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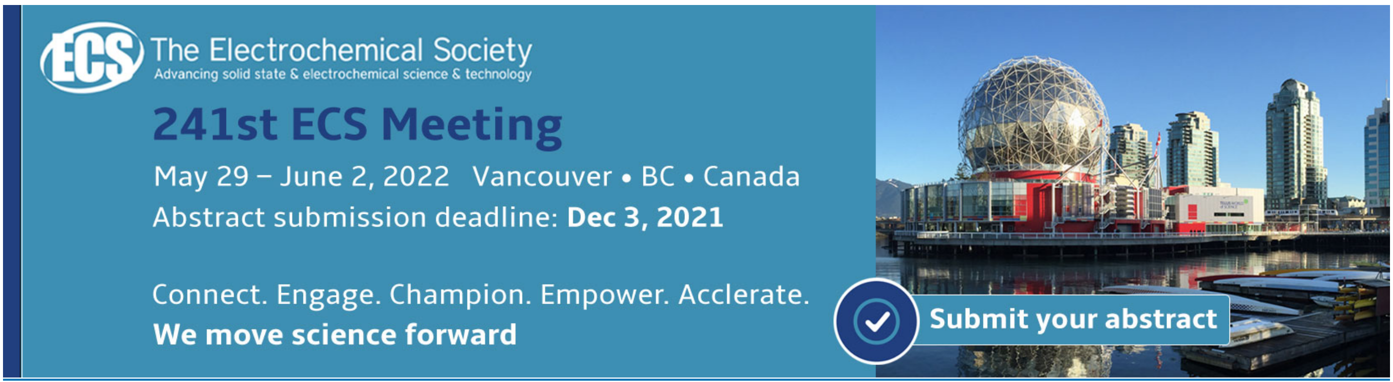

A Comparative Study of the EWMA and Double EWMA Control Schemes

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Abstract. A control scheme is the most excellent and irreplaceable tool in Statistical Process Control because of its effectiveness in controlling and monitoring the process, which indirectly helps ensure that the quality of a product is controlled. The EWMA scheme is one of the popular memory-type schemes used to control a normally distributed process. Therefore, various studies and improvements are performed on this scheme and are available in the literature. One of the significant improvements of the EWMA scheme is its extension to the double EWMA scheme, which has a better performance in detecting a small to moderate shift in the process than the traditional EWMA scheme. In order to study and compare the performance of the EWMA and double EWMA schemes in a comprehensive way, other run-length properties, such as the standard deviation and percentiles of the run-length, are also evaluated, rather than solely evaluating the average run-length results. Furthermore, the exact shift size in the process is usually not known in real life. Thus, the expected average run-length and expected median run-length metrics are used to evaluate the overall performances of these two schemes in a range of shift sizes. Then, the implementation of these two schemes is illustrated with a dataset. Last, some concluding remarks are made.

1. Introduction

"Customer is always right" is one of the popular definitions of quality. Therefore, manufacturers have to continuously improve their manufacturing process to at least meet the customers' satisfaction and expectations. Shewhart [1] highlighted that it is nearly impossible that all the products produced are identical because there exist non-assignable causes of variation. Therefore, it is more practical that the manufacturer ensures that the products are uniform and controlled. In order to achieve this objective, a control scheme, which is the most powerful tool in Statistical Process Control (SPC), is employed to identify the assignable cause and correct the issue.

Many control schemes are available in the literature, and one may categorise them into two main types, i.e., the memoryless and memory-type schemes. The main difference between these two types of schemes is their ability to detect a different shift size. For instance, a memoryless Shewhart scheme has a better performance in detecting a large shift size. In contrast, the memory-type scheme is able to detect small to moderate disturbances effectively [2]. There is a variety of memory-type schemes, where the cumulative sum (CUSUM) [3] and exponentially weighted moving average (EWMA) [4] schemes are the most well-known among them. This paper presents a comparative study between the EWMA scheme and its extension, i.e., the double EWMA (DEWMA) scheme. In detail, a DEWMA scheme can be categorised into two types, i.e., a one-lambda [5] and a two-lambda [6] DEWMA scheme.

The average run-length (ARL) is the widely-employed metric in evaluating the performance of a scheme. However, the performance evaluation of a scheme solely based on the ARL received a lot of critique from the researchers because the run-length distribution of a scheme is positively skewed. Hence, some of the researchers evaluate the performance of a scheme by studying the median run-length (MRL) [7, 8] and the run-length percentiles [9]. Nevertheless, the ARL and MRL metrics are only applicable in evaluating the performance of a scheme if the exact shift size in the process is known, which is impractical in the real-life scenario. To this end, quality practitioners prefer using a scheme that performs well in overall, i.e., within a range of possible shift sizes they considered. Therefore, the use of the expected ARL ($EARL$) [10, 11] and expected MRL ($EMRL$) [12, 13] metrics to evaluate the performance of a scheme in a specific range of shift sizes received a lot of attention from the researchers.

