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Citation for published version:
Liu, X, Zang, M, Wang, Y, Dai, J & Wang, W 2021, Noise limits of polar-interferogram estimation at low light levels. in Q He, D-S Kim & C-F Li (eds), *Quantum and Nonlinear Optics VIII.*, 119050S, Proceedings of SPIE, vol. 11905, SPIE, SPIE/COS Photonics Asia 2021, Nantong, China, 10/10/21. https://doi.org/10.1117/12.2599838

Digital Object Identifier (DOI):
10.1117/12.2599838

Link:
Link to publication record in Heriot-Watt Research Portal

Document Version:
Publisher's PDF, also known as Version of record

Published In:
Quantum and Nonlinear Optics VIII

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Xin Liu, Mengmeng Zang, Ying Wang, Jun Dai, Wei Wang, "Noise limits of polar-interferogram estimation at low light levels," Proc. SPIE 11905, Quantum and Nonlinear Optics VIII, 119050S (9 October 2021); doi: 10.1117/12.2599838

Event: SPIE/COS Photonics Asia, 2021, Nantong, Jiangsu, China

Noise limits of polar-interferogram estimation at low light levels

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ABSTRACT

In many applications using light for practical purposes, the accuracy of optical experiment is limited most fundamentally by the finite amount of light utilized in the measurements. Special emphasis should be put on interferometry aimed at measuring the parameters of simple fringe patterns due to the following reasons. First, fringe parameter measurement provides a relatively well defined and tractable example of application of the theory. The desired parameters are easily defined, and methods for their measurement are readily devised based on common sense. Second, fringe parameter measurement is central to all problems involving coherence. The fundamental descriptors of light waves utilized in coherence theory are in fact measurable parameters of fringes. By examining the limitations to fringe parameter measurement, we are actually examining the limitations to the measurability of coherence itself.

In this paper, fundamental limitations of estimating the amplitudes and phases of polar-interferograms recorded at low light levels are investigated. By modeling the receiver as a spatial array of photon-counting detectors, results are obtained that permit specification of the minimum number of photoevents required for estimation of fringe parameters to a given accuracy. Both a discrete Fourier-transform estimator and an optimum joint maximum-likelihood estimator are considered to specify the limiting performance of all unbiased estimators in terms of the collected light flux.

Keywords: Polar-interferogram, amplitude and phase estimation, polar-coherence measurement

I. MATHEMATICAL MODEL OF FRINGE PARAMETER ESTIMATION

As shown in Fig. 1, the receiver is modeled as a one-dimensional array of N contiguous, identical photo-emissive detectors. We can neglect the smoothing effect caused by finite detector size, when assuming all detectors have a transverse dimension that is much smaller than the fringe period[1].

We assume two-beam interference with quasi-monochromatic, plane-polarized beams of thermal light. Light in which two beams interfere all can be polarized light in the X direction or Y direction. According to these conditions, the irradiance function of polarization interference fringe has the following form:

$$I_{pq}(\xi) = I_{pq}\left[1+V_{pq}\cos(2\pi f_0 \xi + \phi_{pq})\right] + I_{pq}, \quad 0 \le \xi \le L \tag{1}$$

In Eqs(1) above, p and q all can be x or y. And x and y represent polarized light in the X direction and the Y direction, respectively.
The first I_{pq} represents constant signal-dependent, and last I_{pq} represents background irradiances. V_{pq} means the signal-dependent visibility. f_0 means the spatial frequency of the polarization interference fringe, and ϕ_{pq} means its spatial

Quantum and Nonlinear Optics VIII, edited by Qiongyi He, Dai-Sik Kim,
Chuan-Feng Li, Proc. of SPIE Vol. 11905, 119050S · © 2021 SPIE
CCC code: 0277-786X/21/$21 · doi: 10.1117/12.2599838

phase. The classical visibility, *V*, of the polarization interference fringe is simply $V = V_{pq} I_{pq} / (I_{pq} + I_{pq})$, where, I_{pq} has the same meaning as above.

