^j\{\pi^*g, \Phi_A\}_{\omega^{-1}}) s_A^*\{\pi^*f, (\Phi_A)_j\}_{\omega^{-1}} - s_A^*(\tilde{\partial}^j\{\pi^*f, \Phi_A\}_{\omega^{-1}}) s_A^*\{\pi^*g, (\Phi_A)_j\}_{\omega^{-1}} \\ & \quad + s_A^*\{\pi^*s_A^*\{\pi^*f, \Phi_A\}_{\omega^{-1}}, \pi^*g\}_{\omega^{-1}} - s_A^*\{\pi^*s_A^*\{\pi^*g, \Phi_A\}_{\omega^{-1}}, \pi^*f\}_{\omega^{-1}} . \end{aligned} \quad (3.11)$$

We now apply the relation (A.3) from Appendix A to the right-hand side of (3.11) to get

$$\delta_f^\theta(\delta_g^\theta A) - \delta_g^\theta(\delta_f^\theta A) = s_A^*\{\{\pi^*f, \Phi_A\}_{\omega^{-1}}, \pi^*g\}_{\omega^{-1}} - s_A^*\{\{\pi^*g, \Phi_A\}_{\omega^{-1}}, \pi^*f\}_{\omega^{-1}} . \quad (3.12)$$

Using the Jacobi identity in the right-hand side of (3.12), we finally end up with

$$\delta_f^\theta(\delta_g^\theta A) - \delta_g^\theta(\delta_f^\theta A) = s_A^*\{\{\pi^*f, \pi^*g\}_{\omega^{-1}}, \Phi_A\}_{\omega^{-1}} . \quad (3.13)$$

Since

$$\{\pi^*f, \pi^*g\}_{\omega^{-1}} = t\pi^*\{f, g\}_\theta ,$$

we conclude that the gauge transformations (3.8) close the Lie algebra

$$[\mathbb{1} + \delta_f^\theta, \mathbb{1} + \delta_g^\theta]A = \delta_{t\{f, g\}_\theta}^\theta A ,$$

as required. \square

Remark 3.14. The field dependent term $df + t s_A^*\gamma(df, \cdot)$ in (3.9) can be thought of as defining a ‘twisted exterior derivative’ of the gauge parameter f required to close the Poisson gauge algebra when the bivector θ is no longer constant, cf. Examples 2.20 and 3.6. Indeed, the gauge closure condition (3.4) for (3.9) implies the local symplectic embedding equations (2.15) [44]. The precise algebraic meaning of this term will be explained in Section 5, and we shall see some explicit examples in Section 6. \triangleleft

4 Almost Poisson gauge transformations

In this section we consider the case of a generic almost Poisson bivector θ on M with non-vanishing Jacobiator, i.e. $II \neq 0$. In this case, we may again formulate the notion of an *almost Poisson gauge theory* as an appropriate semi-classical limit of a noncommutative gauge theory, which is constructed by nonassociative deformation quantization. The inherent associativity of the symplectic embeddings discussed in Section 2.5 will be needed to ensure that the gauge transformations in an almost Poisson gauge theory close a (strict) Lie algebra, despite their origin in a nonassociative algebra.

The aim of this section is to generalize Proposition 3.7 to the case of generic almost Poisson manifolds. The construction of gauge transformations and gauge algebras in these instances is considerably more involved than in the case of Poisson structures from Section 3. We proceed in two steps: We first set up the problem of defining a suitable notion of almost Poisson gauge transformations and their closure condition, and subsequently prove that solutions to this problem exist and are explicitly computable.

4.1 Formulation of the gauge algebra

It is clear that for a non-trivial Jacobiator $\Pi \neq 0$, a direct generalization of Definition 3.2 is not possible, because $(C^\infty(M), \{\cdot, \cdot\}_\theta)$ is no longer a Lie algebra. This is already manifested in a violation of the Lie algebra homomorphism property (3.1), which for an H -twisted Poisson structure is modified to

$$[X_f, X_g] = X_{\{f,g\}_\theta} + \theta^\sharp H(X_f, X_g, \cdot) .$$

In particular, the gauge closure condition must change substantially. Drawing on results from the L_∞ -algebra approach to noncommutative gauge theories [43], it is clear what needs to be done: the space of gauge parameters should be enlarged to include ‘field dependent’ gauge parameters, such that the commutator of two gauge transformations is again a gauge transformation but with a field dependent gauge parameter. We shall discuss the precise relation between our approach here based on symplectic embeddings and the approach based on L_∞ -algebras in Section 5 below.

Let us first explain precisely what we mean by a ‘field dependent’ gauge parameter in our setting. We will work with 1-jets of sections of vector bundles, see e.g. [61].

Definition 4.1. Let M be a manifold and $U \subseteq M$ an open subset. Let J^1T^*U be the first order jet space of the cotangent bundle over U . A *field dependent gauge parameter* is a smooth function in $C^\infty(J^1T^*U)$.

Concretely, in local coordinates a field dependent gauge parameter is specified by a local function $f[x, A] = f(x, A(x), \partial A(x))$ depending on a gauge field $A \in \Omega^1(U)$, viewed as a section $s_A : U \rightarrow T^*U$, and its first order derivatives. Via pullback along the vector jet bundle $J^1T^*U \rightarrow U$, this contains the usual space $C^\infty(U)$ of (field independent) gauge parameters from Section 3. In what follows we will also use pullbacks along the affine jet bundle $J^1T^*U \rightarrow T^*U$. In particular, this enables us to view the space of gauge fields $\Omega^1(U)$ as a subspace of the sections $\Gamma(J^1T^*U)$ of the vector jet bundle via prolongation; in other words, gauge fields are precisely the integrable sections of $J^1T^*U \rightarrow U$.

With this notion in place, we are now in a position to generalize Definition 3.2. The idea is to enlarge the Lie algebra action $C^\infty(U) \times \Omega^1(U) \rightarrow \Omega^1(U)$ from Section 3 to a suitable affine action $C^\infty(J^1T^*U) \times \Gamma(J^1T^*U) \rightarrow \Gamma(J^1T^*U)$.

Definition 4.2. Let (M, θ) be an almost Poisson manifold. A *(local) almost Poisson gauge transformation* on an open subset $U \subseteq M$ is an affine action of $C_{\text{pol}}^\infty(J^1T^*U)[[t]]$ on the space $\Gamma_{\text{pol}}(J^1T^*U)[[t]]$ of the form

$$(f, A) \mapsto A + \delta_f^\theta A$$

for $(f, A) \in C^\infty(U) \times \Omega^1(U)$, such that:

- (a) The assignment $f \mapsto \delta_f^\theta$ is \mathbb{C} -linear and defines a deformation of an abelian gauge transformation:

$$\delta_f^\theta A|_{t=0} = \delta_f^0 A = \text{d}f ,$$

which satisfies the derivation property

$$\delta_{f_g}^\theta A = g \delta_f^\theta A + f \delta_g^\theta A$$

over the algebra $C^\infty(U)$, and closes the *almost Poisson gauge algebra*

$$[\mathbb{1} + \delta_f^\theta, \mathbb{1} + \delta_g^\theta]A = \delta_t^\theta \llbracket f, g \rrbracket_\theta(A) A, \quad (4.3)$$

where $\llbracket f, g \rrbracket_\theta \in C_{\text{pol}}^\infty(J^1 T^* U)[[t]]$;

- (b) The assignment $(f, g) \mapsto \llbracket f, g \rrbracket_\theta$ is \mathbb{C} -bilinear, skew-symmetric and defines a deformation of the almost Poisson bracket:

$$\llbracket f, g \rrbracket_\theta|_{t=0} = \{f, g\}_\theta,$$

which satisfies the relation

$$\text{Cyc}_{f,g,h}(t \llbracket h, \llbracket f, g \rrbracket_\theta(A) \rrbracket_\theta(A) + \delta_h^\theta \llbracket f, g \rrbracket_\theta(A)) = 0 \quad (4.4)$$

for consistency with the Jacobi identity for the commutator algebra (4.3), where we use the definition (3.10) for gauge variations; and

- (c) For any fixed $f \in C^\infty(U)$, the map $\llbracket f, \cdot \rrbracket_\theta$ is a derivation of the commutative algebra $C_{\text{pol}}^\infty(J^1 T^* U)[[t]]$.

When θ is a Poisson structure, i.e. the Jacobi identity for the bracket $\{\cdot, \cdot\}_\theta$ holds, Definition 3.2 is a special case of Definition 4.2, but the latter allows more freedom to accommodate situations where the Jacobi identity is violated. Nevertheless, a direct generalization of Proposition 3.7 is still not possible for generic almost Poisson bivectors θ ; in the language of [46], the auxiliary variables p_i can no longer be consistently eliminated by the constraints $\Phi_A = 0$ on a symplectic embedding of an almost Poisson manifold. One can still repeat most of the proof of Proposition 3.7 in the present case to arrive again at the expression (3.13). However, due to (2.24), the key difference from the case of a Poisson bivector is that the cosymplectic bracket

$$\{\pi^* f, \pi^* g\}_{\omega^{-1}} = t \underline{\theta}(d\pi^* f, d\pi^* g) \neq t \pi^* \{f, g\}_\theta$$

is not a gauge parameter since it depends on the coordinates in the normal directions to the zero section $U \subset T^* U$ through its local dependence on $\underline{\theta}^{ij}(x, p)$. Similarly, the restriction of

$$\{\pi^* A, \pi^* f\}_{\omega^{-1}} = t \underline{\theta}^\otimes(D\pi^* A, d\pi^* f) \neq t \pi^* \{A, f\}_\theta$$

to $\text{im}(s_A)$ involves a field dependent gauge transformation through its local dependence on $\underline{\theta}^{ij}(x, A(x))$.

Using the relation (A.4) from Appendix A in the right-hand side of (3.13), we can represent the commutator of two gauge transformations defined by (3.8) in the form

$$\begin{aligned} \delta_f^\theta(\delta_g^\theta A) - \delta_g^\theta(\delta_f^\theta A) &= s_A^* \{ \pi^* s_A^* \{ \pi^* f, \pi^* g \}_{\omega^{-1}}, \Phi_A \}_{\omega^{-1}} \\ &\quad + s_A^* (\tilde{\partial}^j \{ \pi^* f, \pi^* g \}_{\omega^{-1}}) s_A^* \{ (\Phi_A)_j, \Phi_A \}_{\omega^{-1}}. \end{aligned} \quad (4.5)$$

While the expression $s_A^* \{ \pi^* f, \pi^* g \}_{\omega^{-1}}$ defines a field dependent gauge parameter, the second line of (4.5) prevents the gauge transformations defined by (3.8) from closing a Lie algebra. To overcome this problem we need to ‘correct’ the gauge variation (3.8) in order to absorb this term, which is the content of

Proposition 4.6. *Let (T^*M, ω) be a local symplectic embedding of an almost Poisson manifold (M, θ) , and let $U \subseteq M$ be an open subset. For $f, g \in C^\infty(U)$ and $A \in \Omega^1(U)$, define $\llbracket f, g \rrbracket_\theta \in C^\infty_{\text{pol}}(J^1T^*U)[[t]]$ by*

$$t \llbracket f, g \rrbracket_\theta(A) = s_A^* \{ \pi^* f, \pi^* g \}_{\omega^{-1}} - s_A^* \{ \Phi_A(L_f), \Phi_A(L_g) \}_{\omega^{-1}} , \quad (4.7)$$

where $L_f \in \mathfrak{X}_{\text{pol}}^{\text{h}}(J^1T^*U)[[t]] \subset \mathfrak{X}_{\text{pol}}^{\text{h}}(T^*U)[[t]]$ are horizontal vector fields $L_f = L_f^i \partial_i$ on the jet space of T^*U satisfying the recursion relations

$$\begin{aligned} L_t^i \llbracket f, g \rrbracket_\theta(A) &= \delta_f^\theta L_g^i - \delta_g^\theta L_f^i + s_A^* (\tilde{\partial}^i \{ \pi^* f, \pi^* g \}_{\omega^{-1}}) + L_f^j L_g^k s_A^* (\tilde{\partial}^i \{ (\Phi_A)_j, (\Phi_A)_k \}_{\omega^{-1}}) \\ &\quad + s_A^* \{ \pi^* f, L_g^i \}_{\omega^{-1}} - s_A^* \{ \pi^* g, L_f^i \}_{\omega^{-1}} \\ &\quad + s_A^* \{ L_f^i, \Phi_A(L_g) \}_{\omega^{-1}} - s_A^* \{ L_g^i, \Phi_A(L_f) \}_{\omega^{-1}} \\ &\quad + L_g^j s_A^* (\tilde{\partial}^i \{ \pi^* f, (\Phi_A)_j \}_{\omega^{-1}}) - L_f^j s_A^* (\tilde{\partial}^i \{ \pi^* g, (\Phi_A)_j \}_{\omega^{-1}}) , \end{aligned} \quad (4.8)$$

with

$$\delta_f^\theta L_g^i(A) := L_g^i(A + \delta_f^\theta A) - L_g^i(A) \quad (4.9)$$

and

$$\delta_f^\theta A := s_A^* \{ \pi^* f + \Phi_A(L_f), \Phi_A \}_{\omega^{-1}} . \quad (4.10)$$

Then the gauge variations (4.10) close the almost Poisson gauge algebra

$$[\mathbb{1} + \delta_f^\theta, \mathbb{1} + \delta_g^\theta] A = \delta_t^\theta \llbracket f, g \rrbracket_\theta(A) A \quad (4.11)$$

and $(f, A) \mapsto A + \delta_f^\theta A$ is an almost Poisson gauge transformation.

Proof. The proof of (4.11) utilizes the derivation property of the almost Poisson bracket together with the fact that, since $\Phi_A = 0$ on the constraint locus $\text{im}(s_A) \subset T^*U$, the gauge variations (4.10) can be expressed as

$$\delta_f^\theta A = s_A^* \{ \pi^* f, \Phi_A \}_{\omega^{-1}} + L_f^i s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega^{-1}} , \quad (4.12)$$

while the field dependent gauge parameter $\llbracket f, g \rrbracket_\theta \in C^\infty_{\text{pol}}(J^1T^*U)[[t]]$ from (4.7) can be determined as

$$t \llbracket f, g \rrbracket_\theta(A) = s_A^* \{ \pi^* f, \pi^* g \}_{\omega^{-1}} - L_f^i L_g^j s_A^* \{ (\Phi_A)_i, (\Phi_A)_j \}_{\omega^{-1}} . \quad (4.13)$$

The details are deferred to Appendix B.

To show that the Jacobi identity (4.4) is satisfied by (4.7), we note that the left-hand side of (4.4) in this case reads

$$\begin{aligned} \text{Cyc}_{f,g,h} &\left(s_A^* \{ \pi^* h, \pi^* s_A^* \{ \pi^* f, \pi^* g \}_{\omega^{-1}} \}_{\omega^{-1}} - s_A^* \{ \pi^* h, \pi^* s_A^* \{ \Phi_A(L_f), \Phi_A(L_g) \}_{\omega^{-1}} \}_{\omega^{-1}} \right. \\ &\quad - s_A^* \{ \Phi_A(L_h), \Phi_A(L_t \llbracket f, g \rrbracket_\theta(A)) \}_{\omega^{-1}} + s_A^* (\tilde{\partial}^i \{ \pi^* f, \pi^* g \}_{\omega^{-1}}) s_A^* \{ \pi^* h + \Phi_A(L_h), (\Phi_A)_i \}_{\omega^{-1}} \\ &\quad - (\delta_h^\theta L_f^i) s_A^* \{ (\Phi_A)_i, \Phi_A(L_g) \}_{\omega^{-1}} - (\delta_h^\theta L_g^i) s_A^* \{ \Phi_A(L_f), (\Phi_A)_i \}_{\omega^{-1}} \\ &\quad + s_A^* \{ \pi^* s_A^* \{ \pi^* h + \Phi_A(L_h), \Phi_A(L_f) \}_{\omega^{-1}}, \Phi_A(L_g) \}_{\omega^{-1}} \\ &\quad \left. + s_A^* \{ \Phi_A(L_f), \pi^* s_A^* \{ \pi^* h + \Phi_A(L_h), \Phi_A(L_g) \}_{\omega^{-1}} \}_{\omega^{-1}} \right) . \end{aligned} \quad (4.14)$$

Now we use the expression for $L_t \llbracket f, g \rrbracket_\theta(A)$ from (4.8) and take into account the cyclic sum over f , g and h . Many terms cancel, while the remaining terms combine using the formulas (A.2), (A.5), and (A.6) from Appendix A. After these simplifications the expression (4.14) becomes

$$\begin{aligned} \text{Cyc}_{f,g,h} & \left(s_A^* \{ \pi^* h, \{ \pi^* f, \pi^* g \}_{\omega^{-1}} \}_{\omega^{-1}} + s_A^* \{ \{ \Phi_A(L_f), \Phi_A(L_g) \}_{\omega^{-1}}, \pi^* h \}_{\omega^{-1}} \right. \\ & + s_A^* \{ \{ \Phi_A(L_g), \pi^* h \}_{\omega^{-1}}, \Phi_A(L_f) \}_{\omega^{-1}} + s_A^* \{ \{ \pi^* h, \Phi_A(L_f) \}_{\omega^{-1}}, \Phi_A(L_g) \}_{\omega^{-1}} \\ & \left. + 2 s_A^* \{ \{ \Phi_A(L_f), \Phi_A(L_g) \}_{\omega^{-1}}, \Phi_A(L_h) \}_{\omega^{-1}} \right) . \end{aligned}$$

This expression now vanishes identically as a consequence of the Jacobi identity for the cosymplectic bracket. Thus the gauge transformations (4.10) form a Lie algebra.

Finally, the linearity and derivation requirements of Definition 4.2 will follow from the corresponding properties of the cosymplectic bracket and by constructing the vector fields L_f as $C^\infty(J^1 T^* U)$ -linear duals of the one-forms df , so that

$$L_f g = g L_f + f L_g$$

and linearity in f is immediate. \square

Remark 4.15. By virtue of their role in cancelling the unwanted terms in the commutator (4.5), we shall refer to the horizontal vector fields L_f as *Lagrangian multipliers*. \triangleleft

Remark 4.16. One can compare with the Poisson gauge transformations (3.9), and in particular see the effects of the modifications by the Lagrangian multiplier vector fields, by using (2.24) together with the notation of Remark 3.5 to write the almost Poisson gauge transformations (4.12) as

$$\begin{aligned} \delta_f^\theta A & = df + t s_A^* \gamma(df, \cdot) + t s_A^* \{ \pi^* A, \pi^* f \}_\theta + t s_A^* \underline{\theta}^\otimes (\pi^* DA(L_f^\otimes), \pi^* DA) + dA(L_f, \cdot) \\ & + t s_A^* \gamma^\otimes(DA, L_f) - t s_A^* \gamma(DA(L_f^\otimes), \cdot) , \end{aligned}$$

for $f \in C^\infty(U)$ and $A \in \Omega^1(U)$, where $L_f^\otimes := L_f \otimes \cdot$. Similarly, the modification of the almost Poisson bracket $\{f, g\}_\theta$ to the field dependent bracket (4.13) can be written as

$$\begin{aligned} \llbracket f, g \rrbracket_\theta(A) & = s_A^* \{ \pi^* f, \pi^* g \}_\theta - s_A^* \underline{\theta} (\pi^* DA(L_f^\otimes), \pi^* DA(L_g^\otimes)) - t^{-1} dA(L_f, L_g) \\ & + s_A^* \gamma(DA(L_f^\otimes), L_g) - s_A^* \gamma(DA(L_g^\otimes), L_f) , \end{aligned}$$

for $f, g \in C^\infty(U)$ and $A \in \Omega^1(U)$. \triangleleft

4.2 Existence of Lagrangian multipliers

Proposition 4.6, which establishes sufficiency conditions under which local symplectic embeddings lead to almost Poisson gauge transformations, is only meaningful of course if we can establish the existence of the vector fields L_f , i.e. the existence of solutions to the defining recursion equations determined by (4.7)–(4.10). We shall now demonstrate that they can be constructed recursively for $f \in C^\infty(U)$ to any order in t .