In this paper, we study and compare the in-control (IC) performance of the EWMA-mean (denoted as $E\bar{X}$) scheme and its extension DEWMA-mean (abbreviated as $DE\bar{X}$) scheme by evaluating the schemes with IC-ARL (ARL_0), IC-standard deviation of the run-length ($SDRL$), abbreviated as $SDRL_0$, and some IC-percentiles. Further, we also compare the performance of these two schemes in a specific range of the shift sizes by assessing the $EARL$ and $EMRL$ metrics. In Section 2, the theoretical framework of the $E\bar{X}$ and $DE\bar{X}$ schemes are explained. Then, some results and findings of the run-length properties of the two schemes are revealed in Section 3. Next, Section 4 demonstrates the monitoring procedures with a dataset. Last, we give some concluding remarks in Section 5.

2. The $E\bar{X}$ and $DE\bar{X}$ Schemes

Suppose that the quality characteristic of a process, denoted as X is collected sequentially in a sample with size n. We denote the t^{th} random sample as $\boldsymbol{X_{tn}} = \{x_{t1}, x_{t2}, \dots, x_{tn}\}$, where $t = 1,2, \dots$, such that $\boldsymbol{X_{in}}$ and $\boldsymbol{X_{jn}}$ $(i \neq j)$ are independent. Furthermore, we assume that all the observations x_{ti}, where $i = 1,2, \dots, n$ are normally distributed with mean $\mu_X + \delta$ and variance σ_X^2, such that the standard values of the process are μ_X and σ_X^2. We expect that $\delta = 0$ if the process is IC; otherwise, the process is out-of-control (OOC) if $\delta \neq 0$.

Then, the t^{th} plotting statistic of the $E\bar{X}$ scheme is defined as

$$E\bar{X}_t = \lambda \bar{X}_t + (1-\lambda) E\bar{X}_{t-1}, \tag{1}$$

where $0 < \lambda \leq 1$ is the smoothing parameter of the scheme, $\bar{X}_t = \frac{1}{n}(x_{t1} + x_{t2} + \dots + x_{tn})$, and $E\bar{X}_0 = \mu_X$. Then, the mean and variance of the $E\bar{X}$ scheme corresponding to the t^{th} sample are $\mu_{E\bar{X}_t} = \mu_X$ and $\sigma^2_{E\bar{X}_t} = \frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2t}]\frac{\sigma_X^2}{n}$, respectively [2]. Therefore, the control limits of the $E\bar{X}$ scheme are defined as

$$\begin{array}{ll} \text{Lower Control Limit:} & L_{E\bar{X}}(t) = \mu_X - K_E \sigma_{E\bar{X}_t} \\ \text{Centre Line:} & CL = \mu_X \\ \text{Upper Control Limit:} & U_{E\bar{X}}(t) = \mu_X + K_E \sigma_{E\bar{X}_t} \end{array}, \tag{2}$$

where K_E is the charting constant of the $E\bar{X}$ scheme.

On the other hand, the t^{th} plotting statistic of the $DE\bar{X}$ scheme is defined as [6]

$$DE\bar{X}_t = \lambda_2 E\bar{X}_t + (1-\lambda_2)DE\bar{X}_{t-1}, \tag{3}$$

where $E\bar{X}_t = \lambda_1\bar{X}_t + (1-\lambda_1)E\bar{X}_{t-1}$ and $DE\bar{X}_0 = \mu_X$. Note that $0 < \lambda_1 \le 1$ and $0 < \lambda_2 \le 1$ are the two smoothing parameters of the $DE\bar{X}$ scheme. However, Zhang and Chen [6] concluded that the two-lambda DEWMA scheme is not performing better than the one-lambda DEWMA scheme. So, we assume that $\lambda_1 = \lambda_2 = \lambda$ for the $DE\bar{X}$ scheme here, and thus, its plotting statistic is

$$\begin{aligned} E\bar{X}_t &= \lambda\,\bar{X}_t + (1-\lambda)E\bar{X}_{t-1} \\ DE\bar{X}_t &= \lambda\,E\bar{X}_t + (1-\lambda)DE\bar{X}_{t-1}. \end{aligned} \tag{4}$$