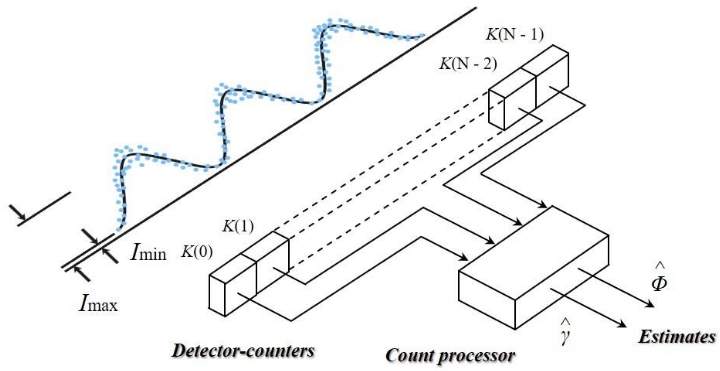

Fig. 1. Detection and estimation system assumed for the amplitude interferometer. γ and ϕ are estimates of the fringe visibility and phase.

As shown in fig.1, we also make two assumptions:

1. Each detector-counter is a counting circuit. In a T-second measurement interval, they need to count the number of photoevents registered by that detector.

2. All detector-counters have identical quantum counting efficiencies $0 \le \eta \le 1$.

In Fig.1, the count processor will process the outputs of *N* detector-counters which can be defined as an *N*-dimensional count vector n. Finally, processing through count processor, the unknown polarization interference fringe parameters are yield.

$\bar{n}_{pq}(j)$ is the mean number of photoevents which is registered by the *j*th detector-counter. Due to the degeneracy parameter of the light is low, so $\bar{n}_{pq}(j)$ need to satisfy

$$\bar{n}_{pq}(j) \ll T\Delta v,. \tag{2}$$

Because the count fluctuations are predominantly Poisson shot noise, the *j*th detector-counter registers a Poisson random variable, with mean $\bar{n}_{pq}(j)$ and probability-mass function

$$P_{n(j)}(k) = \begin{cases} \dfrac{\left[\bar{n}(j)\right]^k}{k!} \exp[-\bar{n}(j)], & k = 0,1,... \\ 0, & otherwise \end{cases} \tag{3}$$

The count mean $\bar{n}_{pq}(j)$ can be expressed as Equation (4).

$$\bar{n}_{pq}(j) = \frac{\eta A T I_{pq}(j)}{h\bar{v}}, \tag{4}$$

h is Planck's constant. η is the count-detector quantum efficiency. *A* is the area of one detector element. $\bar{v}$ is the mean optical frequency, and $I_{pq}(j)$ is the classical irradiance incident on the *j*th detector.

m_0 means several periods of the polarization interference fringe, and it is an integer number. The array of length *L* is equal to m_0 / f_0 or m_0 / T_0, T_0 means periods of the polarization interference fringe.

$\triangle\xi$ means the spacing of the contiguous detectors, and $\triangle\xi = L / N$ .Considering $f_0 = m_0 / L$ and $\triangle\xi = L / N$, according to Eqs. (1) and (4) we obtain equivalent expressions of $\bar{n}_{pq}(j)$, as shown in Equation(5).

$$\begin{aligned} \bar{n}_{pq}(j) &= x_s\left[1 + V_{pq}\cos\left(\frac{2\pi j m_0}{N} + \phi_{pq}\right)\right] + x_b \\ &= x_t\left[1 + V_{pq}\cos\left(\frac{2\pi j m_0}{N} + \phi_{pq}\right)\right], \qquad j = 0,1,\ldots,N-1 \end{aligned} \tag{5}$$

The parameters x_s and x_b are the spatially averaged mean signal and background counts per detector, obtained by substituting the first I_{pq} and last I_{pq} into Equation (4), and the parameter x_t is the sum of x_s and x_b.

N_s, as shown in Equation(6), is mean number of photoevents registered by the entire detector array. It is a important parameter.

$$N_s = N x_s = \left[\sum_{j=0}^{N-1} \bar{n}_{pq}(j)\right]_{x_b=0} \tag{6}$$

In order to make the sampling rate of fringe much higher than its Nyquist rate, we make a assumptions that $m_0 << N/2$.

