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Chiral expansion and Macdonald deformation of two-dimensional Yang-Mills theory

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Abstract

We derive the analog of the large N Gross-Taylor holomorphic string expansion for the refinement of q-deformed $U(N)$ Yang-Mills theory on a compact oriented Riemann surface. The derivation combines Schur-Weyl duality for quantum groups with the Etingof-Kirillov theory of generalized quantum characters which are related to Macdonald polynomials. In the unrefined limit we reproduce the chiral expansion of q-deformed Yang-Mills theory derived by de Haro, Ramgoolam and Torrielli. In the classical limit $q = 1$, the expansion defines a new β-deformation of Hurwitz theory wherein the refined partition function is a generating function for certain parameterized Euler characters, which reduce in the unrefined limit $\beta = 1$ to the orbifold Euler characteristics of Hurwitz spaces of holomorphic maps. We discuss the geometrical meaning of our expansions in relation to quantum spectral curves and β-ensembles of matrix models arising in refined topological string theory.

Contents

1 Introduction and summary

The OSV conjecture relates the partition function of a four-dimensional BPS black hole in Type IIA string theory compactified on a Calabi-Yau threefold X with the A-model topological string amplitudes on X [48]. The black hole partition function counts BPS microstates and can be used for calculation of black hole entropy. On the topological string theory side, if the OSV conjecture is valid, it may be used for a non-perturbative definition of the topological string theory. In the case of a local Calabi-Yau threefold X which the total space of rank two holomorphic vector bundle over a Riemann surface Σ, the counting of BPS states reduces to the computation of the partition function of q-deformed $U(N)$ Yang-Mills theory on Σ (first introduced in [38]), and for large N it factorizes into a coupling of chiral and antichiral partition functions of q-deformed $SU(N)$ Yang-Mills theory; the string theory dual of this two-dimensional gauge theory is topological string theory [4]. From a mathematical perspective, this duality states that the Gromov-Witten invariants of X can be computed in terms of Hurwitz numbers of branched covers of Σ [7] (see also [8, 9, 10, 52]).

The refinement of the OSV conjecture states that the refined black hole partition function has a large N dual which is captured by refined topological string amplitudes [2]. The refined black hole partition function, which counts BPS states with spin, reduces to the partition function of a refined version of two-dimensional q-deformed $U(N)$ Yang-Mills theory, and it factorizes into a coupling of chiral and antichiral partition functions of refined q-deformed $SU(N)$ Yang-Mills theory analogously to the unrefined case. This refined q-deformed two-dimensional gauge theory is called the (q,t)-deformation or Macdonald deformation of Yang-Mills theory, and its string theory dual is refined topological string theory. On open surfaces, the duals of boundary characters of the (q,t)-deformed gauge theory on the refined topological string theory side correspond to insertions of Lagrangian D3-branes in one of the fibers which encircle the boundary; the special Wilson loop observables in Yang-Mills theory correspond to Lagrangian D-branes wrapping a one-cycle in the fiber [2]. In the present paper we develop a dual *closed* string expansion of this gauge theory in the large N limit.

The (q,t)-deformed Yang-Mills theory is also closely related to various gauge theories in higher dimensions. It is related to refined Chern-Simons theory on a circle bundle over the Riemann surface Σ [3, 1, 39]. The topological limit of the partition function was considered in [24] where it was interpreted as the superconformal index for a certain $\mathcal{N}=2$ gauge theory of class $\mathcal{S}$ in four dimensions. These topological versions also arise in geometric engineering of supersymmetric gauge theories in string theory. In particular, the partition function of the two-dimensional topological field theory version of the Macdonald deformation is equal to the partition function of a five-dimensional $\mathcal{N}=1$ gauge theory [2, 35].

The large N phase structure of the Macdonald deformation on the sphere was studied in [39, 27], and related to refined black hole entropy and topological string theory. In this paper we derive the large N expansion of (q,t)-deformed Yang-Mills theory which is the analog of the Gross-Taylor holomorphic string expansion of ordinary Yang-Mills theory. Ordinary two-dimensional Yang-Mills theory can be solved exactly in the lattice formulation [45, 50]. Using Schur-Weyl duality, in [30, 31, 32] it was shown that the chiral part of the partition function can be manipulated into a series consisting of delta-functions over symmetric groups, which count equivalence classes of branched covers of the Riemann surface Σ in terms of Hurwitz numbers, see e. g. [14, 43]. It was further shown in [14, 13] that the chiral series is a generating function for orbifold Euler characters of Hurwitz spaces of holomorphic maps with fixed two-dimensional target space Σ; hence two-dimensional Yang-Mills theory is dual to a closed two-dimensional topological string theory with string coupling $g_{\text{str}} = \frac{1}{N}$. This expansion was extended to the q-deformed case by de Haro, Ramgoolam and Torrielli in [33]. In this instance the $U(N)$ characters and dimensions are replaced by quantum characters and quantum dimensions for the quantized universal enveloping algebra of $U(N)$, and hence for large N the quantum version of Schur-Weyl duality [36] can be applied to the chiral partition function to manipulate it into a series of delta-functions on Hecke algebras of type A [33]. These expansions can also be applied to observables corresponding to open surfaces and Wilson loops.

The purpose of the present paper is to extend the large N chiral expansion of [33] to the refined case. Our constructions are based on the feature that, from a mathematical perspective, the refinement uses the Etingof-Kirillov theory of characters for intertwining operators [21], called "generalized characters", which are related to Jack and Macdonald polynomials in a similar way that ordinary characters are related to Schur polynomials. Using the fact that Macdonald polynomials can be written as vector-valued characters of the underlying quantum group, we can apply quantum Schur-Weyl duality to these characters analogously to the q-deformed case to obtain a more general and complex expansion into delta-functions on Hecke algebras. While we borrow throughout from the wealth of results already derived in [33], along the way we clarify and extend these results in various directions. In particular, we explicitly clarify the definitions of certain central elements from [33], we define a new Fourier-type transformation on quantum group characters to characters of central elements of the Hecke algebra, and we prove a relation between refined and ordinary Littlewood-Richardson coefficients at large N.

Neither the q-deformed nor the refined expansions admit straightforward worldsheet interpretations (much like the situation with refined topological string theory). As such, our constructions may have some independent mathematical interest in light of the property that the undeformed chiral theory is the worldsheet field theory for classical Hurwitz spaces of branched covers. To shed some light on this perspective, we shall study a special classical limit of the (q,t)-deformed gauge theory in which $t = q^\beta$ and $q \to 1$ with fixed β. In this limit the Macdonald polynomials reduce to the Jack symmetric functions, and the underlying Hecke algebra reduces to the ordinary group algebra of the symmetric group. This limit of the (q,t)-deformed partition function yields a new kind of deformation of the usual Hurwitz theory of branched covering maps of Riemann surfaces, which to the best of our knowledge has not appeared in the literature before. In particular, we interpret the limiting partition function as a generating function for certain *parameterized* Euler characters which at $\beta = 1$ reduce to the usual orbifold Euler characters of Hurwitz spaces.

Amongst the various speculations about the worldsheet interpretations of these deformed two-dimensional gauge theories that are given in [33] and in the following, let us present one more to close this introductory section. The enumeration of branched covers of a Riemann surface Σ can be obtained by computing the partition function of a two-dimensional lattice gauge theory on a cellular decomposition of Σ with gauge group the symmetric group [40, 5]; this model is related to chiral two-dimensional Yang-Mills theory as discussed by [40]. It is then tempting to speculate that the q-deformation of chiral two-dimensional Yang-Mills theory is obtained by using instead a model whose gauge group is based on the corresponding Hecke algebra of type A.

The outline of the remainder of this paper is as follows. In §2 we review various aspects of the definition and computation of the partition function of (q,t)-deformed two-dimensional Yang-Mills theory, pointing out its various geometrical incarnations within refined topological string theory in the related settings of M-theory, higher-dimensional supersymmetric gauge theories, and deformed matrix models. In §3 we exploit the quantum version of Schur-Weyl duality developed by [33] to rewrite the (q,t)-deformed dimension factors appearing in the partition function in terms of generalized characters of the underlying Hecke algebras; our final result is summarised in Proposition 3.77. In §4 we compute the chiral expansion of the refined topological partition function; our final result is summarised by Proposition 4.25. We develop in particular the β-deformation of classical Hurwitz theory alluded to above; our main result is summarised in Proposition 4.46. In §5 we extend these considerations to topological partition functions on open Riemann surfaces and to Wilson loop observables. Finally, four appendices at the end of the paper contain technical details and definitions which are refered to throughout the main text: Appendix A recalls the definition of the pertinent quantum groups, Appendix B describes the corresponding Hecke algebras, Appendix C derives some new identities amongst characters of symmetric groups and supersymmetric Schur polynomials, and Appendix D derives new Fourier-type transformations from quantum groups to central elements of inductive limits of Hecke algebras.

2 Macdonald deformation of Yang-Mills theory in two dimensions

The Macdonald deformation of two-dimensional Yang-Mills theory is a two-parameter deformation of the usual two-dimensional gauge theory. It can be thought of as a refinement of the well-known q-deformation, or alternatively as a quantum deformation of the classical β-deformation which can be characterised in certain cases by β-ensembles of random matrix models. In this section we consider the partition function defining the gauge theory and its geometrical interpretations in the context of refined topological string amplitudes; more general amplitudes will be studied in §5.

2.1 Combinatorial definition

Let $\mathfrak{g}$ be the Lie algebra of a connected Lie group G of rank N. Let $\mathcal{R}$ be the root system of $\mathfrak{g}$ and $\mathcal{R}_+$ the system of positive roots; similarly let $\Lambda \cong \mathbb{Z}^N$ be the weight lattice of $\mathfrak{g}$ with dominant weights Λ_+. We fix an invariant bilinear form $(-,-)$ on $\mathfrak{g}$, usually the Killing form. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha \tag{2.1}$$

be the Weyl vector of $\mathfrak{g}$ labelling the trivial representation; we shall often assume that the rank N is such that $\rho \in \mathbb{Z}^N$, which in particular can be supposed in the large N expansion that we consider in the following.

The partition function for the Macdonald deformation of Yang-Mills theory with gauge group G on a closed oriented Riemann surface Σ_h of genus $h \geq 0$ can be written as a generalization of the Migdal-Rusakov heat kernel expansion given by [3, 1, 2]

$$Z_h(q,t;p) = \sum_{\lambda \in \Lambda_+} \frac{\dim_{q,t}(R_\lambda)^{2-2h}}{(g_\lambda)^{1-h}}\, q^{\frac{p}{2}\,(\lambda,\lambda)}\, t^{p\,(\rho,\lambda)} \ , \tag{2.2}$$

where the sum runs over all irreducible unitary representations R_λ of G which are parameterized by dominant weights $\lambda \in \Lambda_+$. Here the degree $p \in \mathbb{Z}$ and the deformation parameters $q,t \in \mathbb{C}^*$ satisfy $|q| < 1$ and $|t| < 1$ in order to ensure that the series (2.2) has a non-zero radius of convergence; we shall sometimes assume $q,t \in (0,1)$ for convenience. For simplicity of presentation, below we shall write some formulas for the case when the refinement parameter

$$\beta = \frac{\log t}{\log q} \tag{2.3}$$

is a positive integer, and then extend our final results to arbitrary $\beta \in \mathbb{C}$ by analytic continuation. The refined quantum dimension of the representation R_λ is

$$\dim_{q,t}(R_\lambda) = \prod_{m=0}^{\beta-1} \prod_{\alpha \in \mathcal{R}_+} \frac{\big[(\lambda + \beta\,\rho, \alpha) + m\big]_q}{\big[(\beta\,\rho, \alpha) + m\big]_q} \ , \tag{2.4}$$

where

$$[x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \tag{2.5}$$

for $x \in \mathbb{R}$ is a q-number. The Macdonald metric is given by

$$g_\lambda = \frac{1}{N!} \prod_{m=0}^{\beta-1} \prod_{\alpha \in \mathcal{R}_+} \frac{\big[(\lambda + \beta\,\rho, \alpha) + m\big]_q}{\big[(\lambda + \beta\,\rho, \alpha) - m\big]_q} \ . \tag{2.6}$$

In this paper we shall specialize to the unitary gauge group $G = U(N)$. In this case there are convenient combinatorial expressions available for the dimension and metric factors. The Weyl vector is $\rho = \big(\frac{N-1}{2}, \dots, -\frac{N-1}{2}\big)$ and the dominant weights $\lambda \in \Lambda_+$ are parameterized by partitions with at most N parts $\lambda = (\lambda_1, \dots, \lambda_N)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$; they are in a one-to-one correspondence with Young diagrams $Y_\lambda \subset (\mathbb{Z}_{>0})^2$ with at most N rows. Then the refined quantum dimension and Macdonald metric have the equivalent forms

$$\begin{aligned} \dim_{q,t}(R_\lambda) &= t^{\frac{1}{2}(\|\lambda^{\rm t}\|-N\,|\lambda|)} \prod_{(i,j)\in Y_\lambda} \frac{1-t^{N-i+1}\,q^{j-1}}{1-t^{\lambda^{\rm t}_j-i+1}\,q^{\lambda_i-j}} \ , \\ g_\lambda &= g_\emptyset \prod_{(i,j)\in Y_\lambda} \frac{1-t^{\lambda^{\rm t}_j-i}\,q^{\lambda_i-j+1}}{1-t^{\lambda^{\rm t}_j-i+1}\,q^{\lambda_i-j}}\, \frac{1-t^{N-i+1}\,q^{j-1}}{1-t^{N-i}\,q^{j}} \ , \end{aligned} \tag{2.7}$$

where $|\lambda| := \sum_{i=1}^N \lambda_i$ and $\|\lambda\| := \sum_{i=1}^N \lambda_i^2$, while

$$g_\emptyset = \frac{1}{N!} \prod_{m=0}^{\beta-1} \ \prod_{1\le i<j\le N} \frac{\big[\beta\,(j-i)+m\big]_q}{\big[\beta\,(j-i)-m\big]_q} \ . \tag{2.8}$$

The products in (2.7) run over all boxes (i,j) of the Young diagram Y_λ with $1\le i\le N$, $1\le j\le \lambda_i$, and $\lambda^{\rm t}$ corresponds to the transposed Young diagram, i.e. $\lambda^{\rm t}_i$ is the number of entries $\le i$ in Y_λ.

2.2 M-theory interpretation

The parameters of the Macdonald deformation are related to the equivariant parameters

$$\epsilon_1 = \frac{1}{\sqrt{\beta}}\, g_s \qquad \text{and} \qquad \epsilon_2 = -\sqrt{\beta}\, g_s \tag{2.9}$$

of the Ω-background [16] through

$$q = {\rm e}^{\,-\epsilon_1} \qquad \text{and} \qquad t = {\rm e}^{\,\epsilon_2} \ , \tag{2.10}$$

where g_s is the topological string coupling constant. The refined A-model topological string theory that gives rise to the two-dimensional gauge theory is defined on the non-compact Calabi-Yau threefold which is the total space of the $\mathbb{C}^2$-fibration $\mathcal{O}_{\Sigma_h}(p+2h-2)\oplus\mathcal{O}_{\Sigma_h}(-p)$ over the Riemann surface Σ_h. The complete geometrical picture of the (q,t)-deformed gauge theory involves M-theory on a particular 11-dimensional manifold [2], which we now describe.

Let us start with five-dimensional $\mathcal{N}=1$ supersymmetric gauge theory on the total space of the flat affine bundle $\mathbb{C}^2\times_{\mathbb{Z}}\mathbb{R}\to\mathbb{T}$ over a circle $\mathbb{T}=S^1$ of radius r, where the $\mathbb{Z}$-action is given by

$$(z,w,x) \ \longmapsto \ (q^n\,z, t^{-n}\,w, x+2\pi\,r\,n) \tag{2.11}$$

for $(z,w)\in\mathbb{C}^2$, $x\in\mathbb{R}$ and $n\in\mathbb{Z}$. Here $(q,t)\in(\mathbb{C}^*)^2$ is regarded as an element of the maximal torus of the complex spin cover $Spin(4,\mathbb{C}) = SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ of the four-dimensional rotation group; the pair (q,t^{-1}) is the holonomy of a flat $Spin(4,\mathbb{C})$-connection on the bundle $\mathbb{C}^2\times_{\mathbb{Z}}\mathbb{R}$. Correspondingly, one must also turn on an $SL(2,\mathbb{C})$ R-symmetry twist

$$T := \begin{pmatrix} \sqrt{q\,t^{-1}} & 0 \\ 0 & \sqrt{q^{-1}\,t} \end{pmatrix} \tag{2.12}$$

in order to preserve the topological supersymmetry which is broken when $\beta\neq1$. We can embed this gauge theory into the M-theory compactification which is the total space of the bundle over $\mathbb{T}$ given by the quotient

$$\mathrm{Tot}\big(\mathcal{O}_{\Sigma_h}(p+2h-2)\oplus\mathcal{O}_{\Sigma_h}(-p)\big) \ \times_{\mathbb{Z}} \ \mathbb{C}^2 \ \times_{\mathbb{Z}} \ \mathbb{R} \tag{2.13}$$

where $n \in \mathbb{Z}$ acts on $\mathbb{C}^2 \times \mathbb{R}$ as in (2.11) and rotates the fibre coordinates $(u,v) \in \mathbb{C}^2$ of $\mathcal{O}_{\Sigma_h}(p+2h-2) \oplus \mathcal{O}_{\Sigma_h}(-p)$ by the matrix T^n. The low energy effective action of this supersymmetric M-theory background geometrically engineers the Ω-deformed $\mathcal{N}=1$ supersymmetric $U(1)$ gauge theory with h adjoint hypermultiplets on $\mathbb{C}^2 \times S^1$, where the integer p corresponds to the level of the five-dimensional Chern-Simons term; the corresponding partition function is the generating function for the equivariant Hirzebruch genus of a certain holomorphic vector bundle $\mathfrak{E}_{h,p}$ on the moduli space of $U(1)$ instantons on $\mathbb{C}^2$ [12].

Let us now consider the M-theory background in which the $\mathbb{C}^2$ factor in (2.13) is replaced with the hyper-Kähler Taub-NUT space TN, which is a local S^1-fibration over $\mathbb{R}^3$, where the $\mathbb{Z}$-action on the local complex coordinates $(z,w) \in \mathbb{C}^2$ near the tip of TN is as in (2.11). We embed the five-dimensional $\mathcal{N}=1$ gauge theory into the $(2,0)$ superconformal theory compactified on the fibre $\mathbb{T}' = S^1$ of the Taub-NUT space, which is the low energy limit of the theory on a single M5-brane with worldvolume that is locally given by

$$\mathbb{T}' \ \times \ \mathrm{Tot}\big(\mathcal{O}_{\Sigma_h}(-p)\big) \ \times \ \mathbb{T} \ . \tag{2.14}$$

After collapsing the M-theory circle $\mathbb{T}'$ and equivariantly localizing with respect to the $\mathbb{C}^*$-scaling action induced by (2.12) along the fibres of the line bundle $\mathcal{O}_{\Sigma_h}(-p)$, we thereby construct the (q,t)-deformed gauge theory on Σ_h; the corresponding partition function is related to the generating function for the Hirzebruch genus of the moduli space of $U(N)$ instantons on the local ruled surface $\mathrm{Tot}\big(\mathcal{O}_{\Sigma_h}(-p)\big)$ over Σ_h [2, 39].

The Ω-background symmetry $(\epsilon_1,\epsilon_2) \mapsto (-\epsilon_2,-\epsilon_1)$ corresponds to the inversion symmetry $\beta \mapsto \frac{1}{\beta}$ of the refinement parameter together with the rank change $N \mapsto \beta\,(N-1)+1$. It acts on the Macdonald deformation parameters as $(q,t) \mapsto (t,q)$ which corresponds to the symmetry $p \mapsto 2-2h-p$ that exchanges the two line bundle summands of the Calabi-Yau fibration over Σ_h.

2.3 Quantum spectral curves and β-ensembles

We can also define the $(2,0)$ theory by wrapping M5-branes on the six-manifold $\Sigma_h \times \mathbb{C}^2$ in (2.13), equipped with a non-trivial fibration of $\mathbb{C}^2$ over Σ_h which specifies the Ω-background [16]. In this case Σ_h acquires an interpretation as the base of a branched covering by the Seiberg-Witten curve of a four-dimensional $\mathcal{N}=2$ gauge theory of class $\mathcal{S}$, which can in turn be regarded as the spectral curve of an associated Hitchin system [25, 26] that is quantized via a suitable deformation; the five-dimensional gauge theories compactified on a circle of radius r lead to a relativistic (q-deformed or difference) version of this Hitchin system. For $p=1$, the bound state of N M5-branes is described by an N-sheeted branched covering of Σ_h given by

$$\Sigma_{\mathrm{SW}} = \Big\{ (x,z) \in \mathrm{Tot}\big(\mathcal{O}_{\Sigma_h}(-1)\big) \ \Big| \ x^N + \sum_{j=2}^{N} \mathrm{t}_j(z)\, x^{N-j} = 0 \Big\} \ , \tag{2.15}$$

where t_j is a $(j,0)$-differential on Σ_h.

Generally, the Seiberg-Witten curve is an affine curve characterized by an algebraic relation of the form $P(x,y)=0$ for $(x,y) \in \mathbb{C}^2$. Turning on the Ω-background lifts this relation to a differential equation $P(\widehat{x},\widehat{y})\psi = 0$ which quantizes the coordinate algebra $\mathbb{C}[x,y]$ to the Weyl algebra $\mathbb{C}[\hbar]\langle\,\widehat{x},\widehat{y}\,\rangle$ defined by the commutation relations

$$[\,\widehat{x},\widehat{y}\,] = -\hbar \qquad \text{with} \quad \hbar = \sqrt{\beta} - \frac{1}{\sqrt{\beta}} = \frac{\epsilon_1+\epsilon_2}{g_s} \ . \tag{2.16}$$

We can represent $\widehat{x}$ as the multiplication operator by $x \in \mathbb{C}$ and $\widehat{y}$ as the differential operator $\hbar\,\partial_x$. This differential equation is interpreted as a "quantum curve" [17]: The differential $\lambda_\hbar =$

$\hbar\,\partial_x \log\psi(x,\hbar)\,\mathrm{d}x$ is a "quantum" differential generating the "quantum" periods of the quantized Riemann surface which in the unrefined limit $\hbar = 0$ coincides with the meromorphic differential $\lambda_0 = y(x)\,\mathrm{d}x$ of the original Seiberg-Witten curve (with $y = y(x)$ depending on x through the equation $P(x,y) = 0$). Thus refinement corresponds to a system of differential equations satisfied by the partition functions of the four-dimensional gauge theory. In the five-dimensional gauge theory, the quantum spectral curve is instead given by a difference equation $P(\,\widehat{X},\widehat{Y}\,)\psi = 0$, where the difference operators $\widehat{X} = \mathrm{e}^{\,r\,\widehat{x}}$ and $\widehat{Y} = \mathrm{e}^{\,r\,\widehat{y}}$ obey $\widehat{X}\,\widehat{Y} = \underline{q}\,\widehat{Y}\,\widehat{X}$ with $\underline{q} = \mathrm{e}^{\,-r^2\,\hbar}$.