Then, the mean and variance of the one-lambda $DE\bar{X}$ scheme corresponding to the t^{th} sample are $\mu_{DE\bar{X}_t} = \mu_X$ and $\sigma^2_{DE\bar{X}_t} = \frac{\lambda}{(2-\lambda)^3}[A_t + B_t]\frac{\sigma_X^2}{n}$, respectively [6]. Note that $A_t = 1 + (1-\lambda)^2 - (t^2+2t+1)(1-\lambda)^{2t}$ and $B_t = (2t^2+2t-1)(1-\lambda)^{2t+2} - t^2(1-\lambda)^{2t+4}$. Hence, the control limits of the $DE\bar{X}$ scheme are defined as below:

$$\begin{aligned} &\text{Lower Control Limit:} && L_{DE\bar{X}}(t) = \mu_X - K_{DE}\sigma_{DE\bar{X}_t} \\ &\text{Centre Line:} && CL = \mu_X \quad , \\ &\text{Upper Control Limit:} && U_{DE\bar{X}}(t) = \mu_X + K_{DE}\sigma_{DE\bar{X}_t} \end{aligned} \tag{5}$$

where K_{DE} is the charting constant of the $DE\bar{X}$ scheme.

The general procedures used to obtain the run-length properties for the two schemes studied in this paper are shown below [14]:

Step I. A subgroup that consists of 5000 random numbers having a fixed sample size of n is simulated from a standard normal distribution sequentially. Then, the plotting statistics for the $E\bar{X}$ and $DE\bar{X}$ schemes, i.e., $E\bar{X}_t$ (Equation (1)) and $DE\bar{X}_t$ (Equation (4)), respectively, are computed.

Step II. Plot the $E\bar{X}_t$ and $DE\bar{X}_t$ statistics against their respective control limits given in Equations (2) and (5). The process is declared as OOC if $E\bar{X}_t$ (or $DE\bar{X}_t$) is plotted beyond the control limits. Then, the program will be stopped, and the point t is recorded.

Step III. Steps I to II are repeated 50,000 times. Then, the run-length metrics, such as ARL, $SDRL$, and run-length percentiles, are obtained from all the recorded t values.

3. Results and Findings

Applying the procedures explained in Section 2, the charting constants K_E and K_{DE} are computed by considering $\lambda \in \{0.05, 0.10, 0.20, 0.30, 0.50\}$ with an IC-ARL (ARL_0) ≈ 370. For the IC performance study, without loss of generality, we assume $n = 5$, $(\mu_X, \sigma_X^2) = (0, 1)$, and $\delta = 0$, then the charting constants and some IC run-length properties, i.e., the ARL_0, $SDRL_0$, and some run-length percentiles are presented in Table 1.

From Table 1, one may easily find that the IC performance of the $E\bar{X}$ scheme is better than that of the $DE\bar{X}$ scheme, in terms of a lower $SDRL_0$ value and higher values for the low percentiles, say 5$^{\text{th}}$ and 25$^{\text{th}}$ percentiles (lower early false alarm rate). However, when the value of λ increases, their difference in the IC performance becomes less significant. Generally, for the two schemes, it is found that ARL_0 falls between the 50$^{\text{th}}$ and 75$^{\text{th}}$ percentiles, supporting the fact that the run-length distribution of the scheme is positively skewed.

Table 1. Charting constants and IC run-length properties when $ARL_0 \approx 370$.

λ	$E\bar{X}$ Scheme		$DE\bar{X}$ Scheme	
	K_E	ARL_0 ($SDRL_0$) 5th, 25th, 50th, 75th, 95th IC-Percentiles	K_{DE}	ARL_0 ($SDRL_0$) 5th, 25th, 50th, 75th, 95th IC-Percentiles
0.05	2.521	370.34 (387.45) 8, 95, 250, 520, 1149	1.962	370.42 (420.50) 2, 64, 236, 529, 1218
0.10	2.713	370.38 (377.13) 14, 104, 254, 514, 1128	2.248	370.99 (393.43) 4, 91, 249, 522, 1149
0.20	2.863	370.48 (373.61) 18, 107, 255, 511, 1122	2.535	370.43 (377.97) 13, 102, 254, 515, 1133
0.30	2.926	370.07 (370.40) 19, 107, 256, 513, 1114	2.700	370.97 (375.36) 16, 105, 255, 513, 1120
0.50	2.979	370.55 (369.98) 19, 107, 258, 513, 1109	2.887	370.16 (369.49) 19, 107, 256, 514, 1114