Ⅱ THE DISCRETE FOURIER TRANSFORM AS A PARAMETER ESTIMATOR OF POLARIZATION INTERFERENCE FRINGE

The Discrete Fourier Transform is an uncomplicated polarization interference fringe-parameter estimator. For an incident polarization interference fringe, it contains m_0 periods across the detector array. At index m_0, the phase and amplitude of the DFT can be used to yield the estimates of peak amplitude and the phase of this incident polarization interference fringe. The sequence {$n(j)$: j=0, 1..., N-1} is the *N* complex DFT components. It can be defined by equation (7).

$$X_{pq}(m) = \frac{1}{N}\sum_{j=0}^{N-1} n_{pq}(j)\exp[-i2\pi jm/N] \qquad m = 0,1,\ldots,N-1 \tag{7}$$

The real part of $X(m)$ can be expressed as equation (8). The imaginary parts can be express as equation (9).

$$R_{pq}(m)=\frac{1}{N}\sum_{j=0}^{N-1}n_{pq}(j)\cos\left(\frac{2\pi jm}{N}\right) \tag{8}$$

$$I_{pq}(m)=-\frac{1}{N}\sum_{j=0}^{N-1}n_{pq}(j)\sin\left(\frac{2\pi jm}{N}\right) \tag{9}$$

Here $\Delta\phi$ is restricted to the primary interval $\left(-\pi,\pi\right)$

In equation (8) and (9), m means a frequency index and it is restricted to [0, $N-1$]. The actual spatial frequency of the m_{th} component is $f_m = m/L$. $X(m)$ can be expressed in the polar form, as shown in equation (10).

$$X_{pq}(m)=r_{pq}(m)\exp[i\theta_{pq}(m)] \tag{10}$$

$r_{pq}(m)$ and $\theta_{pq}(m)$ can be expressed as equation (11) and equation (12), respectively.

$$r_{pq}(m)=\left|X_{pq}(m)\right|=\left[R_{pq}^2(m)+I_{pq}^2(m)\right]^{\frac{1}{2}} \tag{11}$$

$$\theta_{pq}(m)\equiv\arg X_{pq}(m)=\tan^{-1}\left[\frac{I_{pq}(m)}{R_{pq}(m)}\right] \tag{12}$$

Provided that get the statistical distribution of $r(m)$ and $\theta(m)$, we can evaluate the performance of the DFT as a polarization interference fringe -parameter estimator. For m ≠ 0, when we expand the characteristic function about R_{pq}(m) and I_{pq}(m), we can find that the DFT quadrature components are well approximated by gaussian random variables provided that $(N_{x_t})^{1/2} \gg 1$.[2]

The previous article assumes that the array contains an integer number of polarization interference fringe periods. According to this assumption, we use appropriate trigonometric identities can prove that[4]

$$\overline{R}_{pq}(m_0)=\frac{1}{N}\sum_{j=0}^{N-1}\overline{n}_{pq}(j)\cos(\frac{2\pi jm_0}{N})=\frac{x_sV_s}{2}\cos\phi_{pq} \tag{13}$$

$$\overline{I}_{pq}(m_0)=\frac{1}{N}\sum_{j=0}^{N-1}\overline{n}_{pq}(j)\sin(\frac{2\pi jm_0}{N})=\frac{x_sV_s}{2}\sin\phi_{pq} \tag{14}$$

A. DFT Phase Estimation

$\phi\left(\mathbf{n};\mathrm{DFT}\right)$ means the measured phase $\theta_{pq}(m_0)$ of the DFT component at index m_0. Due to noise phasor $\rho_{pq}(m_0)$, there is an error between the measured phase value, $\theta_{pq}(m_0)$, and the true phase of the polarization interference fringe.