We can immediately compute the leading order contribution to L_f by using (2.25) to write the leading contribution to the second line of (4.5) as

$$s_A^* (\tilde{\partial}^j \{ \pi^* f, \pi^* g \}_{\omega^{-1}}) s_A^* \{ (\Phi_A)_j, \Phi_A \}_{\omega^{-1}} = t^2 dA(\Pi(df, dg, \cdot), \cdot) + O(t^3) .$$

At this order, this term can be cancelled by taking

$$L_f = -\frac{t^2}{2} \Pi(df, A, \cdot) + O(t^3) \quad (4.17)$$

in (4.10), which from (4.7) can be straightforwardly seen to yield

$$\llbracket f, g \rrbracket_\theta(A) = \{f, g\}_\theta - t \Pi(df, dg, A) + O(t^2) .$$

To find the higher order contributions, we will work locally. The Lagrangian multipliers $L_f = L_f^i \partial_i \in \mathfrak{X}_{\text{pol}}^h(J^1 T^* U)[[t]]$ should be linear in the gauge parameter $f \in C^\infty(U)$, and will generally depend on the gauge fields $A \in \Omega^1(U)$ and also on their derivatives ∂A , so we write their local forms as

$$L_f^i = t^2 L^{ij}(x, A(x), \partial A(x)) \partial_j f(x) ,$$

for $x \in U$ and functions $L^{ij} \in C_{\text{pol}}^\infty(J^1 T^* U)[[t]]$. In the following we write $\partial_A^l = \partial/\partial A_l$ and $\partial_A^{k;l} = \partial/\partial(\partial_k A_l)$ for derivatives in local coordinates on the jet space. We can then formulate

Proposition 4.18. *The solution to the recursion relations (4.8) is given by*

$$L^{ij}(A, \partial A) = \Lambda^{ij}(A) + \sum_{n=1}^{\infty} (-t^2)^n \Lambda^{ik_1}(A) \partial_{k_1} A_{m_1} \Lambda^{m_1 k_2}(A) \\ \times \dots \Lambda^{m_{n-1} k_n}(A) \partial_{k_n} A_{m_n} \Lambda^{m_n j}(A) , \quad (4.19)$$

where the function $\Lambda^{ij} \in s_A^* C_{\text{pol}}^\infty(T^* U)[[t]] \subset C_{\text{pol}}^\infty(J^1 T^* U)[[t]]$ does not depend on the derivatives ∂A and satisfies local first order differential equations on the jet space given by

$$t (\partial_A^i \Lambda^{kj} - \partial_A^j \Lambda^{ki}) + t^3 (\Lambda^{lj} \partial_l \Lambda^{ki} - \Lambda^{li} \partial_l \Lambda^{kj}) \\ = -\partial_A^k \underline{\theta}^{ij}(A) + t^2 (\Lambda^{kl} \partial_l \underline{\theta}^{ij}(A) + \Lambda^{li} \partial_A^k \gamma_l^j(A) - \Lambda^{lj} \partial_A^k \gamma_l^i(A) \\ + \underline{\theta}^{jl}(A) \partial_l \Lambda^{ki} - \underline{\theta}^{il}(A) \partial_l \Lambda^{kj} + \gamma_l^j(A) \partial_A^l \Lambda^{ki} - \gamma_l^i(A) \partial_A^l \Lambda^{kj}) \\ + t^4 (\gamma_m^l(A) \Lambda^{mi} \partial_l \Lambda^{kj} - \gamma_m^l(A) \Lambda^{mj} \partial_l \Lambda^{ki}) , \quad (4.20)$$

with $\underline{\theta}^{ij}(A) := s_A^* \underline{\theta}^{ij}$ and $\gamma_i^j(A) := s_A^* \gamma_i^j$.

Proof. First we introduce

$$L^{ij}(A, \partial A) = \Lambda^{ij}(A) + \bar{\Lambda}^{ij}(A, \partial A) , \quad (4.21)$$

where $\Lambda^{ij}(A)$ is a function of the gauge fields A only, whereas $\bar{\Lambda}^{ij}(A, \partial A)$ depends on both gauge fields A and their first order derivatives ∂A . To obtain an equation for the function $\Lambda^{ij}(A)$, we substitute (4.21) in (4.8) and collect the terms which contain only first order derivatives of gauge parameters $\partial_i f \partial_j g$ but do not contain derivatives of the gauge fields ∂A .

The left-hand side of (4.8) reads

$$L_t^i \llbracket f, g \rrbracket_\theta(A) = t^2 L^{im} \partial_m (s_A^* \{\pi^* f, \pi^* g\}_{\omega^{-1}} - L_f^j L_g^k s_A^* \{(\Phi_A)_j, (\Phi_A)_k\}_{\omega^{-1}}) , \quad (4.22)$$

and it contributes $t^2 \Lambda^{lk} \partial_k \underline{\theta}^{ij}(A)$ to (4.20). The contributions on the right-hand side of (4.8) from

$$\delta_f^\theta L_g^i = t^2 \partial_A^l L^{ij} (s_A^* \{\pi^* f, (\Phi_A)_l\}_{\omega^{-1}} + t^2 L^{km} s_A^* \{(\Phi_A)_k, (\Phi_A)_l\}_{\omega^{-1}} \partial_k f) \partial_j g \\ + t^2 \partial_A^{p;l} L^{ij} \partial_p (s_A^* \{\pi^* f, (\Phi_A)_l\}_{\omega^{-1}} + t^2 L^{km} s_A^* \{(\Phi_A)_k, (\Phi_A)_l\}_{\omega^{-1}} \partial_k f) \partial_j g , \quad (4.23)$$

and from

$$s_A^* (\tilde{\partial}^i \{ \pi^* f, \pi^* g \}_{\omega^{-1}}) = t \partial_A^i \underline{\theta}^{jk}(A) \partial_j f \partial_k g ,$$

together yield

$$t (\delta_l^i + t \gamma_l^i(A)) \partial_A^l \Lambda^{kj} - t (\delta_l^j + t \gamma_l^j(A)) \partial_A^l \Lambda^{ki} + \partial_A^k \underline{\theta}^{ij}(A) .$$

The terms coming from

$$L_g^j s_A^* (\tilde{\partial}^i \{ \pi^* f, (\Phi_A)_j \}_{\omega^{-1}}) = t^3 L^{lj} \partial_A^i \gamma_l^k(A) \partial_k f \partial_j g - t^3 L^{lj} \partial_A^i \underline{\theta}^{km}(A) \partial_k f \partial_j g \partial_m A_l$$

are given by

$$t^2 \Lambda^{lj} \partial_A^k \gamma_l^i(A) - t^2 \Lambda^{li} \partial_A^k \gamma_l^j(A) .$$

There is no contribution from $L_f^j L_g^k s_A^* (\tilde{\partial}^i \{ (\Phi_A)_j, (\Phi_A)_k \}_{\omega^{-1}})$ on the right-hand side of (4.8) since all of its terms contain derivatives of the gauge fields ∂A . Finally, the contributions from the second and third lines of (4.8):

$$s_A^* \{ \pi^* f, L_g^i \}_{\omega^{-1}} = t^3 \underline{\theta}^{kl}(A) \partial_k f \partial_l (L^{ij} \partial_j g) , \quad (4.24)$$

$$s_A^* \{ L_f^i, \Phi_A(L_g) \}_{\omega^{-1}} = t^4 L^{kp} ((\delta_k^l + t \gamma_k^l(A)) \partial_l (L^{ij} \partial_j f) - t \underline{\theta}^{ml}(A) \partial_m A_k \partial_l (L^{ij} \partial_j f)) \partial_p g ,$$

read as

$$t^2 \underline{\theta}^{il}(A) \partial_l \Lambda^{kj} - t^2 \underline{\theta}^{jl}(A) \partial_l \Lambda^{ki} - t^3 (\delta_m^l + t \gamma_m^l(A)) \Lambda^{mi} \partial_l \Lambda^{kj} + t^3 (\delta_m^l + t \gamma_m^l(A)) \Lambda^{mj} \partial_l \Lambda^{ki} .$$

Altogether the contributions from such terms results in the differential equation (4.20).

The contributions to (4.8) containing second order derivatives of gauge parameters $\partial_i \partial_k f \partial_j g$ and $\partial_i f \partial_j \partial_k g$ should vanish separately. Let us analyse the contribution with $\partial_i \partial_k f \partial_j g$. The contribution from the left-hand side of (4.8) is given by (4.22) and reads

$$t^3 (L^{mk} \underline{\theta}^{ij}(A) - t^3 L^{mk} L^{pi} L^{lj} s_A^* \{ (\Phi_A)_p, (\Phi_A)_l \}_{\omega^{-1}}) \partial_i \partial_k f \partial_j g .$$

On the right-hand side of (4.8), the contribution from $\delta_f^\theta(L_g^i) - \delta_g^\theta(L_f^i)$ comes from the second line of (4.23) and reads

$$t^2 \partial_A^{k;l} L^{mj} (\delta_l^i + t \gamma_l^i(A) - t \underline{\theta}^{in}(A) \partial_n A_l + t^2 L^{pi} s_A^* \{ (\Phi_A)_p, (\Phi_A)_l \}_{\omega^{-1}}) \partial_i \partial_k f \partial_j g .$$

The only remaining terms on the right-hand side of (4.8) which contain second order derivatives of f and g come from the second and third lines, and from (4.24) one finds

$$t^3 (L^{pk} \underline{\theta}^{ij}(A) + t L^{pi} L^{nj} (\delta_n^k + t \gamma_n^k(A)) + t^2 L^{pi} L^{nj} \underline{\theta}^{mk}(A) \partial_m A_n) \partial_i \partial_k f \partial_j g .$$

Thus demanding that (4.8) hold for such contributions leads to the equations

$$\begin{aligned} & (\partial_A^{k;l} L^{mj} + t^2 L^{mk} L^{lj}) (\delta_l^i + t \gamma_l^i(A) - t \underline{\theta}^{in}(A) \partial_n A_l + t^2 L^{pi} s_A^* \{ (\Phi_A)_p, (\Phi_A)_l \}_{\omega^{-1}}) \\ & + (\partial_A^{i;l} L^{mj} + t^2 L^{mi} L^{lj}) (\delta_l^k + t \gamma_l^k(A) - t \underline{\theta}^{kn}(A) \partial_n A_l + t^2 L^{pk} s_A^* \{ (\Phi_A)_p, (\Phi_A)_l \}_{\omega^{-1}}) = 0 . \end{aligned}$$

This is satisfied if

$$\partial_A^{k;l} L^{ij} + t^2 L^{ik} L^{lj} = 0 , \quad (4.25)$$

or taking into account (4.21) we may rewrite it as

$$\partial_A^{k;l} \bar{\Lambda}^{ij} + t^2 (\Lambda^{ik} + \bar{\Lambda}^{ik}) (\Lambda^{lj} + \bar{\Lambda}^{lj}) = 0 .$$

This equation is solved by

$$\bar{\Lambda}^{ij}(A, \partial A) = \sum_{n=1}^{\infty} (-t^2)^n \Lambda^{ik_1}(A) \Lambda^{m_1 k_2}(A) \cdots \Lambda^{m_{n-1} k_n}(A) \Lambda^{m_n j}(A) \partial_{k_1} A_{m_1} \cdots \partial_{k_n} A_{m_n} ,$$

which yields the expansion (4.19).

To complete the proof we need to verify that the remaining terms in (4.8) cancel due to (4.25) and (4.20). This is tedious but completely straightforward: for example, the term

$$-t^6 L^{mk} \partial_k (L^{li} L^{pj}) s_A^* \{(\Phi_A)_l, (\Phi_A)_p\}_{\omega^{-1}} \partial_i f \partial_j g$$

coming from (4.22) precisely cancels the two terms

$$t^4 \partial_A^{k;l} L^{mj} \partial_k L^{pi} s_A^* \{(\Phi_A)_p, (\Phi_A)_l\}_{\omega^{-1}} (\partial_i f \partial_j g - \partial_i g \partial_j f)$$

coming from (4.23). □

It is now straightforward to construct the functions Λ^{ij} recursively from (4.20), and we have

Corollary 4.26. *The functions $\Lambda^{ij} \in s_A^* C_{\text{pol}}^{\infty}(T^*U)[[t]]$ can be calculated order by order in t as the formal power series*

$$\Lambda^{ij}(A) = -\frac{1}{2} \Pi^{ijk} A_k - \sum_{n=1}^{\infty} \frac{t^n}{n+2} \Upsilon^{ij|k_1 \cdots k_n} A_l A_{k_1} \cdots A_{k_n} ,$$

where the functions $\Upsilon^{ij|k_1 \cdots k_n} \in C^{\infty}(U)$ are skew-symmetric in their indices jl , and are given by explicit expressions involving the components of the almost Poisson bivector θ , its Jacobiator Π , and their derivatives.

Proof. We recall the expansions (2.14) (satisfying the local symplectic embedding equations (2.16) and (2.17)) and (2.25), and write $\Lambda^{ij}(A)$ as a formal power series of the form

$$\Lambda^{ij} = \sum_{n=1}^{\infty} t^{n-1} \Lambda_n^{ij} \quad \text{with} \quad \Lambda_n^{ij} = \Lambda^{ij|k_1 \cdots k_n} A_{k_1} \cdots A_{k_n} ,$$

where $\Lambda^{ij|k_1 \cdots k_n} \in C^{\infty}(U)$ for each $n \geq 1$. At first non-trivial order the differential equation (4.20) reads

$$\partial_A^i \Lambda_1^{kj} - \partial_A^j \Lambda_1^{ki} = \Pi^{ijk} ,$$

which has solution $\Lambda_1^{ij} = -\frac{1}{2} \Pi^{ijk} A_k$ as expected from (4.17). At higher orders t^{n-1} for $n \geq 2$, the differential equation (4.20) can be schematically represented as

$$\partial_A^i \Lambda_n^{kj} - \partial_A^j \Lambda_n^{ki} = \Upsilon_n^{kij} \quad \text{with} \quad \Upsilon_n^{kij} = \Upsilon^{kij|k_1 \cdots k_{n-1}} A_{k_1} \cdots A_{k_{n-1}} , \quad (4.27)$$

where on the right-hand side Υ_n^{kij} is constructed from the previously determined lower order solutions Λ_m^{kj} with $m < n$, and $\Upsilon^{kij|k_1 \cdots k_{n-1}} \in C^{\infty}(U)$. The integrability condition for the equation (4.27) reads

$$\partial_A^k \Upsilon_n^{lij} + \partial_A^i \Upsilon_n^{ljk} + \partial_A^j \Upsilon_n^{lki} = 0 . \quad (4.28)$$

At second order, after careful calculation and simplification one obtains

$$\begin{aligned} \Upsilon^{ilj|k} = & -2\theta^{lj|ki} + \frac{1}{2}\Pi^{ikm}\partial_m\theta^{lj} + \frac{1}{4}\Pi^{ilm}\partial_m\theta^{jk} - \frac{1}{4}\Pi^{ijm}\partial_m\theta^{lk} \\ & + \frac{1}{4}\Pi^{jkm}\partial_m\theta^{li} - \frac{1}{4}\Pi^{lkm}\partial_m\theta^{ji} + \frac{1}{2}\theta^{lm}\partial_m\Pi^{ijk} - \frac{1}{2}\theta^{jm}\partial_m\Pi^{ilk} , \end{aligned} \quad (4.29)$$

where the function $\theta^{lj|ki}$ determines the order t^2 contribution to the expansion (2.25). The integrability condition (4.28) reads

$$\Upsilon^{ilj|k} + \Upsilon^{ikl|j} + \Upsilon^{ijk|l} = 0 . \quad (4.30)$$

Substituting the explicit form of $\Upsilon^{ilj|k}$ from (4.29) into (4.30) we arrive at

$$2(\theta^{lj|ki} + \theta^{kl|ji} + \theta^{jk|li}) - F^{ljk i} = 0 , \quad (4.31)$$

with

$$\begin{aligned} F^{ljk i} = & \Pi^{kmi}\partial_m\theta^{lj} + \Pi^{jmi}\partial_m\theta^{kl} + \Pi^{lmi}\partial_m\theta^{jk} + \theta^{km}\partial_m\Pi^{lji} \\ & + \theta^{jm}\partial_m\Pi^{kli} + \theta^{lm}\partial_m\Pi^{jki} + \frac{1}{2}\Pi^{ljm}\partial_m\theta^{ki} + \frac{1}{2}\Pi^{klm}\partial_m\theta^{ji} + \frac{1}{2}\Pi^{jkm}\partial_m\theta^{li} . \end{aligned}$$

The expression (4.31) is exactly the equation from which the function $\theta^{lj|ki}$ was determined in [42], as given explicitly in (2.27), whose integrability condition is precisely the integrability identity (2.2) for the Jacobiator Π . So the integrability condition (4.30) is indeed satisfied and

$$\Lambda_2^{il} = -\frac{1}{3}\Upsilon^{ilj|k}A_jA_k .$$

This recursive construction can be extended in exactly the same way to higher order calculations. The integrability condition (4.28) is always satisfied as a consequence of the definition of the functions Υ_n^{kij} from (4.20) and the construction of the symplectic embedding for the almost Poisson structure θ from [42]. \square

5 Homotopy algebra actions

The reader versed in the BV formalism will have already noticed a strong resemblance between our approach to almost Poisson gauge theories based on symplectic embeddings and the approach to generalized gauge symmetries based on L_∞ -algebras. In this framework, the classical BRST construction amounts to quotienting a Poisson algebra of smooth functions by the ideal of functions which vanish on a constraint locus through a process of homological reduction to achieve the gauge closure condition. In this language, the concept of field dependent gauge transformation that we introduced in Section 4.1 is very natural.

In this section we will show that our almost Poisson gauge symmetries, which do not arise from any Lie algebra action, in fact arise from actions of L_∞ -algebras. For this, we provide a dictionary between the symplectic embeddings proposed in the present paper and the corresponding L_∞ -algebras of generalized gauge transformations [27]. This can be achieved by identifying the gauge variations defined by (4.10), and the closure bracket which is determined in (4.7), with the corresponding objects defined in terms of L_∞ -algebras. As an interesting application of these constructions, we obtain a new perspective on the deformation quantization of the exterior algebra of differential forms on an arbitrary almost Poisson manifold, whose semi-classical limit is described by a P_∞ -algebra.