For the OOC performance study, we compute the $ARL(\delta)$ and $MRL(\delta)$ values, for $\delta \in \{0.10, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 2.00, 2.50, 3.00\}$. Note that we do not present the $ARL(\delta)$ and $MRL(\delta)$ results due to space constraints, and our interest is to compare the performance of EWMA and DEWMA schemes using $EARL$ and $EMRL$ metrics. Next, we compute the $EARL$ and $EMRL$ metrics for the two schemes with Equations (6) and (7), and then compare their performances in a specific range of $[\delta_1, \delta_2]$.

$$EARL = \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} ARL(\delta)\ d\delta \tag{6}$$

and

$$EMRL = \frac{1}{\delta_2 - \delta_1} \int_{\delta_1}^{\delta_2} MRL(\delta)\ d\delta. \tag{7}$$

Here, we consider three scenarios, i.e., (i) an overall possible shift size $[0, 3]$; (ii) a possible small shift size $[0, 1]$; and (iii) a possible moderate to large shift size $[1, 3]$. The $EARL$ and $EMRL$ results are tabulated in Tables 2 and 3, respectively.

Table 2. $EARL$ values for a specific range of shift sizes when $ARL_0 \approx 370$.

λ	Case I: $[0, 3]$		Case II: $[0, 1]$		Case III: $[1, 3]$	
	$E\bar{X}$ Scheme	$DE\bar{X}$ Scheme	$E\bar{X}$ Scheme	$DE\bar{X}$ Scheme	$E\bar{X}$ Scheme	$DE\bar{X}$ Scheme
0.05	12.241	11.356	34.400	31.906	1.166	1.086
0.10	13.524	12.302	38.167	34.660	1.207	1.128
0.20	15.654	13.790	44.478	39.020	1.246	1.179
0.30	17.555	15.193	50.137	43.161	1.266	1.212
0.50	21.330	17.947	61.398	51.338	1.297	1.253

Table 3. $EMRL$ values for a specific range of shift sizes when $ARL_0 \approx 370$.

λ	Case I: $[0, 3]$		Case II: $[0, 1]$		Case III: $[1, 3]$	
	$E\bar{X}$ Scheme	$DE\bar{X}$ Scheme	$E\bar{X}$ Scheme	$DE\bar{X}$ Scheme	$E\bar{X}$ Scheme	$DE\bar{X}$ Scheme
0.05	9.039	8.048	25.000	22.150	1.063	1.000
0.10	9.948	8.981	27.725	24.825	1.063	1.063
0.20	11.357	10.073	31.700	28.100	1.188	1.063
0.30	12.574	10.932	35.350	30.675	1.188	1.063
0.50	15.175	12.857	43.150	36.200	1.188	1.188

From Tables 2 and 3, it is obvious that the $DE\bar{X}$ scheme is always outperforming (or at least as good as) the $E\bar{X}$ scheme for various possible shift sizes. When we consider an overall possible shift size (Case I), it is noticed that the $EARL$ and $EMRL$ values of the $DE\bar{X}$ scheme are always smaller than that of the $E\bar{X}$ scheme, for all λ values. However, if we consider a moderate to large shift in the process (Case III), we can notice that the $EARL$ and $EMRL$ of the $DE\bar{X}$ scheme are just slightly smaller than that of the $E\bar{X}$ scheme, for all λ values, except the $EMRL$ value when $\lambda = 0.50$. Most importantly, we can see the significant superiority of the $DE\bar{X}$ scheme over the $E\bar{X}$ scheme if we consider a small shift size (Case II), such that both the $EARL$ and $EMRL$ values of the $DE\bar{X}$ scheme are significantly smaller than that of the $E\bar{X}$ scheme.

4. An Illustrative Example

In this paper, we illustrate the implementation of the $E\bar{X}$ and $DE\bar{X}$ schemes with a dataset discussed in Montgomery [2] (see page 415 of the book). In this example, the standard values of the process are $\mu_X = 10$ and $\sigma_X = 1$, with a subgroup size of $n = 1$, such that the first twenty observations are randomly selected from an IC sample, while the remaining ten observations are chosen at random from an OOC sample. Here, we considered $ARL_0 \approx 370$ and $\lambda = 0.20$ for both schemes. Based on these ARL_0 and λ values, the charting constants computed when $n = 1$ are $K_E = 2.863$ and $K_{DE} = 2.535$, for the $E\bar{X}$ and $DE\bar{X}$ schemes, respectively, which are similar to the charting constants we obtained when $n = 5$.