In order to understand this point from the perspective of two-dimensional gauge theory, following [2, 53, 39] we define shifted weights $n_i = \lambda_i + \beta\,\rho_i$ for $i = 1,\dots,N$ and rewrite the partition function (2.2) for $G = U(N)$ in the form (up to overall normalization)

$$Z_h(q,t;p) = \sum_{n\in\mathbb{Z}_0^N} \Delta_{q,t}\big(\mathrm{e}^{\,\epsilon_1\,n}\big)^{1-h}\,\Delta_{q,t}\big(\mathrm{e}^{\,-\epsilon_1\,n}\big)^{1-h}\ \mathrm{e}^{\,-\frac{p\,\epsilon_1}{2}\,(n,n)}\ , \tag{2.17}$$

where $\mathbb{Z}_0^N$ is the set of N-vectors of integers $n = (n_1,\dots,n_N)$ for which $n_i \neq n_j$ for all $i \neq j$, and the Macdonald measure $\Delta_{q,t}(x)$ is defined in Appendix D. Let us consider the case of genus $h = 0$. Since $\Delta_{q,t}(1^N) = 0$, we can then sum over all $n \in \mathbb{Z}^N$ and following [51] we rewrite the partition function on the sphere as

$$Z_0(q,t;p) = \prod_{i=1}^N\ \int_0^\infty \frac{\mathrm{d}_q x_i}{x_i}\ \mathrm{e}^{\,-\frac{p}{2\epsilon_1}\,\log^2 x_i}\ \Delta_{q,t}(x)\,\Delta_{q,t}(x^{-1})\ , \tag{2.18}$$

where the multiple Jackson q-integral is defined by

$$\prod_{i=1}^N\ \int_0^\infty \frac{\mathrm{d}_q x_i}{x_i}\ f(x) := (1-q)^N \sum_{n\in\mathbb{Z}^N} f(q^n) \tag{2.19}$$

for a continuous function $f(z)$ on $(\mathbb{C}^*)^N$ (provided the multiple series is absolutely convergent).

The rewriting (2.18) demonstrates that the Macdonald deformation of two-dimensional gauge theory can be described as a generalized Gaussian matrix model in the q-deformed β-ensemble of random matrix theory. In particular, for $p = 1$ the indeterminancy of the moment problem for the Stieltjes-Wigert distribution implies that the discrete and continuous matrix models are equivalent [53]. In this case the geometrical setup of §2.2 greatly simplifies: The relevant Calabi-Yau fibration is the conifold geometry $\mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\big)$, the vector bundle $\mathfrak{E}_{0,1}$ is trivial [12, 35], while the surface $\mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^1}(-1)\big)$ is the blow-up of $\mathbb{C}^2$ at two points (with boundary the three-sphere S^3). This equivalence implies that we may replace the Jackson integral in (2.18) with an ordinary Riemann-Lebesgue integral, and by setting $z_i := \log x_i$ we may write

$$Z_0(q,t;1) = \prod_{i=1}^N\ \int_{-\infty}^\infty \mathrm{d}z_i\ \mathrm{e}^{\,-\frac{1}{2\epsilon_1}\,z_i^2}\ \Delta_{q,t}\big(\mathrm{e}^{\,z}\big)\,\Delta_{q,t}\big(\mathrm{e}^{\,-z}\big)\ , \tag{2.20}$$

which up to normalization coincides with the Stieltjes-Wigert matrix model for refined Chern-Simons theory on S^3 [3]. However, for $p \neq 1$ such an equivalence ceases to hold and one must work directly with the discrete matrix model (2.18) for generic values of $p \in \mathbb{Z}$.

Let us now consider the classical limit $q \to 1$ defined by taking the limits $\epsilon_1 \to 0$, $p \to \infty$ while keeping fixed the refinement parameter β (so that also $\epsilon_2 \to 0$) and the parameter $a := \epsilon_1\,p$; in this limit the (q,t)-deformed gauge theory generally reduces to a β-deformation of ordinary Yang-Mills theory on a Riemann surface of area a. The right-hand side of (2.19) is a multiple infinite Riemann sum, which for $q \to 1^-$ formally converges to $\prod_{i=1}^N \int_0^\infty \frac{\mathrm{d}x_i}{x_i}\, f(x)$. By rescaling $z_i \to \epsilon_1\, z_i$, up to normalization we then find the partition function

$$\begin{aligned} \widetilde{Z}_0(a,\beta) \ &:= \ \lim_{q\to 1}\ \lim_{p\to\infty}\ Z_0(q,t;p)\Big|_{t=q^\beta\,,\,p=-\frac{a}{\log q}} \\ &= \ \prod_{i=1}^N\ \int_{-\infty}^\infty \mathrm{d}z_i\ \mathrm{e}^{\,-\frac{a}{2}\,z_i^2}\ \prod_{m=0}^{\beta-1}\ \prod_{i<j}\ \big((z_i-z_j)^2-m^2\big)\ . \end{aligned} \tag{2.21}$$

The planar limit is defined by taking

$$\tau_1 = \epsilon_1\,N \qquad \text{and} \qquad \tau_2 = \epsilon_2\,N \tag{2.22}$$

large but fixed for $N \to \infty$. In this limit the refinement parameter $\beta = -\frac{\tau_2}{\tau_1}$ is finite, hence the product in (2.21) is finite and $m\,\epsilon_1 = \frac{m\,\tau_1}{N} \to 0$. By rescaling $z_i \to N\,z_i$ as finite variables at large N, it follows that the planar limit of the partition function (2.21) is given by

$$\widetilde{Z}{}_0^{\mathrm{pl}}(\mu,\beta) = \prod_{i=1}^N\ \int_{-\infty}^\infty \mathrm{d}z_i\ \mathrm{e}^{\,-\frac{\mu}{2}\,z_i^2}\ \Delta(z)^{2\beta}\ , \tag{2.23}$$

where $\mu := p\,\tau_1\,N$ and

$$\Delta(z) = \prod_{i<j}\,(z_i-z_j) \tag{2.24}$$

is the Vandermonde determinant. Thus in this limit the weak coupling phase of the two-dimensional gauge theory can be described by a Gaussian matrix model in the classical β-ensembles of random matrix theory. In particular, $\log \widetilde{Z}{}_0^{\mathrm{pl}}(\mu,\beta)$ coincides with the partition function of two-dimensional $c=1$ string theory at radius $R=\beta$; this generalizes the usual identification of the conifold geometry with the $c=1$ string at the self-dual radius for $\beta=1$. In this formulation the symmetry $\beta \to \frac{1}{\beta}$ is manifest [16] and corresponds to T-duality invariance of the string theory. For later use and comparison, let us run through the details of this identification.

The matrix integral (2.23) is a special case of the Selberg integral which can be evaluated analytically in terms of Mehta's formula

$$\widetilde{Z}{}_0^{\mathrm{pl}}(\mu,\beta) = \Big(\,\frac{\sqrt{2\pi}}{\mu^{\frac{1}{2}\,((N-1)\,\beta+1)}}\,\Big)^N\ \prod_{i=1}^N\ \frac{\Gamma(1+\beta\,i)}{\Gamma(1+\beta)}\ . \tag{2.25}$$

One can show that [6]

$$\prod_{i=1}^N\ \Gamma(1+\beta\,i) = \big(\,\sqrt{2\pi}\,\beta^{\frac{1}{2}\,((N-1)\,\beta+1)}\,\big)^N\ \Gamma(1+N\,\beta)\,\Gamma(N)\,\Gamma_2\big(N;-\beta^{-1},-1\big)\ , \tag{2.26}$$

where $\Gamma_2(\tau_2;\epsilon_1,\epsilon_2)$ is the Barnes double gamma-function defined via

$$\log\Gamma_2(\tau_2;\epsilon_1,\epsilon_2) = -\ \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\ \frac{1}{\Gamma(s)}\ \int_0^\infty \frac{\mathrm{d}t}{t}\ t^s\ \frac{\mathrm{e}^{\,-\tau_2\,t}}{\big(1-\mathrm{e}^{\,\epsilon_1\,t}\big)\,\big(1-\mathrm{e}^{\,\epsilon_2\,t}\big)} \tag{2.27}$$

which is the double zeta-function regularization of the infinite product $\prod_{m,n\geq 0}\,(\tau_2-m\,\epsilon_1-n\,\epsilon_2)$. To obtain the large N expansion of the partition function (2.23) we use the asymptotic expansion [46, Appendix E]

$$\begin{aligned} \log\Gamma_2(\tau_2;\epsilon_1,\epsilon_2) \ &= \ \frac{1}{\epsilon_1\,\epsilon_2}\,\big(\tfrac{1}{2}\,\tau_2^2\,\log\tau_2-\tfrac{3}{4}\,\tau_2^2\big)+\frac{\epsilon_1+\epsilon_2}{\epsilon_1\,\epsilon_2}\,\big(\tau_2\,\log\tau_2-\tau_2\big) \\ &\quad +\frac{\epsilon_1^2+\epsilon_2^2+3\,\epsilon_1\,\epsilon_2}{12\,\epsilon_1\,\epsilon_2}\,\log\tau_2-\sum_{n=3}^\infty\,\frac{d_n(\epsilon_1,\epsilon_2)\,\tau_2^{2-n}}{n\,(n-1)\,(n-2)}\ , \end{aligned} \tag{2.28}$$

where the series coefficients $d_n(\epsilon_1,\epsilon_2)$ are defined through the generating function

$$\frac{t^2}{\big(1-\mathrm{e}^{\,\epsilon_1\,t}\big)\,\big(1-\mathrm{e}^{\,\epsilon_2\,t}\big)}=\sum_{n=0}^\infty\,\frac{1}{n!}\,d_n(\epsilon_1,\epsilon_2)\,t^n\ . \tag{2.29}$$

Introducing the Bernoulli numbers B_m through the generating function

$$\frac{s}{1-\mathrm{e}^{\,s}}=-\sum_{m=0}^\infty\,\frac{1}{m!}\,B_m\,s^m \tag{2.30}$$

with $B_0=1$, $B_1=\frac12$, $B_2=\frac16$ and $B_k=0$ for all $k>1$ odd, by comparing series expansions we find

$$d_n(\epsilon_1,\epsilon_2)=g_s^{n-2}\,\sum_{k=0}^n\,\binom{n}{k}\,(-1)^{k-1}\,B_k\,B_{n-k}\,\beta^{k-\frac n2} \tag{2.31}$$

where we have used the relations (2.9). By dropping overall prefactors, for the free energy $\widetilde{F}_0^{\rm pl}(\tau_2,\beta):=-\log\widetilde{Z}_0^{\rm pl}(\tau_2,\beta)=-\log\Gamma_2(\tau_2;\beta^{-1/2}\,g_s,\beta^{1/2}\,g_s)$ in the large N limit this gives the asymptotic expansion

$$\begin{aligned}\widetilde{F}_0^{\rm pl}(\tau_2,\beta)&=\frac{1}{g_s^2}\,\big(\tfrac34\,\tau_2^2-\tfrac12\,\tau_2^2\,\log\tau_2\big)+\frac{1}{g_s}\,\big(\tfrac{1}{\sqrt\beta}-\sqrt\beta\,\big)\,\big(\tau_2-\tau_2\,\log\tau_2\big)\\&\quad+\chi_0(\beta)\,\log\tau_2+\sum_{n=1}^\infty\,\chi_n(\beta)\,\Big(\frac{g_s}{\tau_2}\Big)^n\ ,\end{aligned} \tag{2.32}$$

where

$$\begin{aligned}\chi_0(\beta)&=-\frac14+\frac{\beta^{-1}}{12}+\frac{\beta}{12}\ ,\\ \chi_n(\beta)&=(n-1)!\,\sum_{k=0}^{n+2}\,\frac{(-1)^{k-1}\,B_k\,B_{n+2-k}}{k!\,(n+2-k)!}\,\beta^{k-\frac n2-1}\qquad\text{for}\quad n\geq1\ .\end{aligned} \tag{2.33}$$

Note that the expansion parameter is

$$\frac{g_s}{\tau_2}=\frac{1}{\sqrt\beta\,N}\ . \tag{2.34}$$

In the unrefined limit $\beta=1$, the identity

$$\frac{1}{\big(1-\mathrm{e}^{\,s}\big)\,\big(1-\mathrm{e}^{\,-s}\big)}=-\frac{\mathrm{d}}{\mathrm{d}s}\,\frac{1}{1-\mathrm{e}^{\,s}} \tag{2.35}$$

together with (2.30) imply that $d_n(g_s,-g_s)=0$ for n odd while $d_{2g}(g_s,-g_s)=g_s^{2g-2}\,(2g-1)\,B_{2g}$ for $g\geq1$, so that the non-vanishing coefficients

$$\chi_0(1)=-\frac{1}{12}\qquad\text{and}\qquad\chi_{2g-2}(1)=\frac{B_{2g}}{2g\,(2g-2)}\quad(g>1) \tag{2.36}$$

coincide with the orbifold Euler characteristics $\chi_{\rm orb}(\mathcal{M}_g)$ of the Riemann moduli spaces $\mathcal{M}_g$ of genus $g\geq1$ complex curves, i.e. the Euler character of $\mathcal{M}_g$ calculated by resolving its orbifold singularities. On the other hand, for $\beta=2$ one can use the identity

$$\frac{1}{\big(1-\mathrm{e}^{\,s}\big)\,\big(1-\mathrm{e}^{\,-2s}\big)}=-\frac12\,\frac{\mathrm{d}}{\mathrm{d}s}\,\frac{1}{1-\mathrm{e}^{\,s}}+\frac12\,\frac{1}{1-\mathrm{e}^{\,-2s}}-\frac12\,\frac{1}{1-\mathrm{e}^{\,-s}} \tag{2.37}$$

together with (2.30) to infer that

$$\chi_{2g-1}(2) = \sqrt{2}\, 2^{-g}\, \frac{\left(2^{2g-2} - \frac{1}{2}\right) B_{2g}}{2g\,(2g-1)} \qquad \text{and} \qquad \chi_{2g-2}(2) = 2^{-g}\, \chi_{\rm orb}(\mathcal{M}_g) \tag{2.38}$$

for $g \geq 1$, so that the coefficients $\chi_{2g-1}(2)$ are proportional to the orbifold Euler characteristics of the moduli spaces of certain real algebraic curves of genus g [28]. Thus in this case refinement corresponds to the replacement of $\chi_{\rm orb}(\mathcal{M}_g)$ with the *parameterized* Euler characters $\chi_n(\beta)$ [42, 41], which provide a geometric parameterization that interpolates between the orbifold Euler characters of the moduli spaces of closed oriented Riemann surfaces at $\beta = 1$ and closed unoriented Riemann surfaces with crosscap at $\beta = 2$. In other words, the string theory at $\beta = 2$ can be regarded as the orientifold of the string theory at $\beta = 1$. From this perspective, it is natural to expect that the generic β-deformed Euler characters $\chi_n(\beta)$ themselves describe characteristic classes of some related moduli spaces [28].

The expansion (2.32) governs the leading order behaviour of the dual refined B-model topological string amplitudes on the mirror of the conifold geometry, which is the cotangent bundle T^*S^3. Generally, these local Calabi-Yau geometries are described by algebraic equations of the form

$$u\, v + F(x, y) = 0 \tag{2.39}$$

in $\mathbb{C}^4$, where the equation $F(x, y) = 0$ describes an affine curve Σ in $\mathbb{C}^2$ and a (local) function $y(x)$ which determines a meromorphic differential $\lambda = y(x)\,\mathrm{d}x$ giving the periods of Σ. In the Gaussian matrix model (2.23) at $\beta = 1$, the Riemann surface Σ is the corresponding rational spectral curve which is given by a double cover of the y-plane with $F(x, y) = x^2 - y^2 + m = 0$, where m is the Kähler parameter of the resolved conifold. After a simple change of variables this spectral curve can be regarded as the holomorphic curve $F(z, w) = z\, w - m = 0$ in $\mathbb{C}^2$, which after refinement quantizes to the differential operator $F(\,\widehat{z}, \widehat{w}\,) = \hbar\, z\, \partial_z - m$ [22]; the quantum curve in this case is the canonical example of a D-module [17] and it can be regarded as a differential equation for certain correlators in the matrix model [16]. After q-deformation, the quantum spectral curve for the conifold is naturally described by a difference equation (rather than a differential equation) with difference operator

$$F(\,\widehat{X}, \widehat{Y}\,) = \left(1 - \underline{q}^{-1/2}\, \widehat{X}\,\right) \widehat{Y} - \left(1 - \underline{Q}\; \underline{q}\, \widehat{X}\,\right) , \tag{2.40}$$

where $\underline{Q} := \mathrm{e}^{-r^2\, m}$; it can be thought of as a differential equation for the partition functions of refined topological string theory [11]. It is then natural to expect that a similar quantum spectral curve governs the matrix model (2.18) of the q-deformed β-ensemble that represents the (q, t)-deformed gauge theory, along the lines of [54]. In the following, these lines of reasoning will be applied to the closed string chiral expansion of the (q, t)-deformed two-dimensional gauge theory to give geometrical interpretations of the Macdonald deformation in terms of contributions from deformed characteristic classes associated to quantum Riemann surfaces.

3 Generalized quantum characters as Hecke characters

In this section we develop a combinatorial description of the dimension factors for the quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_N)$ appearing in (2.2) in terms of characters of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$ of type A_{n-1}; our final result is summarised in Proposition 3.77. For this, we shall use quantum Schur-Weyl duality between $\mathcal{U}_q(\mathfrak{gl}_N)$ and $\mathsf{H}_q(\mathfrak{S}_n)$. See Appendix A for relevant definitions and properties of quantum groups which are used throughout, and Appendix B for those pertaining to Hecke algebras.

The (unrefined) q-deformation of the standard two-dimensional Yang-Mills theory can be obtained by replacing representations of G with quantum group modules, as in [33, 53]. Refinement then corresponds to the β-deformation wherein (quantum) characters are replaced with generalized characters; see e.g. [34, §2] and [53, §6.1]. If V, W are finite-dimensional representations of $\mathcal{U}_q(\mathfrak{gl}_N)$, and $\Phi : V \to V \otimes W$ is a non-zero intertwining operator for $\mathcal{U}_q(\mathfrak{gl}_N)$, then the vector-valued function

$$\chi_\Phi(U) = \mathrm{Tr}_V\big(\Phi\, U\big) \tag{3.1}$$

on the maximal torus $T \subset G$ is called a *generalized character*. Contrary to the classical case $q = 1$, if the representation W is non-trivial then $\chi_\Phi(U)$ is not invariant under the action of the Weyl group $\mathfrak{S}_N$ on T. Since the operator Φ preserves weight, the vector $\chi_\Phi(U)$ actually takes values in the weight zero subspace $W_0 \subset W$.

To compute the generalized character explicitly, let V^* denote the dual $\mathcal{U}_q(\mathfrak{gl}_N)$-module, and let v_i, v^i be dual bases for V, V^*. We can then identify Φ with an intertwiner $\Phi : V^* \otimes V \to W$ and

$$\chi_\Phi(U) = \Phi\big(v^i \otimes U v_i\big) \ , \tag{3.2}$$

where throughout we use the Einstein summation convention for repeated upper and lower indices. Since $v^i \otimes v_i = (\mathbb{1}_{V^*} \otimes q^{-(\rho,H)})\, \mathbf{1}_\mathbb{C}$, where $H = (H_1, \dots, H_N)$ are the Cartan generators of $\mathfrak{gl}_N$ and $\mathbf{1}_\mathbb{C} = \imath(1)$ with $\imath : \mathbb{C} \to V^* \otimes V$ an embedding of $\mathcal{U}_q(\mathfrak{gl}_N)$-modules, we can also write the generalized character as

$$\chi_\Phi(U) = \Phi\big((\mathbb{1}_{V^*} \otimes q^{-(\rho,H)}\, U)\mathbf{1}_\mathbb{C}\big) \ . \tag{3.3}$$

In the special instance where $W = \mathbb{C}$ is the trivial representation of $\mathcal{U}_q(\mathfrak{gl}_N)$ and $\Phi : V \to V$ is the identity operator, so that $\Phi : V^* \otimes V \to \mathbb{C}$ is the canonical dual pairing, then $\chi_{\mathbb{1}_V}(U) = \chi_V(U) = \mathrm{Tr}_V(U)$ is the usual character of U in the representation V.

Now let $V = R_\lambda$ for fixed $\lambda \in \Lambda_+$ and $W = W_{\beta-1}$ for fixed $\beta \in \mathbb{Z}_{>0}$ where

$$W_{\beta-1} := R_{\omega_1}^{\odot(\beta-1)\,N} \otimes (\det)^{-(\beta-1)} \tag{3.4}$$

is the q-deformation of the traceless $(\beta-1)\,N$-th symmetric power of the first fundamental representation $R_{\omega_1} = \mathbb{C}^N$ of G, which is a finite-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{gl}_N)$ with highest weight $(\beta-1)\,N\,\omega_1 - (\beta-1)\,(1,\dots,1) = (\beta-1)\,(N-1,-1,\dots,-1)$. By [21, Lemma 1], the space of intertwining operators $\mathrm{Hom}_{\mathcal{U}_q(\mathfrak{gl}_N)}\big(R_{\lambda'}, R_{\lambda'} \otimes W_{\beta-1}\big)$ for $\mathcal{U}_q(\mathfrak{gl}_N)$ is one-dimensional if $\lambda' = \lambda_\beta := \lambda + (\beta-1)\,\rho$ for a highest weight λ and zero otherwise; recall that λ is a dominant weight of $\mathfrak{gl}_N$ if and only if it is of the form

$$\lambda = a\,(1,\dots,1) + \sum_{i=1}^N n_i\,\omega_i \tag{3.5}$$

for some $n_i \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{C}$, where $\omega_i = (1^i\ 0^{N-i})$, $i = 1,\dots,N$ are the fundamental weights of $\mathfrak{gl}_N$. It follows that a non-zero $\mathcal{U}_q(\mathfrak{gl}_N)$-homomorphism $\Phi_\lambda : R_{\lambda_\beta} \to R_{\lambda_\beta} \otimes W_{\beta-1}$ is unique up to normalization. As the weight zero subspace $\big(W_{\beta-1}\big)_0$ is one-dimensional, the corresponding generalized character

$$\chi_{\Phi_\lambda}(U) := \mathrm{Tr}_{R_{\lambda_\beta}}\big(\Phi_\lambda\, U\big) \tag{3.6}$$

can be regarded as taking values in $\mathbb{C}$. By [21, Theorem 1], if λ is a partition these generalized characters are given in terms of the monic form $M_\lambda(x;q,t)$ of the Macdonald polynomials at $t = q^\beta$, where $U = \mathrm{e}^{(z,H)}$ and $x = \mathrm{e}^{z}$. We choose the normalization of Φ_λ and the identification $\big(W_{\beta-1}\big)_0 \cong \mathbb{C}$ in such a way so that

$$\chi_{\Phi_\lambda}(U) = \frac{M_\lambda(x;q,t)}{\sqrt{g_\lambda}} \ . \tag{3.7}$$

In the unrefined limit $\beta = 1$, we have $g_\lambda = 1$ and the Macdonald polynomials reduce to the Schur polynomials $M_\lambda(x;q,q) = s_\lambda(x)$ (independently of q), which coincide with the ordinary characters $\chi_{R_\lambda}(U) = \mathrm{Tr}\,_{R_\lambda}(U)$ of the irreducible representation R_λ.