5.1 L_∞ -algebras and generalized gauge symmetries

Let us start by fixing definitions and conventions. A (*flat*) L_∞ -algebra (also known as a *strong homotopy Lie algebra*) is a graded vector space V over a field of characteristic zero together with a sequence of linear maps $\ell_n : V^{\otimes n} \rightarrow V$ of degree $2 - n$, for $n = 1, 2, \dots$, which satisfy two properties. Firstly, ℓ_n are graded skew-symmetric,

$$\ell_n(\dots, v_i, v_{i+1}, \dots) = -(-1)^{|v_i||v_{i+1}|} \ell_n(\dots, v_{i+1}, v_i, \dots)$$

for all $v_1, \dots, v_n \in V$, where $|v|$ denotes the degree of a homogeneous element $v \in V$. Secondly, the linear map defined by

$$\mathcal{L}_n(v_1 \otimes \dots \otimes v_n) := \sum_{k=1}^n \frac{(-1)^{k(n-k)}}{k!(n-k)!} \ell_{n-k+1}(\ell_k(v_1, \dots, v_k), v_{k+1}, \dots, v_n) \quad (5.1)$$

vanishes on the image of the alternatization map

$$\text{Alt}_n(v_1 \otimes \dots \otimes v_n) = \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)},$$

where the sum runs over permutations σ of degree n with sign $\text{sgn}(\sigma)$. We denote the alternatization of the map (5.1) by $\mathcal{J}_n = \mathcal{L}_n \circ \text{Alt}_n : V^{\otimes n} \rightarrow V$. Then the relations $\mathcal{J}_n = 0$ are called *homotopy Jacobi identities*; when evaluated explicitly on elements they involve cumbersome sign factors which are determined by the usual Koszul sign rules for skew-symmetric maps of graded vector spaces.

In particular, the first relation $\mathcal{J}_1 = 0$ implies that ℓ_1 is a differential making (V, ℓ_1) into a cochain complex, while $\mathcal{J}_2 = 0$ implies that ℓ_1 is a (graded) derivation of the bracket ℓ_2 , i.e. ℓ_2 is a cochain map. The third relation $\mathcal{J}_3 = 0$ then implies that the bracket ℓ_2 obeys the Jacobi identity up to exact terms, i.e. ℓ_2 induces a (graded) Lie bracket on the cohomology of ℓ_1 . Differential graded Lie algebras can be regarded as L_∞ -algebras where $\ell_n = 0$ for all $n \geq 3$.

Generalized gauge symmetries of classical field theories on a manifold M which are irreducible can be encoded in 2-term L_∞ -algebras [27, 34]. For the purposes of the present paper, the underlying graded vector space encoding the kinematical data on an open subset $U \subseteq M$ has non-zero homogeneous subspaces only in degrees 0 and 1, and is given by

$$V = C^\infty(U) \oplus \Omega^1(U), \quad (5.2)$$

with the grading provided by differential form degree. The gauge variations are then encoded by an L_∞ -structure $(\ell_n)_{n=1}^\infty$ on V with $\ell_{n+1}(f, A^{\otimes n}) \in \Omega^1(U)$, for $f \in C^\infty(U)$ and $A \in \Omega^1(U)$, as the formal power series

$$\begin{aligned} \delta_f A &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+1}(f, A^{\otimes n}) \\ &= \ell_1(f) + t \ell_2(f, A) - \frac{t^2}{2} \ell_3(f, A, A) + O(t^3) \end{aligned} \quad (5.3)$$

in $\Gamma_{\text{pol}}(J^1 T^* U)[[t]]$. For “standard” gauge symmetries which only involve a finite number of brackets, one can set $t = 1$ and avoid the use of formal power series as well as jet coordinates altogether, but the constructions of this paper necessitate their usage in general.

For the graded vector space (5.2), since ℓ_n is of degree $2 - n$, it vanishes unless its entries contain at least one and at most two gauge parameters. Hence the only non-trivial homotopy Jacobi identities for the L_∞ -structure on V involve two and three gauge parameters, that is, $\mathcal{J}_{n+2}(f, g, A^{\otimes n}) = 0$ and $\mathcal{J}_{n+3}(f, g, h, A^{\otimes n}) = 0$. It was shown in [7, 27, 34] that the homotopy Jacobi identities involving two gauge parameters imply the closure of the symmetry variations

$$[\mathbb{1} + \delta_f, \mathbb{1} + \delta_g]A = \delta_t \llbracket f, g \rrbracket(A)A ,$$

with the formal power series

$$\begin{aligned} \llbracket f, g \rrbracket(A) &= - \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+2}(f, g, A^{\otimes n}) \\ &= -\ell_2(f, g) - t \ell_3(f, g, A) + \frac{t^2}{2} \ell_4(f, g, A, A) + O(t^3) \end{aligned} \quad (5.4)$$

in $C_{\text{pol}}^\infty(J^1 T^* U)[[t]]$. The homotopy Jacobi identities involving three gauge parameters then guarantee that the strict Jacobi identities

$$\text{Cyc}_{f,g,h} [\mathbb{1} + \delta_h, [\mathbb{1} + \delta_f, \mathbb{1} + \delta_g]]A = 0$$

hold for any triple of gauge variations.

The following explicit formulas are useful for concrete checks of the homotopy Jacobi identities for gauge transformations.

Lemma 5.5. *For $f, g, h \in C^\infty(U)$ and $A \in \Omega^1(U)$, the L_∞ -relations $\mathcal{J}_{n+2}(f, g, A^{\otimes n}) = 0$ and $\mathcal{J}_{n+3}(f, g, h, A^{\otimes n}) = 0$ for $n \geq 1$ are given explicitly by*

$$\begin{aligned} \mathcal{J}_{n+2}(f, g, A^{\otimes n}) &= \ell_1(\ell_{n+2}(f, g, A^{\otimes n})) - (-1)^n \ell_{n+2}(\ell_1(f), g, A^{\otimes n}) - (-1)^n \ell_{n+2}(f, \ell_1(g), A^{\otimes n}) \\ &\quad + \sum_{i=1}^n (-1)^{(i+1)(n-i+1)} \left[\binom{n}{i-1} \ell_{n-i+2}(\ell_{i+1}(f, g, A^{\otimes i-1}), A^{\otimes n-i+1}) \right. \\ &\quad \left. + (-1)^i \binom{n}{i} \ell_{n-i+2}(\ell_{i+1}(f, A^{\otimes i}), g, A^{\otimes n-i}) \right. \\ &\quad \left. - (-1)^i \binom{n}{i} \ell_{n-i+2}(\ell_{i+1}(g, A^{\otimes i}), f, A^{\otimes n-i}) \right] , \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} \mathcal{J}_{n+3}(f, g, h, A^{\otimes n}) &= \text{Cyc}_{f,g,h} \left((-1)^n \ell_{n+3}(\ell_1(f), g, h, A^{\otimes n}) + (-1)^n \ell_2(\ell_{n+2}(f, g, A^{\otimes n}), h) \right. \\ &\quad \left. + \sum_{i=1}^n (-1)^{(i+1)(n-i)} \left[\binom{n}{i} \ell_{n-i+3}(\ell_{i+1}(f, A^{\otimes i}), g, h, A^{\otimes n-i}) \right. \right. \\ &\quad \left. \left. - (-1)^i \binom{n}{i-1} \ell_{n-i+3}(\ell_{i+1}(f, g, A^{\otimes i-1}), h, A^{\otimes n-i+1}) \right] \right) . \end{aligned} \quad (5.7)$$

Proof. The alternatization of the sum $\mathcal{L}_{n+2}(f, g, A^{\otimes n})$ in (5.1) involves, for fixed n and k , a sum over $\binom{n+2}{k} = (n+2)!/k!(n+2-k)!$ inequivalent splittings of permutations $\sigma \in S_{n+2}$ [34]. If $k = 1$ the contribution to (5.1) is $\ell_{n+2}(\ell_1(f), g, A^{\otimes n}) + \ell_{n+2}(f, \ell_1(g), A^{\otimes n})$, and when $k = n+2$

the contribution is $\ell_1(\ell_{n+2}(f, g, A^{\otimes n}))$; note that terms such as $\ell_{n+2}(f, g, \ell_1(A), A^{\otimes n-1})$ do not appear as $\ell_1(A) = 0$ for degree reasons. If $k \neq 1, n+2$ then, taking into account that there are n identical entries A , the sum over inequivalent splittings of permutations $\sigma \in S_{n+2}$ will contain $\binom{n}{k-2}$ contributions of the type $\ell_{n-k+3}(\ell_k(f, g, A^{\otimes k-2}), A^{\otimes n-k+2})$, $\binom{n}{k-1}$ contributions of each type $\ell_{n-k+3}(\ell_k(f, A^{\otimes k-1}), g, A^{\otimes n-k+1})$ and $\ell_{n-k+3}(\ell_k(g, A^{\otimes k-1}), f, A^{\otimes n-k+1})$, and also $\binom{n}{k}$ elements $\ell_{n-k+3}(\ell_k(A^{\otimes k}), f, g, A^{\otimes n-k})$; these latter contributions however vanish as $\ell_k(A^{\otimes k}) = 0$ for degree reasons. Altogether, the total number of such contributions is

$$\binom{n}{k-2} + \binom{n}{k-1} + \binom{n}{k-1} + \binom{n}{k} = \binom{n+2}{k},$$

which is exactly the number of all contributions to the sum over $\sigma \in S_{n+2}$. This results in (5.6), where the sign factor $(-1)^i$ in the last two lines appears from moving f and g from the second argument to the $(i+1)$ -th argument of the $(n-i+2)$ -bracket. Following the same line of argument we find (5.7). \square

5.2 L_∞ -structures on Poisson gauge algebroids

It is instructive to start with the simpler case of Poisson gauge transformations from Section 3, where we know $\llbracket f, g \rrbracket(A) = \{f, g\}_\theta$ to all orders and a Poisson gauge symmetry corresponds to a Lie algebra structure on the graded vector space V . Geometrically, this structure is encoded in the action algebroid

$$C^\infty(U) \times \Omega^1(U) \longrightarrow \Omega^1(U)$$

corresponding to the Lie algebra $(C^\infty(U), \{\cdot, \cdot\}_\theta)$ and the Lie module $\Omega^1(U)$ over $C^\infty(U)$; for the moment we drop the formal power series extension to streamline the presentation. This is the Lie algebroid with trivial vector bundle $C^\infty(U) \times \Omega^1(U)$ over $\Omega^1(U)$ (regarded as a Fréchet space with the weak Whitney C^∞ -topology), whose anchor map $\rho : C^\infty(U) \times \Omega^1(U) \rightarrow T\Omega^1(U)$ sends a pair (f, A) to the gauge variation $\delta_f^\theta A$ of Definition 3.2, and whose Lie bracket is induced by the Poisson bracket $\{\cdot, \cdot\}_\theta$ and the gauge variations. Generally, any Lie algebroid gives rise to a cochain complex (its Chevalley-Eilenberg algebra), and in the case of an action algebroid this yields the BRST complex of a classical field theory, which is dual to a gauge L_∞ -algebra (see e.g. [35]).

In fact, the proper reinstatement of the formal power series extension, as required by the symplectic embedding approach, into this Lie algebroid perspective necessitates the use of L_∞ -algebras. For this, let us note the following useful geometric way of thinking about the ‘Taylor expansion’ of the twisting one-forms $s_A^* \gamma(df, \cdot)$ in (3.9) which are induced by (2.14). Without loss of generality we can assume that the functions $\gamma_i^{j|i_1 \dots i_n} \in C^\infty(U)$ are symmetric in their last n indices, and introduce the $(n+1, 1)$ -tensor fields $\gamma^{(n)} \in \Omega^1(U, \mathfrak{X}(U) \otimes \mathfrak{X}(U)^{\odot n})$ which in local coordinates read as

$$\gamma^{(n)} = n! \gamma_i^{j|i_1 \dots i_n}(x) \partial_j \otimes (\partial_{i_1} \odot \dots \odot \partial_{i_n}) \otimes dx^i.$$

Then

$$s_A^* \gamma(df, \cdot) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \gamma^{(n)}(df, A^{\otimes n}) \quad (5.8)$$

as a formal power series in $\Omega^1(U)[[t]]$.

Proposition 5.9. *Let (T^*M, ω) be a local symplectic embedding of a Poisson manifold (M, θ) , and let $U \subseteq M$ be an open subset. Then there is an L_∞ -structure $(\ell_n^\theta)_{n=1}^\infty$ on $C^\infty(U) \oplus \Omega^1(U)$, unique up to L_∞ -quasi-isomorphism, which induces the Poisson gauge algebroid constructed by Proposition 3.7 with non-vanishing maps on coincident degree 1 entries given by*

$$\begin{aligned}\ell_1^\theta(f) &= df , \\ \ell_2^\theta(f, g) &= -\{f, g\}_\theta , \\ \ell_2^\theta(f, A) &= \{A, f\}_\theta + \gamma^{(1)}(df, A) , \\ \ell_{n+1}^\theta(f, A^{\otimes n}) &= (-1)^{\frac{n(n-1)}{2}} \gamma^{(n)}(df, A^{\otimes n}) ,\end{aligned}$$

and skew-symmetry imposed by definition, for $n \geq 2$, $f, g \in C^\infty(U)$, and $A \in \Omega^1(U)$.

Proof. The brackets ℓ_n^θ follow from comparing (3.4) with (5.4) and (3.9) with (5.3) order by order in t using (5.8). The fact that our symplectic embeddings define Poisson gauge algebras in the sense of Definition 3.2, as we proved in Proposition 3.7, then implies that the statement is a special instance of [27, Theorem 2] for the case of gauge transformations which are generated by Lie algebra actions (see [27, Section 6]).

The uniqueness statement follows from noting that different completions of the brackets to non-coincident gauge field entries, using the homotopy Jacobi identities, are related by invertible field redefinitions $\chi : \Gamma_{\text{pol}}(J^1 T^* U)[[t]] \rightarrow \Gamma_{\text{pol}}(J^1 T^* U)[[t]]$, called Seiberg-Witten maps, which leave invariant the gauge parameters f and define deformations of the gauge fields: $\chi(A)|_{t=0} = A$ for $A \in \Omega^1(U)$. We define the new gauge transformations $\chi(A) \mapsto \chi(A) + \hat{\delta}_f^\theta \chi(A)$ by setting

$$\hat{\delta}_f^\theta := \chi \circ \delta_f^\theta \circ \chi^{-1} .$$

Then the field redefinition maps gauge orbits onto gauge orbits:

$$\chi(A + \delta_f^\theta A) = \chi(A) + \hat{\delta}_f^\theta \chi(A) ,$$

and it preserves the Poisson gauge algebra:

$$[\mathbb{1} + \hat{\delta}_f^\theta, \mathbb{1} + \hat{\delta}_g^\theta] \chi(A) = \chi \circ [\mathbb{1} + \delta_f^\theta, \mathbb{1} + \delta_g^\theta] A = \chi \circ \delta_{t\{f, g\}_\theta}^\theta A = \hat{\delta}_{t\{f, g\}_\theta}^\theta \chi(A) .$$

In [10] it was shown that the Seiberg-Witten maps χ correspond to L_∞ -quasi-isomorphisms which describe the arbitrariness in the definition of the related L_∞ -algebras in the L_∞ -bootstrap approach [11]. \square

Example 5.10. Let $M = \mathbb{R}^d$ with a constant Poisson structure θ . By Example 2.20, in this case we can take $\gamma^{(n)} = 0$ for all $n \geq 1$. Then the only non-vanishing maps are

$$\ell_1^\theta(f) = df , \quad \ell_2^\theta(f, g) = -\{f, g\}_\theta \quad \text{and} \quad \ell_2^\theta(f, A) = \{A, f\}_\theta .$$

Thus in this case the symplectic embedding of (\mathbb{R}^d, θ) in $(T^*\mathbb{R}^d, \omega_0 + t\theta^*)$ makes $C^\infty(\mathbb{R}^d) \oplus \Omega^1(\mathbb{R}^d)$ into a differential graded Lie algebra, with Lie bracket given by the Poisson bracket $\{\cdot, \cdot\}_\theta$. This describes the algebroid of Poisson gauge transformations from Example 3.6. \triangleleft

Remark 5.11. The brackets $\ell_2^\theta(f, A)$ in Proposition 5.9 encode the failure of the exterior derivative d in being a derivation of the Poisson algebra in general. The homotopy Jacobi identity $\mathcal{J}_2^\theta(f, g) = 0$ reads

$$d\{f, g\}_\theta = \{df, g\}_\theta + \{f, dg\}_\theta + \gamma^{(1)}(dg, df) - \gamma^{(1)}(df, dg) ,$$

which using (2.16) can be written locally in components as

$$\partial_i\{f, g\}_\theta = \{\partial_i f, g\}_\theta + \{f, \partial_i g\}_\theta + \partial_i \theta^{jk} \partial_j f \partial_k g .$$

This is just the familiar violation of the Leibniz rule for non-constant Poisson bivectors θ .

Using $\ell_{n+2}^\theta(f, g, A^{\otimes n}) = 0$ for $n \geq 1$, we can demonstrate explicitly that the higher Jacobi identities are equivalent to the local equations (2.15) for a symplectic embedding of a Poisson manifold, as an illustration of the tight relationship between our symplectic embeddings and the corresponding L_∞ -structures. For this, we use Lemma 5.5 and write (5.6) as

$$\begin{aligned} \mathcal{J}_{n+2}^\theta(f, g, A^{\otimes n}) &= \ell_{n+1}^\theta(\ell_2^\theta(f, g), A^{\otimes n}) - (-1)^n \ell_{n+2}^\theta(\ell_1^\theta(f), g, A^{\otimes n}) - (-1)^n \ell_{n+2}^\theta(f, \ell_1^\theta(g), A^{\otimes n}) \\ &\quad - n \left(\ell_{n+1}^\theta(\ell_2^\theta(f, A), g, A^{\otimes n-1}) - \ell_{n+1}^\theta(\ell_2^\theta(g, A), f, A^{\otimes n-1}) \right. \\ &\quad \left. + \ell_2^\theta(\ell_{n+1}^\theta(f, A^{\otimes n}), g) - \ell_2^\theta(\ell_{n+1}^\theta(g, A^{\otimes n}), f) \right) \\ &\quad - \sum_{i=3}^n (-1)^{(i+1)(n-i)} \binom{n}{i-1} \left(\ell_{n-i+3}^\theta(\ell_i^\theta(f, A^{\otimes i-1}), g, A^{\otimes n-i+1}) \right. \\ &\quad \left. - \ell_{n-i+3}^\theta(\ell_i^\theta(g, A^{\otimes i-1}), f, A^{\otimes n-i+1}) \right) . \end{aligned} \quad (5.12)$$

We substitute the brackets from Proposition 5.9 in (5.12), and after some careful simplification in local coordinates its components become

$$\begin{aligned} \mathcal{J}_{n+2}^\theta(f, g, A^{\otimes n})_i &= n! (-1)^{\frac{(n+1)(n+2)}{2}} \\ &\quad \times \left((n+1) (\gamma_i^{k|li_2 \dots i_{n+1}} - \gamma_i^{l|ki_2 \dots i_{n+1}}) \right. \\ &\quad \left. + \sum_{m=1}^n (n-m+1) (\gamma_j^{l|i_1 \dots i_m} \gamma_i^{k|j i_{m+1} \dots i_{n+1}} - \gamma_j^{k|i_1 \dots i_m} \gamma_i^{l|j i_{m+1} \dots i_{n+1}}) \right. \\ &\quad \left. - \theta^{kj} \partial_j \gamma_i^{l|i_2 \dots i_{n+1}} + \theta^{lj} \partial_j \gamma_i^{k|i_2 \dots i_{n+1}} - \gamma_i^{j|i_2 \dots i_{n+1}} \partial_j \theta^{lk} \right) \\ &\quad \times \partial_k f \partial_l g A_{i_2} \dots A_{i_{n+1}} . \end{aligned}$$

This vanishes as a consequence of (2.17), and thus the closure condition for the gauge algebra implies that the brackets ℓ_n^θ indeed define an L_∞ -algebra. In other words, the homotopy Jacobi identities are equivalent to the Poisson integrability condition $[\omega^{-1}, \omega^{-1}] = 0$, as written in (2.8) and (2.26) (with $\underline{\theta}^{ij}(x, p) = \theta^{ij}(x)$); algebraically, the brackets follow from a standard higher derived bracket construction [69], as we shall see in Section 5.4.