The $E\bar{X}$ and $DE\bar{X}$ schemes are, respectively, illustrated in Figures 1 and 2. From Figure 1, it is noticed that none of the plotting statistics plotted beyond the control limits, suggesting that the process is IC. In contrast, the $DE\bar{X}$ scheme (Figure 2) signals two OOC alarms, i.e., at the 29$^{\text{th}}$ and 30$^{\text{th}}$ samples, which implies that the process is OOC. In addition, there is an increasing trend displayed by the $DE\bar{X}$ plotting statistics starting from the 22$^{\text{nd}}$ test sample, which is another indication of OOC. Hence, in this example, we observe the superiority of the $DE\bar{X}$ scheme over the $E\bar{X}$ scheme because the $DE\bar{X}$ scheme is able to capture the OOC signal in the process, whereas the $E\bar{X}$ scheme cannot.

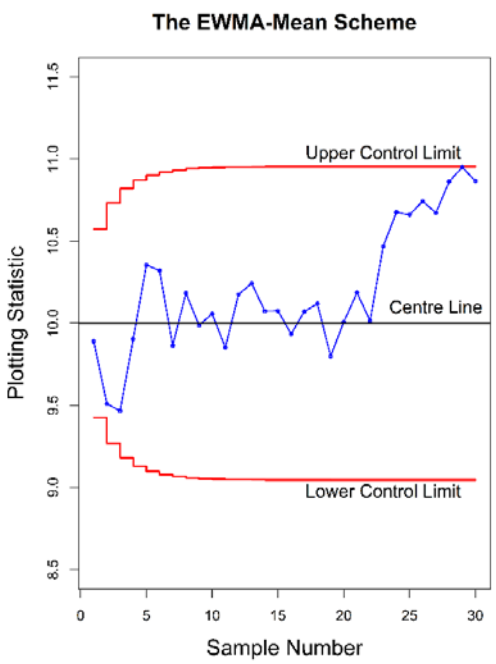

Figure 1. The $E\bar{X}$ scheme when $ARL_0 \approx 370$ and $\lambda = 0.20$.

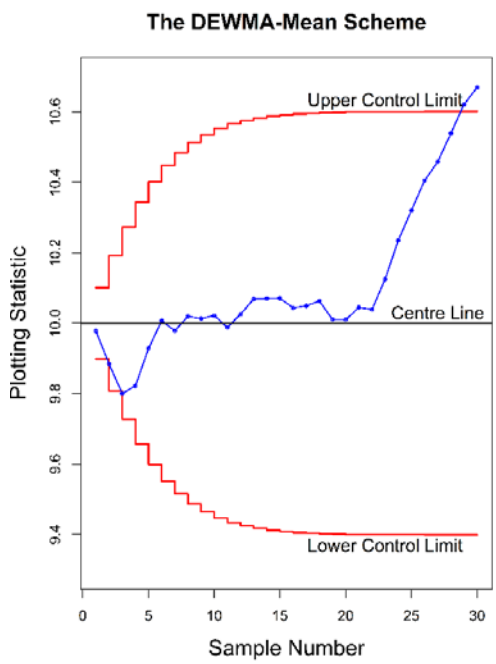

Figure 2. The $DE\bar{X}$ scheme when $ARL_0 \approx 370$ and $\lambda = 0.20$.

5. Concluding Remarks

In most of the scenarios, especially in terms of practical applications, quality practitioners prefer the memory-type scheme over the memoryless Shewhart-type scheme due to the strength of the memory-type schemes in detecting small to moderate disturbances in the process. In this paper, we study and compare the performance of two memory-type schemes, namely the $E\bar{X}$ and $DE\bar{X}$ schemes by performing Monte-Carlo simulation. We observe that the $E\bar{X}$ scheme exhibits a relatively better IC performance than the $DE\bar{X}$ scheme, because the latter has a slightly higher early false alarm rate due to the lower values in the 5th and 25th IC-percentiles. On the flip side, in terms of the OOC performance, we notice the superiority of the $DE\bar{X}$ scheme over the $E\bar{X}$ scheme, especially in detecting a small shift in the process mean. By evaluating the $EARL$ and $EMRL$ metrics, in general, we conclude that the $DE\bar{X}$ scheme always requires a lesser sample to signal an OOC alarm in a specific range of shift sizes, as compared to the $E\bar{X}$ scheme. Also, we observe the supremacy of the $DE\bar{X}$ scheme in the illustrative example, such that it is able to detect an OOC signal in the process; whereas the $E\bar{X}$ scheme is unable to detect any OOC signal.

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