3.2 Quantum Schur-Weyl duality

For $n \geq 2$, the actions of $\mathcal{U}_q(\mathfrak{gl}_N)$ and $\mathsf{H}_q(\mathfrak{S}_n)$ on $R_{\omega_1}^{\otimes n}$ are given respectively by the iterated coproduct $\Delta^{n-1} = (\Delta \otimes \mathbb{1}^{\otimes (n-1)}) \circ \cdots \circ (\Delta \otimes \mathbb{1}) \circ \Delta$ (see Appendix A) and by q-deformation of permutation of the factors (to be specified in §3.3 below). These actions commute, and so $R_{\omega_1}^{\otimes n}$ is a representation of the product $\mathcal{U}_q(\mathfrak{gl}_N) \times \mathsf{H}_q(\mathfrak{S}_n)$ which is completely reducible to the form

$$R_{\omega_1}^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda_+^n} R_\lambda \otimes r_\lambda \ . \tag{3.8}$$

Here $\Lambda_+^n \subset \Lambda_+$ is the set of partitions λ of n with at most N parts, i.e. $|\lambda| = n$, R_λ is the corresponding irreducible representation of $\mathcal{U}_q(\mathfrak{gl}_N)$, and r_λ is the representation of $\mathsf{H}_q(\mathfrak{S}_n)$ associated with λ. Letting P_λ denote the quantum Young projector for the representation r_λ, one has

$$P_\lambda R_{\omega_1}^{\otimes n} \cong R_\lambda \otimes r_\lambda \ . \tag{3.9}$$

Using quantum Schur-Weyl duality we can write the generalized characters $\chi_{\Phi_\lambda}(U)$ for $\lambda_\beta \in \Lambda_+^n$ as combinatorial expansions over the symmetric group $\mathfrak{S}_n$ involving characters of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$. For this, we introduce a $\mathcal{U}_q(\mathfrak{gl}_N)$-intertwiner

$$\Phi_n \ : \ R_{\omega_1}^{\otimes n} \ \longrightarrow \ R_{\omega_1}^{\otimes n} \otimes W_{\beta-1} \tag{3.10}$$

for each $n \geq 0$, which can be defined in the following (non-canonical) way: As a $\mathcal{U}_q(\mathfrak{gl}_N)$-module the vector space $R_{\omega_1}^{\otimes n}$ decomposes into irreducible unitary representations as

$$R_{\omega_1}^{\otimes n} = \bigoplus_{\lambda \in \Lambda_+^n} R_\lambda^{\oplus d_\lambda(1)} \tag{3.11}$$

where $d_\lambda(1) = \dim(r_\lambda)$. We can use the projector property $\sum_{\lambda \in \Lambda_+^n} P_\lambda = \mathbb{1}_{R_{\omega_1}^{\otimes n}}$ to write

$$\Phi_n = \sum_{\lambda,\mu \in \Lambda_+^n} \big(P_\mu \otimes \mathbb{1}_{W_{\beta-1}}\big)\, \Phi_n\, P_\lambda \ , \tag{3.12}$$

with

$$\big(P_\mu \otimes \mathbb{1}_{W_{\beta-1}}\big)\, \Phi_n\, P_\lambda := \delta_{\lambda,\mu} \sum_{\lambda \in \Lambda_+^n} \Phi_{\lambda_{\beta-2}} \otimes \mathbb{1}_{r_\lambda} \ , \tag{3.13}$$

where we used (3.11) and $\Phi_{\lambda_{\beta-2}} \in \mathrm{Hom}_{\mathcal{U}_q(\mathfrak{gl}_N)}\big(R_\lambda, R_\lambda \otimes W_{\beta-1}\big)$. In the large N limit, if λ is a dominant weight then so are λ_β and $\lambda_{\beta-2}$, and thus $\mathrm{Hom}_{\mathcal{U}_q(\mathfrak{gl}_N)}\big(R_\lambda, R_\lambda \otimes W_{\beta-1}\big)$ is non-zero and one-dimensional if $\lambda \in \Lambda_+$. This gives an identification of underlying linear transformations

$$\Phi_n = \bigoplus_{\lambda \in \Lambda_+^n} \Phi_{\lambda_{\beta-2}} \otimes \mathbb{1}_{r_\lambda} \ . \tag{3.14}$$

By [33, eq. (2.28)], the quantum Young projector can be written explicitly as the central element of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$ given by

$$P_\lambda = \frac{d_\lambda(q)}{q^{\frac{n\,(n-1)}{4}}\,[n]_q!} \sum_{\sigma \in \mathfrak{S}_n} q^{-\ell(\sigma)}\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma^{-1})\big)\ \mathsf{h}(\sigma) \ , \tag{3.15}$$

where $\ell(\sigma)$ is the length of the permutation $\sigma \in \mathfrak{S}_n$, $\mathsf{h}(\sigma) \in \mathsf{H}_q(\mathfrak{S}_n)$ is the Hecke algebra element associated to $\sigma \in \mathfrak{S}_n$ and $\chi_{r_\lambda}\big(\mathsf{h}(\sigma^{-1})\big)$ are the characters of the irreducible representation r_λ of the Hecke algebra; here $d_\lambda(q) = \dim_q(r_\lambda)$ is a q-deformation of the dimension of the symmetric group representation r_λ, see Appendix B. Then we evaluate the trace $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,P_\lambda\big)$ in two different ways. Firstly, using (3.9) and (3.11) along with (3.2) we easily get

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,P_\lambda\big) \ = \ \mathrm{Tr}_{R_\lambda}(\Phi_{\lambda_{\beta-2}}\,U)\,d_\lambda(1) \ = \ \chi_{\Phi_{\lambda_{\beta-2}}}(U)\,d_\lambda(1) \ . \tag{3.16}$$

Secondly, we substitute the explicit expansion (3.15), and hence for any weight $\lambda \in \Lambda_+^n$ we can express the vector-valued trace as

$$\mathrm{Tr}_{R_\lambda}(\Phi_{\lambda_{\beta-2}}\,U) \ = \ \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_\lambda(q)}{d_\lambda(1)} \sum_{\sigma \in \mathfrak{S}_n} q^{-\ell(\sigma)}\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma^{-1})\big)\ \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{h}(\sigma)\big) \ . \tag{3.17}$$

It will prove useful later on to derive directly a formula for the inverse of this transformation of characters, generalizing [33, eq. (2.21)].

Lemma 3.18 $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{h}(\sigma)\big) = \sum_{\lambda \in \Lambda_+^n} \chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)\ \chi_{\Phi_{\lambda_{\beta-2}}}(U)$.

Proof: Starting from the projector and centrality properties of P_λ in the Hecke algebra along with (3.13) we compute

$$\begin{aligned}
\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{h}(\sigma)\big) \ &= \sum_{\lambda,\mu \in \Lambda_+^n} \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big((P_\mu \otimes \mathbb{1}_{W_{\beta-1}})\,(P_\mu \otimes \mathbb{1}_{W_{\beta-1}})\,\Phi_n\,P_\lambda\,U\,\mathsf{h}(\sigma)\,P_\lambda\big) \\
&= \sum_{\lambda \in \Lambda_+^n} \mathrm{Tr}_{R_\lambda \otimes r_\lambda}\Big(\big(\Phi_{\lambda_{\beta-2}} \otimes \mathbb{1}_{r_\lambda}\big)\,\big(U \otimes \mathsf{h}(\sigma)\big)\Big) \\
&= \sum_{\lambda \in \Lambda_+^n} \mathrm{Tr}_{R_\lambda}\big(\Phi_{\lambda_{\beta-2}}\,U\big)\,\mathrm{Tr}_{r_\lambda}\big(\mathsf{h}(\sigma)\big) \ = \sum_{\lambda \in \Lambda_+^n} \mathrm{Tr}_{R_\lambda}\big(\Phi_{\lambda_{\beta-2}}\,U\big)\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)
\end{aligned} \tag{3.19}$$

as required. ■

By multiplying the left-hand side and the right-hand side of the character formula in Lemma 3.18 with $q^{-\ell(\sigma)}\,\chi_{r_{\lambda'}}\big(\mathsf{h}(\sigma^{-1})\big)$, summing over all permutations $\sigma \in \mathfrak{S}_n$ and using the orthogonality relations for Hecke characters [33]

$$\sum_{\sigma \in \mathfrak{S}_n} q^{-\ell(\sigma)}\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)\,\chi_{r_{\lambda'}}\big(\mathsf{h}(\sigma^{-1})\big) = \delta_{\lambda,\lambda'}\ q^{\frac{n\,(n-1)}{4}}\,[n]_q!\,\frac{d_\lambda(1)}{d_\lambda(q)} \ , \tag{3.20}$$

we arrive at the expression (3.17).

We are finally ready to derive our Hecke character expansion for the refined quantum dimensions (2.4). Firstly we note that the refined quantum dimension (2.4) and the Macdonald metric (2.6) are both invariant under any shift of the dominant weight λ by the maximal partition $(1^N) := (1, \dots, 1)$ of length N, i.e.

$$\dim_{q,t}(R_{\lambda + a\,(1^N)}) = \dim_{q,t}(R_\lambda) \qquad \text{and} \qquad g_{\lambda + a\,(1^N)} = g_\lambda \ . \tag{3.21}$$

We shall assume that a is an integer. The refined quantum dimension is obtained by the specialization $U = t^{(\rho,H)}$ in the generalized characters (3.7), i.e.

$$\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} = \chi_{\Phi_\lambda}\big(q^{\beta\,(\rho,H)}\big) = \mathrm{Tr}_{\,R_{\lambda_\beta}}\big(\Phi_\lambda\, q^{\beta\,(\rho,H)}\big) \ . \tag{3.22}$$

We wish to substitute in the expansion (3.17), but the Hecke characters and dimensions are only defined for partitions, whereas λ_β is not necessarily a partition. Hence we use the shift symmetry (3.21) to get a partition $\lambda_\beta + a\,(1^N)$, which is true as long as $a \geq \frac{N-1}{2}\,(\beta - 1)$. For definiteness we use the lowest value

$$a = \tfrac{N-1}{2}\,(\beta - 1) \tag{3.23}$$

which for large N can be regarded as integral. In the large N expansion that we consider later on, we will typically also consider the limit $\beta \to 1$ such that the quantity $a\,N$ is finite. Then we get

$$\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} = \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_{\lambda_\beta + a\,(1^N)}(q)}{d_{\lambda_\beta + a\,(1^N)}(1)} \sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)}\,\chi_{r_{\lambda_\beta + a\,(1^N)}}\big(\mathsf{h}(\sigma^{-1})\big)\ \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\, q^{\beta\,(\rho,H)}\,\mathsf{h}(\sigma)\big) \tag{3.24}$$

for $\lambda_\beta + a\,(1^N) \in \Lambda_+^n$, i.e. $\lambda \in \Lambda_+^{n-a\,N}$. We can easily check that this expression agrees with the anticipated formula for the quantum dimension $\dim_q(R_\lambda) = \mathrm{Tr}_{\,R_\lambda}\big(q^{(\rho,H)}\big)$ in the unrefined limit: In the limit $\beta = 1$, we have $\lambda_\beta = \lambda$, $a = 0$, and the intertwiners Φ_λ and Φ_n become identity operators on finite-dimensional modules, so that (3.24) coincides with the q-dimension formula from [33, eq. (2.33)].

To manipulate the sum in (3.24), let us first explicitly specify how the Hecke operators act. From [37] we can express $\mathsf{h}(\sigma) \in \mathsf{H}_q(\mathfrak{S}_n)$ as a product of generators g_i, $i = 1, \dots, n-1$ in the same way that we express $\sigma \in \mathfrak{S}_n$ in the form of a reduced word; we say that $\mathsf{h}(\sigma) \in \mathsf{H}_q(\mathfrak{S}_n)$ is a reduced word if $\sigma \in \mathfrak{S}_n$ is a reduced word. From [33], g_i acts on $R_{\omega_1}^{\otimes n}$ as the braiding operator

$$\mathsf{g} = q^{1/2}\ \check{\mathsf{R}} \tag{3.25}$$

on the tensor product $R_{\omega_1} \otimes R_{\omega_1}$ in the $(i, i+1)$ slot of $R_{\omega_1}^{\otimes n}$, where $\check{\mathsf{R}} = \mathsf{P}\,\mathsf{R}$ with P the flip operator $\mathsf{P}(v \otimes w) = w \otimes v$; here R is the FRT quantum R-matrix [23]

$$\mathsf{R} = q^{1/2} \sum_{i=1}^N H_i \otimes H_i + \sum_{i\neq j} H_i \otimes H_j + \big(q^{1/2} - q^{-1/2}\big) \sum_{i>j} E_{ij} \otimes E_{ji} \ , \tag{3.26}$$

where $H_i = E_{ii}$ and E_{ij} for $i, j = 1, \dots, N$ act on the standard basis $\{e_k\} \subset R_{\omega_1} = \mathbb{C}^N$ of the fundamental representation as

$$E_{ij} e_k = \delta_{jk}\ e_i \ . \tag{3.27}$$

Let us define the *(q,t)-trace* of an element $x \in \mathsf{H}_q(\mathfrak{S}_n)$ by $\mathrm{Tr}_{\,q,t}(x) := \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\, U\, x\big)$, where $U = t^{(\rho,H)}$ and Φ_n is the intertwiner (3.10). This terminology is justified by

Lemma 3.28 *The (q,t)-trace is cyclic:* $\mathrm{Tr}_{\,q,t}(x\,y)=\mathrm{Tr}_{\,q,t}(y\,x)$ *for all* $x,y\in \mathsf{H}_q(\mathfrak{S}_n)$.

Proof: Since the reduced words $\{\mathsf{h}(\sigma)\}_{\sigma\in\mathfrak{S}_n}$ form a basis of $\mathsf{H}_q(\mathfrak{S}_n)$, we can express $\mathrm{Tr}_{\,q,t}(x\,y)$ as a linear combination of (q,t)-traces $\mathrm{Tr}_{\,q,t}\big(\mathsf{h}(\sigma)\,\mathsf{h}(\tau)\big)$, and therefore we only need to show that $\mathrm{Tr}_{\,q,t}\big(\mathsf{h}(\sigma)\,\mathsf{h}(\tau)\big)=\mathrm{Tr}_{\,q,t}\big(\mathsf{h}(\tau)\,\mathsf{h}(\sigma)\big)$ for all $\sigma,\tau\in\mathfrak{S}_n$. We first prove that Φ_n and the Hecke algebra generators g_i commute. Let us consider a fixed element f_j of a basis $\{f_i\}\subset W_{\beta-1}$, with corresponding dual basis $\{f^i\}$. If we restrict the codomain of Φ_n to f_j, then ${}_{f_j}|\Phi_n\in\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_N)}\big(R_{\omega_1}^{\otimes n}\big)$ and using quantum Schur-Weyl duality we can decompose this restriction as

$$ {}_{f_j}|\Phi_n=\bigoplus_{\lambda\in\Lambda_+^n}\big(\,{}_{f_j}|\Phi_n\,\big)\big|_{R_\lambda}\otimes\mathbb{1}_{r_\lambda}\ , \tag{3.29} $$

where $(\,{}_{f_j}|\Phi_n\,)|_{R_\lambda}\in\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_N)}(R_\lambda)$. It follows that ${}_{f_j}|\Phi_n$ acts on the Hecke algebra representation r_λ as the identity, and so ${}_{f_j}|\Phi_n$ and g_i commute. Note that g_i commutes with $U^{\otimes n}$, because both P and $\mathsf{R}\in\mathcal{U}_q(\mathfrak{gl}_N)\otimes\mathcal{U}_q(\mathfrak{gl}_N)$ commute with $t^{(\rho,H)}\otimes t^{(\rho,H)}$. Thus we get

$$ \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\,{}_{f_j}|\Phi_n\,U\,x\,y\,\big)=\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\,{}_{f_j}|\Phi_n\,U\,y\,x\,\big) \tag{3.30} $$

and hence

$$ \begin{aligned} \mathrm{Tr}_{\,q,t}(x\,y) &= \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,x\,y\big)\\ &= f^j\big(\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}(\,{}_{f_j}|\Phi_n\,U\,x\,y\,)\big)\ f_j\\ &= \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,y\,x\big)\ =\ \mathrm{Tr}_{\,q,t}(y\,x)\ , \end{aligned} \tag{3.31} $$

as required. ∎

We can use Lemma 3.28 to truncate the reduced words $\mathsf{h}(\sigma)$ to minimal words in the sum (3.24). A word is said to be *minimal* if it is both reduced and contains no generators g_i more than once. Using the Hecke relations on g_i and cyclicity of the (q,t)-trace, we can then truncate the sum in (3.24) to conjugacy classes T in $\mathfrak{S}_n$ and express the (q,t)-trace $\mathrm{Tr}_{\,q,t}\big(\mathsf{h}(\sigma)\big)$ for any $\sigma\in T$ as the (q,t)-trace $\mathrm{Tr}_{\,q,t}\big(\mathsf{h}(m_T)\big)$ of the minimal word $m_T\in T$ in the conjugacy class [37]. Hence following the derivation of [33, eq. (2.38)], we can write the expansion (3.24) as

$$ \frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}}=\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_{\lambda_\beta+a\,(1^N)}(q)}{d_{\lambda_\beta+a\,(1^N)}(1)}\sum_{T\in\mathfrak{S}_n^\vee}q^{-\ell(T)}\,\chi_{r_{\lambda_\beta+a\,(1^N)}}(C_T)\ \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\,q^{\beta\,(\rho,H)}\,\mathsf{h}(m_T)\big) \tag{3.32} $$

where $\ell(T)$ is the length of the permutation $m_T\in\mathfrak{S}_n$ and C_T are the same central elements of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$ as in [33]. To write C_T explicitly, we need to express an arbitrary character of the Hecke algebra in terms of characters of minimal words [37] using Lemma 3.28 and the Hecke relations (B.2) from Appendix B as

$$ \chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)=\sum_{T\in\mathfrak{S}_n^\vee}\alpha_T(\sigma)\,\chi_{r_\lambda}\big(\mathsf{h}(m_T)\big)\ , \tag{3.33} $$

where the expansion coefficients $\alpha_T(\sigma)$ do not depend on the representation r_λ. Then the central element C_T is defined by

$$ C_T=q^{\ell(T)}\sum_{\sigma\in\mathfrak{S}_n}q^{-\ell(\sigma)}\,\alpha_T(\sigma^{-1})\,\mathsf{h}(\sigma)\ . \tag{3.34} $$

The quantum Young projector (3.15) can be rewritten as

$$ P_\lambda=\frac{d_\lambda(q)}{q^{\frac{n\,(n-1)}{4}}\,[n]_q!}\sum_{T\in\mathfrak{S}_n^\vee}q^{-\ell(T)}\,\chi_{r_\lambda}\big(\mathsf{h}(m_T)\big)\,C_T\ . \tag{3.35} $$

This transformation can be inverted to express C_T in terms of the central elements P_λ, because the determinant of the transformation matrix is non-zero. To see that the determinant of $\chi_{r_\lambda}\big(\mathsf{h}(m_T)\big)$ is non-zero, we use the orthogonality relation (3.20) and expand it into minimal words using (3.33). Then we take the determinant of the obtained expression and use multiplicativity of the determinant to find that it is non-vanishing. Hence the centrality of the elements C_T follows from the centrality of the projectors.

3.4 (q,t)-traces of minimal words

We are left with the problem of evaluating the (q,t)-traces $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\, q^{\beta\,(\rho,H)}\, \mathsf{h}(m_T)\big)$ of minimal words; we shall follow the strategy of [33, Appendix B]. For $n=1$, the R-matrix $\check{\mathsf{R}}$ acts trivially while $U e_i = t^{\rho_i}\, e_i$, where $\rho_i = \frac{N+1}{2} - i$. Using

$$\dim_{q,t}(R_{\omega_1}) = [N]_t = \mathrm{Tr}_{R_{\omega_1}}(U) \qquad \text{and} \qquad g_{\omega_1} = g_\emptyset\ \frac{[N]_t}{\big[\beta\,(N-1)+1\big]_q} \tag{3.36}$$

where generally

$$[N]_{t^k} := \frac{[k\,\beta\,N]_q}{[k\,\beta]_q}\ , \tag{3.37}$$

we then easily find for the (q,t)-trace

$$\mathrm{Tr}_{R_{\omega_1}}\big(\Phi_1\, U\big) = \frac{\dim_{q,t}(R_{\omega_1})}{\sqrt{g_{\omega_1}}} = \bigg(\frac{[N]_t\,\big[\beta\,(N-1)+1\big]_q}{g_\emptyset}\bigg)^{1/2}\ . \tag{3.38}$$

Note that here the intertwining operator $\Phi_1 = \Phi_{\omega_1} : R_{\omega_1} \to R_{\omega_1} \otimes W_{\beta-1}$ simply acts in the (q,t)-trace of 1 to rescale the normalization of the trace of U by the Macdonald measure factor $(g_{\omega_1})^{-1/2}$. In the unrefined limit $\beta = 1$, this expression reduces as expected to the quantum dimension of the fundamental representation $\dim_q(R_{\omega_1}) = [N]_q$. Below we shall also need the generalization of the trace formula in (3.36) to arbitrary powers U^k, $k \in \mathbb{Z}_{>0}$, which is given by

$$\mathrm{Tr}_{R_{\omega_1}}\big(U^k\big) = [N]_{t^k}\ . \tag{3.39}$$

Next we set $n=2$. We can use the FRT formula (3.26) for the R-matrix to compute

$$\big(U^k \otimes \mathbb{1}_{R_{\omega_1}}\big)\, \check{\mathsf{R}}(e_i \otimes e_j) = t^{k\,\rho_j}\, e_j \otimes e_i + t^{k\,\rho_j}\,\big(q^{1/2}-1\big)\,\delta_{ij}\, e_i \otimes e_j + t^{k\,\rho_i}\,\big(q^{1/2}-q^{-1/2}\big)\,\theta_{ij}\, e_i \otimes e_j \tag{3.40}$$

for any $k \in \mathbb{Z}_{>0}$, where $\theta_{ij} := 1$ if $i<j$ and $\theta_{ij} := 0$ otherwise. From this expression one can easily derive the partial traces

$$\big(\mathrm{Tr}_{R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}\big)\Big(\big(U^k \otimes \mathbb{1}_{R_{\omega_1}}\big)\,\mathsf{g}_1\Big) = t^{k\,\frac{N+1}{2}}\ \frac{q-1}{t^k-1}\ \mathbb{1}_{R_{\omega_1}} + \frac{t^k-q}{t^k-1}\ U^k \tag{3.41}$$

as operators in $\mathcal{U}_q(\mathfrak{gl}_N)$ acting on the fundamental representation R_{ω_1}. In the unrefined limit $t=q$ at $k=1$, this operator reduces to $q^{\frac{N+1}{2}}\,\mathbb{1}_{R_{\omega_1}}$ as in [33, eq. (B.5)]; in the general case it is also diagonal but no longer proportional to the identity operator.