This discussion illustrates the necessity of the L_∞ -algebra formulation even in the case of Poisson gauge transformations for non-constant bivectors θ , despite the fact that they are still defined by an underlying Lie algebra action. \triangleleft

Remark 5.13. We can now give a homotopy algebraic answer to the question of uniqueness of symplectic embeddings for a given Poisson manifold (M, θ) . Similarly to the uniqueness statement of Proposition 5.9, different local symplectic embeddings (T^*M, ω) correspond to field redefinitions of the brackets of the corresponding L_∞ -algebras, which are related to one another through L_∞ -quasi-isomorphisms [10, 43]. \triangleleft

5.3 L_∞ -structures on almost Poisson gauge algebroids

The general case of almost Poisson gauge transformations from Section 4 is markedly different from the Poisson case, as now there is no underlying Lie algebra structure on V and hence no corresponding action algebroid. However, suppressing momentarily the formal power series extension again as well as the jet coordinate dependence, there is still an underlying Lie algebroid with vector bundle

$$C^\infty(U) \longrightarrow \mathcal{E}(U) \longrightarrow \Omega^1(U)$$

whose fibre over a gauge field $A \in \Omega^1(U)$ is $s_A^* C^\infty(T^*U) \simeq C^\infty(U)$. A field dependent gauge parameter can then be identified with a section of this bundle. The bracket and anchor map of this gauge algebroid are induced by the bracket $[\cdot, \cdot]_\theta$ and the gauge variation $(f, A) \mapsto \delta_f^\theta A$ of Definition 4.2.

The corresponding L_∞ -structure making this Lie algebroid picture precise is induced by the symplectic embedding construction of Section 4 and may be presented in the following geometric way. For this, we introduce tensor fields whose local components are the coefficients of the formal power series expansions of Sections 2.4 and 4.2, in order to write the gauge variations and brackets of Remark 4.16 as formal power series analogously to (5.8). Assuming without loss of generality that the functions $\theta^{ij|i_1 \dots i_n} \in C^\infty(U)$ for $n \geq 2$ are symmetric in their last n indices, we introduce $n+2$ -tensors $\theta^{(n)} \in \mathfrak{X}^2(U, \mathfrak{X}(U)^{\odot n})$ in local coordinates by

$$\theta^{(n)} := n! \theta^{ij|i_1 \dots i_n}(x) (\partial_i \wedge \partial_j) \otimes (\partial_{i_1} \odot \dots \odot \partial_{i_n}) .$$

We set $\theta^{(0)} := \theta$ and $\theta^{(1)} := -\Pi$. Then

$$s_A^* \underline{\theta}(\pi^* \cdot, \pi^* \cdot) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \theta^{(n)}(\cdot, \cdot, A^{\otimes n}) \quad (5.14)$$

as a formal power series in $\mathfrak{X}^2(U)[[t]]$.

Similarly, taking $\Upsilon^{ijk|i_1 \dots i_{n-1}} \in C^\infty(U)$ for $n \geq 2$ to be symmetric in their last n indices, we introduce $n+2$ -tensors $\Upsilon^{(n)} \in \mathfrak{X}(U, \mathfrak{X}(U) \otimes \mathfrak{X}(U)^{\odot n})$ in local coordinates by

$$\Upsilon^{(n)} := n! \Upsilon^{ijk|i_1 \dots i_{n-1}}(x) \partial_i \otimes \partial_j \otimes (\partial_k \odot \partial_{i_1} \odot \dots \odot \partial_{i_{n-1}}) .$$

We set $\Upsilon^{(1)} := \Pi$. By the construction of Section 4.2, the vector fields $\Upsilon^{(n)}(df, A^{\otimes n}) \in \mathfrak{X}(U)$ completely determine the formal power series expansions of the Lagrangian multipliers $L_f \in \mathfrak{X}_{\text{pol}}^h(J^1 T^*U)[[t]]$ through

$$L_f = - \sum_{n=1}^{\infty} \frac{t^{n+1}}{(n+1)!} L^{(n)}(df, A^{\otimes n}) , \quad (5.15)$$

where

$$\begin{aligned} L^{(n)}(df, A^{\otimes n}) := & \Upsilon^{(n)}(df, A^{\otimes n}) + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\substack{l_1, \dots, l_{k+1} \geq 1 \\ l_1 + \dots + l_{k+1} = n-k}} \frac{(n+1)!}{(l_1+1)! \dots (l_{k+1}+1)!} \\ & \times \Upsilon^{(l_1)} \left(\Upsilon^{(l_2) \otimes} (DA, \dots \Upsilon^{(l_k) \otimes} (DA, \Upsilon^{(l_{k+1}) \otimes} (DA, df, A^{\otimes l_{k+1}}), A^{\otimes l_k}), \dots A^{\otimes l_2}), A^{\otimes l_1} \right) \end{aligned}$$

with the sum omitted for $n = 1, 2$.

Proposition 5.16. *Let (T^*M, ω) be a local symplectic embedding of an almost Poisson manifold (M, θ) with Jacobiator Π , and let $U \subseteq M$ be an open subset. Then there is an L_∞ -structure $(\ell_n^\theta)_{n=1}^\infty$ on $C^\infty(U) \oplus \Omega^1(U)$, unique up to L_∞ -quasi-isomorphism, which induces the almost Poisson gauge algebroid constructed by Proposition 4.6 with non-vanishing maps on coincident degree 1 entries given as follows. The brackets involving a single gauge parameter are given by*

$$\begin{aligned} \ell_1^\theta(f) &= df, \\ \ell_2^\theta(f, A) &= \{A, f\}_\theta + \gamma^{(1)}(df, A), \\ \ell_3^\theta(f, A^{\otimes 2}) &= -\gamma^{(2)}(df, A^{\otimes 2}) + 2\Pi^\otimes(df, A, DA) + dA(\Pi(df, A, \cdot), \cdot), \\ \ell_{n+1}^\theta(f, A^{\otimes n}) &= (-1)^{\frac{n(n-1)}{2}} \left[\gamma^{(n)}(df, A^{\otimes n}) - n\theta^{(n-1)\otimes}(df, DA, A^{\otimes n-1}) \right. \\ &\quad \left. - dA(L^{(n-1)}(df, A^{\otimes n-1}), \cdot) \right. \\ &\quad \left. + \sum_{k=2}^{n-1} \binom{n}{k} \left(\gamma^{(n-k)}(DA(L^{(k-1)\otimes}(df, A^{\otimes k-1})), A^{\otimes n-k}) \right. \right. \\ &\quad \left. \left. - \gamma^{(n-k)\otimes}(DA, A^{\otimes n-k}, L^{(k-1)}(df, A^{\otimes k-1})) \right. \right. \\ &\quad \left. \left. - (n-k)\theta^{(n-k-1)\otimes}(DA(L^{(k-1)\otimes}(df, A^{\otimes k-1})), DA, A^{\otimes n-k-1}) \right) \right], \end{aligned}$$

for $n \geq 3$. The brackets involving two gauge parameters are given by

$$\begin{aligned} \ell_2^\theta(f, g) &= -\{f, g\}_\theta, \\ \ell_3^\theta(f, g, A) &= \Pi(df, dg, A), \\ \ell_4^\theta(f, g, A^{\otimes 2}) &= \theta^{(2)}(df, dg, A^{\otimes 2}), \\ \ell_5^\theta(f, g, A^{\otimes 3}) &= \theta^{(3)}(df, dg, A^{\otimes 3}) - \frac{3}{2}dA(\Pi(df, A, \cdot), \Pi(dg, A, \cdot)), \\ \ell_{n+2}^\theta(f, g, A^{\otimes n}) &= -(-1)^{\frac{n(n-1)}{2}} \\ &\times \left[\theta^{(n)}(df, dg, A^{\otimes n}) - \sum_{k=1}^{n-2} \frac{1}{k+1} \binom{n}{k} dA(L^{(k)}(df, A^{\otimes k}), L^{(n-k-1)}(dg, A^{\otimes n-k-1})) \right. \\ &\quad \left. + \sum_{k=3}^{n-1} \frac{1}{k+1} \binom{n}{k} \sum_{l=1}^{k-2} \binom{k+1}{l+1} \left(\gamma^{(n-k)}(DA(L^{(l)\otimes}(df, A^{\otimes l})), A^{\otimes n-k}, L^{(k-l-1)}(dg, A^{\otimes k-l-1})) \right. \right. \\ &\quad \left. \left. - \gamma^{(n-k)}(DA(L^{(l)\otimes}(dg, A^{\otimes l})), A^{\otimes n-k}, L^{(k-l-1)}(df, A^{\otimes k-l-1})) \right) \right. \\ &\quad \left. - (n-k)\theta^{(n-k-1)}(DA(L^{(l)\otimes}(df, A^{\otimes l})), DA(L^{(k-l-1)\otimes}(dg, A^{\otimes k-l-1})), A^{\otimes n-k-1}) \right], \end{aligned}$$

for $n \geq 4$. In these expressions skew-symmetry between gauge parameters and gauge fields is imposed by definition, for $f, g \in C^\infty(U)$ and $A \in \Omega^1(U)$.

Proof. The proof is completely analogous to the proof of Proposition 5.9. The brackets ℓ_n^θ now follow from a straightforward but laborious comparison of the gauge variation and bracket of Remark 4.16 with (5.3) and (5.4), respectively, order by order in t using (5.8), (5.14) and (5.15). The fact that Proposition 4.6 constructs an almost Poisson gauge algebra, in the sense of Definition 4.2, then implies that the statement is a special instance of [27, Theorem 2] for the general case of field dependent gauge transformations. \square

Remark 5.17. When θ is a Poisson structure, then the tensors Π , $\theta^{(n)}$ for $n \geq 2$ and $L^{(k)}$ all vanish, and the L_∞ -structure of Proposition 5.16 reduces to the L_∞ -structure of Proposition 5.9. The brackets ℓ_3^θ in Proposition 5.16 coincide exactly with the brackets defined in [11] (after taking into account that the Jacobiator Π in [11] differs from ours by a factor $\frac{1}{3}$). Moreover, the brackets $\ell_{n+2}^\theta(f, g, A^{\otimes n})$ for $n = 0, 1, 2$ coincide with the brackets obtained in [43]. In particular, the homotopy Jacobi identity $\mathcal{J}_3^\theta(f, g, h) = 0$ for $f, g, h \in C^\infty(U)$ reads

$$\text{Cyc}_{f,g,h} \{f, \{g, h\}_\theta\}_\theta = \frac{1}{2} \Pi(df, dg, dh) ,$$

which is the familiar violation of the strict Jacobi identity for an almost Poisson bracket with non-vanishing Jacobiator Π . However, for $n > 2$ the brackets $\theta^{(n)}(df, dg, A^{\otimes n})$ are corrected by terms involving contributions from the Lagrangian multiplier vector fields and do not on their own constitute the correct L_∞ -structure required by the gauge closure condition (4.3) at higher orders, contrary to the conjecture of [42]. We shall consider an explicit example in Section 6 below. \triangleleft

5.4 P_∞ -algebras of exterior differential forms

Let us now discuss some potential applications of our constructions to deformation quantization. A central problem in the formulation of noncommutative gauge theories is to find an extension of a star-product, which quantizes an almost Poisson manifold (M, θ) , to the de Rham complex $(\Omega^\bullet(M), d)$. Even at the purely kinematical level this problem is non-trivial, as the vector space action mentioned in Remark 3.5 does not generally make $\Omega^1(U)[[\hbar]]$ into a $C^\infty(U)[[\hbar]]$ -bimodule. The problem can be traced back to the semi-classical limit: the differential graded algebra $\Omega^\bullet(M)$ is not generally a Poisson algebra, even when θ is a Poisson bivector field. In the case of symplectic manifolds, the problem of endowing $\Omega^\bullet(M)$ with the structure of a differential graded Poisson algebra is discussed in e.g. [5, 20, 30, 54]; the general construction depends on the choice of an almost symplectic connection or of a contravariant connection. Here we shall present an alternative treatment based on the constructions of this paper which is more general: it does not require the auxiliary data of a connection and works for any almost Poisson bivector field θ .

Our main observation is that a symplectic embedding, which always exists locally, naturally induces a P_∞ -structure on the de Rham complex of the underlying almost Poisson manifold. A P_∞ -algebra is a graded commutative algebra \mathcal{A} together with an L_∞ -structure $\{\ell_n\}_{n=1}^\infty$ such that the differential $\ell_1 : \mathcal{A} \rightarrow \mathcal{A}$ is a derivation of degree 1 and, for each $n \geq 2$ and fixed elements $a_1, \dots, a_{n-1} \in \mathcal{A}$, the map $a \mapsto \ell_n(a_1, \dots, a_{n-1}, a)$ is a derivation of degree $2-n + \sum_{i=1}^{n-1} |a_i|$; this is a strong homotopy version of a Poisson algebra. P_∞ -algebras were introduced by Cattaneo and Felder in their approach to quantization of coisotropic submanifolds of Poisson manifolds [17]. Just as a Poisson algebra can be regarded as the semi-classical limit of an associative algebra in deformation quantization, P_∞ -algebras arise as semi-classical limits of A_∞ -algebras.

Since a symplectic embedding (T^*M, ω) of a generic almost Poisson structure θ similarly involves a Lagrangian submanifold of a symplectic manifold, we can offer a homotopy algebraic explanation for the meaning of the symplectic structure ω on T^*M away from the zero section, as well as a new perspective on the relation between our symplectic embeddings and the semi-classical limit of a deformation quantization of θ . Let C be any manifold. The general result of [17, Proposition 2.1] then constructs a P_∞ -structure on the graded commutative algebra

$\Gamma(C, \wedge^\bullet E)$ of sections of the exterior algebra of any vector bundle $E \rightarrow C$ whose total space is a Poisson manifold.

Applying this result to $E = T^*M$ with a local symplectic embedding ω of the almost Poisson structure θ , we find that the almost Poisson bracket on $C^\infty(M)$ can be viewed as part of a P_∞ -structure on the de Rham complex of the manifold M , induced by the Poisson structure ω^{-1} on T^*M .

Proposition 5.18. *Let (T^*M, ω) be a local symplectic embedding of an almost Poisson manifold (M, θ) with Jacobiator Π , and let $U \subseteq M$ be an open subset. Then there is a P_∞ -structure $\{\varrho_n^\theta\}_{n=1}^\infty$ on the exterior algebra $\Omega^\bullet(U)$ of differential forms on U defined as follows. On generators of $\Omega^\bullet(U)$, the non-vanishing brackets are defined by*

$$\begin{aligned}
\varrho_1^\theta(f) &= df, & \varrho_1^\theta(\alpha) &= d\alpha, \\
\varrho_2^\theta(f, g) &= \{f, g\}_\theta, & \varrho_2^\theta(\alpha, \beta) &= \{\alpha, \beta\}_\theta - \gamma^{(1)\otimes}(\mathrm{D}\alpha, \beta) - \gamma^{(1)\otimes}(\mathrm{D}\beta, \alpha), \\
\varrho_2^\theta(f, \alpha) &= \{f, \alpha\}_\theta - \gamma^{(1)}(df, \alpha), & \varrho_3^\theta(f, g, \alpha) &= \Pi(df, dg, \alpha), \\
\varrho_n^\theta(\alpha_1, \dots, \alpha_n) &= \frac{(-1)^n}{(n-1)!} \left(\sum_{i=1}^n \gamma^{(n-1)\otimes}(\mathrm{D}\alpha_i, \alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n) \right. \\
&\quad \left. - 2(n-1) \sum_{i < j} \theta^{(n-2)\otimes}(\mathrm{D}\alpha_i, \mathrm{D}\alpha_j, \alpha_1, \dots, \widehat{\alpha}_i, \dots, \widehat{\alpha}_j, \dots, \alpha_n) \right), \\
\varrho_{n+1}^\theta(f, \alpha_1, \dots, \alpha_n) &= \frac{1}{n!} \gamma^{(n)}(df, \alpha_1, \dots, \alpha_n) \\
&\quad + \frac{1}{(n-1)!} \sum_{i=1}^n \theta^{(n-1)\otimes}(df, \mathrm{D}\alpha_i, \alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n), \\
\varrho_{n+2}^\theta(f, g, \alpha_1, \dots, \alpha_n) &= \frac{(-1)^n}{n!} \theta^{(n)}(df, dg, \alpha_1, \dots, \alpha_n), \tag{5.19}
\end{aligned}$$

for $n \geq 2$, $f, g \in C^\infty(U)$, and $\alpha, \beta, \alpha_1, \dots, \alpha_n \in \Omega^1(U)$, where a hat indicates omission of the corresponding entry. The brackets are then defined on higher degree forms by uniquely extending (5.19) to linear maps $\varrho_n^\theta : \Omega^\bullet(U)^{\otimes n} \rightarrow \Omega^\bullet(U)$ as polyderivations.