The phase error is expressed as $\Delta\phi_{pq} = \theta_{pq}(m_0) - \phi_{pq}$. According to communication theory, the phase error probability-density function is defined by equation (15).

$$P_{\Delta\phi}(\Delta\phi_{pq}) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}\gamma^2(m_0)\cos\Delta\phi_{pq}\right] \times\Theta\left[\frac{1}{\sqrt{2}}\gamma(m_0)\cos\Delta\phi_{pq}\right]\exp\left[-\frac{1}{2}\gamma(m_0)\sin^2\Delta\phi_{pq}\right] \tag{15}$$

$\Delta\phi_{pq}$ is restricted to the primary interval $\left(-\pi,\pi\right)$.

Θ means an error integral of the form, as shown in equation (16). $\gamma(m_0)$ is a parameter, which means an rms signal-to-noise ratio. $\gamma(m_0)$ is given by equation (17).

$$\Theta\left(b\right) = \left(2\pi\right)^{-\frac{1}{2}}\int_{-\infty}^{b}\exp\left(-\frac{y^2}{2}\right)dy \tag{16}$$

$$\gamma\left(m_0\right) \equiv \left[\frac{\overline{R}^2\left(m_0\right)+\overline{I}^2\left(m_0\right)}{\sigma^2\left(m_0\right)}\right] = V_s\left[N_s/2\right]^{\frac{1}{2}}\left[1+\frac{x_b}{x_s}\right]^{-\frac{1}{2}} \tag{17}$$

For $\gamma(m_0) = 0$, $P_{\Delta\phi}(\Delta\phi_{pq})$ is uniformly distributed on $\left(-\pi,\pi\right)$,but when $\gamma(m_0)$ tends to infinity, $P_{\Delta\phi}(\Delta\phi_{pq})$ will become a Dirac function at zero error.[17] Due to the mean of $\Delta\phi_{pq}$ is zero for every value of $\gamma(m_0)$, the phase estimate $\theta_{pq}(m_0)$ is unbiased.

$\left\langle\Delta\phi_{pq}^2\right\rangle$ means the mean-square phase error (in radians squared) and it is defined by equation (18).

$$\left\langle\Delta\phi_{pq}^2\right\rangle \cong \frac{1}{\gamma^2\left(m_0\right)} = \frac{2\left(1+x_b/x_s\right)}{V_s^2 N_s} \tag{18}$$

N_s represents the minimum value of registered signal photoevents on the array, defined by equation (19). When N_s reaches a certain value, we can achieve an rms error of at most $\left\langle\Delta\phi_{pq}^2\right\rangle^{\frac{1}{2}}$ radians.

$$N_s = \frac{2\left(1+x_b/x_s\right)}{V_s^2\left\langle\Delta\phi_{pq}^2\right\rangle} \tag{19}$$

B. DFT Fringe-Amplitude Estimation

a_{pq} is the peak of the polarization interference fringe, and $a_{pq} = x_s V_s$. In this section, we discuss and calculate the errors in estimating polarization interference fringe amplitude using a DFT estimator.

$a_{pq}(\boldsymbol{n};\mathrm{DFT})$ is the DFT estimate of polarization interference fringe amplitude. It is expressed as equation (20).

$$a_{pq}(\boldsymbol{n};\mathrm{DFT}) \equiv 2r_{pq}(m_0) = 2\left[R_{pq}^2(m_0) + I_{pq}^2(m_0)\right]^{\frac{1}{2}} \tag{20}$$

In equation (20), the amplitude of the polarization interference fringe is complex exponential components, and the DFT determines it. Therefore, the factor of 2 arises in the above equation.

a_{pq} is proportional to the length of a constant phasor plus a Rayleigh-noise phasor. Consequently, a_{pq} is Rician distribution, and its probability density function is defined by equation (21).[18]

$$p(a_{pq}) = \frac{N a_{pq}}{x_s + x_b} \exp\left\{-N \frac{\left[(a_{pq})^2 + (x_s V_s)^2\right]}{4(x_s + x_b)}\right\} \times I_0\left[\frac{V_s N_s a_{pq}}{2(x_s + x_b)}\right] \tag{21}$$