Let us decompose the representation $W_{\beta-1}$ into its one-dimensional weight subspaces $\big(W_{\beta-1}\big)_\alpha \cong \mathbb{C}w^\alpha$; in particular, the isomorphism $\big(W_{\beta-1}\big)_0 \cong \mathbb{C}$ is given by mapping $w^0 \mapsto 1$. Then one can find explicit formulas for the matrix elements of $\Phi_2 : R_{\omega_1}^{\otimes 2} \to R_{\omega_1}^{\otimes 2} \otimes W_{\beta-1}$ in the following way: We write

$$\Phi_2(e_i \otimes e_j) = {P_{ij}}^{kl}{}_\alpha\ e_k \otimes e_l \otimes w^\alpha\ . \tag{3.42}$$

Then the condition that Φ_2 is an intertwiner can be rewritten as a system of linear equations for the expansion coefficients ${P_{ij}}^{kl}{}_\alpha$. Since Φ_2 is uniquely determined up to the normalization in (3.7), this linear system has a unique solution which determines ${P_{ij}}^{kl}{}_\alpha$ as a rational function in $q^{1/2}$ and $t^{1/2}=q^{\beta/2}$; with (3.7) the intertwining operators $\Phi_\lambda : R_{\lambda_\beta} \to R_{\lambda_\beta}\otimes W_{\beta-1}$ are normalized in such a way that if v_{λ_β} is a highest weight vector of R_{λ_β}, then the component of $\Phi_\lambda(v_{\lambda_\beta})$ in $R_{\lambda_\beta}\otimes \big(W_{\beta-1}\big)_0$ is $(g_{\lambda_\beta})^{-1/2}\, v_{\lambda_\beta}\otimes w^0$. Setting ${P_{ij}}^{kl} := {P_{ij}}^{kl}{}_0$, the (q,t)-trace of the generator g_1 can then be written as

$$\begin{aligned}
\mathrm{Tr}_{R_{\omega_1}^{\otimes 2}}\big(\Phi_2\, U\, \mathsf{g}_1\big) \;&:=\; \big(\mathrm{Tr}_{R_{\omega_1}}\otimes \mathrm{Tr}_{R_{\omega_1}}\big)\big(\Phi_2\,(U\otimes U)\,\mathsf{g}_1\big) \\
&=\; q^{1/2}\, t^{N+1}\Big(\sum_{i,j=1}^N t^{-i-j}\, {P_{ji}}^{ij} + \big(q^{1/2}-1\big)\sum_{i=1}^N t^{-2i}\, {P_{ii}}^{ii} \\
&\qquad\qquad + \big(q^{1/2}-q^{-1/2}\big)\sum_{i<j} t^{-i-j}\, {P_{ij}}^{ij}\Big)\ . \qquad (3.43)
\end{aligned}$$

In the unrefined limit $\beta=1$, we have ${P_{ij}}^{kl}=\delta_i{}^k\,\delta_j{}^l$ and it is easy to check that this expression reduces to $q^{\frac{N+1}{2}}\,[N]_q$ as in [33, eq. (B.5)]. In the general case we have

Lemma 3.44 ${P_{ij}}^{kl} = (g_{\omega_1})^{-1}\ \delta_i{}^k\,\delta_j{}^l$.

Proof: By $\mathcal{U}_q(\mathfrak{gl}_N)$-equivariance we have the relations

$$\Phi_2\,\Delta(H_p) = \Delta^2(H_p)\,\Phi_2 \qquad \text{for} \quad p=1,\dots,N \qquad (3.45)$$

as operators on $R_{\omega_1}^{\otimes 2}\to R_{\omega_1}^{\otimes 2}\otimes W_{\beta-1}$, where $\Delta(H_p)=H_p\otimes \mathbb{1}+\mathbb{1}\otimes H_p$ and $\Delta^2(H_p)=(\Delta\otimes\mathbb{1})\Delta(H_p)$. We evaluate both sides of this equation on a generic basis element $e_i\otimes e_j$ of $R_{\omega_1}\otimes R_{\omega_1}$, and denote the action of the Cartan generators on the weight subspaces of $W_{\beta-1}$ as $H_p w^\alpha = \alpha_p\, w^\alpha$ with $\alpha_p\in\mathbb{C}$. Then the equality can be written as

$$\begin{aligned}
&\big({P_{pj}}^{kl}{}_\alpha\,\delta_{pi} + {P_{ip}}^{kl}{}_\alpha\,\delta_{pj}\big)\ e_k\otimes e_l\otimes w^\alpha \qquad (3.46)\\
&\qquad = {P_{ij}}^{kl}{}_\alpha\,\big(\delta_{pk}\ e_p\otimes e_l + \delta_{pl}\ e_k\otimes e_p + \alpha_p\ e_k\otimes e_l\big)\otimes w^\alpha\ .
\end{aligned}$$

In particular, for the weight $\alpha=0$ component we obtain the constraints

$${P_{ij}}^{kl}\ (\delta_{pk}+\delta_{pl}-\delta_{pi}-\delta_{pj}) = 0 \qquad (3.47)$$

for all $i,j,k,l,p=1,\dots,N$. The tensor ${P_{ij}}^{kl}=\delta_i{}^k\,\delta_j{}^l$ solves this equation, and it is the unique solution up to normalization. The normalization is found as above by observing that the intertwiner Φ_2 acts in the (q,t)-trace as a multiple of the identity operator $\mathbb{1}_{R_{\omega_1}}\otimes\mathbb{1}_{R_{\omega_1}}$, with proportionality constant $(g_{\omega_1})^{-1}$. ■

Using Lemma 3.44 we can straightforwardly express the (q,t)-trace (3.43) in terms of q-numbers as

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes 2}}\big(\Phi_2\, U\, \mathsf{g}_1\big) = \frac{\big[\beta\,(N-1)+1\big]_q}{g_\emptyset}\,\Big(t^{\frac{N+1}{2}}\,\frac{q-1}{t-1}+\frac{t-q}{t-1}\,\frac{[N]_{t^2}}{[N]_t}\Big)\ . \qquad (3.48)$$

Next we have to generalise this expression to the (q,t)-trace of the connected minimal word $\mathsf{g}_1\,\mathsf{g}_2\cdots\mathsf{g}_{n-1}$ for $n\ge 2$. Defining

$$\Phi_n(e_{i_1}\otimes\cdots\otimes e_{i_n}) = {P_{i_1\cdots i_n}}^{j_1\cdots j_n}{}_\alpha\ e_{j_1}\otimes\cdots\otimes e_{j_n}\otimes w^\alpha \qquad (3.49)$$

and setting ${P_{i_1\cdots i_n}}^{j_1\cdots j_n} := {P_{i_1\cdots i_n}}^{j_1\cdots j_n}{}_0$, by a completely analogous argument to that used in the proof of Lemma 3.44 one obtains the constraints

$${P_{i_1\cdots i_n}}^{j_1\cdots j_n}\ (\delta_{p\,j_1}+\cdots+\delta_{p\,j_n}-\delta_{p\,i_1}-\cdots-\delta_{p\,i_n}) = 0 \qquad (3.50)$$

for all $p, i_1, \dots, i_n, j_1, \dots, j_n = 1, \dots, N$, and $P_{i_1 \cdots i_n}{}^{j_1 \cdots j_n} = \delta_{i_1}{}^{j_1} \cdots \delta_{i_n}{}^{j_n}$ solves this; thus again Φ_n acts in the (q,t)-trace as a multiple of the identity operator $\mathbb{1}_{R_{\omega_1}}^{\otimes n}$ with the normalization determined as before to be $(g_{\omega_1})^{-n/2}$, and we find

$$P_{i_1 \cdots i_n}{}^{j_1 \cdots j_n} = (g_{\omega_1})^{-n/2} \ \delta_{i_1}{}^{j_1} \cdots \delta_{i_n}{}^{j_n} \ . \tag{3.51}$$

We can use this result together with the partial trace formula (3.41) and the traces (3.39) to calculate the (q,t)-trace

$$\begin{aligned} \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}} \big(\Phi_n \, U \, \mathsf{g}_1 \, \mathsf{g}_2 \cdots \mathsf{g}_{n-1}\big) \ &:= \ \big(\mathrm{Tr}_{\,R_{\omega_1}}\big)^{\otimes n} \big(\Phi_{\omega_1}^{\otimes n} \, U^{\otimes n} \, \mathsf{g}_1 \, \mathsf{g}_2 \cdots \mathsf{g}_{n-1}\big) \\ &= \ (g_{\omega_1})^{-n/2} \ \mathrm{Tr}_{\,R_{\omega_1}} \Big(U \, \big(\mathrm{Tr}_{\,R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}\big) \big((U \otimes \mathbb{1}_{R_{\omega_1}}) \, \mathsf{g}_1\big) \\ &\qquad \cdots \ \big(\mathrm{Tr}_{\,R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-2)}\big) \big((U \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-2)}) \, \mathsf{g}_1\big) \ \big(\mathrm{Tr}_{\,R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-1)}\big) \big((U \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-1)}) \, \mathsf{g}_1\big)\Big) \end{aligned} \tag{3.52}$$

recursively in n. To simplify the derivation, let us introduce the short-hand notation

$$\xi_k := t^{k \, \frac{N+1}{2}} \ \frac{q-1}{t^k - 1} \qquad \text{and} \qquad \varphi_k := \frac{t^k - q}{t^k - 1} \ . \tag{3.53}$$

Using (3.41), we then compute the first partial trace from (3.52) as

$$\mathcal{O}_1 \ := \ \big(\mathrm{Tr}_{\,R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-1)}\big) \big((U \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-1)}) \, \mathsf{g}_1\big) = \xi_1 \, \mathbb{1}_{R_{\omega_1}}^{\otimes (n-1)} + \varphi_1 \, U \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-2)} \ . \tag{3.54}$$

The factors of U can be cyclically permuted in the partial traces, so for $0 \leq m \leq n-1$ the m-th partial trace

$$\begin{aligned} \mathcal{O}_m \ \ := \ \ & \big(\mathrm{Tr}_{\,R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-m)}\big) \big((U \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-m)}) \, \mathsf{g}_1\big) \\ & \cdots \ \big(\mathrm{Tr}_{\,R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-2)}\big) \big((U \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-2)}) \, \mathsf{g}_1\big) \ \big(\mathrm{Tr}_{\,R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-1)}\big) \big((U \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-1)}) \, \mathsf{g}_1\big) \end{aligned} \tag{3.55}$$

can be written as

$$\mathcal{O}_m = \sum_{k=0}^{m} f_k^{(m)}[\xi, \varphi] \ U^k \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-m-1)} \ , \tag{3.56}$$

where $f_k^{(m)}[\xi, \varphi]$ for $0 \leq k \leq m$ are polynomials in ξ_l and φ_l with $l \leq k$. We derive a recursion relation for $f_k^{(m)}$ inductively by writing

$$\mathcal{O}_{m+1} = \sum_{k=0}^{m} f_k^{(m)} \ \big(\mathrm{Tr}_{\,R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-m-1)}\big) \big((U^{k+1} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-m-1)}) \, \mathsf{g}_1\big) \tag{3.57}$$

and using (3.41) to get

$$\mathcal{O}_{m+1} = \sum_{k=0}^{m} f_k^{(m)} \, \xi_{k+1} \, \mathbb{1}_{R_{\omega_1}}^{\otimes (n-m-1)} + \sum_{k=0}^{m} f_k^{(m)} \, \varphi_{k+1} \, U^{k+1} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes (n-m-2)} \ . \tag{3.58}$$

Comparing this with the expansion (3.56) for $\mathcal{O}_{m+1}$ yields the recursion relations

$$f_0^{(m+1)} \ = \ \sum_{k=0}^{m} f_k^{(m)} \, \xi_{k+1} \ , \tag{3.59}$$

$$f_{k+1}^{(m+1)} \ = \ f_k^{(m)} \, \varphi_{k+1} \tag{3.60}$$

with initial condition $f_0^{(0)} = 1$.

For $k > 0$ we can use (3.60) to express $f_k^{(m)}$ entirely in terms of $f_0^{(m)}$ as

$$f_k^{(m)} = \varphi_1 \cdots \varphi_k \; f_0^{(m-k)} = \frac{q^{k+1}}{q-1} \; \frac{\left(q^{-1};t\right)_{k+1}}{(t;t)_k} \; f_0^{(m-k)} \; , \tag{3.61}$$

where we introduced the q-Pochhammer symbols

$$(a;q)_k := \prod_{l=0}^{k-1} \left(1 - a\, q^l\right) \qquad \text{for} \quad 0 < k \leq \infty \tag{3.62}$$

and $(a;q)_0 := 1$. Using (3.59) we can express $f_0^{(m)}$ recursively as

$$f_0^{(m)} = \sum_{k=0}^{m-1} f_0^{(k)} \, \phi_{m-k} \; , \tag{3.63}$$

where we defined

$$\phi_k \; := \; \xi_k \, \varphi_1 \cdots \varphi_{k-1} = -t^{k \, \frac{N+1}{2}} \, q^k \, \frac{\left(q^{-1};t\right)_k}{(t;t)_k} \; . \tag{3.64}$$

It is easy to see that the solution to this recursion with $f_0^{(0)} = 1$ is given by an expansion into partitions of m as

$$f_0^{(m)} = \sum_{\lambda \in \Lambda_+^m} L_\lambda \, \phi_\lambda \; , \tag{3.65}$$

where this formula should be understood in the large N limit as it involves a sum over *all* partitions of m. Here $\phi_\lambda := \prod_{i=1}^{\ell(\lambda)} \phi_{\lambda_i}$ with $\ell(\lambda)$ the length of the partition λ, and the combinatorial weight

$$L_\lambda = \frac{\ell(\lambda)!}{\left|\mathrm{Aut}(\lambda)\right|} \tag{3.66}$$

is the number of distinguishable orderings of λ (e.g. $L_{(2,1)} = 2$ and $L_{(1,1)} = 1$), where

$$\left|\mathrm{Aut}(\lambda)\right| = \prod_{i=1}^{|\lambda|} m_i(\lambda)! \tag{3.67}$$

is the order of the automorphism group of λ consisting of permutations $\sigma \in \mathfrak{S}_{\ell(\lambda)}$ such that $\lambda_{\sigma(i)} = \lambda_i$ for all i, and $m_i(\lambda)$ is the number of parts of λ equal to i. For example, the first four terms are given by

$$\begin{aligned} f_0^{(1)} &= \phi_1 \; , \\ f_0^{(2)} &= \phi_2 + \phi_1^2 \; , \\ f_0^{(3)} &= \phi_3 + 2\,\phi_1\,\phi_2 + \phi_1^3 \; , \\ f_0^{(4)} &= \phi_4 + 2\,\phi_1\,\phi_3 + \phi_2^2 + 3\,\phi_1^2\,\phi_2 + \phi_1^4 \; . \end{aligned} \tag{3.68}$$

We can finally evaluate the (q,t)-trace of the connected minimal word using (3.52) and (3.56) to write

$$\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}} \left(\Phi_n \, U \, \mathsf{g}_1 \, \mathsf{g}_2 \cdots \mathsf{g}_{n-1}\right) \; = \; (g_{\omega_1})^{-n/2} \; \mathrm{Tr}_{\,R_{\omega_1}} \left(U \, \mathcal{O}_{n-1}\right) \; = \; (g_{\omega_1})^{-n/2} \sum_{k=0}^{n-1} f_k^{(n-1)} \; \mathrm{Tr}_{\,R_{\omega_1}} \left(U^{k+1}\right) \; , \tag{3.69}$$

and using (3.39), (3.61) and (3.65) we get

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{g}_1\,\mathsf{g}_2\cdots\mathsf{g}_{n-1}\big)=(g_{\omega_1})^{-n/2}\,\frac{q^n}{q-1}\,\sum_{k=0}^{n-1}t^{(n-1-k)\,\frac{N+1}{2}}\,\frac{\big(q^{-1};t\big)_{k+1}}{(t;t)_k}\,\zeta_{n-1-k}(q,t)\ [N]_{t^{k+1}} \tag{3.70}$$

where

$$\begin{aligned}\zeta_0(q,t)\ &:=\ 1\ ,\\ \zeta_m(q,t)\ &:=\ \sum_{\lambda\in\Lambda_+^m}(-1)^{\ell(\lambda)}\,L_\lambda\,\prod_{i=1}^{\ell(\lambda)}\frac{\big(q^{-1};t\big)_{\lambda_i}}{(t;t)_{\lambda_i}}\qquad\text{for}\quad m>0\ .\end{aligned} \tag{3.71}$$

It is straightforward to show that this expression reduces to (3.48) for $n=2$, while for $n=3$ it reads as

$$\begin{aligned}\mathrm{Tr}_{R_{\omega_1}^{\otimes 3}}\big(\Phi_3\,U\,\mathsf{g}_1\,\mathsf{g}_2\big)\ &=\ (g_{\omega_1})^{-3/2}\left(t^{N+1}\,\frac{q-1}{t-1}\left(\frac{q-1}{t-1}+\frac{t-q}{t^2-1}\right)[N]_t+t^{\frac{N+1}{2}}\,\frac{(q-1)(t-q)}{(t-1)^2}\,[N]_{t^2}\right.\\ &\qquad\left.+\,\frac{(t-q)\,\big(t^2-q\big)}{(t-1)\,\big(t^2-1\big)}\,[N]_{t^3}\right)\ .\end{aligned} \tag{3.72}$$

Let us look at the unrefined limit $q=t$ of the expression (3.70). In this case $(q^{-1};q)_{k+1}=0$ for $k>0$ from the definition (3.62), so only the $k=0$ term in (3.70) is non-zero and the sum over partitions in (3.71) receives a non-vanishing contribution from only the maximal partition $\lambda=(1^m)$ with $\ell(\lambda)=m$ parts, and $L_{(1,\dots,1)}=1$, so that

$$\zeta_m(q,q)=\prod_{i=1}^m\frac{1}{q}=q^{-m}\qquad\text{for}\quad m\geq 0\ . \tag{3.73}$$

Then the $q=t$ limit of the (q,t)-trace formula (3.70) becomes

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{g}_1\,\mathsf{g}_2\cdots\mathsf{g}_{n-1}\big)\Big|_{q=t}\ =\ q^{(n-1)\,\frac{N+1}{2}}\ [N]_q \tag{3.74}$$

which coincides with the unrefined quantum trace formula of [33, eq. (B.6)]. The ensuing simplicity of the unrefined limit as compared to the general case (3.70) is explained in terms of the combinatorics of symmetric functions in Appendix C.

We can now use (3.70) to evaluate $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{h}(m_T)\big)$. If the conjugacy class T is composed of permutations which have $\mu_i(T)$ cycles of length i, then $n=\sum_i i\,\mu_i(T)$ and we get

$$\begin{aligned}&\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{h}(m_T)\big)\\ &\quad=\ (g_{\omega_1})^{-n/2}\,\frac{q^n\,t^{n\,\frac{N+1}{2}}}{(q-1)^{\sum_i\mu_i(T)}}\,\prod_{i=1}^n\Big(\sum_{k=0}^{i-1}t^{-(k+1)\,\frac{N+1}{2}}\,\frac{\big(q^{-1};t\big)_{k+1}}{(t;t)_k}\,\zeta_{i-1-k}(q,t)\ [N]_{t^{k+1}}\Big)^{\mu_i(T)}\ .\end{aligned} \tag{3.75}$$

Let us rewrite this formula in terms of the partitions $\mu=\mu(T)$ which parameterize the conjugacy classes $T=T_\mu$ as

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{h}(m_{T_\mu})\big)=(g_{\omega_1})^{-n/2}\,\frac{q^n\,t^{n\,\frac{N+1}{2}}}{(q-1)^{\ell(\mu)}}\,\prod_{i=1}^{\ell(\mu)}\Big(\sum_{k=1}^{\mu_i}t^{-k\,\frac{N+1}{2}}\,\frac{\big(q^{-1};t\big)_k}{(t;t)_{k-1}}\,\zeta_{\mu_i-k}(q,t)\ [N]_{t^k}\Big) \tag{3.76}$$

where $\ell(\mu)=\sum_i\mu_i(T)$ is the length of the partition μ.

We can finally substitute the formula (3.76) into (3.32) to get the main result of this section.

Proposition 3.77 *The refined quantum dimensions can be expressed as*

$$\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} = \frac{q^{-\frac{n\,(n-5)}{4}}\, t^{n\,\frac{N+1}{2}}}{(g_{\omega_1})^{n/2}\,[n]_q!}\,\frac{d_{\lambda_\beta+a\,(1^N)}(q)}{d_{\lambda_\beta+a\,(1^N)}(1)}\sum_{\mu\in\Lambda_+^n}\frac{q^{-\ell^*(\mu)}}{(q-1)^{\ell(\mu)}}\,\chi_{r_{\lambda_\beta+a\,(1^N)}}(C_\mu)$$
$$\times\prod_{i=1}^{\ell(\mu)}\Big(\sum_{k=1}^{\mu_i} t^{-k\,\frac{N+1}{2}}\,\frac{(q^{-1};t)_k}{(t;t)_{k-1}}\,\zeta_{\mu_i-k}(q,t)\ [N]_{t^k}\Big)$$

for $\lambda\in\Lambda_+^{n-a\,N}$, where $\ell^*(\mu)=\sum_i(i-1)\,\mu_i = n-\ell(\mu)$ is the colength of the partition μ (the complement to its length) which coincides with the length of the permutation (the minimal number of generators) that belongs to the conjugacy class labelled by μ, and the central Hecke algebra element $C_\mu := C_{T_\mu}$ is defined by (3.34). *The coefficients $\zeta_m(q,t)$ are defined in* (3.71) *and* (3.66).