Proof. Since ϱ_n^θ is of degree $2-n$, it vanishes on elements of degree 0 or 1 except in the cases given in (5.19). Its structural form is merely a translation of the statement of [17, Theorem 2.2] to this situation. That statement tells us that the components of the P_∞ -structure on $\Omega^\bullet(U)$ are the Taylor series expansion coefficients, in the transverse coordinates to the zero section $U \subset T^*U$, of the cosymplectic bivector field (2.24). In local coordinates where $\alpha = \alpha_i dx^i \in \Omega^1(U)$, and vector fields $X = X^i \partial_i \in \mathfrak{X}(U)$ are regarded as fibre-linear functions $X^i p_i$ on T^*U , these are given by

$$\begin{aligned}
\varrho_{n+2}^\theta(f, g, \alpha_1, \dots, \alpha_n) &= (-1)^n \alpha_{i_1} \cdots \alpha_{n i_n} \tilde{\partial}^{i_1} \cdots \tilde{\partial}^{i_n} \{\pi^* f, \pi^* g\}_{\omega^{-1}} \Big|_{p=0}, \\
\varrho_{n+1}^\theta(f, \alpha_1, \dots, \alpha_n)(X) &= \alpha_{i_1} \cdots \alpha_{n i_n} \tilde{\partial}^{i_1} \cdots \tilde{\partial}^{i_n} \{\pi^* f, X^i p_i\}_{\omega^{-1}} \Big|_{p=0}, \\
\varrho_n^\theta(\alpha_1, \dots, \alpha_n)(X, Y) &= (-1)^n \alpha_{i_1} \cdots \alpha_{n i_n} \tilde{\partial}^{i_1} \cdots \tilde{\partial}^{i_n} \{X^i p_i, Y^j p_j\}_{\omega^{-1}} \Big|_{p=0}.
\end{aligned}$$

Substituting the series expansions (2.14) and (2.25) at $t = 1$, and using the Koszul formula for the de Rham differential, after some calculation we arrive at the formulas (5.19). One can

check directly that these brackets extend to a P_∞ -structure: the homotopy Jacobi identities are equivalent to the Poisson integrability condition $[\omega^{-1}, \omega^{-1}] = 0$, as written in (2.8) and (2.26), and indeed the brackets follow algebraically from a standard higher derived bracket construction (see [17, Section 2.6]). \square

Example 5.20. Let $M = \mathbb{R}^d$ with a constant Poisson structure θ . In this case all tensors Π , $\gamma^{(n)}$ and $\theta^{(n)}$ for $n \geq 1$ vanish in Proposition 5.18, and the only non-zero brackets are given by

$$\varrho_1^\theta(\xi) = d\xi \quad \text{and} \quad \varrho_2^\theta(\xi, \zeta) = \{\xi, \zeta\}_\theta ,$$

for all $\xi, \zeta \in \Omega^\bullet(\mathbb{R}^d)$. In this case we recover the well-known realization of $\Omega^\bullet(\mathbb{R}^d)$ as a differential graded Poisson algebra. In general, however, the P_∞ -algebra of Proposition 5.18 involves infinitely-many non-zero brackets on the exterior algebra of differential forms, even for Poisson bivectors. \triangleleft

Remark 5.21. An L_∞ -algebroid is a Lie algebroid together with an L_∞ -structure on its Chevalley-Eilenberg algebra whose differential is the Lie algebroid differential; it is a P_∞ -algebroid if the brackets of the L_∞ -structure are polyderivations. Hence an equivalent way of stating Proposition 5.18 is that a symplectic embedding makes the tangent Lie algebroid over an almost Poisson manifold into a P_∞ -algebroid. In this language, Proposition 5.18 is essentially a special case of [57, Theorem 9.4] when (M, θ) is a Poisson manifold. \triangleleft

Remark 5.22. Proposition 5.18 implies a new homotopy algebraic construction of a deformation quantization of the exterior algebra $\Omega^\bullet(U)$ in the direction of a generic almost Poisson bracket. Since M is a Lagrangian submanifold of the local symplectic embedding (T^*M, ω) , and since for our local symplectic embeddings the cohomological obstructions of [17, Corollary 3.3] are trivial, we may apply the result of [17, Theorem 3.2] to quantize the P_∞ -algebra of Proposition 5.18 to an A_∞ -structure on $\Omega^\bullet(U)[[\hbar]]$ over $\mathbb{R}[[\hbar]]$, which is a deformation of the exterior product on $\Omega^\bullet(U)$ and whose semi-classical limit induces the P_∞ -structure $\{\varrho_n^\theta\}_{n=1}^\infty$ in the following sense. The alternatization of the structure maps of this A_∞ -algebra, which are polydifferential operators, define an L_∞ -structure $\{\varrho_n^*\}_{n=1}^\infty$ on $\Omega^\bullet(U)[[\hbar]]$. Then $\varrho_n^\theta = \frac{1}{\hbar} \varrho_n^*|_{\hbar=0}$, and in this sense we may regard the structure maps $\{\varrho_n^\theta\}_{n=1}^\infty$ as a semi-classical limit of $\{\varrho_n^*\}_{n=1}^\infty$. These considerations are based on a version of Kontsevich's formality theorem for the case of Lagrangian submanifolds of a symplectic manifold. The role of homotopy algebras in nonassociative deformation quantization of twisted Poisson structures was anticipated by [55], and the present observation makes this precise for arbitrary almost Poisson manifolds. The details are beyond the scope of this paper and will be explored elsewhere. \triangleleft

Poisson gauge algebras. In the case of a Poisson structure θ , the structure maps of Proposition 5.18, truncated to degrees 0 and 1 with $\theta^{(n)} = 0$ for all $n \geq 1$ and $\alpha = \beta = \alpha_1 = \dots = \alpha_n = A$ for all $n \geq 1$, essentially agree with those of Proposition 5.9 up to numerical factors. This suggests that the L_∞ -structure of Proposition 5.9 is compatible with a graded commutative algebra structure on $V = C^\infty(U) \oplus \Omega^1(U)$, such that the Poisson bracket on $C^\infty(U)$ is part of a P_∞ -structure on the 2-term cochain complex (V, d) . This expectation turns out to be correct and is the content of

Proposition 5.23. *Let (M, θ) be a Poisson manifold and $U \subseteq M$ an open subset. Let $\mathcal{A} = C^\infty(U) \oplus \Omega^1(U)$ be the graded commutative algebra with multiplication defined by truncating the*

product on the exterior algebra $\Omega^\bullet(U)$ at degree 1, that is,

$$f \cdot g = f g, \quad f \cdot A = f A \quad \text{and} \quad A \cdot B = 0$$

for all $f, g \in C^\infty(U)$ and $A, B \in \Omega^1(U)$. Then the L_∞ -structure of Proposition 5.9 turns \mathcal{A} into a P_∞ -algebra.

Proof. This follows from the fact that the subalgebra of Proposition 5.18 obtained by truncation to degrees 0 and 1, and subsequent replacement of θ with $-\theta$, is precisely the stated L_∞ -structure on \mathcal{A} . It is also an easy direct check using the Leibniz rules for the exterior derivative d , the Poisson bracket on $C^\infty(U)$, and the $\Omega^1(U)$ -valued bracket of Remark 3.5. \square

The significance of this observation is that it offers a natural path towards a homotopy algebraic construction of noncommutative gauge transformations beyond the semi-classical level. Similarly to the discussion of Remark 5.22, the P_∞ -algebra of Proposition 5.23 can be quantized to an A_∞ -structure on $\mathcal{A}[[\hbar]]$ over $\mathbb{R}[[\hbar]]$ which is a deformation of the algebra structure on \mathcal{A} and whose semi-classical limit induces the P_∞ -structure $\{\ell_n^\theta\}_{n=1}^\infty$ through the alternatization $\{\ell_n^*\}_{n=1}^\infty$ of the structure maps of this A_∞ -algebra: $\ell_n^\theta = \frac{1}{\hbar} \ell_n^*|_{\hbar=0}$. Then the Poisson gauge algebra is a semi-classical limit of a noncommutative gauge algebra which is organized by the ‘‘quantum’’ L_∞ -structure $\{\ell_n^*\}_{n=1}^\infty$, making the discussion at the beginning of Section 3 somewhat more precise. While this is an interesting approach to the construction of noncommutative gauge symmetries, it also lies beyond the scope of the present paper and we leave it for future investigation.

Almost Poisson gauge algebras. The situation is much more complicated in the case of a general almost Poisson bivector θ . Now, because of the presence of the non-vanishing brackets ρ_{n+2}^θ in Proposition 5.18 involving two functions and forms of higher degree, the corresponding truncation does not determine a subalgebra, as this would violate the homotopy Jacobi identities. The difference between the L_∞ -structure maps of Propositions 5.16 and 5.18 (aside from numerical factors) lies entirely in the terms involving the Lagrangian multipliers of the almost Poisson gauge algebroid, whose inclusion restores the homotopy Jacobi identities. As we will now show, these terms violate the derivation properties of the original P_∞ -algebra from Proposition 5.18.

For this, let us study in detail the derivation properties of the brackets $\ell_{n+2}^\theta(f, g, A^{\otimes n})$ and $\ell_{n+1}^\theta(f, A^{\otimes n})$ from Proposition 5.16 with the same graded commutative algebra \mathcal{A} as in Proposition 5.23. First of all, for the multiplication of gauge parameters it is clear that the derivation properties

$$\begin{aligned} \ell_{n+2}^\theta(f \cdot h, g, A^{\otimes n}) &= f \cdot \ell_{n+2}^\theta(h, g, A^{\otimes n}) + \ell_{n+2}^\theta(f, g, A^{\otimes n}) \cdot h, \\ \ell_{n+1}^\theta(f \cdot h, A^{\otimes n}) &= f \cdot \ell_{n+2}^\theta(h, A^{\otimes n}) + \ell_{n+2}^\theta(f, A^{\otimes n}) \cdot h \end{aligned}$$

hold.

For the multiplication of gauge fields by gauge parameters, consider first the brackets involving two gauge parameters. Since brackets with three gauge parameters vanish for degree reasons, the desired derivation property is just $C^\infty(U)$ -linearity

$$\ell_{n+2}^\theta(f, g, h \cdot A, A^{\otimes n-1}) = h \cdot \ell_{n+2}^\theta(f, g, A^{\otimes n}). \quad (5.24)$$

This essentially means that the field dependent gauge parameter $\llbracket f, g \rrbracket_\theta$ from Proposition 4.6, when evaluated on the argument $h \cdot A$, should not depend on the derivatives of h . The potentially problematic terms are the ones involving the Lagrangian multiplier vector fields. However, the functions $L^{ij}(A)$ constructed in Corollary 4.26 satisfy

$$L^{ij}(A) A_i = 0 , \quad (5.25)$$

and together with Proposition 4.18 it now easily follows that

$$L^{ij}(A, \partial(h A)) = L^{ij}(A, h \partial A) .$$

Moreover, from (4.19) the identity (5.25) also implies

$$L^{ij}(A, \partial A) A_i = 0 ,$$

and as a consequence the terms with derivatives of h disappear from $s_{hA}^* \{ \Phi_{hA}(L_f), \Phi_{hA}(L_g) \}_{\omega^{-1}}$. It follows that all terms involving derivatives of h also disappear from $\llbracket f, g \rrbracket_\theta(h A)$ and this implies (5.24). In other words, the map $A \mapsto \ell_{n+2}^\theta(f, g, \alpha_1, \dots, \alpha_{n-1}, A)$ is a derivation, for all $n \geq 1$ and $\alpha_1, \dots, \alpha_{n-1} \in \Omega^1(U)$.

For the brackets with a single gauge parameter the desired derivation property reads

$$\ell_{n+1}^\theta(f, h \cdot A, A^{\otimes n-1}) = h \cdot \ell_{n+1}^\theta(f, A^{\otimes n}) + (-1)^{n-1} \ell_{n+1}^\theta(f, h, A^{\otimes n-1}) \cdot A . \quad (5.26)$$

This relation is more complicated since it involves brackets of different nature and different graded symmetry. In particular, since the bracket with two gauge parameters is skew-symmetric in f and h , the relation (5.26) implies the consistency condition

$$\ell_{n+1}^\theta(f, h \cdot A, A^{\otimes n-1}) + \ell_{n+1}^\theta(h, f \cdot A, A^{\otimes n-1}) = h \cdot \ell_{n+1}^\theta(f, A^{\otimes n}) + f \cdot \ell_{n+1}^\theta(h, A^{\otimes n}) . \quad (5.27)$$

For $n = 1$ it is easy to see that the bracket ℓ_2^θ from Proposition 5.16 satisfies (5.26), while for $n = 2$ the relevant brackets are

$$\begin{aligned} \ell_3^\theta(f, h, A) &= \Pi(df, dh, A) , \\ \ell_3^\theta(f, A, B) &= -\gamma^{(2)}(df, A, B) + \Pi^\otimes(df, A, DB) + \Pi^\otimes(df, B, DA) \\ &\quad + \frac{1}{2} dA(\Pi(df, B, \cdot), \cdot) + \frac{1}{2} dB(\Pi(df, A, \cdot), \cdot) , \end{aligned}$$

for $f, h \in C^\infty(U)$ and $A, B \in \Omega^1(U)$. One then calculates explicitly

$$\begin{aligned} \ell_3^\theta(f, h \cdot A, A) &= -h \cdot \gamma^{(2)}(df, A, A) + h \cdot \Pi^\otimes(df, A, DA) + \Pi^\otimes(df, A, dh \otimes A + h DA) \\ &\quad + \frac{1}{2} (dh \wedge A)(\Pi(df, A, \cdot), \cdot) + \frac{1}{2} h \cdot dA(\Pi(df, A, \cdot), \cdot) \\ &\quad + \frac{1}{2} h \cdot dA(\Pi(df, A, \cdot), \cdot) \\ &= h \cdot (-\gamma^{(2)}(df, A, A) + 2 \Pi^\otimes(df, A, DA) + dA(\Pi(df, A, \cdot), \cdot)) \\ &\quad + \Pi(df, A, dh) \cdot A + \frac{1}{2} \Pi(df, A, dh) \cdot A - \frac{1}{2} dh \cdot \Pi(df, A, A) \\ &= h \cdot \ell_3^\theta(f, A, A) - \frac{3}{2} \ell_3^\theta(f, h, A) \cdot A , \end{aligned}$$

showing that already for $n = 2$ the derivation property (5.26) is violated. Starting from the next order $n = 3$, even the consistency condition (5.27) is violated. In the case of a Poisson structure the higher brackets all satisfy (5.27) because of their \mathcal{A} -linearity, but for generic almost Poisson structures the derivation property imposes severe restrictions on the L_∞ -algebra. We summarise the present discussion in

Proposition 5.28. *Let (M, θ) be an almost Poisson manifold with non-zero Jacobiator Π , and let $U \subseteq M$ be an open subset. Then the L_∞ -structure of Proposition 5.16 and the graded commutative algebra structure of Proposition 5.23 do not combine into a compatible P_∞ -structure on $C^\infty(U) \oplus \Omega^1(U)$.*

Remark 5.29. We do not solve the problem of finding a P_∞ -structure on the almost Poisson gauge algebroid in this paper, but let us briefly mention some possible ways that one may proceed.

As we have shown above, the derivation properties are violated by precisely the terms involving the Lagrangian multipliers L_f , which were introduced to cancel the second term in the commutator of two gauge transformations in (4.5). Instead of including L_f , one could try to cure the problem by identifying this term as the contribution from a non-zero homogeneous subspace V_2 in degree 2. This would lead to a 3-term L_∞ -algebra, which is likely a P_∞ -algebra under a suitable extension of the truncated exterior product on $C^\infty(U) \oplus \Omega^1(U)$. However, the gauge theory interpretation of the space V_2 is not clear, and this appears to be a general feature of physical systems based on almost Poisson algebras: to maintain all desirable features one inevitably needs to introduce some auxiliary (unphysical) degrees of freedom, such that the elimination of these auxiliary variables comes at the price of losing some of the desired properties (see [46] for an example of this). In the present situation, we lose the Leibniz rule for the L_∞ -structure, but still retain a well-defined gauge algebra.

Another possibility would be to work instead with a *curved* L_∞ -algebra. A *curving* of an L_∞ -structure $\{\ell_n\}_{n=1}^\infty$ on a graded vector space V is an additional map ℓ_0 of degree 2 from the ground field into V , which intertwines with the higher brackets through the homotopy Jacobi identities extended to $\{\ell_n\}_{n=0}^\infty$; then ℓ_1 is no longer a differential and one loses the underlying cochain complex. In our situation, it may be possible to redefine the brackets ℓ_n^θ (via an L_∞ -quasi-isomorphism) to absorb the violation of the Leibniz rule, which may then violate the standard L_∞ -relations, but may instead form a curved L_∞ -algebra. However, a curved L_∞ -algebra also necessarily contains a non-zero homogeneous subspace V_2 in degree 2, whose meaning at the purely kinematic level of gauge symmetries is not clear.

Finally, one could work with a weaker notion of P_∞ -structure, where the graded commutative product is also replaced by a sequence of higher products such that the commutativity of the product and the Leibniz rule hold only up to homotopy. It would be interesting to explore all of these modifications of the L_∞ -structure exhibited in Proposition 5.16 in order to fully elucidate the structure of the almost Poisson gauge algebroid, and its extension to noncommutative and nonassociative gauge transformations. \triangleleft

Remark 5.30. Our constructions of P_∞ -structures above differ in several ways from the treatments of [11, Section 4.3] and [44, Section 4], where an L_∞ -structure on the de Rham complex, truncated at degree 2, was constructed for generic almost Poisson structures.

Firstly, the L_∞ -structure of Proposition 5.18 differs for brackets involving forms of degree higher than one: for example, the 2-bracket of $E \in \Omega^2(U)$ and $f \in C^\infty(U)$ is simply the almost Poisson bracket in [11, 44], while in our case the 2-bracket is

$$\varrho_2^\theta(f, E) = \{f, E\}_\theta - 2\gamma^{(1)\otimes}(df, E) .$$

On the one hand the brackets of [11, 44] do not define a P_∞ -structure, while on the other hand the brackets of Proposition 5.18 are not designed to close an arbitrary gauge algebra. For a

Poisson bivector θ , the P_∞ -algebra of Proposition 5.18 contains the Poisson gauge algebra of Proposition 5.9, and following the standard L_∞ -algebra formulation of field theories [34, 35], it also contains the higher degree spaces needed to formulate the dynamics of a particular Poisson gauge theory. For an almost Poisson structure, one can use the P_∞ -algebra in this way to formulate a gauge algebra without the need of Lagrangian multipliers, at the price of obtaining a closure condition that involves the field equations in addition to the field dependent gauge transformations [34, 35]. For example, in $d = 3$ dimensions the brackets appear to define an almost Poisson Chern-Simons theory which is different from that of [11, 44]; it would be interesting to further develop this field theory, which in our symplectic embedding approach can be written down concisely and explicitly to all orders using the brackets (5.19).

Secondly, an A_∞ -structure on $\mathcal{A}[[\hbar]]$ is sketched in [11, Appendix B], where the first few multiplication maps are deduced up to order \hbar^2 . However, this structure is different from what is proposed above, as in their case the classical limit reduces the A_∞ -algebra to the *differential* graded commutative algebra (\mathcal{A}, d) , whereas here we propose that the classical limit should simply be the algebra \mathcal{A} without further structure. In other words, even the differential of the A_∞ -structure should be considered as part of the deformation quantization of the graded commutative algebra \mathcal{A} . \triangleleft

5.5 Comparison with the L_∞ -bootstrap

According to the prescription of the L_∞ -bootstrap approach to constructing almost Poisson gauge algebroids [11], one starts with the natural structure maps

$$\ell_1(f) = df \quad \text{and} \quad \ell_2(f, g) = -\{f, g\}_\theta ,$$

and attempts to construct the rest of the L_∞ -structure by consistently solving the homotopy Jacobi identities order by order. Let us briefly comment on the benefits of our approach to deformations of gauge transformations, based on symplectic embeddings, over the approach based on the L_∞ -bootstrap, which was used in [43] to propose recursion relations for the construction of the gauge L_∞ -algebras:

- In the L_∞ -bootstrap approach, one can define contributions to the gauge transformations order by order in the formal deformation parameter t . Symplectic embeddings are more appropriate for computing explicit all orders expressions, which are sometimes asymptotic expansions of analytic functions known in closed form. We will illustrate this in several examples in Section 6 below.
- Following the method proposed in [43], one can recursively construct the brackets of the form $\ell_{n+1}(f, \ell_1(g), \ell_1(h), \dots)$ from previously defined brackets at lower orders. In the case of Poisson deformations of gauge theories, there is no problem in restoring $\ell_{n+1}(f, A^{\otimes n})$ from the given brackets $\ell_{n+1}(f, \ell_1(g), \ell_1(h), \dots)$. However, in the case of almost Poisson deformations the situation is much more complicated, as the passage from the brackets $\ell_{n+1}(f, \ell_1(g), \ell_1(h), \dots)$ to $\ell_{n+1}(f, A^{\otimes n})$ is extremely non-trivial. This problem is circumvented when working with symplectic embeddings.
- From purely technical and calculational standpoints, the approach of this paper based on symplectic embeddings is significantly simpler than the bootstrap approach proposed in [43].