In equation (21), when $\hat{a}_s \geq 0$, I_0 will be a modified Bessel function of the first kind, zero-order. When $\gamma(m_0) \gg 2$, $\hat{a}_s$ is an approximately unbiased estimate and as $\gamma(m_0)$ continues to approach infinity, $\hat{a}_s$ is an asymptotically impartial estimate. Thus, as long as the $\gamma(m_0)$ is large enough, the $\hat{a}_s$'s mean-square error can be express as equation (22).

$$\left\langle \left[a_{pq}(\mathbf{n};\mathbf{DFT}) - a_{pq}\right]^2 \right\rangle \cong \frac{2(x_s + x_b)}{N} \tag{22}$$

R is the rms signal-to-noise ratio of the DFT amplitude estimate. Provide values of $\gamma(m_0)$ is large, R can be expressed as equation (23)

$$R = \left\{\frac{\left\langle a_{pq}(\mathbf{n};\mathbf{DFT})\right\rangle^2}{\left\langle \left|a_{pq}(\mathbf{n};\mathbf{DFT}) - a_{pq}\right|^2 \right\rangle}\right\}^{\frac{1}{2}} \cong \left\{\frac{N(x_s V_s)^2}{2(x_s + x_b)}\right\}^{\frac{1}{2}} = \gamma(m_0) \tag{23}$$

$\left(\mathrm{p} = R^{-1} \times 100\right)\%$ is an rms accuracy. According to Eqs. (17) and (23), N_s represents the minimum value of registered signal photoevents on the array, defined by equation (24). When N_s reaches a certain value, we can achieve an rms accuracy of true value of amplitude.

$$N_s = \frac{2R^2\left(1 + x_b / x_s\right)}{V_s^{\ 2}} \tag{24}$$

III. JOINT-MAXIMUM-LIKELIHOOD ESTIMATION

This section will discuss the joint-maximum-likelihood(JML) estimation in estimating polarization interference fringe amplitude and comparison with the DFT estimate.

The count vector **n** is independent Poisson random variables. Given the observed count vector **n**, the JML estimators can operate on this vector to solve the polarization interference fringe-parameter values. [2.3]

We assume that α is polarization interference fringe parameters that contain phase and amplitude. The N elements of the vector **n** are conditional independence. Under these conditions, the likelihood function can be expressed as equation (25).

$$P\left(\mathbf{n} \mid \alpha\right) = \prod_{j=0}^{N-1} P\left(n_j \mid \alpha\right) = \prod_{j=0}^{N-1} \frac{\left[\bar{n}_j\left(\alpha\right)\right]^{n_j}}{n_j\,!} \exp\left[-\bar{n}_j\left(\alpha\right)\right] \tag{25}$$

Then, we maximize the log likelihood function, as shown in equation (26)

$$\Lambda\left(\mathbf{n} \mid \alpha\right) = \ln P\left(\boldsymbol{n} \mid \boldsymbol{\alpha}\right) = \sum_{j=0}^{N-1}\left[-\ln n_j\,! - \bar{n}_j\left(\boldsymbol{\alpha}\right) + n_j \ln \bar{n}_j\left(\alpha\right)\right] \tag{26}$$

In Equation (3-2), the first term is not related to α. Therefore, we can simplify Equation (26) as Equation (27).

$$\Lambda_0\left(\mathbf{n} \mid \boldsymbol{\alpha}\right) = \sum_{j=0}^{N-1}\left[-\bar{n}_j\left(\boldsymbol{\alpha}\right) + n_j \ln \bar{n}_j\left(\alpha\right)\right] \tag{27}$$

where, based on Equation (5), we write $\bar{n}_{pqj}\left(\boldsymbol{\alpha}\right)$ in the form

$$\bar{n}_{pqj}\left(\boldsymbol{\alpha}\right) = x_t\left[1 + V_{pq}\cos\left(\frac{2\pi\, jm_0}{N} + \phi_{pq}\right)\right], \ j = 0,1,\ldots,N-1 \tag{28}$$

with $x_t=x_s+x_b$, $V=x_sV_{pq}/(x_s+x_b)$, and $x_tV=x_sV_{pq}=\mathrm{a}_{\mathrm{pq}}$