It easy to see that this refined quantum dimension formula reduces at $\beta=1$ to the quantum dimension formula from [33, eq. (2.36)].

4 Chiral expansion of the partition function

To explore the relations between the refinement of q-deformed Yang-Mills theory on Σ_h and a refined topological string theory, we consider the topological limit of the gauge theory which is the limit of degree $p=0$. In this section we will study the partition function which from (2.2) is given by

$$Z_h(q,t;0)=\sum_{n=0}^{\infty}\sum_{\lambda\in\Lambda_+^n}\Big(\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}}\Big)^{2-2h}\ , \tag{4.1}$$

analogously to [33]. The chiral expansion is the asymptotic large N expansion defined by dropping the constraint $\ell(\lambda)\le N$ on the lengths of the partitions λ; our main results are summarised in Propositions 4.25 and 4.46.

4.1 Generalised quantum Ω-factors

Let us begin by rewriting the refined quantum dimension from Proposition 3.77 in a simpler condensed form. We define the element

$$\Omega_n(q,t)=\frac{t^{n\,\frac{N+1}{2}}}{\big([N]_t\big)^n}\sum_{\mu\in\Lambda_+^n}\Big(\frac{q}{q-1}\Big)^{\ell(\mu)}\prod_{i=1}^{\ell(\mu)}\Big(\sum_{k=1}^{\mu_i}t^{-k\,\frac{N+1}{2}}\,\frac{(q^{-1};t)_k}{(t;t)_{k-1}}\,\zeta_{\mu_i-k}(q,t)\ [N]_{t^k}\Big)\,C_\mu\ . \tag{4.2}$$

This is a sum of central elements C_μ of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$, so $\Omega_n(q,t)$ is also central. The normalization is chosen so that the identity is the leading term at large N. For this, we note that, under the assumptions $q,t\in(0,1)$, the largest terms of $\Omega_n(q,t)$ in the large N expansion come from rectangular partitions of the form $\mu=(m,\dots,m)$ which give the leading term

$$\Big(\frac{q\,(q^{-1};t)_m}{(q-1)\,(t;t)_{m-1}}\Big)^{\ell(\mu)}\ . \tag{4.3}$$

For $\beta\ge1$ and $k\in\mathbb{Z}_{>0}$ we have

$$\Big|\,\frac{1-q^{-1}\,t^k}{1-t^k}\,\Big|=\Big|\,\frac{1-t^{k-\frac{1}{\beta}}}{1-t^k}\,\Big|<1\ . \tag{4.4}$$

This implies that the absolute value of (4.3) is less than 1, unless $\mu = (1,\dots,1)$ in which case it is equal to 1. Hence the maximal partition $(1,\dots,1)$ is the leading term which corresponds to the identity permutation, and we can write

$$\Omega_n(q,t) \,=\, 1 \,+\, \Omega_n'(q,t) \ , \tag{4.5}$$

where $\Omega_n'(q,t)$ has the same form as $\Omega_n(q,t)$ except that the sum runs over all non-maximal partitions μ of n.

The $q=t$ limit of (4.2) coincides with the unrefined element

$$\Omega_n(q,q) = \sum_{\mu\in\Lambda_+^n} q^{\frac{N-1}{2}\,\ell(\mu)}\,\big([N]_q\big)^{-\ell^*(\mu)}\,C_\mu \tag{4.6}$$

from [33, eq. (2.46)]. As we discuss further below, the power $\big([N]_q\big)^{-\ell^*(\mu)}$ appearing here suggests an interpretation of $\Omega_n(q,q)$ in terms of branch points on Σ_h in a topological string theory of worldsheet branched covers of the target Riemann surface Σ_h, with string coupling $g_{\rm str} = \frac{1}{[N]_q}$.

With this new notation we can write the result of Proposition 3.77 as

$$\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} = \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\,\frac{d_{\lambda_\beta+a\,(1^N)}(q)}{d_{\lambda_\beta+a\,(1^N)}(1)}\,\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^n\,\chi_{r_{\lambda_\beta+a\,(1^N)}}\big(\Omega_n(q,t)\big) \ . \tag{4.7}$$

This formula is very similar to the unrefined one from [33, eq. (2.45)], except that in our case the expansion parameter is

$$\frac{[N]_t}{\sqrt{g_{\omega_1}}} = \bigg(\frac{\big[N\big]_t\,\big[\beta\,(N-1)+1\big]_q}{g_\emptyset}\bigg)^{1/2} \tag{4.8}$$

and $g_\emptyset$ is of order 1 in the large N limit. This expansion parameter respects the Ω-background symmetry $(\epsilon_1,\epsilon_2)\mapsto(-\epsilon_2,-\epsilon_1)$ described in §2.2; however, fixing $p=0$ breaks this symmetry of the topological partition function in the ensuing large N expansion.

4.2 Chiral series

We next use the following fact to evaluate the powers of the refined quantum dimensions: If C is any central element of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$ and $\sigma\in\mathfrak{S}_n$, then by [33, eq. (2.44)] one has

$$\chi_{r_\lambda}(C)\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big) \,=\, d_\lambda(1)\,\chi_{r_\lambda}\big(C\,\mathsf{h}(\sigma)\big) \ . \tag{4.9}$$

This implies

$$\begin{aligned} Z_h(q,t;0) \;=\; \sum_{n=a\,N}^\infty\ \sum_{\lambda\in\Lambda_+^{n-a\,N}} \bigg(\frac{q^{-\frac{n\,(n-1)}{4}}\,d_{\lambda_\beta+a\,(1^N)}(q)}{[n]_q!}\bigg)^{2-2h}\,\frac{1}{d_{\lambda_\beta+a\,(1^N)}(1)}\,\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)} \\ \times\ \chi_{r_{\lambda_\beta+a\,(1^N)}}\big(\Omega_n(q,t)^{2-2h}\big) \ . \end{aligned} \tag{4.10}$$

Because of our normalization (4.5), the element $\Omega_n(q,t)$ is always formally invertible in the large N expansion.

By [33, Appendix A.1] we have

$$\bigg(\frac{[n]_q!}{q^{-\frac{n\,(n-1)}{4}}\,d_{\lambda_\beta+a\,(1^N)}(q)}\bigg)^2 = \frac{1}{d_{\lambda_\beta+a\,(1^N)}(1)}\,\sum_{\sigma,\tau\in\mathfrak{S}_n} q^{-\ell(\sigma)-\ell(\tau)}\,\chi_{r_{\lambda_\beta+a\,(1^N)}}\big(\mathsf{h}(\sigma)\,\mathsf{h}(\tau)\,\mathsf{h}(\sigma^{-1})\,\mathsf{h}(\tau^{-1})\big) \tag{4.11}$$

which yields

$$Z_h(q,t;0) \;=\; \sum_{n=a\,N}^{\infty}\ \sum_{\lambda\in\Lambda_+^{n-a\,N}}\ \frac{1}{d_{\lambda_\beta+a\,(1^N)}(1)}\Big(\frac{q^{-\frac{n\,(n-1)}{4}}\,d_{\lambda_\beta+a\,(1^N)}(q)}{[n]_q!}\Big)^2\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)}$$
$$\times \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n} q^{-\sum_i\,(\ell(\sigma_i)+\ell(\tau_i))}\,\chi_{r_{\lambda_\beta+a\,(1^N)}}\Big(\Omega_n(q,t)^{2-2h}\ \prod_{i=1}^h \mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big) \tag{4.12}$$

where we used (4.9) and the centrality property of [33, Appendix A.2]. Following [33, Appendix A.3] we introduce the element

$$D_n \;=\; \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\ \sum_{k=0}^{\infty}(-1)^k \sum_{\substack{\sigma_1,\dots,\sigma_k\in\mathfrak{S}_n\\ \sigma_i\neq 1}} q^{-\sum_i\,\ell(\sigma_i)}\ \prod_{j=1}^k \mathsf{h}(\sigma_j^{-1})\,\mathsf{h}(\sigma_j) \tag{4.13}$$

in the completed Hecke algebra $\widehat{\mathsf{H}}_q(\mathfrak{S}_n)$. We can then express the q-dimension $d_{\lambda_\beta+a\,(1^N)}(q)$ in terms of the character of D_n as

$$d_{\lambda_\beta+a\,(1^N)}(q) \;=\; \chi_{r_{\lambda_\beta+a\,(1^N)}}(D_n)\ . \tag{4.14}$$

Since D_n is a series in central elements of $\mathsf{H}_q(\mathfrak{S}_n)$ by [33, Appendix A.1], it is central in $\widehat{\mathsf{H}}_q(\mathfrak{S}_n)$ and using (4.9) we get

$$Z_h(q,t;0) \;=\; \sum_{n=a\,N}^{\infty} \frac{q^{-\frac{n\,(n-1)}{2}}}{\big([n]_q!\big)^2}\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)} \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n} q^{-\sum_i\,(\ell(\sigma_i)+\ell(\tau_i))}$$
$$\times \sum_{\lambda\in\Lambda_+^{n-a\,N}} d_{\lambda_\beta+a\,(1^N)}(q)\,\chi_{r_{\lambda_\beta+a\,(1^N)}}\Big(D_n\,\Omega_n(q,t)^{2-2h}\ \prod_{i=1}^h \mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big)\ . \tag{4.15}$$

We define a delta-function on Hecke algebras analogously to [33] by

$$\delta\big(\mathsf{h}(\sigma)\big) \;=\; \begin{cases} 1 & \text{if} \quad \sigma \;=\; 1\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{4.16}$$

and extended over $\mathsf{H}_q(\mathfrak{S}_n)$ by $\mathbb{C}$-linearity. It can be expressed as the sum of characters of $\mathsf{H}_q(\mathfrak{S}_n)$ given by

$$\delta\big(\mathsf{h}(\sigma)\big) = \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\ \sum_{\lambda\in\Lambda_+^n} d_\lambda(q)\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma)\big)\ . \tag{4.17}$$

To write the partition function in terms of delta-functions as in the unrefined case, we have to take the sum over all partitions of n. There is a bijection between partitions $\alpha\in\Lambda_+^n$ such that $\alpha_i \geq (\beta-1)\,\rho_i + a \;=\; (\beta-1)\,(N-i)$ for $i=1,\dots,N$ and partitions in $\Lambda_+^{n-a\,N}$. Thus we need to construct a step function $\Theta_n(\beta)$ on partitions that cuts off the contributions involving smaller partitions and allows us to sum over all $\alpha\in\Lambda_+^n$; it is defined by the property

$$\chi_\alpha\big(\Theta_n(\beta)\big) \;=\; \begin{cases} d_\alpha(1) & \text{if} \quad \alpha_i \geq (\beta-1)\,(N-i)\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{4.18}$$

for $\alpha \in \Lambda_+^n$. The sum of quantum Young projectors (3.15) given by

$$\Theta_n(\beta) \ = \sum_{\substack{\mu\in\Lambda_+^n \\ \mu_i \geq (\beta-1)\,(N-i)}} P_\mu \tag{4.19}$$

fulfills this criterion, and it is a central element of $\mathsf{H}_q(\mathfrak{S}_n)$ because the projectors are central.

We now insert $\big(d_{\lambda_\beta+a\,(1^N)}(1)\big)^{-1}\,\chi_{\lambda_\beta+a\,(1^N)}\big(\Theta_n(\beta)\big) = 1$ in (4.15) and using (4.9) we get

$$\begin{aligned} Z_h(q,t;0) \ &= \ \sum_{n=a\,N}^\infty \frac{q^{-\frac{n\,(n-1)}{2}}}{\big([n]_q!\big)^2}\,\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)} \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n} q^{-\sum_i\,(\ell(\sigma_i)+\ell(\tau_i))} \\ &\quad \times \sum_{\lambda\in\Lambda_+^{n-a\,N}} d_{\lambda_\beta+a\,(1^N)}(q)\,\chi_{r_{\lambda_\beta+a\,(1^N)}}\Big(\Theta_n(\beta)\,D_n\,\Omega_n(q,t)^{2-2h}\,\prod_{i=1}^h \mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big)\ . \end{aligned} \tag{4.20}$$

A partition $\alpha \in \Lambda_+^n$ satisfies $\alpha_i \geq (\beta-1)\,(N-i)$ for $i = 1,\dots,N$ if and only if it can be written as $\alpha = \lambda_\beta + a\,(1^N)$ for some $\lambda \in \Lambda_+^{n-a\,N}$. The contributions involving $\alpha_i < (\beta-1)\,(N-i)$ for some i in the partition function vanish because of the step function $\Theta_n(\beta)$. Hence we can shift the summation range and sum over all $\alpha \in \Lambda_+^n$ to get

$$\begin{aligned} Z_h(q,t;0) \ &= \ \sum_{n=a\,N}^\infty \frac{q^{-\frac{n\,(n-1)}{2}}}{\big([n]_q!\big)^2}\,\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)} \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n} q^{-\sum_i\,(\ell(\sigma_i)+\ell(\tau_i))} \\ &\qquad \times \sum_{\alpha\in\Lambda_+^n} d_\alpha(q)\,\chi_{r_\alpha}\Big(\Theta_n(\beta)\,D_n\,\Omega_n(q,t)^{2-2h}\,\prod_{i=1}^h \mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big)\ , \end{aligned} \tag{4.21}$$

and then using the expression for the delta-function from (4.17) we arrive at

$$\begin{aligned} Z_h(q,t;0) \ &= \ \sum_{n=\lceil a\,N\rceil}^\infty \Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h)}\,\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!} \\ &\quad \times \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n} \delta\Big(\Theta_n(\beta)\,D_n\,\Omega_n(q,t)^{2-2h}\,\prod_{i=1}^h q^{-\ell(\sigma_i)-\ell(\tau_i)}\,\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1})\Big)\ . \end{aligned} \tag{4.22}$$

Now we can analytically continue β away from integer values, because this expansion only depends on β in the q-numbers and in a, and we can choose a larger value for a; the smallest choice is a_0 such that $a_0\,N = \lceil a\,N\rceil$, and $\lceil a\,N\rceil$ also vanishes in the unrefined limit. The restriction on the sum in the definition of the step function $\Theta_n(\beta)$ from (4.19) can also be continued in a straightforward way. This expansion is the refined version of [33, eq. (2.56)]; it is a refined quantum deformation of the large N chiral Gross-Taylor expansion. The Hecke element $\Theta_n(\beta)$ does not have any unrefined analog, as it reduces to $\Theta_n(\beta=1) = \sum_{\mu\in\Lambda_+^n} P_\mu = 1$ by (3.15) together with the expression for the delta-function from (4.17) and we recover the unrefined expansion.

Finally, using (4.5) we can expand the Ω-factors $\Omega_n(q,t)^{2-2h}$ in the completion $\widehat{\mathsf{H}}_q(\mathfrak{S}_n)$ via the power series

$$\Omega_n(q,t)^{2-2h} \ = \ \sum_{L=0}^\infty d(2-2h,L)\ \Omega_n'(q,t)^L\ , \tag{4.23}$$

where

$$d(m,L) \ = \ \frac{\Gamma(m+1)}{\Gamma(L+1)\,\Gamma(m-L+1)}\ . \tag{4.24}$$

As explained in [14, §7.1.2], the binomial coefficient $d(2-2h,L)$ is the Euler characteristic $\chi(\Sigma_{h,L})$ of the configuration space of L points on the Riemann surface Σ_h, i.e. the L-tuples of distinct points on Σ_h modulo the natural action of the permutation group $\mathfrak{S}_L$. In this way we arrive at

Proposition 4.25 *The chiral series for the partition function of topological (q,t)-deformed Yang-Mills theory on Σ_h is given by*

$$\begin{aligned} Z_h(q,t;0) \ &= \ \sum_{n=\lceil a\,N\rceil}^\infty \ \sum_{L=0}^\infty \ \big(g_{\omega_1}\big)^{n\,(h-1)} \ \big([N]_t\big)^{n\,(2-2h-L)} \ \frac{\chi(\Sigma_{h,L})}{[n]_q!} \ q^{-\frac{n\,(n-1)}{4}} \ t^{n\,L\,\frac{N+1}{2}} \\ &\quad \times \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n} \ \prod_{l=1}^L \ \sum_{\substack{\mu^l\in\Lambda_+^n\\ \mu^l\neq(1^n)}} \Big(\frac{q}{q-1}\Big)^{\ell(\mu^l)} \ \prod_{i=1}^{\ell(\mu^l)} \Big(\sum_{k=1}^{\mu_i^l} t^{-k\,\frac{N+1}{2}} \ \frac{\big(q^{-1};t\big)_k}{(t;t)_{k-1}} \ \zeta_{\mu_i^l-k}(q,t) \ [N]_{t^k}\Big) \\ &\qquad \times \delta\Big(\Theta_n(\beta)\,D_n\,C_{\mu^1}\cdots C_{\mu^L} \ \prod_{j=1}^h q^{-\ell(\sigma_j)-\ell(\tau_j)} \ \mathsf{h}(\sigma_j)\,\mathsf{h}(\tau_j)\,\mathsf{h}(\sigma_j^{-1})\,\mathsf{h}(\tau_j^{-1})\Big) \ . \end{aligned}$$

Here the central Hecke algebra elements $\Theta_n(\beta)$ and D_n are defined in (4.19) *and* (4.13)*.*

This is a refined quantum deformation of the symmetric group enumeration of covering maps of the Riemann surface Σ_h, analogously to the description in terms of quantum spectral curves discussed in §2.3. In particular, following [33] it is tempting to suppose that this expansion is captured by a balanced topological string theory [15] with target space the M-theory compactification described in §2.2, which would naturally compute Euler characters of certain moduli spaces of curves in this background. We elaborate further on these points below.

In the unrefined limit $q=t$, the asymptotic expansion of Proposition 4.25 becomes

$$\begin{aligned} Z_h(q,q;0) \ &= \ \sum_{n,L=0}^\infty \ \frac{\chi(\Sigma_{h,L})}{[n]_q!} \ \sum_{\substack{\mu^1,\dots,\mu^L\in\Lambda_+^n\\ \mu^l\neq(1^n)}} \ \big([N]_q\big)^{n\,(2-2h-L)+\sum_l\ell(\mu^l)} \ q^{-\frac{n\,(n-3)}{4}+n\,L\,\frac{N+1}{2}-\frac{N-1}{2}\,\sum_l\ell(\mu^l)} \\ &\quad \times \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n} \ \delta\Big(D_n\,C_{\mu^1}\cdots C_{\mu^L} \ \prod_{j=1}^h q^{-\ell(\sigma_j)-\ell(\tau_j)} \ \mathsf{h}(\sigma_j)\,\mathsf{h}(\tau_j)\,\mathsf{h}(\sigma_j^{-1})\,\mathsf{h}(\tau_j^{-1})\Big) \ , \end{aligned} \tag{4.26}$$

independently of the parts of the partitions $\mu^1,\dots,\mu^L$. When q is a power of a prime, as explained in [33] it is tempting to interpret this expansion in terms of the enumeration of algebraic curves over the (finite) field $\mathbb{F}_q$, rather than $\mathbb{C}$. As further evidence, we note that q-deformation in this regard can be interpreted as replacing the symmetric group $\mathfrak{S}_n$ with the general linear group $GL_n(\mathbb{F}_q)$, which sends a transposition σ_i to a reflection in $GL_n(\mathbb{F}_q)$, i.e. the fixed space $(\mathbb{F}_q^n)^{\sigma_i}$ is a hyperplane in $\mathbb{F}_q^n\cong\mathbb{F}_{q^n}$ (as $\mathbb{F}_q$-vector spaces). The irreducible representations u_λ of $GL_n(\mathbb{F}_q)$ are also parameterized by partitions λ of n and have dimensions $\dim(u_\lambda)=(q;q)_n\,\dim_q(R_\lambda)$ [29], while the character of a semisimple reflection σ in $GL_n(\mathbb{F}_q)$ (characterised as having $\det(\sigma)\in\mathbb{F}_q^*\setminus\{1\}$) is given by

$$\chi_{u_\lambda}(\sigma)=\frac{\det(\sigma)}{[n]_q} \ \sum_{(i,j)\in Y_\lambda} q^{j-i} \ . \tag{4.27}$$

It would be interesting to understand if the corresponding refinements can be likewise regarded in terms of generalised characters.

4.3 β-deformed Hurwitz theory

To understand better the geometrical effect of refinement as it occurs in the expansion of Proposition 4.25, let us consider the classical limit $q \to 1$ with β fixed. In this limit the Macdonald polynomials reduce to the Jack polynomials which are ordinary generalized characters of irreducible $U(N)$ representations [21], and the Hecke algebra reduces to the ordinary group algebra of the symmetric group $\mathbb{C}[\mathfrak{S}_n]$. It is straightforward to show that the Ω-factors reduce to

$$\lim_{q\to 1} \Omega_n(q,t)\big|_{t=q^\beta} = \sum_{\mu\in\Lambda_+^n} \frac{\varDelta_\mu(\beta)}{N^{\ell^*(\mu)}}\, C_\mu \ , \tag{4.28}$$

where $C_\mu = \sum_{\sigma\in T_\mu} \sigma$ and we have defined

$$\varDelta_\mu(\beta) = \prod_{i=1}^{\ell(\mu)} \Big(\sum_{k=1}^{\mu_i} \gamma_k(\beta) \sum_{\lambda\in\Lambda_+^{\mu_i-k}} \beta^{-\ell(\lambda)}\, \frac{\ell(\lambda)!}{z_\lambda} \prod_{j=1}^{\ell(\lambda)} \gamma_{\lambda_j}(\beta) \Big) \ , \tag{4.29}$$

with

$$\begin{aligned} \gamma_1(\beta) &= 1 \ , \\ \gamma_k(\beta) &= \prod_{l=1}^{k-1} \frac{\beta\, l-1}{\beta\, l} = \frac{\Gamma(k-\frac1\beta)}{\Gamma(1-\frac1\beta)\,\Gamma(k)} \qquad \text{for} \quad k>1 \ . \end{aligned} \tag{4.30}$$

The integer

$$z_\lambda = \prod_{i=1}^{|\lambda|} i^{m_i(\lambda)}\, m_i(\lambda)! \tag{4.31}$$

is the order of the stabilizer, under conjugation, of any element of the conjugacy class T_λ. In the unrefined limit, $\varDelta_\mu(\beta) \to 1$ as $\beta \to 1$ and (4.28) coincides with the unrefined Ω-factor from [14, eq. (6.5)], in which case the weights of the sum depend only on the colengths $\ell^*(\mu) = n - \ell(\mu)$ of the partitions $\mu \in \Lambda_+^n$ (the same is true of the unrefined q-deformed Ω-factors (4.6)). In marked contrast, for $\beta \neq 1$ the weights depend explicitly on the parts of the partition μ through the combinatorial coefficients $\varDelta_\mu(\beta)$.