- The L_∞ -bootstrap approach describes a linearization of the complete A_∞ -structure of non-commutative gauge variations, which loses part of the information required to completely determine the semi-classical limit of the full noncommutative gauge transformations. The missing information is a derivation property, which requires a P_∞ -algebra to describe the semi-classical limit, and this is naturally captured by our symplectic embedding construction.

6 Examples

In previous sections we already looked at the two simplest examples of Poisson bivector fields: the case of a manifold M with the trivial Poisson structure $\theta_0 = 0$, for which the symplectic embedding is given by the associated symplectic groupoid (T^*M, ω_0) for M , and $M = \mathbb{R}^d$ with a constant Poisson structure $\theta(x) = \theta$, whose symplectic embedding is given by the strict deformation of the cotangent bundle $T^*\mathbb{R}^d$ with symplectic form $\omega = \omega_0 + t\theta^*$; the corresponding Poisson gauge transformations were described in Example 3.6 and the associated L_∞ -structure on the gauge algebroid in Example 5.10. The purpose of this final section is to extend these basic examples to some more complicated examples, and in particular to consider an example of an almost Poisson structure. These cases will generally involve symplectic embeddings of (M, θ) given by a formal deformation of the cotangent bundle of M , while the (almost) Poisson gauge algebroid is correspondingly described by an L_∞ -algebra which in general is no longer simply a differential graded Lie algebra and involves infinitely many brackets.

6.1 Linear Poisson structures

We consider first a large class of Poisson structures whereby the symplectic embedding can be described as a *strict* deformation of (T^*M, ω_0) . Let \mathfrak{g} be a Lie algebra of dimension d with structure constants in a given basis denoted by f_k^{ij} . On $M = \mathbb{R}^d$ we can define the linear Poisson bivector field [1]

$$\theta^{ij}(x) = f_k^{ij} x^k, \quad (6.1)$$

which, by regarding \mathbb{R}^d as the dual of the Lie algebra \mathfrak{g} , is the Kirillov-Kostant Poisson structure on \mathfrak{g}^* . In this case the function $\Sigma(p, p', x)$ in (2.29) is a generating function for the Dynkin series for the Baker-Campbell-Hausdorff formula for the Lie algebra $\mathfrak{g} = (\mathbb{R}^d)^*$; if G is any Lie group whose Lie algebra is \mathfrak{g} , then an integrating symplectic groupoid is $T^*G \simeq G \ltimes \mathfrak{g}^*$, regarded as the action groupoid with respect to the coadjoint action, see e.g. [16]. Using the polydifferential representation constructed in [26, 29, 49] one finds, in the notation of Section 2.2, that a local symplectic embedding of the Poisson structure (6.1) can be written as

$$\gamma_j^i(p) = \sum_{n=1}^{\infty} \frac{t^{n-1} B_n}{n!} f_{k_1}^{ij_1} f_{k_2}^{k_1 j_2} \dots f_j^{k_{n-1} j_n} p_{j_1} \dots p_{j_n}, \quad (6.2)$$

where B_n are the Bernoulli numbers ($B_n = -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \dots$ for $n = 1, 2, 3, 4, 5, \dots$).

Suppose that the structure constants of \mathfrak{g} are chosen such that the $d \times d$ matrix M , with elements $M_l^i(p) := f_k^{ij_1} f_l^{k j_2} p_{j_1} p_{j_2}$, is diagonalizable for all transverse coordinates p . Following [49], one may then construct an analytic function $\gamma_j^i(p)$ whose Taylor expansion around

$t = 0$ coincides with the asymptotic series (6.2). For this, we observe that (6.2) can be rewritten as

$$\gamma_j^i(p) = -\frac{1}{2} f_j^{il} p_l + \frac{1}{t} \mathcal{X}(-t^2 M/2)_j^i, \quad (6.3)$$

where $\mathcal{X}(M)_j^i$ is the matrix-valued function with

$$\mathcal{X}(u) = \sqrt{\frac{u}{2}} \cot \sqrt{\frac{u}{2}} - 1 = \sum_{n=1}^{\infty} \frac{(-2)^n B_{2n} u^n}{(2n)!},$$

and the power series converges for $u \in \mathbb{C}$ with $|u| < \frac{1}{2}$. Since M is diagonalizable, there exists a non-degenerate $d \times d$ matrix S such that

$$M = S D S^{-1},$$

where D is the diagonal matrix whose entries are the eigenvalues $\lambda_1(p), \dots, \lambda_d(p)$ of M on the diagonal. Thus (6.3) becomes

$$\gamma_j^i(p) = -\frac{1}{2} f_j^{il} p_l + \frac{1}{t} [S \mathcal{X}(-t^2 D/2) S^{-1}]_j^i,$$

where

$$\mathcal{X}(D) = \begin{pmatrix} \mathcal{X}(\lambda_1) & & \\ & \ddots & \\ & & \mathcal{X}(\lambda_d) \end{pmatrix}.$$

Let us now consider two particular examples.

Example 6.4. Let $\mathfrak{g} = \mathfrak{su}(2)$ with the Lie-Poisson structure

$$\theta^{ij}(x) = 2 \varepsilon^{ij}_k x^k$$

on $\mathfrak{su}(2)^* = \mathbb{R}^3$, where ε^{ijk} is the Levi-Civita symbol in three dimensions, and we used the standard Euclidean inner product on \mathbb{R}^3 to raise and lower indices: $\varepsilon^{ij}_k := \varepsilon^{ijl} \delta_{lk}$; the factor of 2 is just for convenience. In this case the generalized Bopp shift (2.30) is given by [49]

$$\pi_\theta(x, p)^i = x^i - t \varepsilon^{ij}_k p_j x^k + t^2 \chi(t^2 |p|^2) (x^i |p|^2 - p^i p_j x^j),$$

from which one calculates

$$\gamma_j^i(p) = -\varepsilon_j^{ik} p_k + t \chi(t^2 |p|^2) (|p|^2 \delta_j^i - p_j p^i),$$

where

$$\chi(u) = \frac{1}{u} (\sqrt{u} \cot \sqrt{u} - 1) \quad \text{with} \quad \chi(0) = -\frac{1}{3},$$

and $|p|^2 := \delta^{ij} p_i p_j$.

According to (3.9), the corresponding deformation of abelian gauge transformations yields the Poisson gauge transformations [43]

$$\delta_f^\theta A = df + t x \cdot (\nabla A \times \nabla f) + t A \times \nabla f + t^2 \chi(t^2 |A|^2) (|A|^2 df - (A \cdot \nabla f) A),$$

where $A \times \nabla f := *(A \wedge df)$, with $*$ the Hodge duality operator on the Euclidean vector space \mathbb{R}^3 , while $|A|^2 := \delta^{ij} A_i A_j$ and $A \cdot \nabla f := \delta^{ij} A_i \partial_j f$; we also abbreviated $\{A, f\}_\theta := x \cdot (\nabla A \times \nabla f)$

at $x \in \mathbb{R}^3$. This may be verified directly to close the Poisson gauge algebra (3.4) and to have the correct deformation property (3.3). It encodes the gauge symmetry of rotationally invariant Poisson gauge theories, which by Proposition 5.9 is generated by the action of the P_∞ -algebra with infinitely many non-vanishing brackets in coincident gauge field entries given by

$$\begin{aligned}\ell_1^\theta(f) &= df , \\ \ell_2^\theta(f, g) &= -x \cdot (\nabla f \times \nabla g) , \\ \ell_2^\theta(f, A) &= x \cdot (\nabla A \times \nabla f) + A \times \nabla f , \\ \ell_{2n+1}^\theta(f, A^{\otimes 2n}) &= B_{2n} (|A|^2)^{n-1} (|A|^2 df - (A \cdot \nabla f) A) ,\end{aligned}$$

for $n \geq 1$. ◁

Example 6.5. Let \mathfrak{g} be the d -dimensional κ -Minkowski algebra which yields the Kirillov-Kostant structure

$$\theta_a^{ij}(x) = 2 (a^i x^j - a^j x^i) ,$$

parameterized by a fixed constant vector $(a^i) \in \mathbb{R}^d$. For this Poisson structure one finds [50]

$$\gamma_j^i(p) = \left(\frac{1}{t} \sqrt{1 + t^2 \langle a, p \rangle^2} - \frac{1}{t} + \langle a, p \rangle \right) \delta_j^i - a^i p_j ,$$

where $\langle a, p \rangle := a^i p_i$ is the usual pairing between $\mathbb{R}^d = \mathfrak{g}^*$ and \mathfrak{g} . The corresponding deformation of abelian gauge transformations becomes the Poisson gauge transformations

$$\delta_f^{\theta_a} A = \left(\sqrt{1 + t^2 (\iota_a A)^2} + t \iota_a A \right) df + t \{A, f\}_{\theta_a} - t (\iota_a df) A ,$$

where ι_a denotes interior multiplication with the constant vector field $a^i \partial_i$ on \mathbb{R}^d . By expanding the square root in its Taylor series around $t = 0$ using the binomial series

$$\sqrt{1 + u^2} = 1 - \sum_{n=1}^{\infty} \frac{2}{n} \binom{2n-2}{n-1} \left(-\frac{u^2}{4} \right)^n ,$$

we can calculate the non-zero coincident gauge field brackets of the corresponding P_∞ -algebra from Proposition 5.9 to get

$$\begin{aligned}\ell_1^{\theta_a}(f) &= df , \\ \ell_2^{\theta_a}(f, g) &= -\{f, g\}_{\theta_a} , \\ \ell_2^{\theta_a}(f, A) &= \{A, f\}_{\theta_a} + \iota_a(A \wedge df) , \\ \ell_{2n+1}^{\theta_a}(f, A^{\otimes 2n}) &= -\frac{1}{2^{2n-1}} \frac{(2n-2)! (2n)!}{(n-1)! n!} (\iota_a A)^{2n} df ,\end{aligned}$$

for $n \geq 1$. ◁

6.2 Poisson families

We will now show that a rather large class of non-linear Poisson structures on $M = \mathbb{R}^2$ admit symplectic embeddings that yield Poisson gauge algebras whose homotopy algebraic structures are given by differential graded Poisson algebras, though in a different form than what we have

encountered thus far in this paper. The basic idea is to start from the outset with smooth families of bivectors $\theta_t \in \mathfrak{X}^2(\mathbb{R}^2)$, parameterized smoothly by $t \in [0, 1]$. Any such bivector defines a family of Poisson structures on \mathbb{R}^2 by dimensional reasons and can be written as

$$\theta_t^{ij}(x) = \vartheta_t(x) \varepsilon^{ij} , \quad (6.6)$$

where ε^{ij} is the Levi-Civita symbol in two dimensions. We assume that the smooth functions $\vartheta_t(x)$ satisfy two properties:

- (a) $\vartheta_t(x) \neq 0$ for all $x \in \mathbb{R}^2$ and $t \in [0, 1]$; and
- (b) $\vartheta_0(x) = 1$ for all $x \in \mathbb{R}^2$.

From this we can construct a symplectic embedding of (\mathbb{R}^2, θ_t) by finding a one-form $b = b_i(x) dx^i$ on \mathbb{R}^2 whose components solve the first order differential equation

$$1 + t \varepsilon^{ij} \partial_i b_j = \frac{1}{\vartheta_t(x)} , \quad (6.7)$$

together with the Jacobian equation

$$\det(\partial_i b_j) = 0 . \quad (6.8)$$

For given $\vartheta_t(x)$ the solution of (6.7) and (6.8) was constructed in [28], where a first order classical mechanics on the cotangent bundle of \mathbb{R}^2 was proposed whose canonical quantization gives a quantization of Poisson algebras with non-constant bivectors. The action functional of this one-dimensional topological sigma-model is given by

$$\mathcal{S}(X, P) = \int_{\mathbb{R}} \langle P, dX \rangle + \frac{t}{2} (P + X^*b) \wedge d(P + X^*b) ,$$

where $(X, P) : \mathbb{R} \rightarrow T^*\mathbb{R}^2 = \mathbb{R}^2 \times (\mathbb{R}^2)^*$ are (suitably supported) smooth maps, with $\langle \cdot, \cdot \rangle$ the pairing between $(\mathbb{R}^2)^*$ and \mathbb{R}^2 . Hamiltonian reduction of the phase space of this sigma-model by its rank 2 second class constraints defines Dirac brackets which, in our language, yield a symplectic embedding of the family (6.6) with cosymplectic structure

$$\omega_t^{-1} = \frac{t}{2} \vartheta_t(x) \varepsilon^{ij} \partial_i \wedge \partial_j + \frac{1}{2} \gamma_{tj}^i(x) (\partial_i \wedge \tilde{\partial}^j + \tilde{\partial}^j \wedge \partial_i) , \quad (6.9)$$

where

$$\gamma_{tj}^i(x) = \vartheta_t(x) (\delta_j^i - t \varepsilon^{ik} \partial_k b_j(x)) , \quad (6.10)$$

and the Jacobian condition (6.8) ensures the Lagrangian section condition. By condition (a) above this is indeed non-degenerate for all $t \in [0, 1]$, and by condition (b) it coincides with the canonical cosymplectic structure on $T^*\mathbb{R}^2$ at $t = 0$. In particular, it defines a *strict* deformation of the cotangent symplectic groupoid $(T^*\mathbb{R}^2, \omega_0)$ for \mathbb{R}^2 .

The important new feature here, compared to our previous formulation in (2.13), is that the matrix (6.10) does not depend on the transverse coordinates p . Regarding it as a $(1, 1)$ -tensor field on \mathbb{R}^2 , according to (3.8) the deformation of abelian gauge transformations by the Poisson family (6.6) is given by the Poisson gauge transformations

$$\delta_f^{\theta_t} A = \gamma_t(df, \cdot) + t \{A, f\}_{\theta_t} . \quad (6.11)$$

The L_∞ -algebra formulation of these gauge symmetries has the remarkable property given by

Proposition 6.12. *The Poisson gauge transformations (6.11) are generated by the action of the family of differential graded Poisson algebras on $C^\infty(\mathbb{R}^2) \oplus \Omega^1(\mathbb{R}^2)$ with non-vanishing brackets*

$$\ell_1^{\theta_t}(f) = \gamma_t(df, \cdot) , \quad \ell_2^{\theta_t}(f, g) = -\{f, g\}_{\theta_t} \quad \text{and} \quad \ell_2^{\theta_t}(f, A) = \{A, f\}_{\theta_t} ,$$

for $f, g \in C^\infty(\mathbb{R}^2)$ and $A \in \Omega^1(\mathbb{R}^2)$.

Proof. This is a simple consequence of the fact that the components of the Poisson bivector (6.9) do not depend on p , the Jacobi identities for the Poisson brackets, and the Lagrangian zero section condition $\{p_i, p_j\}_{\omega_t^{-1}} = 0$. In coordinates

$$\ell_1^{\theta_t}(f) = \{f, p_i\}_{\omega_t^{-1}} dx^i ,$$

and the differential condition $(\ell_1^{\theta_t})^2 = 0$ follows from the Jacobi identity for the Poisson bracket $\{\cdot, \cdot\}_{\omega_t^{-1}}$ along with $\{p_i, p_j\}_{\omega_t^{-1}} = 0$. Similarly, the derivation property

$$\ell_1^{\theta_t}\{f, g\}_{\theta_t} = \{\ell_1^{\theta_t}(f), g\}_{\theta_t} + \{f, \ell_1^{\theta_t}(g)\}_{\theta_t}$$

holds as a consequence of the Jacobi identity for the bracket $\{\cdot, \cdot\}_{\omega_t^{-1}}$. The homotopy Jacobi identity $\mathcal{J}_3 = 0$ is simply the graded Jacobi identity for the Lie bracket $\{\cdot, \cdot\}_{\theta_t}$, and the derivation properties of the brackets are clear. \square

Remark 6.13. Proposition 6.12 implies that there is no need to introduce higher brackets in the L_∞ -structure on the gauge algebra in this case, provided that one “twists” the differential, which in our previous treatments always coincided with the exterior derivative d , in such a way that the new differential is a derivation of the Poisson algebra corresponding to (6.6). This generalizes Example 5.10 for constant Poisson structures, recovered here in the case that $\vartheta_t(x) = 1$ for all $x \in \mathbb{R}^2$ and $t \in [0, 1]$, for which $b = 0$. The “twisting” here is captured automatically by the symplectic embedding approach to the deformation of gauge transformations that we developed in the present paper. \triangleleft

Example 6.14. Consider the family of rotationally symmetric Poisson structures (6.6) with

$$\vartheta_t(x) = \frac{1}{1 + t|x|^2} ,$$

where $|\cdot|$ is the standard Euclidean norm. In this case, one finds

$$b_i(x) = -\frac{1}{4}|x|^2 \varepsilon_{ij} x^j$$

as a solution to (6.7) and (6.8). The twisted differential from Proposition 6.12 then has components

$$\ell_1^{\theta_t}(f)_i = \frac{1}{1 + t|x|^2} \left(\left(1 + \frac{t}{4}|x|^2\right) \partial_i f + \frac{t}{2} \varepsilon_{ik} x^k x^l \varepsilon_l^j \partial_j f \right) ,$$

for $i = 1, 2$. \triangleleft

6.3 Magnetic Poisson structures

Let $M = T^*Q$ be the cotangent bundle of a d -dimensional manifold Q , with bundle projection $\varpi : M \rightarrow Q$. The manifold M is a symplectic manifold with canonical symplectic two-form which we denote by σ_0 . We write local coordinates on M as $(x^i) = (q^a, q_a^*)$, with $a = 1, \dots, d$ and $i = 1, \dots, 2d$, where (q^a) are local coordinates on Q and (q_a^*) are canonically conjugate coordinates in the normal directions to the zero section $Q \subset M$. We denote the corresponding derivatives by $(\partial_i) = (\partial_a, \partial_a^*)$, where $\partial_a = \partial/\partial q^a$ and $\partial_a^* = \partial/\partial q_a^*$.