We maximize Equation (27), and make it can comparisons with the DFT estimators. Let the parameter vector $\alpha = \left(\phi, xt, a_s\right)$. Then, we obtain Equation (29).

$$\frac{\partial\Lambda_0\left(\mathbf{n} \mid \boldsymbol{\alpha}\right)}{\partial\alpha_i} = 0, \quad i = 1,2,3, \tag{29}$$

where $\alpha_1=\varphi$, $\alpha_1=x_t$, and $\alpha_3 = a_s$. We let V greater than 0 and far less than 1. Then, to uncouple Equation (29), and we can use a small-argument logarithmic approximation ln(1+x). The result is Equations (30), (31), and (32).

$$\phi_{pq}\left(\mathbf{n};\mathrm{JML}\right)=\tan^{-1}\left\{-\sum_{j=0}^{N-1}n_{pqj}\sin\left(\frac{2\pi\, jm_0}{N}\right)\Big/\sum_{j=0}^{N-1}n_{pqj}\cos\left(\frac{2\pi\, jm_0}{N}\right)\right\}, \tag{30}$$

$$x_t\left(\mathbf{n};\mathrm{JML}\right)=\frac{1}{N}\sum_{j=0}^{N-1}n_{pqj}, \tag{31}$$

$$a_s\left(\mathbf{n};\mathrm{JML}\right)=x_t\left(\mathbf{n};\mathrm{JML}\right)\times\left\{\sum_{j=0}^{N-1}n_{pqj}\cos\left[\frac{2\pi\, jm_0}{N}+\phi\left(\mathbf{n};\mathrm{JML}\right)\right]\Big/\sum_{j=0}^{N-1}n_{pqj}\cos^2\left[\frac{2\pi\, jm_0}{N}+\phi\left(\mathbf{n};\mathrm{JML}\right)\right]\right\} \tag{32}$$

Comparing Eqs. (18) with (30), we can conclude that the DFT phase estimator has the same effect as the low-visibility JML phase estimate. Moreover, by substituting Eqs. (30) and (31) into (32), then let that result compare with Equation (33), we can obtain Equations (33).

$$a_{pq}\left(\mathbf{n};\mathrm{JML}\right)=\frac{a_s\left(\mathbf{n};\mathrm{DFT}\right)}{1+\left[C\left(\mathbf{n};\phi\right)\Big/x_t\left(\mathbf{n};\mathrm{JML}\right)\right]}, \tag{33}$$

According to the previous description, provided that signal-dependent visibility V_s=0.1, appreciable background x_b = 0 and the maximum allowable rms phase error is π/18 radians (10°), then, according to equation (19), when we use the DFT estimate, the value of N_s is 6400.

Provided that γ(m0) is not a small value, V_s=0.1 and x_b = 0. Under such conditions, if we want to achieve 10% accuracy in measuring the amplitude of a polarization interference fringe, the N_s need to be greater than or equal to 20000.

Nx_t is the total number of recorded photoevents. In Equation (33), assuming that Nx_t is much larger than unity, we can prove that the value of the bracketed term will be much less than unity. Therefore, when Nx_t is much greater than 1, we find that the DFT and JML estimator of polarization interference fringe amplitude discussed in this paper are the same.

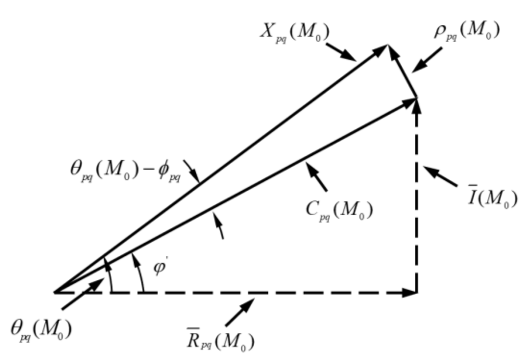

Fig.2. Phasor diagram for the m_0th DFT component

The conclusions described above are quantitative. As shown in Fig.2, signal phasor $C_{pq}(m_0)$ has quadrature components $\overline{R}_{pq}(m_0)$ and $\overline{I}_{pq}(m_0)$, and the $X_{pq}(m_0)$ is equal to a signal phasor $C_{pq}(m_0)$ plus a noise phasor $\rho_{pq}(m_0)$ .Thus, when noise phasor $\rho_{pq}(m_0)$ is present, the problem of estimating the amplitude and phase of the signal phasor $C_{pq}(m_0)$ is equivalent to the problem of estimating the amplitude and phase of the polarization interference fringe.