The expansion into Euler characters given in Proposition 4.25 reduces to

$$\begin{aligned} \widetilde{Z}_h(\beta) &:= \lim_{q\to 1} Z_h(q,t;0)\big|_{t=q^\beta} \\ &= \sum_{n=\lceil a\, N\rceil}^\infty \tilde g(\beta)^{n\,(h-1)} \sum_{L=0}^\infty \frac{\chi(\Sigma_{h,L})}{n!} \prod_{l=1}^L \sum_{\substack{\mu^l\in\Lambda_+^n \\ \mu^l\neq(1^n)}} \varDelta_{\mu^l}(\beta)\, N^{n\,(2-2h)-\sum_{l=1}^L \ell^*(\mu^l)} \\ &\qquad \times \sum_{\lambda\in\Lambda_+^n} \omega_\lambda(\beta) \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n} \delta\Big(C_\lambda\, C_{\mu^1}\cdots C_{\mu^L} \prod_{i=1}^h [\sigma_i,\tau_i]\Big) \ , \end{aligned} \tag{4.32}$$

where $[\sigma,\tau] = \sigma\,\tau\,\sigma^{-1}\,\tau^{-1}$ denotes the *group* commutator and

$$\tilde g(\beta) := \frac{N}{\beta\,(N-1)+1} \prod_{m=0}^{\beta-1} \prod_{1\le i<j\le N} \frac{\beta\,(j-i)+m}{\beta\,(j-i)-m} \ . \tag{4.33}$$

We used (3.35) and (3.36), and defined a new deformation weight

$$\omega_\lambda(\beta) \;=\; \frac{1}{n!} \sum_{\substack{\mu\in\Lambda_+^n \\ \mu_i\geq(\beta-1)\,(N-i)}} d_\mu\,\chi_{r_\mu}(m_{T_\lambda}) \ , \tag{4.34}$$

which reduces to $\delta(C_\lambda)=\delta_{\lambda,(1^n)}$ in the $\beta=1$ limit. Rewriting this expansion entirely as sums over elements of the symmetric group $\mathfrak{S}_n$ reveals that it is a β-deformation of the ordinary chiral Gross-Taylor series in [14, eq. (7.3)], containing an extra class sum, and with extra deformation weights $\mathit{\Delta}_\mu(\beta)$ and $\omega_\lambda(\beta)$. In this expansion the weights depend explicitly on the parts of the partitions and not only on their colengths, although they are decoupled according to distinct partitions.

To make contact with Hurwitz theory, we shall collect terms with a fixed value of the integer

$$d=\sum_{l=1}^L \ell^*(\mu^l) \tag{4.35}$$

and set

$$2g-2=n\,(2h-2)+d \ . \tag{4.36}$$

For each $n\geq\lceil a\,N\rceil$, let us first consider the contribution from the maximal partition $\lambda=(1^n)$, for which $C_{(1^n)}=1$ and

$$\omega_{(1^n)}(\beta) \;=\; \frac{1}{n!} \sum_{\substack{\mu\in\Lambda_+^n \\ \mu_i\geq(\beta-1)\,(N-i)}} (d_\mu)^2 \ . \tag{4.37}$$

Let $H_{h,n}(\mu^1,\dots,\mu^L)$ be the number, weighted by $\frac{1}{n!}$, of $(2h{+}L)$-tuples consisting of permutations $\sigma_i,\tau_i\in\mathfrak{S}_n$, $i=1,\dots,h$ and central elements C_{μ^l} of $\mathfrak{S}_n$ sitting in fixed conjugacy classes T_{μ^l}, $l=1,\dots,L$ such that $C_{\mu^1}\cdots C_{\mu^L}\,\prod_{i=1}^h\,[\sigma_i,\tau_i]=1$; this quantity is called a (combinatorial) Hurwitz number, and it has an explicit combinatorial expression given by the Frobenius-Schur formula [43, Appendix A]

$$H_{h,n}\big(\mu^1,\dots,\mu^L\big)=(n!)^{2h}\sum_{\lambda\in\Lambda_+^n}\frac{1}{(d_\lambda)^{L+2h-2}}\,\prod_{l=1}^L\,\frac{\chi_{r_\lambda}\big(m_{T_{\mu^l}}\big)}{z_{\mu^l}} \ . \tag{4.38}$$

Geometrically, $H_{h,n}(\mu^1,\dots,\mu^L)$ counts the number of n-sheeted branched covers $f:\Sigma_g\to\Sigma_h$ with L fixed branch points $z^1,\dots,z^L\in\Sigma_h$ whose ramification profiles are specified by partitions $\mu^1,\dots,\mu^L\in\Lambda_+^n$ such that $\mu^l=\big(\mu_1^l,\dots,\mu^l_{\ell(\mu^l)}\big)$ are the ramification indices of the preimages $f^{-1}(z^l)$ (encoding how many sheets of the cover $f:\Sigma_g\to\Sigma_h$ come together above the point $z^l\in\Sigma_h$, see e.g. [14, §5]), weighted by $\frac{1}{|\mathrm{Aut}(f)|}$ where $\mathrm{Aut}(f)$ is the automorphism group of the covering map consisting of automorphisms $\alpha:\Sigma_g\to\Sigma_g$ such that $f\circ\alpha=f$; the degree d of such a holomorphic map over $\Sigma_h\setminus\{z^1,\dots,z^L\}$ is given by (4.35), while the genus g of the covering surface is given by the Riemann-Hurwitz formula (4.36). Note that $H_{h,n}(\mu^1,\dots,\mu^L)$ is independent of the branch point positions $z^1,\dots,z^L\in\Sigma_h$ and also of the choice of (fixed) complex structure on Σ_h. Incorporating the contributions from all partitions $\lambda\in\Lambda_+^n$, we then have

$$\begin{aligned}\widetilde{Z}_h(\beta) \;=\; & \sum_{n=\lceil a\,N\rceil}^\infty \tilde g(\beta)^{n\,(h-1)}\,\sum_{\lambda\in\Lambda_+^n}\omega_\lambda(\beta)\,\sum_{d=0}^\infty\Big(\frac1N\Big)^{2g-2}\,\sum_{L=0}^d\chi(\Sigma_{h,L}) \\ & \times \sum_{\substack{\mu^1,\dots,\mu^L\in\Lambda_+^n \\ \mu^l\neq(1^n)\,,\,\sum_{l=1}^L\ell^*(\mu^l)=d}} \mathit{\Delta}_{\mu^1}(\beta)\cdots\mathit{\Delta}_{\mu^L}(\beta)\;H_{h,n}\big(\lambda,\mu^1,\dots,\mu^L\big) \ . \end{aligned}\tag{4.39}$$

From this expression we can infer at least four novel aspects of the closed string expansion of the β-deformation of two-dimensional Yang-Mills theory, interpreted from the geometric point of view of Hurwitz theory:

1. Branched covers of index $n < \lceil a\,N \rceil$ do not contribute to the string expansion. This feature has important ramifications for the planar limit of the gauge theory which we discuss below.

2. The refinement introduces an additional weighting by the quantity (4.33) such that the expansion parameter does not simply distinguish the genera of the covering worldsheets Σ_g. Below we shall replace this weight with its leading term $\tilde{g}(\beta) = \frac{1}{\beta}$ in the large N limit.

3. The string expansion (4.39) generically involves deformations of the enumeration of branched covers $f : \Sigma_g \to \Sigma_h$ in terms of Hurwitz numbers $H_{h,n}(\mu^1, \dots, \mu^L)$ which include an additional marked point with holonomy in the representation (3.4) of $U(N)$; the inclusion of such marked points is the earmark of refinement and is captured by the intertwining operators defining the generalised characters [34, 53]. Accordingly, the Hurwitz numbers $H_{h,n}\big(\lambda, \mu^1, \dots, \mu^L\big)$ account for additional branching over this marked point with ramification profile $\lambda \in \Lambda_+^n$. Due to the deformation weights $\omega_\lambda(\beta)$, for $\beta \neq 1$ their contributions are strongly suppressed in the large N limit.

4. The expansion (4.39) involves *weighted* sums of Hurwitz numbers, with deformation weights $\varDelta_{\mu^l}(\beta)$ and $\omega_\lambda(\beta)$ depending explicitly on the parts of the partitions μ^l and λ which label the winding numbers of closed strings around the branch points in the target space Σ_h. This deformation obstructs a rewriting of the partition function as a generating function of characters of Hurwitz spaces of holomorphic maps $f : \Sigma_g \to \Sigma_h$, as occurs in the unrefined case [14], and as such an interpretation as a balanced topological string theory [15] with string coupling $g_{\rm str} = \frac{1}{N}$ and two-dimensional target space Σ_h is not immediately evident. In fact, this weighting suggests that the string expansion involves contributions from *marked* covers $f^{\rm m} : \Sigma_g \to \Sigma_h$; a marking of a branched cover $f : \Sigma_g \to \Sigma_h$ consists of a marking of each of its branch points z^l for $l = 1, \dots, L$, i.e. a choice of labelling $\big\{w_1^l, \dots, w_{\ell(\mu^l)}^l\big\} = f^{-1}(z^l)$ such that μ_i^l is the ramification index at w_i^l. An automorphism $\alpha : \Sigma_g \to \Sigma_g$ of a marked cover preserves the labels w_i^l, and we denote the corresponding marked cover counts by $H_{h,n}^{\rm m}(\mu^1, \dots, \mu^L)$. The action of the automorphism group ${\rm Aut}(f)$ on the labels of $f^{\rm m}$ gives a group homomorphism

$$
{\rm Aut}(f) \ \longrightarrow \ \prod_{l=1}^L {\rm Aut}\big(\mu^l\big) \tag{4.40}
$$

whose kernel is ${\rm Aut}(f^{\rm m})$ and whose image has index given by the number m of markings of f (up to isomorphism). It follows that $|{\rm Aut}(f)|\, m = |{\rm Aut}(f^{\rm m})| \prod_{l=1}^L |{\rm Aut}(\mu^l)|$, and hence the combinatorial expansion (4.39) can be written in terms of marked Hurwitz numbers as

$$
\begin{aligned}
\widetilde{Z}_h(\beta) \ = \ & \sum_{n=\lceil a\,N \rceil}^\infty \beta^{-n\,(h-1)} \sum_{\lambda\in\Lambda_+^n} \frac{\omega_\lambda(\beta)}{|{\rm Aut}(\lambda)|} \sum_{d=0}^\infty \Big(\frac{1}{N}\Big)^{2g-2} \sum_{L=0}^d \chi(\Sigma_{h,L}) \\
& \times \sum_{\substack{\mu^1,\dots,\mu^L\in\Lambda_+^n \\ \mu^l\neq(1^n)\,,\,\sum_{l=1}^L \ell^*(\mu^l)=d}} \frac{\varDelta_{\mu^1}(\beta)}{\big|{\rm Aut}\big(\mu^1\big)\big|} \cdots \frac{\varDelta_{\mu^L}(\beta)}{\big|{\rm Aut}\big(\mu^L\big)\big|} \ H_{h,n}^{\rm m}\big(\lambda, \mu^1, \dots, \mu^L\big) \ .
\end{aligned} \tag{4.41}
$$

While the refined weights (4.34) have a natural meaning as deformations of the identity (see (4.37)), it would be interesting to understand better the geometrical significance of the combinatorial weights (4.29) in terms of orbifold Euler characteristics of moduli spaces of Riemann surfaces, as suggested by the appearence of the binomial-type coefficients (4.30). We can give further insight into this perspective following the geometric interpretation of refinement from §2.3. Let $\mathcal{H}_{n,d,h,L}$ denote the Hurwitz space of isomorphism classes of n-sheeted branched covers $f:\Sigma_g\to\Sigma_h$ of degree d with L branch points. It has the structure of a discrete fibration

$$\pi_{n,d,h,L}\,:\,\mathcal{H}_{n,d,h,L}\;\longrightarrow\;\Sigma_{h,L} \tag{4.42}$$

over the configuration space of L indistinguishable points on Σ_h, which sends the class of a holomorphic map $f:\Sigma_g\to\Sigma_h$ to the branch locus of f. There is also a natural map

$$\mathcal{H}_{n,d,h,L}\;\longrightarrow\;\mathcal{M}_g \tag{4.43}$$

which sends the class of the cover $f:\Sigma_g\to\Sigma_h$ to the class of the curve Σ_g; the image of $\mathcal{H}_{n,d,h,L}$ under this map is a subvariety of the moduli space $\mathcal{M}_g$ of genus g curves. Recall from §2.3 that, in the planar limit of the gauge theory on the sphere, refinement can be interpreted geometrically as replacing the orbifold Euler characters $\chi_{\rm orb}(\mathcal{M}_g)$ with the parameterized Euler characters (2.33). It is natural to think of pulling back the corresponding characteristic classes under the map (4.43), and for fixed N we define the *parameterized Euler character*

$$\begin{aligned}\chi_{n,d,h,L}(\beta)\;:=\;\sum_{\lambda\in\Lambda_+^n}\frac{\omega_\lambda(\beta)}{\beta^{n\,(h-1)}}\;\sum_{\substack{\mu^1,\dots,\mu^L\in\Lambda_+^n\\ \mu^l\neq(1^n)\,,\,\sum_{l=1}^L\ell^*(\mu^l)=d}}\;\varDelta_{\mu^1}(\beta)\cdots\varDelta_{\mu^L}(\beta)\\ \times\,\chi(\Sigma_{h,L})\,H_{h,n}\big(\lambda,\mu^1,\dots,\mu^L\big)\end{aligned} \tag{4.44}$$

for $n\geq\lceil a\,N\rceil$. Via the fibration (4.42), in the unrefined limit it reduces to the orbifold Euler character

$$\chi_{n,d,h,L}(1)=\chi_{\rm orb}(\mathcal{H}_{n,d,h,L}) \tag{4.45}$$

of the singular variety $\mathcal{H}_{n,d,h,L}$ [14]. Then we can rewrite (4.39) in the more suggestive form

Proposition 4.46 *The chiral series for the partition function of topological β-deformed Yang-Mills theory on Σ_h is the generating function*

$$\widetilde{Z}_h(\beta)=\sum_{n=\lceil a\,N\rceil}^\infty\;\sum_{d=0}^\infty\Big(\frac1N\Big)^{2g-2}\;\sum_{L=0}^d\chi_{n,d,h,L}(\beta)$$

for the parameterized Euler characters (4.44)*, where g is determined from n, h and d by the Riemann-Hurwitz formula* (4.36)*.*

This generalizes the string theory interpretation of the unrefined case [14], wherein the orbifold Euler characters of Hurwitz spaces $\chi_{\rm orb}(\mathcal{H}_{n,d,h,L})$ are replaced under refinement by the parameterized Euler characters $\chi_{n,d,h,L}(\beta)$. As in §2.3, it is natural to expect that these β-deformed characters are themselves associated to characteristic classes of some related moduli spaces; in particular, for $\beta=2$ the deformation weights are given by

$$\varDelta_\mu(2)=2^{\ell^*(\mu)}\prod_{i=1}^{\ell(\mu)}\Big(\sum_{k=1}^{\mu_i}\frac{(2k-3)!!}{(k-1)!}\;\sum_{\lambda\in\Lambda_+^{\mu_i-k}}\frac{\ell(\lambda)!}{z_\lambda}\;\prod_{j=1}^{\ell(\lambda)}\frac{(2\lambda_j-3)!!}{(\lambda_j-1)!}\Big)\ . \tag{4.47}$$

However, in the present case the characters are non-polynomial functions of β; below we will compare their forms explicitly with the parameterized Euler characters (2.33).

For the case of a spherical target space $\Sigma_0=\mathbb{P}^1$, certain classes of Hurwitz numbers can be expressed as integrals of psi-classes and Hodge classes over the Deligne-Mumford moduli spaces of punctured curves $\overline{\mathcal{M}}_{g,n}$ [20]. The corresponding partition functions are annihilated by the differential operator of a quantum curve, see e.g. [19] for a review; it would be interesting to see if there is a similar quantum spectral curve underlying the partition function $\widetilde{Z}_0(\beta)$. On the other hand, orbifold Hurwitz numbers lead to partition functions which are annihilated by the difference operator of a quantum curve [19], and it would be interesting to understand the general (q,t)-deformed partition function $Z_0(q,t;0)$ also in this context.

4.4 Planar limit

In the planar limit of §2.3, the pertinent generalised Selberg integrals can also be expressed in terms of Jack symmetric functions [28], which gives a geometrical meaning to the refinement parameter β as a combinatorial invariant of non-orientability for maps of graphs into surfaces. Let us now compare the leading term of the partition function (4.22) for $h \geq 2$ in the classical limit $q = 1$ with the parameterized Euler characteristics (2.33). This amounts to setting the Ω-factors $\Omega_n(q,t)$ to 1 and keeping only the $n = a\,N$ term of the sum in (4.22), which yields

$$Z_h^{\rm pl}(q,t;0) \,=\, \frac{q^{-\frac{a\,N\,(a\,N-1)}{4}}}{[a\,N]_q!}\,\big(d_{(\beta-1)\,\rho+a\,(1^N)}(q)\big)^{2-2h}\,\Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{a\,N\,(2-2h)}\ . \tag{4.48}$$

In the classical limit this becomes

$$\widetilde{Z}_h^{\rm pl}(\beta) = \frac{1}{(a\,N)!}\,\big(d_{(\beta-1)\,\rho+a\,(1^N)}\big)^{2-2h}\,\Big(\frac{N^4}{\tilde g(\beta)}\Big)^{a\,N\,(1-h)}\ , \tag{4.49}$$

where we used (3.36). We can rewrite (4.33) in the form

$$\tilde g(\beta) = \frac{N}{\beta\,(N-1)+1}\,\frac{\displaystyle\prod_{i=1}^N\,\frac{\Gamma(\beta\,i)}{\Gamma(\beta)}}{\displaystyle\prod_{m=0}^{\beta-1}\,\prod_{i=1}^{N-1}\,\frac{\beta^i\,\Gamma\big(i+1-\frac{m}{\beta}\big)}{\Gamma\big(1-\frac{m}{\beta}\big)}}\ , \tag{4.50}$$

and using the dimension formula (B.6) from Appendix B we can write the dimension of the symmetric group representation corresponding to the partition $(\beta-1)\,\rho+a\,(1^N)$ as

$$d_{(\beta-1)\,\rho+a\,(1^N)} = \frac{\beta^{\frac{N\,(N-1)}{2}}\,\Gamma(a\,N+1)\,G(N+1)}{\displaystyle\prod_{i=1}^{N-1}\,\Gamma(1+\beta\,i)}\ , \tag{4.51}$$

where $G(z)$ is the Barnes G-function with the property that $G(N+1) = \prod_{i=1}^N\,\Gamma(i)$. The appearence of this Barnes function suggests, following [47], that our asymptotic expansion could be related to refined topological closed string theory on the resolved conifold geometry.

The corresponding free energy $\widetilde{F}_h^{\rm pl}(\beta) := -\log\widetilde{Z}_h^{\rm pl}(\beta)$ can be expanded as a power series in β, whose coefficients are combinations of Bernoulli numbers, in much the same way that we dealt with the partition function (2.25). The resulting expansion is somewhat complicated, so we content ourselves with an integral representation from which the expansion is straightforwardly extracted. For this, we use the integral formula for the gamma-function [41, eq. (3.6)]

$$\log\Gamma(z) = \int_0^\infty\,\frac{{\rm d}x}{x}\,\frac{1}{{\rm e}^{\,x}-1}\,\big((z-1)\,(1-{\rm e}^{\,-x})+{\rm e}^{\,-x\,(z-1)}-1\big)\ , \tag{4.52}$$

which holds for $\mathfrak{Re}(z) > 0$. After some calculation, one infers the free energy

$$\begin{aligned} \widetilde{F}_h^{\rm pl}(\beta) &= (h-1)\big(N\,(N-1)\,\log\beta - \tfrac12\,a\,N^2\,(N-1)\,\beta\,\log\beta - 3\,a\,N\,\log N \\ &\qquad - a\,N\,\log(\beta\,(N-1)+1)\big) + \int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{1}{\mathrm{e}^{\,x}-1}\,\mathcal{F}_h^{\beta,N}(x)\ , \end{aligned} \tag{4.53}$$

where we have defined

$$\begin{aligned} \mathcal{F}_h^{\beta,N}(x) &= a\,N\,\big(1+\tfrac{\beta}{2}\,N\,(N-1)\,(1-h)\big)\,\big(1-\mathrm{e}^{\,-x}\big) \\ &\quad + (2h-2)\,\Big(\frac{1-\mathrm{e}^{\,-\beta\,(N-1)\,x}}{1-\mathrm{e}^{\,\beta\,x}} - \frac{1-\mathrm{e}^{\,-(N-1)\,x}}{1-\mathrm{e}^{\,x}}\Big) + (2h-1)\,\big(\mathrm{e}^{\,-a\,N\,x}-1\big) \\ &\quad + a\,N\,(h-1)\,\Big(\frac{\mathrm{e}^{\,-\beta\,(N-1)\,x}+\mathrm{e}^{\,x}-\mathrm{e}^{\,-(\beta\,N-1)\,x}-1}{1-\mathrm{e}^{\,\beta\,x}} - N+1 \\ &\qquad + N\,\mathrm{e}^{\,-(\beta-1)\,x} + \tfrac{\beta}{2}\,N\,(N-1)\,\mathrm{e}^{\,-\beta\,x}\,\big(\mathrm{e}^{\,x}-1\big)\Big)\ . \end{aligned} \tag{4.54}$$

In the $\beta = 1$ limit (which also induces $a = 0$) the planar free energy vanishes, as we expect from (4.48).

The power series expansion in β can now be obtained by using the generating function (2.30) to expand the denominators $(1-\mathrm{e}^{\,\beta\,x})^{-1}$ and the integral identities of [41, Appendix A]. For example, we can readily compute the contribution

$$-\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{1}{\mathrm{e}^{\,x}-1}\,\frac{1-\mathrm{e}^{\,-\beta\,(N-1)\,x}}{1-\mathrm{e}^{\,\beta\,x}} = \sum_{n=0}^\infty \mathcal{F}_n^N\,\beta^n \tag{4.55}$$

where

$$\begin{aligned} \mathcal{F}_0^N &= (N-1)\,\big(\tfrac{1}{\varepsilon}+\tfrac12\,\log\varepsilon+\tfrac12\,(\gamma-\log 2\pi)\big)\ , \\ \mathcal{F}_n^N &= \zeta(n)\,(n-1)!\,\sum_{k=0}^n (N-1)^{k+1}\,\frac{B_{n-k}}{(k+1)!\,(n-k)!} \qquad \text{for} \quad n\geq 1\ . \end{aligned} \tag{4.56}$$

Here $\varepsilon \to 0^+$ gives the leading one-loop linear and logarithmic divergences, γ is the Euler-Mascheroni constant, and $\zeta(z)$ is the Riemann zeta-function. These formulas explicitly illustrate the analytic dependence of the parameterized Euler characteristics (4.44) on the refinement parameter β, as compared to the polynomial characters (2.33).