Let $B \in \Omega^2(Q)$ be an arbitrary two-form on the base manifold Q . Its pullback to M deforms the symplectic structure σ_0 to an almost symplectic form

$$\sigma_B = \sigma_0 - \varpi^* B$$

which is closed if and only if B is a closed two-form on Q . The inverse $\theta_B = \sigma_B^{-1}$ is an H -twisted Poisson structure on M , with twisting three-form $H \in \Omega^3(M)$ given by

$$H = \varpi^* dB .$$

The bivector θ_B defines a *magnetic Poisson structure* on M ; the terminology comes from monopole physics where the two-form B plays the role of a magnetic field. In local coordinates, where $B = \frac{1}{2} B_{ab}(q) dq^a \wedge dq^b$ and $H = \frac{1}{3!} H_{abc}(q) dq^a \wedge dq^b \wedge dq^c$, the bivector θ_B reads

$$\theta_B = \frac{1}{2} (\partial_a \wedge \partial_a^* + \partial_a^* \wedge \partial_a) + \frac{1}{2} B_{ab}(q) \partial_a^* \wedge \partial_b^* ,$$

and the Jacobiator

$$\Pi_B = [\theta_B, \theta_B] = \frac{1}{3!} H_{abc}(q) \partial_a^* \wedge \partial_b^* \wedge \partial_c^*$$

has non-vanishing components only in the normal directions to the zero section $Q \subset M$.

Deformation quantization of such twisted Poisson manifolds was originally considered in [55] using Kontsevich's formalism. Their higher geometric quantization was developed in [13] by regarding the three-form H as the curvature of a trivial gerbe on M , and the extension to non-trivial gerbes is discussed in [14]; see [67] for a review of the different perspectives to quantization of magnetic Poisson structures. In the language of Remark 2.28, the almost cosymplectic structure σ_B^{-1} is equivalent to the canonical cosymplectic structure σ_0^{-1} by means of a B -transformation, which leads to many simplifications in its symplectic embedding construction: many components of the tensors $\gamma^{(n)}$ and $\theta^{(n)}$ of the symplectic embedding vanish as various combinations of derivatives acting on the components of θ_B and Π_B , which are functions on the base manifold Q , are identically zero; in particular $\Pi_B^{ijk} \partial_k \theta_B^{lm} = 0$, see e.g. [45].

Linear magnetic Poisson structures. We specialise now to $Q = \mathbb{R}^d$ with the *linear* magnetic Poisson structure defined by the two-form

$$B_{ab}(q) = \frac{1}{2} H_{abc} q^a ,$$

where H is a constant three-form on \mathbb{R}^d . This is the twisted Poisson analog of a constant Poisson structure: In this case the components of the Jacobiator Π_B are constant and the generalized Bopp shift (2.30) reads

$$\pi_{\theta_B}(x, p)^i = x^i - \frac{\hbar}{2} \theta_B^{ij}(x) p_j ,$$

so $\gamma^{(n)} = 0$ and $\theta^{(n)} = 0$ for all $n \geq 2$ [46]. We denote the fibre coordinates of T^*M by $(p_i) = (\xi_a, \xi_*^a)$. In the notation of Sections 2.2 and 2.4, the non-zero components of the symplectic embedding are then given by

$$\gamma_{ab}(\xi_*) = -\frac{1}{2} H_{abc} \xi_*^c, \quad \underline{\theta}_b^a = -\underline{\theta}_a^b = \delta_b^a \quad \text{and} \quad \underline{\theta}_{ab}(q, \xi_*) = H_{abc} (q^c - t \xi_*^c).$$

For the corresponding deformation of abelian gauge transformations, we note that the tensors $\Upsilon^{(n)}$ determined from Corollary 4.26 also vanish for all $n \geq 2$, so that in this case $\Lambda^{ij}(A) = -\frac{1}{2} \Pi_B^{ijk} A_k$. This depends only on the transverse components of the gauge fields $A \in \Omega^1(M)$ to the zero section $Q \subset M$, so we decompose gauge fields as

$$A = A_i(x) dx^i = \alpha_a(q, q^*) dq^a + \alpha_*^a(q, q^*) dq_a^* \quad (6.15)$$

and obtain explicitly

$$A_{ab}(\alpha_*) = -\frac{1}{2} H_{abc} \alpha_*^c$$

as the only non-zero components of $\Lambda^{ij}(A)$. The Lagrangian multipliers L_f are constructed from Proposition 4.18 as formal power series which are now given explicitly to all orders by the constant Jacobiator Π_B on M . We can write them in terms of an analytic function on the jet space $J^1 T^*M$ whose Taylor series around $t = 0$ coincides with the asymptotic series (4.19). For this, we introduce the $2d \times 2d$ matrix \mathbf{M} with elements

$$\mathbf{M}_b^a(\alpha_*, \partial_* \alpha_*) = -H_{bce} \alpha_*^c \partial_*^a \alpha_*^e$$

and observe that the non-vanishing components in (4.19) can be rewritten as

$$L_{ab}(\alpha_*, \partial_* \alpha_*) = -\frac{1}{2} H_{ace} \alpha_*^e \sum_{n=0}^{\infty} \left(\frac{t^2}{2}\right)^n (\mathbf{M}^n)_b^c = -\frac{1}{2} H_{ace} \alpha_*^e [(\mathbb{1} - \frac{t^2}{2} \mathbf{M})^{-1}]_b^c. \quad (6.16)$$

The Lagrangian multipliers

$$L_f = L_{f_a} \partial_*^a \quad \text{with} \quad L_{f_a} = t^2 L_{ab}(\alpha_*, \partial_* \alpha_*) \partial_*^b f$$

are then transverse to $Q \subset M$, depend only on the transverse components of the gauge fields and their normal derivatives, and are determined entirely by the normal derivatives of gauge parameters $f \in C^\infty(M)$.

From Remark 4.16, the corresponding deformation of abelian gauge transformations is given by the almost Poisson gauge transformations

$$\delta_f^{\theta_B} A = \delta_f^{\theta_B} \alpha_a dq^a + \delta_f^{\theta_B} \alpha_*^a dq_a^*,$$

where

$$\begin{aligned} \delta_f^{\theta_B} \alpha_a &= \partial_a f + t \{\alpha_a, f\}_{\theta_B} + \frac{t}{2} H_{abc} \alpha_*^c \partial_*^b f + t^2 H_{bce} \alpha_*^c \partial_*^e f \partial_*^b \alpha_a \\ &\quad + \frac{t^2}{2} H_{bln} \alpha_*^n \partial_*^k f [(\mathbb{1} - \frac{t^2}{2} \mathbf{M})^{-1}]_k^l (\partial_a \alpha_*^b - \partial_*^b \alpha_a + \frac{t}{2} H_{ace} \alpha_*^e \partial_*^c \alpha_*^b + t \{\alpha_a, \alpha_*^b\}_{\theta_B} \\ &\quad \quad \quad + t^2 H_{cem} \alpha_*^m \partial_*^c \alpha_*^b \partial_*^e \alpha_a), \\ \delta_f^{\theta_B} \alpha_*^a &= \partial_*^a f + t \{\alpha_*^a, f\}_{\theta_B} + t^2 H_{bce} \alpha_*^c \partial_*^e f \partial_*^b \alpha_*^a \\ &\quad + \frac{t^2}{2} H_{bln} \alpha_*^n \partial_*^k f [(\mathbb{1} - \frac{t^2}{2} \mathbf{M})^{-1}]_k^l (\partial_*^a \alpha_*^b - \partial_*^b \alpha_*^a + t \{\alpha_*^a, \alpha_*^b\}_{\theta_B} \\ &\quad \quad \quad + t^2 H_{cem} \alpha_*^m \partial_*^c \alpha_*^b \partial_*^e \alpha_*^a). \end{aligned} \quad (6.17)$$

These satisfy the almost Poisson gauge algebra (4.3) with the field dependent gauge parameter

$$\begin{aligned} \llbracket f, g \rrbracket_{\theta_B}(\alpha_*) &= \{f, g\}_{\theta_B} - t H_{abc} \alpha_*^c \partial_*^a f \partial_*^b g \\ &\quad - \frac{t^3}{4} H_{bsn} H_{crm} \alpha_*^n \alpha_*^m \partial_*^k f \partial_*^p g \left[(\mathbb{1} - \frac{t^2}{2} M)^{-1} \right]_k^s \left[(\mathbb{1} - \frac{t^2}{2} M)^{-1} \right]_p^r \\ &\quad \times \left(\partial_*^b \alpha_*^c - \partial_*^c \alpha_*^b + t \{ \alpha_*^b, \alpha_*^c \}_{\theta_B} - t^2 H_{ael} \alpha_*^l \partial_*^a \alpha_*^b \partial_*^e \alpha_*^c \right). \end{aligned} \quad (6.18)$$

Notice that, apart from the terms involving the almost Poisson brackets $\{ \cdot, \cdot \}_{\theta_B}$, the gauge variations of the transverse components α_*^a in (6.17) as well as the brackets (6.18) are determined completely by normal components to the embedding $Q \subset M$.

The L_∞ -algebra of these almost Poisson gauge symmetries is given by Proposition 5.16, with infinitely many non-vanishing brackets which in this example greatly simplify due to the vanishing of most tensors in the symplectic embedding and in the Lagrangian multipliers. They can be read off directly from (6.17) and (6.18) by reinstating the formal power series expansion (6.16). For the coincident gauge field brackets involving a single gauge parameter, we use the decomposition (6.15) to similarly write $\ell_{n+1}^{\theta_B}(f, A^{\otimes n}) \in \Omega^1(M)$ in component form

$$\ell_{n+1}^{\theta_B}(f, A^{\otimes n}) = \lambda_{n+1}^{\theta_B}(f, A^{\otimes n})_a dq^a + \lambda_{n+1}^{\theta_B}(f, \alpha_*^{\otimes n})_*^a dq_a^*,$$

where

$$\begin{aligned} \lambda_1^{\theta_B}(f)_a &= \partial_a f, \\ \lambda_2^{\theta_B}(f, A)_a &= \{ \alpha_a, f \}_{\theta_B} + \frac{1}{2} H_{abc} \alpha_*^c \partial_*^b f, \\ \lambda_3^{\theta_B}(f, A^{\otimes 2})_a &= H_{blk} \alpha_*^l \partial_*^k f (\partial_a \alpha_*^b - 3 \partial_*^b \alpha_a), \\ \lambda_4^{\theta_B}(f, A^{\otimes 3})_a &= 3 H_{blk} \alpha_*^l \partial_*^k f (\{ \alpha_a, \alpha_*^b \}_{\theta_B} + \frac{1}{2} H_{ace} \alpha_*^e \partial_*^c \alpha_*^b), \\ \lambda_{2n-1}^{\theta_B}(f, A^{\otimes 2n-2})_a &= -\frac{(2n-2)!}{2^{n-1}} H_{blm} H_{kb_1c_1} H_{a_1b_2c_2} \cdots H_{a_{n-3}b_{n-2}c_{n-2}} \alpha_*^m \alpha_*^{b_1} \cdots \alpha_*^{b_{n-2}} \partial_*^k f \\ &\quad \times \partial_*^{a_1} \alpha_*^{c_1} \cdots \partial_*^{a_{n-3}} \alpha_*^{c_{n-3}} \partial_*^l \alpha_*^{c_{n-2}} (\partial_a \alpha_*^b - 3 \partial_*^b \alpha_a), \\ \lambda_{2n}^{\theta_B}(f, A^{\otimes 2n-1})_a &= -\frac{(2n-1)!}{2^{n-1}} H_{blm} H_{kb_1c_1} H_{a_1b_2c_2} \cdots H_{a_{n-3}b_{n-2}c_{n-2}} \alpha_*^m \alpha_*^{b_1} \cdots \alpha_*^{b_{n-2}} \partial_*^k f \\ &\quad \times \partial_*^{a_1} \alpha_*^{c_1} \cdots \partial_*^{a_{n-3}} \alpha_*^{c_{n-3}} \partial_*^l \alpha_*^{c_{n-2}} (\{ \alpha_a, \alpha_*^b \}_{\theta_B} + \frac{1}{2} H_{ace} \alpha_*^e \partial_*^c \alpha_*^b), \end{aligned}$$

and

$$\begin{aligned} \lambda_1^{\theta_B}(f)_*^a &= \partial_*^a f, \\ \lambda_2^{\theta_B}(f, \alpha_*)_*^a &= \{ \alpha_*^a, f \}_{\theta_B}, \\ \lambda_3^{\theta_B}(f, \alpha_*^{\otimes 2})_*^a &= H_{blk} \alpha_*^l \partial_*^k f (\partial_*^a \alpha_*^b - 3 \partial_*^b \alpha_*^a), \\ \lambda_4^{\theta_B}(f, \alpha_*^{\otimes 3})_*^a &= 3 H_{blk} \alpha_*^l \partial_*^k f \{ \alpha_*^a, \alpha_*^b \}_{\theta_B}, \\ \lambda_{2n-1}^{\theta_B}(f, \alpha_*^{\otimes 2n-2})_*^a &= -\frac{(2n-2)!}{2^{n-1}} H_{blm} H_{kb_1c_1} H_{a_1b_2c_2} \cdots H_{a_{n-3}b_{n-2}c_{n-2}} \alpha_*^m \alpha_*^{b_1} \cdots \alpha_*^{b_{n-2}} \partial_*^k f \\ &\quad \times \partial_*^{a_1} \alpha_*^{c_1} \cdots \partial_*^{a_{n-3}} \alpha_*^{c_{n-3}} \partial_*^l \alpha_*^{c_{n-2}} (\partial_*^a \alpha_*^b - 3 \partial_*^b \alpha_*^a), \\ \lambda_{2n}^{\theta_B}(f, \alpha_*^{\otimes 2n-1})_*^a &= -\frac{(2n-1)!}{2^{n-1}} H_{blm} H_{kb_1c_1} H_{a_1b_2c_2} \cdots H_{a_{n-3}b_{n-2}c_{n-2}} \alpha_*^m \alpha_*^{b_1} \cdots \alpha_*^{b_{n-2}} \partial_*^k f \\ &\quad \times \partial_*^{a_1} \alpha_*^{c_1} \cdots \partial_*^{a_{n-3}} \alpha_*^{c_{n-3}} \partial_*^l \alpha_*^{c_{n-2}} \{ \alpha_*^a, \alpha_*^b \}_{\theta_B}, \end{aligned} \quad (6.19)$$

for $n \geq 3$. For the coincident gauge field brackets involving two gauge parameters, after some further tedious calculation and simplification we obtain for the first few brackets

$$\begin{aligned}
\ell_2^{\theta_B}(f, g) &= -\{f, g\}_{\theta_B}, \\
\ell_3^{\theta_B}(f, g, \alpha_*) &= H_{abc} \alpha_*^c \partial_*^a f \partial_*^b g, \\
\ell_4^{\theta_B}(f, g, \alpha_*^{\otimes 2}) &= 0, \\
\ell_5^{\theta_B}(f, g, \alpha_*^{\otimes 3}) &= -\frac{3}{2} H_{bke} H_{cpm} \alpha_*^e \alpha_*^m \partial_*^k f \partial_*^p g (\partial_*^b \alpha_*^c - \partial_*^c \alpha_*^b), \\
\ell_6^{\theta_B}(f, g, \alpha_*^{\otimes 4}) &= -6 H_{bke} H_{cpm} \alpha_*^e \alpha_*^m \partial_*^k f \partial_*^p g \{\alpha_*^b, \alpha_*^c\}_{\theta_B}, \\
\ell_7^{\theta_B}(f, g, \alpha_*^{\otimes 5}) &= -30 H_{bke} H_{cpm} H_{arl} \alpha_*^e \alpha_*^m \partial_*^k f \partial_*^p g \\
&\quad \times \left(\partial_*^a \alpha_*^b \partial_*^r \alpha_*^c + \frac{1}{2} \partial_*^b \alpha_*^a \partial_*^r \alpha_*^c + \frac{1}{2} \partial_*^a \alpha_*^b \partial_*^c \alpha_*^r \right), \tag{6.20}
\end{aligned}$$

together with the higher order brackets

$$\begin{aligned}
\ell_{2n}^{\theta_B}(f, g, \alpha_*^{\otimes 2n-2}) &= \frac{(2n-2)!}{2^{n-1}} H_{bse} H_{crm} \alpha_*^e \alpha_*^m \alpha_*^{b_1} \dots \alpha_*^{b_{n-3}} \partial_*^k f \partial_*^p g \{\alpha_*^b, \alpha_*^c\}_{\theta_B} \\
&\quad \times \left(H_{a_1 b_2 c_2} \dots H_{a_{n-4} b_{n-3} c_{n-3}} \partial_*^{a_1} \alpha_*^{c_1} \dots \partial_*^{a_{n-4}} \alpha_*^{c_{n-4}} \right. \\
&\quad \quad \times \left(\delta_k^s H_{pb_1 c_1} \partial_*^r \alpha_*^{c_{n-3}} + \delta_p^r H_{kb_1 c_1} \partial_*^s \alpha_*^{c_{n-3}} \right) \\
&\quad + \sum_{l=1}^{n-4} H_{kb_1 c_2} H_{a_1 b_2 c_2} \dots H_{a_{l-1} b_l c_l} \\
&\quad \quad \times H_{pb_{l+1} c_{l+1}} H_{a_{l+1} b_{l+2} c_{l+2}} \dots H_{a_{n-4} b_{n-3} c_{n-3}} \\
&\quad \quad \times \partial_*^{a_1} \alpha_*^{c_1} \dots \partial_*^{a_{l-1}} \alpha_*^{c_{l-1}} \partial_*^s \alpha_*^{c_l} \\
&\quad \quad \times \partial_*^{a_{l+1}} \alpha_*^{c_{l+1}} \dots \partial_*^{a_{n-4}} \alpha_*^{c_{n-4}} \partial_*^r \alpha_*^{c_{n-3}} \left. \right), \tag{6.21}
\end{aligned}$$

and

$$\begin{aligned}
\ell_{2n+1}^{\theta_B}(f, g, \alpha_*^{\otimes 2n-1}) &= -\frac{(2n-1)!}{2^{n-1}} H_{bse} H_{crm} \alpha_*^e \alpha_*^m \alpha_*^{b_1} \dots \alpha_*^{b_{n-2}} \partial_*^k f \partial_*^l g \\
&\quad \times \left(H_{a_1 b_2 c_2} \dots H_{a_{n-4} b_{n-3} c_{n-3}} \partial_*^{a_1} \alpha_*^{c_1} \dots \partial_*^{a_{n-4}} \alpha_*^{c_{n-4}} \right. \\
&\quad \times \left(\frac{1}{2} H_{a_{n-3} b_{n-2} c_{n-2}} \partial_*^{a_{n-3}} \alpha_*^{c_{n-3}} (\partial_*^b \alpha_*^c - \partial_*^c \alpha_*^b) \right. \\
&\quad \quad \times \left(\delta_k^s H_{pb_1 c_1} \partial_*^r \alpha_*^{c_{n-2}} + \delta_p^r H_{kb_1 c_1} \partial_*^s \alpha_*^{c_{n-2}} \right) \\
&\quad \quad + H_{alb_{n-2}} \partial_*^a \alpha_*^b \partial_*^l \alpha_*^c (\delta_k^s H_{pb_1 c_1} \partial_*^r \alpha_*^{c_{n-3}} + \delta_p^r H_{kb_1 c_1} \partial_*^s \alpha_*^{c_{n-3}}) \left. \right) \\
&\quad + \left(\frac{1}{2} H_{a_{n-3} b_{n-2} c_{n-2}} \partial_*^{a_{n-3}} \alpha_*^{c_{n-3}} \partial_*^r \alpha_*^{c_{n-2}} (\partial_*^b \alpha_*^c - \partial_*^c \alpha_*^b) \right. \\
&\quad \quad \left. + H_{alb_{n-2}} \partial_*^a \alpha_*^b \partial_*^l \alpha_*^c \partial_*^r \alpha_*^{c_{n-3}} \right) \\
&\quad \times \sum_{l=1}^{n-4} H_{kb_1 c_1} H_{a_1 b_2 c_2} \dots H_{a_{l-1} b_l c_l} \\
&\quad \quad \times H_{pb_{l+1} c_{l+1}} H_{a_{l+1} b_{l+2} c_{l+2}} \dots H_{a_{n-4} b_{n-3} c_{n-3}} \\
&\quad \quad \times \partial_*^{a_1} \alpha_*^{c_1} \dots \partial_*^{a_{l-1}} \alpha_*^{c_{l-1}} \partial_*^s \alpha_*^{c_l} \partial_*^{a_{l+1}} \alpha_*^{c_{l+1}} \dots \partial_*^{a_{n-4}} \alpha_*^{c_{n-4}} \\
&\quad + \frac{1}{2} H_{kb_1 c_1} H_{a_1 b_2 c_2} \dots H_{a_{n-4} b_{n-3} c_{n-3}} H_{pb_{n-2} c_{n-2}} \\
&\quad \quad \times \partial_*^{a_1} \alpha_*^{c_1} \dots \partial_*^{a_{n-4}} \alpha_*^{a_{n-4} c_{n-4}} \partial_*^s \alpha_*^{c_{n-3}} \partial_*^r \alpha_*^{c_{n-2}} \left. \right), \tag{6.22}
\end{aligned}$$

for $n \geq 4$.