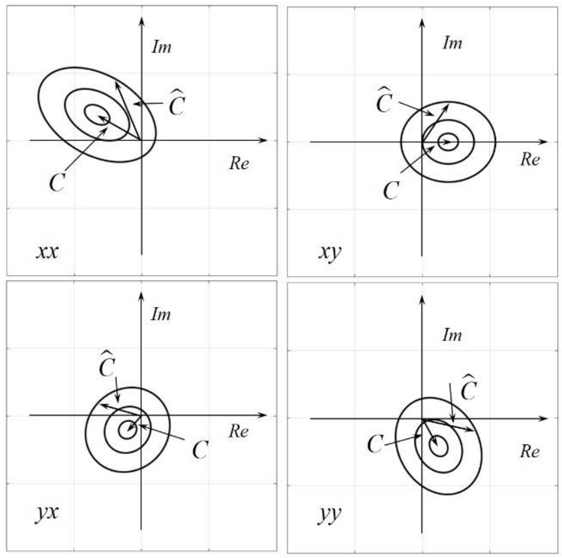

Fig.3 Phasor diagram for polarization interference fringe estimation

Fig. 3 shows polarization interference in four different cases, where *x* and *y* represent horizontally polarized light and vertically polarized light respectively. As shown in Fig.3, the true value of the polarization interference fringe phasor is represented by *C*. And if the true value of the fringe phasor is directed along the positive real axis, the true phase of the polarization interference fringe is zero.

About the point defined by the tip of *C* there exists a "noise cloud," the contours of which can be regarded as contours of constant probability density. These contours are broader in the direction of the real axis than they are in the direction of the imaginary axis. More generally, when the true phase of the fringe is not zero, the contours are elongated in the direction of *C*.

CONCLUSION

This paper studies the basic limitations of estimating the amplitude and phase of polar interferograms at low light levels. We consider the discrete Fourier transform estimator and the optimal joint maximum likelihood estimator to specify the limit performance of all unbiased estimators in terms of collected luminous flux. By modeling the receiver as a spatial array of photon counting detectors, we obtain the results that allow specifying the minimum number of optical events required to estimate polarization interference fringe parameters with a given accuracy.

REFERENCE

[1] W. I. Beavers ; W. D. Swift, "Photoelectric Fringe Strength Measurement," Appl. Opt. 7, 1975-1979 (1968).

[2] G. D. Bergland, "A guided tour of the fast Fourier Transform, " IEEE Spectrum, 6 (7), 41-52 (1969).

[3] Singleton, “An Algorithm for Computing the Mixed Radix Fast Fourier Transform," IEEE Transactions on Au dio and Electroacoustics, 17(2), 93-103 (1969).

[4] Angel, E. S. , "Fast Fourier transform and convolution algorithm." Proceedings of the IEEE 70(5),527 (1982).

[5] K. R. Rao; D. N. Kim; J. J. H wang, "Fast Fourier Transform - Algorithms and Applications". Springer Nether lands, 2010.

[6] T. Yoshizawa; "Fast Fourier Transform and its Applications." Journal of the Society of Instrument & Control Engineers, 8,851-860(1969).

[7] W. Cooley, P. A. W. Lewis, P. D. Welch, "Historical notes on the fast Fourier transform", IEEE Trans. Audio and Electroacoustics, vol.15,76-79(1967).

[8] E. O. Brigham, R. E. Morrow, "The fast Fourier transform", IEEE Spectrum, vol. 4, 63-70(1967).