5 Chiral expansions of defect observables

Two-dimensional Yang-Mills theory also involves observables corresponding to insertions in the partition function of operators supported on real codimension one defects in Σ_h. In this final section we extend the chiral expansion of §4 to these observables.

5.1 Boundaries

We first describe the large N expansion of the refinement of q-deformed Yang-Mills theory on open Riemann surfaces of genus h with b boundaries; via suitable gluing rules they give the building blocks for all Yang-Mills amplitudes. The holonomies along the boundaries are specified by generalized quantum characters of elements $U_i \in T$, $i = 1,\dots,b$, and the partition function is given by

$$Z_{h,b}(q,t;p;U_1,\dots,U_b) = \sum_{\lambda\in\Lambda_+} \frac{\dim_{q,t}(R_\lambda)^{2-2h-b}}{(g_\lambda)^{1-h-b/2}}\ q^{\frac{p}{2}\,(\lambda,\lambda)}\ t^{p\,(\rho,\lambda)}\ \prod_{i=1}^b \chi_{\Phi_\lambda}(U_i)\ . \tag{5.1}$$

Again we consider only the topological gauge theory and study

$$Z_{h,b}(q,t;0;U_1,\dots,U_b) = \sum_{n=0}^{\infty} \sum_{\lambda\in\Lambda_+^n} \Big(\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} \Big)^{2-2h-b} \prod_{i=1}^{b} \chi_{\Phi_\lambda}(U_i) \ . \tag{5.2}$$

This partition function was also considered in [2] but with a different normalization for the boundary characters.

We begin by using the transformation from Appendix D to change to a basis of central elements C_i of $\mathsf{H}_q(\mathfrak{S}_\infty)$ and set

$$\begin{aligned} &Z_{h,b}(q,t;0;C_1,\dots,C_b) \\ &:= \int_{T^b} \prod_{i=1}^{b} [\mathrm{d}U_i]_{q,t} \sum_{n_i=a\,N}^{\infty} \frac{q^{-\frac{n_i\,(n_i-1)}{4}}}{[n_i]_q!} \ \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n_i}} \big(\Phi_{n_i}\,\Theta_{n_i}(\beta)\,C_i\,U_i^\dagger \big) \ Z_{h,b}(q,t;0;U_1,\dots,U_b) \ , \end{aligned} \tag{5.3}$$

where the intertwining operator Φ_{n_i} is defined in §3.2, the step function $\Theta_{n_i}(\beta)$ is defined in (4.19), and the integration measure $[\mathrm{d}U]_{q,t}$ on the maximal torus $T\subset G$ given by (D.9) defines the Macdonald inner product of generalized characters as an integral over holonomies [21]. Since the commutant of the representation of $\mathcal{U}_q(\mathfrak{gl}_N)$ on $R_{\omega_1}^{\otimes n}$ is the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_n)$, the map (5.3) may be regarded as the refined version of the quantum Fourier transformation of the boundary holonomy amplitudes. The generalized characters are orthonormal with respect to the Macdonald inner product, i.e.

$$\int_T [\mathrm{d}U]_{q,t}\ \chi_{\Phi_\lambda}(U)\,\chi_{\Phi_{\lambda'}}(U^\dagger) \ = \ \delta_{\lambda,\lambda'} \tag{5.4}$$

for $\lambda,\lambda'\in\Lambda_+$. Using Lemma 3.18 we can write

$$\begin{aligned} \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}} \big(\Phi_n\,\Theta_n(\beta)\,C_i\,U_i^\dagger \big) \ &= \ \sum_{\lambda\in\Lambda_+^n} \chi_{r_\lambda}\big(\Theta_n(\beta)\,C_i\big)\,\chi_{\Phi_{\lambda_{\beta-2}}}(U_i^\dagger) \\ &= \ \sum_{\mu\in\Lambda_+^{n-a\,N}} \chi_{r_{\mu_\beta+a\,(1^N)}}(C_i)\,\chi_{\Phi_{\mu+a\,(1^N)}}(U_i^\dagger) \ , \end{aligned} \tag{5.5}$$

which yields

$$\begin{aligned} Z_{h,b}(q,t;0;C_1,\dots,C_b) \ &= \ \sum_{n=a\,N}^{\infty} \Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!} \Big)^b \sum_{n'=0}^{\infty} \sum_{\lambda\in\Lambda_+^{n'}} \Big(\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} \Big)^{2-2h-b} \\ &\qquad \times \sum_{\mu_1,\dots,\mu_b\in\Lambda_+^{n-a\,N}} \prod_{i=1}^{b} \chi_{r_{\mu_{i\beta}+a\,(1^N)}}(C_i)\ \delta_{\lambda,\mu_i+a\,(1^N)} \qquad (5.6) \\ &= \ \sum_{n=a\,N}^{\infty} \Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!} \Big)^b \sum_{\mu\in\Lambda_+^{n-a\,N}} \Big(\frac{\dim_{q,t}(R_\mu)}{\sqrt{g_\mu}} \Big)^{2-2h-b} \prod_{i=1}^{b} \chi_{r_{\mu_\beta+a\,(1^N)}}(C_i) \end{aligned}$$

where we used (5.4) and (3.21).

We can now use (4.7) to expand the refined quantum dimensions and from (4.9) we get

$$
\begin{aligned}
Z_{h,b}(q,t;0;C_1,\dots,C_b) \ &= \ \sum_{n=\lceil a\,N\rceil}^\infty \Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n\,(2-2h-b)} \ \Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\Big)^b \\
&\quad \times \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n} \delta\Big(\Theta_n(\beta)\,(E_n)^{b-1}\,\Omega_n(q,t)^{2-2h-b} \\
&\qquad\qquad \times \prod_{i=1}^h q^{-\ell(\sigma_i)-\ell(\tau_i)}\,\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}(\sigma_i^{-1})\,\mathsf{h}(\tau_i^{-1}) \ \prod_{j=1}^b C_j\Big) \ ,
\end{aligned}
\tag{5.7}
$$

where the central element E_n is defined by [33, Appendix A.3]

$$
E_n \ := \ \sum_{\sigma\in\mathfrak{S}_n} q^{-\ell(\sigma)}\,\mathsf{h}(\sigma^{-1})\,\mathsf{h}(\sigma) \tag{5.8}
$$

with the properties

$$
E_n^{-1} \ = \ \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\ D_n \qquad \text{in} \quad \widehat{\mathsf{H}}_q(\mathfrak{S}_n) \tag{5.9}
$$

and

$$
\chi_{r_\lambda}(E_n) \ = \ q^{\frac{n\,(n-1)}{4}}\ [n]_q!\ \frac{d_\lambda(1)^2}{d_\lambda(q)} \qquad \text{for} \quad \lambda\in\Lambda_+^n \ . \tag{5.10}
$$

For fixed n this expression reduces to [33, eq. (3.6)] in the unrefined limit, whereas our derivation gives the full partition function summed over all indices n. In particular, this partition function is a refined quantum deformation of the counting of holomorphic maps with specified monodromies C_j at the boundaries [49]; by expanding the Ω-factors, in the classical limit $q=1$ it can be expressed in terms of parameterized Euler characters as in §4.3.

Let us look at some of the basic amplitudes which are the building blocks for the entire (q,t)-deformed gauge theory. The topological disk amplitude (with puncture of holonomy in the representation (3.4)) is the case $h=0,b=1$ in (5.2) which evaluates to

$$
Z_{0,1}(q,t;0;U) \ = \sum_{n=0}^\infty\ \sum_{\lambda\in\Lambda_+^n} \frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}}\ \chi_{\Phi_\lambda}(U) = \delta_{q,t}\big(U,q^{\beta\,(\rho,H)}\big) \ , \tag{5.11}
$$

where $\delta_{q,t}$ is the delta-function in the measure $[\mathrm{d}U]_{q,t}$. This shows that the wavefunction $\Psi(U)$ for a disk in the topological (q,t)-deformed gauge theory is supported on generalized quantum group holonomies of flat connections on a disk, generalising the unrefined case of [33, eq. (3.7)] wherein $\delta_{q,q}$ is the delta-function in the Haar measure for $U(N)$. Dually, we can represent the disk partition function in a form that depends solely on Hecke algebra quantities by using (5.7) to write

$$
Z_{0,1}(q,t;0;C) \ = \ \sum_{n=\lceil a\,N\rceil}^\infty \Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^n\ \frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\ \delta\big(\Theta_n(\beta)\,\Omega_n(q,t)\,C\big) \ , \tag{5.12}
$$

independently of the central elements (5.8). Similarly, the punctured topological cylinder amplitude is obtained from (5.2) with $h=0,b=2$, giving

$$
Z_{0,2}(q,t;0;U_1,U_2) \ = \sum_{n=0}^\infty\ \sum_{\lambda\in\Lambda_+^n} \chi_{\Phi_\lambda}(U_1)\,\chi_{\Phi_\lambda}(U_2) = \delta_{q,t}(U_1,U_2) \tag{5.13}
$$

with the dual formulation

$$
Z_{0,2}(q,t;0;C_1,C_2) \ = \ \sum_{n=\lceil a\,N\rceil}^\infty \Big(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\Big)^2\ \delta\big(\Theta_n(\beta)\,E_n\,C_1\,C_2\big) \tag{5.14}
$$

independently of the Ω-factors (4.2).

5.2 Wilson loops

The natural closed defect observables of the gauge theory are of course the Wilson loops which correspond to simple closed curves on the surface Σ_h. For definiteness, let us consider the large N expansion of a single Wilson loop in the representation R_λ on a simple oriented closed curve which divides the Riemann surface Σ_h into two faces of genera h_1 and h_2 with $h = h_1 + h_2$. The expectation value of the Wilson loop operator is given by [53]

$$\begin{aligned} W_\lambda(q,t;p;h_1,h_2) &= \sum_{\mu,\nu\in\Lambda_+} \int_T [\mathrm{d}U]_{q,t} \, \frac{\dim_{q,t}(R_\mu)^{1-2h_1}}{(g_\mu)^{\frac12-h_1}} \, \frac{\dim_{q,t}(R_\nu)^{1-2h_2}}{(g_\nu)^{\frac12-h_2}} \ q^{\frac{p}{2}\,(\lambda,\lambda)}\, t^{p\,(\rho,\lambda)} \\ &\qquad \times \chi_{\Phi_\mu}(U)\, \chi_{\Phi_\lambda}(U)\, \chi_{\Phi_\nu}(U^\dagger) \ . \end{aligned} \tag{5.15}$$

In the topological limit $p=0$ we can use the orthonormality relation (5.4) to obtain

$$W_\lambda(q,t;0;h_1,h_2) = \sum_{\mu,\nu\in\Lambda_+} \Big(\frac{\dim_{q,t}(R_\mu)}{\sqrt{g_\mu}}\Big)^{1-2h_1} \Big(\frac{\dim_{q,t}(R_\nu)}{\sqrt{g_\nu}}\Big)^{1-2h_2} \, \widetilde{N}_{\mu\,\lambda}^{\nu} \ , \tag{5.16}$$

where $\widetilde{N}_{\mu\,\lambda}^{\nu}$ are refined fusion coefficients defined by the relation

$$\chi_{\Phi_\mu}(U)\, \chi_{\Phi_\lambda}(U) = \sum_{\nu\in\Lambda_+} \widetilde{N}_{\mu\,\lambda}^{\nu} \ \chi_{\Phi_\nu}(U) \tag{5.17}$$

expressing the completeness of the Macdonald polynomials $M_\lambda(x;q,t)$ in the ring of symmetric functions. Below we compare them to the Littlewood-Richardson coefficients $N_{\mu\,\lambda}^{\nu} \in \mathbb{Z}_{\geq 0}$ which give the multiplicities in the decomposition of tensor products of irreducible $U(N)$-modules as

$$R_\mu \otimes R_\lambda = \bigoplus_{\nu\in\Lambda_+} R_\nu^{\oplus N_{\mu\,\lambda}^{\nu}} \ , \tag{5.18}$$

and the same decomposition is true as $\mathcal{U}_q(\mathfrak{gl}_N)$-modules. To suitably express $\widetilde{N}_{\mu\,\lambda}^{\nu}$ and expand the Wilson loops we need a couple of preliminary lemmata.

Lemma 5.19 *If λ, μ, ν, λ_β, μ_β and ν_β are all partitions, then for large N one has*

$$\widetilde{N}_{\mu\,\lambda}^{\nu} = N_{\mu_\beta\,\lambda_\beta}^{\nu_\beta}$$

in $\mathcal{U}_q(\mathfrak{gl}_N)$.

Proof: Let us consider $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}(\Phi_n\, P_{\lambda_\beta}\, U)$ for $\lambda \in \Lambda_+^n$. The trace takes values in the weight zero subspace of $W_{\beta-1}$ from (3.4), and in this subspace the intertwiner Φ_n acts proportionally to the identity on $R_{\omega_1}^{\otimes n}$ via (3.51). This yields

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}(\Phi_n\, P_{\lambda_\beta}\, U) = (g_{\omega_1})^{-n/2}\ \mathrm{Tr}_{R_{\omega_1}^{\otimes n}}(P_{\lambda_\beta}\, U) \ , \tag{5.20}$$

where the second trace is an ordinary $\mathbb{C}$-valued trace. We can use quantum Schur-Weyl duality and the definition of the quantum Young projectors from (3.9) together with the definition of Φ_n in (3.14) to get

$$d_{\lambda_\beta}(1)\, \chi_{\Phi_\lambda}(U) = d_{\lambda_\beta}(1)\, (g_{\omega_1})^{-n/2}\ \mathrm{Tr}_{R_{\lambda_\beta}}(U) \ . \tag{5.21}$$

It follows that the generalized character and the trace of U differ only by a factor as

$$\mathrm{Tr}_{R_{\lambda_\beta}}(U) = (g_{\omega_1})^{n/2}\, \chi_{\Phi_\lambda}(U) \ . \tag{5.22}$$

Using in addition the definitions of the coefficients (5.18) and (5.17), we then get

$$
\begin{aligned}
\chi_{\Phi_\mu}(U)\,\chi_{\Phi_\lambda}(U) &= (g_{\omega_1})^{-(|\mu|+|\lambda|)/2}\ \mathrm{Tr}_{\,R_{\mu_\beta}}(U)\ \mathrm{Tr}_{\,R_{\lambda_\beta}}(U) \\
&= (g_{\omega_1})^{-(|\mu|+|\lambda|)/2} \sum_{\nu\in\Lambda_+} N^{\nu}_{\mu_\beta\,\lambda_\beta}\ \mathrm{Tr}_{\,R_\nu}(U) \ = \ \sum_{\nu\in\Lambda_+} N^{\nu}_{\mu_\beta\,\lambda_\beta}\ \chi_{\Phi_{\nu_{\beta-2}}}(U)
\end{aligned}
\tag{5.23}
$$

and

$$
\chi_{\Phi_\mu}(U)\,\chi_{\Phi_\lambda}(U) \ = \ \sum_{\nu\in\Lambda_+} \widetilde{N}^{\nu}_{\mu\,\lambda}\,\chi_{\Phi_\nu}(U) \ . \tag{5.24}
$$

The result now follows by taking the inner product (5.4) of $\chi_{\Phi_\mu}(U)\,\chi_{\Phi_\lambda}(U)$ with $\chi_{\Phi_\nu}(U)$ in each of these expressions and comparing the two results. ■

Lemma 5.25 $\widetilde{N}^{\nu}_{\mu\,\lambda} \ = \ \widetilde{N}^{\nu+2a\,(1^N)}_{\mu+a\,(1^N)\ \ \lambda+a\,(1^N)}$.

<u>Proof</u>: We use the shift property of the Macdonald polynomials from [44, §IV, eq. (4.17)] which reads

$$
M_{\lambda+a\,(1^N)}(x;q,t) = x^a\, M_\lambda(x;q,t) \tag{5.26}
$$

where $x^a := (x_1\cdots x_N)^a$. Together with (3.21) this implies

$$
\chi_{\Phi_{\lambda+a\,(1^N)}}(U) = x^a\,\chi_{\Phi_\lambda}(U) \ , \tag{5.27}
$$

where $x = \mathrm{e}^{\,z}$ and $U = \mathrm{e}^{\,(z,H)}$. We then obtain

$$
\chi_{\Phi_{\mu+a\,(1^N)}}(U)\,\chi_{\Phi_{\lambda+a\,(1^N)}}(U) \ = \ x^{2a}\,\chi_{\Phi_\mu}(U)\,\chi_{\Phi_\lambda}(U) \ = \ \sum_{\nu\in\Lambda_+} \widetilde{N}^{\nu}_{\mu\,\lambda}\,\chi_{\Phi_{\nu+2a\,(1^N)}}(U) \tag{5.28}
$$

and

$$
\chi_{\Phi_{\mu+a\,(1^N)}}(U)\,\chi_{\Phi_{\lambda+a\,(1^N)}}(U) \ = \ \sum_{\nu\in\Lambda_+} \widetilde{N}^{\nu}_{\mu+a\,(1^N)\ \ \lambda+a\,(1^N)}\,\chi_{\Phi_\nu}(U) \ . \tag{5.29}
$$

The result now follows by taking the inner product (5.4) of $\chi_{\Phi_{\mu+a\,(1^N)}}(U)\,\chi_{\Phi_{\lambda+a\,(1^N)}}(U)$ with the generalized character $\chi_{\Phi_{\nu+2a\,(1^N)}}(U)$ in each of these expressions and comparing the two results. ■

To work out the large N expansion of the Wilson loop (5.16), we use the expansion of the Littlewood-Richardson coefficients in terms of Hecke characters given by [33, eq. (4.11)]

$$
\begin{aligned}
N^{\nu}_{\mu\,\lambda} \ &= \ \frac{q^{-\frac{n_1\,(n_1-1)}{4}}}{[n_1]_q!}\ \frac{q^{-\frac{n_2\,(n_2-1)}{4}}}{[n_2]_q!}\ \frac{d_\mu(q)}{d_\mu(1)}\ \frac{d_\lambda(q)}{d_\lambda(1)} \sum_{\sigma_1\in\mathfrak{S}_{n_1}}\ \sum_{\sigma_2\in\mathfrak{S}_{n_2}} q^{-\ell(\sigma_1)-\ell(\sigma_2)} \\
&\qquad \times\ \chi_{r_\mu}\big(\mathsf{h}(\sigma_1^{-1})\big)\,\chi_{r_\lambda}\big(\mathsf{h}(\sigma_2^{-1})\big)\,\chi_{r_\nu}\big(\mathsf{h}(\sigma_1)\cdot\mathsf{h}(\sigma_2)\big) \ ,
\end{aligned}
\tag{5.30}
$$

where $|\mu| = n_1$, $|\lambda| = n_2$, $|\nu| = n_1+n_2 =: n$, and $\mathsf{h}(\sigma_1)\cdot\mathsf{h}(\sigma_2)$ acts on $\mathsf{H}_q(\mathfrak{S}_n)$ via $\mathsf{g}_1,\dots,\mathsf{g}_{n_1-1}\in\mathsf{H}_q(\mathfrak{S}_{n_1})\subset\mathsf{H}_q(\mathfrak{S}_n)$ and $\mathsf{g}_{n_1+1},\dots,\mathsf{g}_{n_1+n_2-1}\in\mathsf{H}_q(\mathfrak{S}_{n_2})\subset\mathsf{H}_q(\mathfrak{S}_n)$. We rewrite the expectation value of the Wilson loop (5.16) using (3.21), Lemma 5.25 and Lemma 5.19 to get

$$
\begin{aligned}
W_\lambda(q,t;0;h_1,h_2) \ &= \ \sum_{n_1=a\,N}^{\infty}\ \sum_{n=2a\,N}^{\infty}\ \sum_{\mu\in\Lambda_+^{n_1-a\,N}}\ \sum_{\nu\in\Lambda_+^{n-2a\,N}} N^{\nu_\beta+2a\,(1^N)}_{\mu_\beta+a\,(1^N)\ \ \lambda_\beta+a\,(1^N)} \\
&\qquad \times\ \Big(\frac{\dim_{q,t}(R_{\mu+a\,(1^N)})}{\sqrt{g_{\mu+a\,(1^N)}}}\Big)^{1-2h_1}\ \Big(\frac{\dim_{q,t}(R_{\nu+2a\,(1^N)})}{\sqrt{g_{\nu+2a\,(1^N)}}}\Big)^{1-2h_2} \ ,
\end{aligned}
\tag{5.31}
$$

for $|\lambda| = n_2 - a\,N$. Again we expand a transformed version of the Wilson loop given by

$$W(q,t;0;h_1,h_2;C) \;=\; \frac{q^{-\frac{n_2\,(n_2-1)}{4}}}{[n_2]_q!} \sum_{\lambda\in\Lambda_+^{n_2-a\,N}} \chi_{r_{\lambda_\beta+a\,(1^N)}}(C)\; W_\lambda(q,t;0;h_1,h_2)\ , \tag{5.32}$$

where C is an arbitrary central element of the Hecke algebra $\mathsf{H}_q(\mathfrak{S}_{n_2})$. Using now the expansion of the Littlewood-Richardson coefficients $N_{\mu\,\lambda}^{\nu}$ from (5.30), the expansion of the refined quantum dimensions from (4.7), the character of the central element D_n from (4.14), the definition of the step function $\Theta_n(\beta)$ from (4.19), the properties (4.11) and (4.9), and the definition of the delta-functions on Hecke algebras from (4.17) we finally arrive at the chiral series for Wilson loop observables given by

$$\begin{aligned} &W(q,t;0;h_1,h_2;C)\\ &\quad= \sum_{n_1=\lceil a\,N\rceil}^{\infty}\ \sum_{n=\lceil 2a\,N\rceil}^{\infty} \Big(\frac{[N]_t}{\sqrt{g_{\omega_1}}}\Big)^{n_1\,(1-2h_1)+n\,(1-2h_2)}\ \delta_{n_1+n_2,n}\ \frac{q^{-\frac{n_1\,(n_1-1)}{4}}}{[n_1]_q!}\ \frac{q^{-\frac{n_2\,(n_2-1)}{4}}}{[n_2]_q!}\\ &\quad\times \sum_{\sigma_1\in\mathfrak{S}_{n_1}}\ \sum_{\sigma_2\in\mathfrak{S}_{n_2}} q^{-\ell(\sigma_1)-\ell(\sigma_2)}\,\delta\Big(\Theta_{n_2}(\beta)\,C\,\mathsf{h}\big(\sigma_2^{-1}\big)\Big)\,\delta\Big(\Theta_{n_1}(\beta)\,D_{n_1}\,\Omega_{n_1}(q,t)^{1-2h_1}\,\Pi_{n_1}^{(h_1)}\,\mathsf{h}\big(\sigma_1^{-1}\big)\Big)\\ &\qquad\qquad\times\,\delta\Big(\Theta_n^2(\beta)\,\Omega_n(q,t)^{1-2h_2}\,\Pi_n^{(h_2)}\,\big(\mathsf{h}(\sigma_1)\cdot\mathsf{h}(\sigma_2)\big)\Big)\ , \end{aligned} \tag{5.33}$$

where we have defined

$$\Theta_n^2(\beta) \;=\; \sum_{\substack{\mu\in\Lambda_+^n\\ \mu_i\geq(\beta-1)\,\rho_i+2a}} P_\mu \tag{5.34}$$

and

$$\Pi_n^{(h)} \;=\; \sum_{\sigma_1,\tau_1,\dots,\sigma_h,\tau_h\in\mathfrak{S}_n}\ \prod_{i=1}^h q^{-\ell(\sigma_i)-\ell(\tau_i)}\,\mathsf{h}(\sigma_i)\,\mathsf{h}(\tau_i)\,\mathsf{h}\big(\sigma_i^{-1}\big)\,\mathsf{h}\big(\tau_i^{-1}\big)\ . \tag{5.35}$$

This expression is the refined version of [33, eq. (4.18)]. It is a refined quantum deformation of the counting of covering worldsheets with boundary that maps to the corresponding Wilson graph on Σ_h according to the specified monodromy C [13, 49]; the expansion into parameterized orbifold Euler characters in the classical limit $q = 1$ proceeds as in §4.3.