Remark 6.23. The forms of the transverse gauge transformations in (6.17) and the brackets (6.18), as well as their corresponding L_∞ -structures (6.19) and (6.20)–(6.22), suggest a natural truncation of almost Poisson gauge transformations in this case to gauge parameters and gauge fields along the normal directions to the zero section $Q \subset M$. Let Q^* be the submanifold defined by the equations $q_a = 0$ in M . Given $f, g \in C^\infty(Q^*)$ and $\alpha_* \in \Omega^1(Q^*)$, the pullbacks $\delta_f^{\theta_B} \alpha_*|_{Q^*}$ and $\llbracket f, g \rrbracket_{\theta_B}(\alpha_*)|_{Q^*}$ eliminate only the terms involving almost Poisson brackets $\{\cdot, \cdot\}_{\theta_B}$, leaving a non-trivial dependence on the three-form H which deforms the standard abelian gauge transformations on the manifold Q^* . In [3] it was shown that this pullback operation turns the nonassociative star-product on $C^\infty(M)[[\hbar]]$, which quantizes the twisted Poisson structure θ_B , into a sequence of ‘triproducts’ on $C^\infty(Q^*)[[\hbar]]$, which quantizes the trivector $\Pi_B \in \mathfrak{X}^3(Q^*)$ that for $d = 3$ defines a Nambu-Poisson structure (of degree 3) on $Q^* = \mathbb{R}^3$. This pullback operation can thus be thought of as defining a ‘Nambu-Poisson gauge symmetry’ which ought to be related to a 3-Lie algebra (or equivalently a related Lie 2-algebra) action on $\Omega^1(Q^*)$. Indeed, the pullback of the L_∞ -structure to Q^* does not involve any 2-brackets, as $\ell_2^{\theta_B}(f, \alpha_*)|_{Q^*} = 0$ and $\ell_2^{\theta_B}(f, g)|_{Q^*} = 0$, and should be regarded as the homotopy algebra action underlying this higher Lie algebra gauge symmetry. It would be interesting to work out the details and understand the underlying structures better. \triangleleft

A Cosymplectic brackets on the constraint locus

Let (T^*M, ω) be a local symplectic embedding of an almost Poisson manifold (M, θ) , let $U \subseteq M$ be an open subset, and let $\lambda_0 = p_i dx^i$ be the Liouville one-form on T^*U . Let $f, g, h \in C^\infty(U)$ and $A \in \Omega^1(U)$. In this appendix we derive some useful identities for the cosymplectic brackets on the constraint locus $\text{im}(s_A) \subset T^*U$, where $\Phi_A = (p_i - A_i(x)) dx^i = 0$, which we use in the main text.

We will need the relation

$$\begin{aligned}
s_A^* \{ \pi^* f, \{ \pi^* g, \lambda_0 \}_{\omega^{-1}} \}_{\omega^{-1}} &- s_A^* \{ \pi^* f, \pi^* s_A^* \{ \pi^* g, \lambda_0 \}_{\omega^{-1}} \}_{\omega^{-1}} \\
&= t s_A^* \{ \pi^* f, \gamma(\pi^* dg, \cdot) \}_{\omega^{-1}} - t s_A^* \{ \pi^* f, \pi^* s_A^* \gamma(dg, \cdot) \}_{\omega^{-1}} \\
&= s_A^* (\tilde{\partial}^i \{ \pi^* g, \lambda_0 \}_{\omega^{-1}}) (s_A^* \{ \pi^* f, p_i \}_{\omega^{-1}} - s_A^* \{ \pi^* f, \pi^* A_i \}_{\omega^{-1}}) \\
&= s_A^* (\tilde{\partial}^i \{ \pi^* g, \lambda_0 \}_{\omega^{-1}}) s_A^* \{ \pi^* f, (\Phi_A)_i \}_{\omega^{-1}} . \tag{A.1}
\end{aligned}$$

We also need

$$\begin{aligned}
s_A^* \{ \pi^* f, \{ \pi^* g, \pi^* h \}_{\omega^{-1}} \}_{\omega^{-1}} &- s_A^* \{ \pi^* f, \pi^* s_A^* \{ \pi^* g, \pi^* h \}_{\omega^{-1}} \}_{\omega^{-1}} \\
&= t s_A^* \{ \pi^* f, \{ \pi^* g, \pi^* h \}_{\theta} \}_{\omega^{-1}} - t s_A^* \{ \pi^* f, \pi^* s_A^* \{ \pi^* g, \pi^* h \}_{\theta} \}_{\omega^{-1}} \\
&= s_A^* (\tilde{\partial}^i \{ \pi^* g, \pi^* h \}_{\omega^{-1}}) (s_A^* \{ \pi^* f, p_i \}_{\omega^{-1}} - s_A^* \{ \pi^* f, \pi^* A_i \}_{\omega^{-1}}) \\
&= s_A^* (\tilde{\partial}^i \{ \pi^* g, \pi^* h \}_{\omega^{-1}}) s_A^* \{ \pi^* f, (\Phi_A)_i \}_{\omega^{-1}} . \tag{A.2}
\end{aligned}$$

Now the combination of (A.1) and (A.2) implies

$$\begin{aligned}
s_A^* \{ \pi^* f, \{ \pi^* g, \Phi_A \}_{\omega^{-1}} \}_{\omega^{-1}} &- s_A^* \{ \pi^* f, \pi^* s_A^* \{ \pi^* g, \Phi_A \}_{\omega^{-1}} \}_{\omega^{-1}} \\
&= s_A^* (\tilde{\partial}^i \{ \pi^* g, \Phi_A \}_{\omega^{-1}}) s_A^* \{ \pi^* f, (\Phi_A)_i \}_{\omega^{-1}} . \tag{A.3}
\end{aligned}$$

Next we calculate

$$\begin{aligned}
& s_A^* \{ \{ \pi^* g, \pi^* f \}_{\omega-1}, \Phi_A \}_{\omega-1} - s_A^* \{ \pi^* s_A^* \{ \pi^* g, \pi^* f \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
&= s_A^* (\tilde{\partial}^i \{ \pi^* g, \pi^* f \}_{\omega-1}) (- s_A^* \{ \pi^* A_i, \lambda_0 \}_{\omega-1} - s_A^* \{ p_i, \pi^* A \}_{\omega-1} + s_A^* \{ \pi^* A_i, \pi^* A \}_{\omega-1}) \\
&= s_A^* (\tilde{\partial}^i \{ \pi^* g, \pi^* f \}_{\omega-1}) s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega-1} .
\end{aligned} \tag{A.4}$$

One may also check

$$\begin{aligned}
& s_A^* \{ \{ \pi^* f, (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} - s_A^* \{ \pi^* s_A^* \{ \pi^* f, (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
&= s_A^* (\tilde{\partial}^i \{ \pi^* f, (\Phi_A)_j \}_{\omega-1}) s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega-1} ,
\end{aligned} \tag{A.5}$$

and

$$\begin{aligned}
& s_A^* \{ \{ (\Phi_A)_k, (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} - s_A^* \{ \pi^* s_A^* \{ (\Phi_A)_k, (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
&= s_A^* (\tilde{\partial}^i \{ (\Phi_A)_k, (\Phi_A)_j \}_{\omega-1}) s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega-1} .
\end{aligned} \tag{A.6}$$

B Closure of almost Poisson gauge transformations

In this appendix we prove that the gauge transformations defined in Proposition 4.6 close the almost Poisson gauge algebra (4.11). For this, we use (4.12) to calculate the left-hand side of (4.11), and after rearranging the terms we find

$$\begin{aligned}
& \delta_f^\theta (s_A^* \{ \pi^* g, \Phi_A \}_{\omega-1} + L_g^j s_A^* \{ (\Phi_A)_j, \Phi_A \}_{\omega-1}) - \delta_g^\theta (s_A^* \{ \pi^* f, \Phi_A \}_{\omega-1} + L_f^i s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega-1}) \\
&= -s_A^* \{ \pi^* g + L_g^j (\Phi_A)_j, \pi^* s_A^* \{ \pi^* f, \Phi_A \}_{\omega-1} + L_f^i \pi^* s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega-1} \}_{\omega-1} \\
&\quad - s_A^* (\tilde{\partial}^k \{ \pi^* f, \Phi_A \}_{\omega-1}) (s_A^* \{ \pi^* g, (\Phi_A)_k \}_{\omega-1} + L_g^j s_A^* \{ (\Phi_A)_j, (\Phi_A)_k \}_{\omega-1}) \\
&\quad - L_f^i s_A^* (\tilde{\partial}^k \{ (\Phi_A)_i, \Phi_A \}_{\omega-1}) (s_A^* \{ \pi^* g, (\Phi_A)_k \}_{\omega-1} + L_g^j s_A^* \{ (\Phi_A)_j, (\Phi_A)_k \}_{\omega-1}) \\
&\quad + s_A^* \{ \pi^* f + L_f^i (\Phi_A)_i, \pi^* s_A^* \{ \pi^* g, \Phi_A \}_{\omega-1} + L_g^j \pi^* s_A^* \{ (\Phi_A)_j, \Phi_A \}_{\omega-1} \}_{\omega-1} \\
&\quad + s_A^* (\tilde{\partial}^k \{ \pi^* g, \Phi_A \}_{\omega-1}) (s_A^* \{ \pi^* f, (\Phi_A)_k \}_{\omega-1} + L_f^i s_A^* \{ (\Phi_A)_i, (\Phi_A)_k \}_{\omega-1}) \\
&\quad + L_g^j s_A^* (\tilde{\partial}^k \{ (\Phi_A)_j, \Phi_A \}_{\omega-1}) (s_A^* \{ \pi^* f, (\Phi_A)_k \}_{\omega-1} + L_f^i s_A^* \{ (\Phi_A)_i, (\Phi_A)_k \}_{\omega-1}) \\
&\quad - L_g^j s_A^* \{ \pi^* s_A^* \{ \pi^* f, (\Phi_A)_j \}_{\omega-1} + L_f^i \pi^* s_A^* \{ (\Phi_A)_i, (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
&\quad + L_f^i s_A^* \{ \pi^* s_A^* \{ \pi^* g, (\Phi_A)_i \}_{\omega-1} + L_g^j \pi^* s_A^* \{ (\Phi_A)_j, (\Phi_A)_i \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
&\quad + (\delta_f^\theta L_g^i - \delta_g^\theta L_f^i) s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega-1} .
\end{aligned} \tag{B.1}$$

By explicit calculation and using the formulas (A.3), (A.5) and (A.6) from Appendix A one may check

$$\begin{aligned}
& s_A^* \{ \pi^* f + L_f^i (\Phi_A)_i, \{ \pi^* g + L_g^j (\Phi_A)_j, \Phi_A \}_{\omega-1} \}_{\omega-1} \\
&\quad - s_A^* \{ \pi^* f + L_f^i (\Phi_A)_i, \pi^* s_A^* \{ \pi^* g + L_g^j (\Phi_A)_j, \Phi_A \}_{\omega-1} \}_{\omega-1} \\
&= (s_A^* (\tilde{\partial}^k \{ \pi^* g, \Phi_A \}_{\omega-1}) + L_g^j s_A^* (\tilde{\partial}^k \{ (\Phi_A)_j, \Phi_A \}_{\omega-1} + s_A^* \{ L_g^k, \Phi_A \}_{\omega-1}) \\
&\quad \times (s_A^* \{ \pi^* f, (\Phi_A)_k \}_{\omega-1} + L_f^i s_A^* \{ (\Phi_A)_i, (\Phi_A)_k \}_{\omega-1}) .
\end{aligned}$$

Using this expression we rewrite (B.1) as

$$\begin{aligned}
& s_A^* \{ \pi^* f + L_f^i (\Phi_A)_i, \{ \pi^* g + L_g^j (\Phi_A)_j, \Phi_A \}_{\omega-1} \}_{\omega-1} \\
& \quad - s_A^* \{ \pi^* g + L_g^i (\Phi_A)_i, \{ \pi^* f + L_f^j (\Phi_A)_j, \Phi_A \}_{\omega-1} \}_{\omega-1} \\
& + s_A^* \{ L_f^i, \Phi_A \}_{\omega-1} s_A^* \{ \pi^* g, (\Phi_A)_i \}_{\omega-1} - s_A^* \{ L_g^i, \Phi_A \}_{\omega-1} s_A^* \{ \pi^* f, (\Phi_A)_i \}_{\omega-1} \\
& + L_f^i s_A^* \{ \pi^* s_A^* \{ \pi^* g, (\Phi_A)_i \}_{\omega-1}, \Phi_A \}_{\omega-1} - L_g^i s_A^* \{ \pi^* s_A^* \{ \pi^* f, (\Phi_A)_i \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
& + 2 (L_f^i s_A^* \{ L_g^j, \Phi_A \}_{\omega-1} - L_g^i s_A^* \{ L_f^j, \Phi_A \}_{\omega-1}) s_A^* \{ (\Phi_A)_j, (\Phi_A)_i \}_{\omega-1} \\
& + 2 L_f^i L_g^j s_A^* \{ \pi^* s_A^* \{ (\Phi_A)_j, (\Phi_A)_i \}_{\omega-1}, \Phi_A \}_{\omega-1} + (\delta_f^\theta L_g^i - \delta_g^\theta L_f^i) s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega-1} .
\end{aligned}$$

Using the Jacobi identity in the first two lines and simplifying the remaining lines, we rewrite this expression as

$$\begin{aligned}
& s_A^* \{ \{ \pi^* f + L_f^i (\Phi_A)_i, \pi^* g + L_g^j (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} + (\delta_f^\theta L_g^i - \delta_g^\theta L_f^i) s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega-1} \\
& \quad - s_A^* \{ L_g^j \pi^* s_A^* \{ \pi^* f, (\Phi_A)_j \}_{\omega-1} + L_f^i \pi^* s_A^* \{ (\Phi_A)_i, \pi^* g \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
& \quad + 2 s_A^* \{ L_f^i L_g^j \pi^* s_A^* \{ (\Phi_A)_i, (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} .
\end{aligned}$$

Using $\Phi_A = 0$ on the constraint locus $\text{im}(s_A)$, we rewrite the first line so that this expression becomes

$$\begin{aligned}
& s_A^* \{ \{ \pi^* f, \pi^* g \}_{\omega-1} + L_g^j \{ \pi^* f, (\Phi_A)_j \}_{\omega-1} + L_f^i \{ (\Phi_A)_i, \pi^* g \}_{\omega-1} + L_f^i L_g^j \{ (\Phi_A)_i, (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
& + (s_A^* \{ L_f^k, \pi^* g \}_{\omega-1} + s_A^* \{ \pi^* f, L_g^k \}_{\omega-1} + L_f^i s_A^* \{ (\Phi_A)_i, L_g^k \}_{\omega-1} \\
& \quad + L_g^j s_A^* \{ L_f^k, (\Phi_A)_j \}_{\omega-1}) s_A^* \{ (\Phi_A)_k, \Phi_A \}_{\omega-1} + (\delta_f^\theta L_g^i - \delta_g^\theta L_f^i) s_A^* \{ (\Phi_A)_i, \Phi_A \}_{\omega-1} \\
& \quad - s_A^* \{ L_g^j \pi^* s_A^* \{ \pi^* f, (\Phi_A)_j \}_{\omega-1} + L_f^i \pi^* s_A^* \{ (\Phi_A)_i, \pi^* g \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
& \quad + 2 s_A^* \{ L_f^i L_g^j \pi^* s_A^* \{ (\Phi_A)_i, (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} .
\end{aligned}$$

Applying again the formulas (A.4)–(A.6) from Appendix A in the first line of this expression, after simplification we end up with

$$\begin{aligned}
& s_A^* \{ \pi^* s_A^* \{ \pi^* f, \pi^* g \}_{\omega-1} - L_f^i L_g^j \pi^* s_A^* \{ (\Phi_A)_i, (\Phi_A)_j \}_{\omega-1}, \Phi_A \}_{\omega-1} \\
& + \left(\delta_f^\theta L_g^k - \delta_g^\theta L_f^k + s_A^* (\tilde{\partial}^k \{ \pi^* f, \pi^* g \}_{\omega-1}) + L_g^j s_A^* (\tilde{\partial}^k \{ \pi^* f, (\Phi_A)_j \}_{\omega-1}) \right. \\
& \quad + L_f^i s_A^* (\tilde{\partial}^k \{ (\Phi_A)_i, \pi^* g \}_{\omega-1}) + L_f^i L_g^j s_A^* (\tilde{\partial}^k \{ (\Phi_A)_i, (\Phi_A)_j \}_{\omega-1}) + s_A^* \{ L_f^k, \pi^* g \}_{\omega-1} \\
& \quad \left. + s_A^* \{ \pi^* f, L_g^k \}_{\omega-1} + L_f^i s_A^* \{ (\Phi_A)_i, L_g^k \}_{\omega-1} + L_g^j s_A^* \{ L_f^k, (\Phi_A)_j \}_{\omega-1} \right) s_A^* \{ (\Phi_A)_k, \Phi_A \}_{\omega-1} .
\end{aligned}$$

By (4.13) and (4.8) this is equal to $s_A^* \{ t \llbracket f, g \rrbracket_\theta(A) + L_{t \llbracket f, g \rrbracket_\theta(A)}^k (\Phi_A)_k, \Phi_A \}_{\omega-1}$, which proves the closure formula (4.11).

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