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A Quantum group $\mathcal{U}_q(\mathfrak{gl}_N)$

For a generic value of q, let $\mathcal{U}_q(\mathfrak{gl}_N)$ be the associative algebra over $\mathbb{C}$ with generators E_i, F_i for $i = 1, \dots, N-1$ and $q^{\pm H_i/2}$ for $i = 1, \dots, N$ obeying the relations

$$\begin{aligned}
q^{H_i/2}\, E_i\, q^{-H_i/2} &= q^{1/2}\, E_i \ , \\
q^{H_i/2}\, E_{i-1}\, q^{-H_i/2} &= q^{-1/2}\, E_{i-1} \ , \\
q^{H_i/2}\, F_i\, q^{-H_i/2} &= q^{-1/2}\, F_i \ , \\
q^{H_i/2}\, F_{i-1}\, q^{-H_i/2} &= q^{1/2}\, F_{i-1} \ , \\
\big[q^{H_i/2}, E_j\big] \ = \ \big[q^{H_i/2}, F_j\big] &= 0 \qquad \text{for} \quad j \neq i, i-1 \ , \\
[E_i, F_j] &= \delta_{ij}\, \frac{q^{(H_i-H_{i+1})/2} - q^{-(H_i-H_{i+1})/2}}{q^{1/2} - q^{-1/2}} \ , \\
[E_i, E_j] \ = \ [F_i, F_j] &= 0 \qquad \text{for} \quad |i-j| > 1 \ , \\
E_i^2\, E_j - \big(q^{1/2} + q^{-1/2}\big)\, E_i\, E_j\, E_i + E_j\, E_i^2 &= 0 \qquad \text{for} \quad |i-j| = 1 \ , \\
F_i^2\, F_j - \big(q^{1/2} + q^{-1/2}\big)\, F_i\, F_j\, F_i + F_j\, F_i^2 &= 0 \qquad \text{for} \quad |i-j| = 1 \ .
\end{aligned} \tag{A.1}$$

In the fundamental representation (3.27), we have $H_i = E_{ii}$, $E_i = E_{i\,i+1}$ and $F_i = E_{i+1\,i}$. The coproduct on $\mathcal{U}_q(\mathfrak{gl}_N)$ is defined by

$$\begin{aligned}
\Delta(E_i) &= E_i \otimes q^{-(H_i-H_{i+1})/2} + q^{(H_i-H_{i+1})/2} \otimes E_i \ , \\
\Delta(F_i) &= F_i \otimes q^{-(H_i-H_{i+1})/2} + q^{(H_i-H_{i+1})/2} \otimes F_i \ , \\
\Delta\big(q^{H_i/2}\big) &= q^{H_i/2} \otimes q^{H_i/2} \ .
\end{aligned} \tag{A.2}$$

B Hecke algebra of type A_{n-1}

The symmetric group $\mathfrak{S}_n$ of degree $n \geq 2$ is generated by the elementary transpositions $\sigma_i = (i\ i{+}1)$ for $i = 1, \dots, n-1$ satisfying the relations

$$\sigma_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \sigma_{i+1} \ , \qquad \sigma_i\, \sigma_j = \sigma_j\, \sigma_i \quad \text{for} \quad |i-j| > 1 \qquad \text{and} \qquad \sigma_i^2 = 1 \ . \tag{B.1}$$

The *length* $\ell(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ is the smallest integer r such that there exists $i_1, \dots, i_r$ with $\sigma = \sigma_{i_1} \cdots \sigma_{i_r}$; such an expression is called a decomposition of σ into a *reduced word*. Note that decompositions into reduced words are not unique.

The *Hecke algebra* $\mathsf{H}_q(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ for $n \geq 2$ is the algebra over $\mathsf{H}_q(\mathfrak{S}_0) = \mathsf{H}_q(\mathfrak{S}_1) := \mathbb{C}[q, q^{-1}]$ generated by g_i for $i = 1, \dots, n-1$ with the relations

$$\mathsf{g}_i\, \mathsf{g}_{i+1}\, \mathsf{g}_i = \mathsf{g}_{i+1}\, \mathsf{g}_i\, \mathsf{g}_{i+1} \ , \qquad \mathsf{g}_i\, \mathsf{g}_j = \mathsf{g}_j\, \mathsf{g}_i \quad \text{for} \quad |i-j| > 1 \qquad \text{and} \qquad (\mathsf{g}_i - q)\, (\mathsf{g}_i + 1) = 0 \ . \tag{B.2}$$

The inverse of the generator g_i is

$$\mathsf{g}_i^{-1} = q^{-1}\, \mathsf{g}_i + \big(q^{-1} - 1\big) \ . \tag{B.3}$$

If $\sigma = \sigma_{i_1} \cdots \sigma_{i_r}$ is a decomposition of $\sigma \in \mathfrak{S}_n$ in the form of a reduced word, then we set $\mathsf{h}(\sigma) = \mathsf{g}_{i_1} \cdots \mathsf{g}_{i_r} \in \mathsf{H}_q(\mathfrak{S}_n)$. One can show that $\mathsf{h}(\sigma)$ is independent of the decomposition of σ into a reduced word and that $\big\{\mathsf{h}(\sigma)\big\}_{\sigma \in \mathfrak{S}_n}$ is a $\mathbb{C}[q, q^{-1}]$-basis of the free $\mathbb{C}[q, q^{-1}]$-module $\mathsf{H}_q(\mathfrak{S}_n)$. Given $\sigma, \tau \in \mathfrak{S}_n$ with $\ell(\sigma\, \tau) = \ell(\sigma) + \ell(\tau)$, one has $\mathsf{h}(\sigma)\, \mathsf{h}(\tau) = \mathsf{h}(\sigma\, \tau)$. The algebra $\mathsf{H}_q(\mathfrak{S}_n)$ is a

q-deformation of the group algebra $\mathbb{C}[\mathfrak{S}_n]$; in the classical limit $q=1$ the element $\mathsf{h}(\sigma)$ becomes the permutation σ. The combinatorial identity

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\ell(\sigma)} = q^{\frac{n\,(n-1)}{4}}\,[n]_q! \tag{B.4}$$

expresses a q-deformation of the order of $\mathfrak{S}_n$, where we defined the q-factorial $[n]_q! := [1]_q\cdots[n]_q$.

The irreducible representations r_λ of the symmetric group $\mathfrak{S}_n$ are in one-to-one correspondence with partitions λ of n. In particular, the sign representation $\det = \bigwedge^n R_{\omega_1}^{\otimes n}$ corresponds to the trivial partition $\lambda=(n)$ while the trivial representation corresponds to the maximal partition $\lambda=(1^n)$ with n parts. The splitting (3.8) then gives the decomposition of $R_{\omega_1}^{\otimes n}$ into subrepresentations corresponding to its λ-isotopical components. The q-deformation of the dimension of r_λ is given by

$$d_\lambda(q) = \frac{\prod\limits_{i<j}\big(q^{\ell_i}-q^{\ell_j}\big)}{\prod\limits_{i=1}^{\ell(\lambda)}\big(q-1\big)\big(q^2-1\big)\cdots\big(q^{\ell_i}-1\big)}\ \frac{\big(q-1\big)\big(q^2-1\big)\cdots\big(q^n-1\big)}{q^{\frac{\ell(\lambda)\,(\ell(\lambda)-1)\,(\ell(\lambda)-2)}{6}}}\ , \tag{B.5}$$

where $\ell(\lambda)$ is the length of the partition λ (the number of non-zero λ_i) and $\ell_i=\lambda_i+\ell(\lambda)-i$. In the classical limit $q\to 1$ this expression reduces to the usual dimension formula

$$d_\lambda(1) = d_\lambda := \chi_{r_\lambda}(1) = \frac{n!}{\prod\limits_{i=1}^{\ell(\lambda)}\ell_i!}\ \prod_{1\le i<j\le\ell(\lambda)}\big(\ell_i-\ell_j\big)\ . \tag{B.6}$$

C (q,t)-traces and symmetric functions

In this paper we are interested in the large N expansion of refined $U(N)$ Yang-Mills amplitudes. The computation of the traces $\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{h}(m_T)\big)$ in this limit can be related to some combinatorial identities involving symmetric functions. For this, we shall say that the minimal word $\mathsf{h}(m_T)$ of $\mathsf{H}_q(\mathfrak{S}_n)$ has *connectivity class* $\mu(T)=\big(\mu_1(T),\dots,\mu_n(T)\big)$ if the conjugacy class $T\in\mathfrak{S}_n^\vee$ is parameterized by the partition $\mu(T)$ of n, i.e. any element of T is composed of reduced words with $\mu_i(T)$ cycles of length i. The minimal word m_T in the conjugacy class T has length

$$\ell^*\big(\mu(T)\big) = \sum_{i=1}^n (i-1)\,\mu_i(T)\ , \tag{C.1}$$

and $\ell(\mu(T))=\sum_{i=1}^n\mu_i(T)$ is the total number of cycles in the cycle decomposition of T.

We interpret Lemma 3.18 as an expansion in (normalized) Macdonald polynomials as

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(\Phi_n\,U\,\mathsf{h}(\sigma)\big) = \sum_{\lambda_\beta\in\Lambda_+^n}\chi_{r_{\lambda_\beta}}\big(\mathsf{h}(\sigma)\big)\ \frac{M_\lambda(x;q,t)}{\sqrt{g_\lambda}}\ . \tag{C.2}$$

Let us consider the unrefined limit $\beta=1$, wherein the normalized Macdonald polynomials in (C.2) reduce to Schur polynomials $s_\lambda(x)$. We can then apply [37, Theorem 1 and Definition 1] to get

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\big(U\,\mathsf{h}(m_T)\big) = \prod_{i=1}^n p_{\mu_i(T)}(q;x)\ , \tag{C.3}$$

where

$$p_r(q;x) := \sum_{\substack{a,b=0\\ a+b=r-1}}^{r-1}(-1)^b\,q^a\,s_{(a+1\ 1^b)}(x)\ . \tag{C.4}$$

By [37, Lemma 1] we can equivalently write (C.4) as

$$p_r(q;x) = \frac{1}{q-1} \sum_{\lambda\in\Lambda_+^r} s_\lambda(q|-1)\, s_\lambda(x) \ , \tag{C.5}$$

where $s_\lambda(q|-1)$ is a supersymmetric Schur function [53, Section 4.4]. The sum in (C.5) can be evaluated by using the Cauchy-Binet identity for supersymmetric Schur functions [53, eq. (4.27)]

$$\sum_{\lambda\in\Lambda_+} s_\lambda(x|z)\, s_\lambda(y|w) = \prod_{i,j=1}^N \frac{(1+x_i\,w_j)\,(1+y_i\,z_j)}{(1-x_i\,y_j)\,(1-z_i\,w_j)} \ , \tag{C.6}$$

which at the specializations $z=(0,\dots,0)$, $y=(q,0,\dots,0)$ and $w=(-1,0,\dots,0)$ yields

$$\sum_{\lambda\in\Lambda_+} s_\lambda(q|-1)\, s_\lambda(x) = \prod_{i=1}^N \frac{1-x_i}{1-q\,x_i} \ . \tag{C.7}$$

The sum over partitions of r can in this way be computed by using the homogeneity property $s_\lambda(\alpha\,x) = \alpha^{|\lambda|}\, s_\lambda(x)$ of Schur polynomials and the generating function

$$\sum_{r=1}^\infty \alpha^r \sum_{\lambda\in\Lambda_+^r} s_\lambda(q|-1)\, s_\lambda(x) = \sum_{\lambda\in\Lambda_+} s_\lambda(q|-1)\, s_\lambda(\alpha\,x) \tag{C.8}$$

for $\alpha\in\mathbb{C}$. Using (C.7) we then find

$$p_r(q;x) = \frac{1}{q-1}\,\frac{1}{r!}\,\frac{\partial^r}{\partial\alpha^r}\bigg|_{\alpha=0}\ \prod_{i=1}^N \frac{1-\alpha\,x_i}{1-\alpha\,q\,x_i} \ . \tag{C.9}$$

In particular, the connected minimal word $\mathsf{h}(m_T) = \mathsf{g}_1\,\mathsf{g}_2\cdots\mathsf{g}_{n-1}$ belongs to the connectivity class $\mu(T)=(n)$ and the corresponding trace gives exactly $p_n(q;x)$, so that

$$\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(U\,\mathsf{g}_1\,\mathsf{g}_2\cdots\mathsf{g}_{n-1}\big) = \frac{1}{1-q} \sum_{\lambda\in\Lambda_+^n} s_\lambda(q|-1)\, s_\lambda(x) \ . \tag{C.10}$$

At the specialization $U=q^{(\rho,H)}$ we can compare this formula with the explicit computation of the trace from [33, eq. (B.6)] to arrive at the combinatorial identity

Proposition C.11 $\displaystyle\sum_{\lambda\in\Lambda_+^n} s_\lambda(q|-1)\, s_\lambda(q^\rho) = (q-1)\, q^{(n-1)\,\frac{N+1}{2}}\, [N]_q \ .$

This identity can be compared explicitly with the formula (C.9), in which case the specialization of the product in (C.7) to $x_i = \alpha\,q^{\frac{N+1}{2}-i}$ yields the generating function

$$\sum_{\lambda\in\Lambda_+} s_\lambda(q|-1)\, s_\lambda(\alpha\,q^\rho) = \frac{1-\alpha\,q^{-\frac{N-1}{2}}}{1-\alpha\,q^{\frac{N+1}{2}}} \ . \tag{C.12}$$

Substituting now $p_r(q;q^\rho) = q^{(r-1)\,\frac{N+1}{2}}\,[N]_q$ into (C.3) we get

$$\mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(q^{(\rho,H)}\,\mathsf{h}(m_T)\big) = \prod_{i=1}^n \Big(q^{(i-1)\,\frac{N+1}{2}}\,[N]_q\Big)^{\mu_i(T)} = q^{\frac{N+1}{2}\,\ell^*(\mu(T))}\,\big([N]_q\big)^{\ell(\mu(T))} \tag{C.13}$$

as in [33, eq. (B.7)]. We are not aware of any analogous simplifying identities for generating functions of Macdonald polynomials which could aid in simplifying the (q,t)-traces for $\beta\neq1$.

D Center of $\mathsf{H}_q(\mathfrak{S}_\infty)$

There is a natural embedding of Hecke algebras $\mathsf{H}_q(\mathfrak{S}_n) \hookrightarrow \mathsf{H}_q(\mathfrak{S}_{n+1})$, and so the inductive limit of $\mathsf{H}_q(\mathfrak{S}_n)$ as $n \to \infty$ exists [37]; we write this inductive limit as $\mathsf{H}_q(\mathfrak{S}_\infty)$. We want to find an inductive limit of central elements of the Hecke algebras as well. Firstly we need an embedding of central elements given by a monomorphism

$$\varphi_n \; : \; \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big) \; \lhook\joinrel\longrightarrow \; \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_{n+1})\big) \ , \tag{D.1}$$

where $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ is a linear subspace of the center $Z\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ of the algebra $\mathsf{H}_q(\mathfrak{S}_n)$ such that

$$\sum_i \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\, x\, C_i^{(n)}\, U\big) \; = \; \sum_i \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}}\big(\Phi_n\, x\, C_i^{(n+1)}\, U\big) \ , \tag{D.2}$$

for $U \in T$ and $x \in \mathsf{H}_q(\mathfrak{S}_n)$, where $C_i^{(n)}$ span a linear basis of $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$.

According to [18, Theorem 2.14], the center $Z\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ is the algebra of symmetric polynomials in the *Murphy operators* L_i, $i = 1, \dots, n$, which are defined as

$$\begin{aligned} L_1 &= \mathsf{h}(1) \; = \; 1 \ , \\ L_i &= q^{-(i-1)} \sum_{j=1}^{i-1} q^{j-1}\, \mathsf{h}\big((j\ i)\big) \qquad \text{for} \quad i > 1 \ , \end{aligned} \tag{D.3}$$

where $(i\, j) = \sigma_i \cdots \sigma_{j-2}\, \sigma_{j-1}\, \sigma_{j-2} \cdots \sigma_i$ for $j > i$ is the transposition which interchanges i and j. For example, using the definition of the central elements C_T from (3.34), for $n = 3$ we obtain $C_{(1,1,1)} = 1$, $C_{(2,1)} = q\,(L_2{+}L_3)$ and $C_{(3)} = \frac{q^2}{2}\,(L_2\, L_3{+}L_3\, L_2)$ as homogeneous symmetric polynomials in Murphy operators.

Given a symmetric polynomial $s(L_1, \dots, L_n)$ in $L_i \in \mathsf{H}_q(\mathfrak{S}_n)$, we need to embed it into $\mathsf{H}_q(\mathfrak{S}_{n+1})$, but a symmetric polynomial in n variables is not necessarily a symmetric polynomial in $n + 1$ variables so we need a non-trivial embedding. Because of (D.2) we require

$$\varphi_n\big(s(L_1, \dots, L_n)\big) \; = \; s(L_1, \dots, L_n) \; + \; p(L_1, \dots, L_{n+1}) \ , \tag{D.4}$$

where $p(L_1, \dots, L_{n+1})$ is not necessarily a symmetric polynomial. If $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ is the space of homogeneous symmetric polynomials in Murphy operators, then φ_n is unique and so there exists just one p for every s in (D.4). For example one has

$$\begin{aligned} \varphi_1(1) &= 1 \ , \\ \varphi_2(L_2) &= L_2 + L_3 \ , \\ \varphi_3(L_2\, L_3 + L_3\, L_2) &= L_2\, L_3 + L_3\, L_2 + L_2\, L_4 + L_4\, L_2 + L_3\, L_4 + L_4\, L_3 \ . \end{aligned} \tag{D.5}$$

In the representation $R_{\omega_1}^{\otimes n}$, the Murphy operator L_{n+1} is represented as 0 if $\mathsf{g}_{n+i} \cdot (R_{\omega_1} \otimes \cdots \otimes R_{\omega_1}) = 0$ for $i \geq 0$. Hence we get

$$\varphi_n\big(s(L_1, \dots, L_n)\big) \cdot (R_{\omega_1} \otimes \cdots \otimes R_{\omega_1}) \; = \; s(L_1, \dots, L_n) \cdot (R_{\omega_1} \otimes \cdots \otimes R_{\omega_1}) \ . \tag{D.6}$$

All elements of $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_{n+1})\big)$ either belong to the image of φ_n, or else all of their monomials contain at least one factor L_{n+1} and so are represented as 0 in $R_{\omega_1}^{\otimes n}$.

Using this embedding, we can now take the inductive limit of $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ as $n \to \infty$, which is given by the equivalence classes

$$\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_\infty)\big) = \bigsqcup_{n=1}^{\infty} \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big) \ / \ \sim \tag{D.7}$$

where $x \sim y$ if and only if $y = \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_n(x)$ for $x \in \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$, $y \in \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_m)\big)$ and $m > n$; here the disjoint union over $\widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_n)\big)$ is factorized with a sequence of the embeddings.

Now we can consider the transformation of a function $f(U)$ for $U \in T$ given by

$$f(C) \ = \sum_{n=0}^{\infty} \ \int_T [\mathrm{d}U]_{q,t} \ \mathrm{Tr}_{\,R_{\omega_1}^{\otimes n}} \big(\Phi_n \, y_n \, C \, U\big) \ f(U) \ , \tag{D.8}$$

for $y_n \in \mathsf{H}_q(\mathfrak{S}_n)$ and $C \in \widetilde{Z}\big(\mathsf{H}_q(\mathfrak{S}_\infty)\big)$, provided the series converges. Here we defined the integration measure

$$[\mathrm{d}U]_{q,t} := \frac{(-1)^N}{N!} \prod_{i=1}^{N} \mathrm{d}z_i \ \Delta_{q,t}(x) \, \Delta_{q,t}(x^{-1}) \tag{D.9}$$

where

$$\Delta_{q,t}(x) := t^{-\frac{N\,(N-1)}{2}} \prod_{i<j} \frac{\big(x_i \, x_j^{-1}; q\big)_\infty}{\big(t \, x_i \, x_j^{-1}; q\big)_\infty} = \prod_{m=0}^{\beta-1} \prod_{i<j} \big(q^{-m/2} \ \mathrm{e}^{\,(z_j-z_i)/2} - q^{m/2} \ \mathrm{e}^{\,(z_i-z_j)/2}\big) \tag{D.10}$$

for $\beta \in \mathbb{Z}_{>0}$, with $U = \mathrm{e}^{\,(z,H)}$ and $x = \mathrm{e}^{\,z}$. In the unrefined limit $\beta = 1$, the measure (D.9) reduces to the usual Haar measure

$$[\mathrm{d}U]_q = [\mathrm{d}U]_{q,q} = \frac{1}{N!} \prod_{i=1}^{N} \mathrm{d}z_i \ \Delta_q(x)^2 \tag{D.11}$$

for integration over the maximal torus $T \subset G$, where

$$\Delta_q(x) = \Delta_{q,q}(x) = \prod_{i<j} 2 \sinh\Big(\frac{z_i - z_j}{2}\Big) \tag{D.12}$$

is the Weyl determinant for $G = U(N)$.

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