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Abelian subgroups of two-dimensional Artin groups

Alexandre Martin and Piotr Przytycki

ABSTRACT

We classify abelian subgroups of two-dimensional Artin groups.

1. Introduction

Let S be a finite set and for all $s \neq t \in S$ let $m_{st} = m_{ts} \in \{2, 3, \dots, \infty\}$. The associated Artin group A_S is given by generators and relations:

$$A_S = \langle S \mid \underbrace{sts \cdots}_{m_{st}} = \underbrace{tst \cdots}_{m_{st}} \rangle.$$

Assume that A_S is *two-dimensional*, that is, for all $s, t, r \in S$, we have

$$\frac{1}{m_{st}} + \frac{1}{m_{tr}} + \frac{1}{m_{sr}} \leq 1.$$

We say that A_S is of *hyperbolic type* if the inequality above is strict for all $s, t, r \in S$. The associated Coxeter group W_S is obtained from A_S by adding the relations $s^2 = 1$ for every $s \in S$.

In [9, Theorem D], we classified explicitly all virtually abelian subgroups of A_S of hyperbolic type. In this article, we extend this, using different techniques, to all two-dimensional A_S . Charney and Davis [4, Theorem B] showed that two-dimensional Artin groups satisfy the $K(\pi, 1)$ -conjecture, in particular they are torsion-free and of cohomological dimension 2. Thus all noncyclic virtually abelian subgroups of A_S are virtually \mathbb{Z}^2 . The first step of our classification involves the *modified Deligne complex* Φ (see Section 2) introduced by Charney and Davis [4], and generalising a construction of Deligne for Artin groups of spherical type [5].

THEOREM A. *Let A_S be a two-dimensional Artin group, and let H be a subgroup of A_S that is virtually \mathbb{Z}^2 . Then:*

- (i) H is contained in the stabiliser of a vertex of Φ ; or
- (ii) H is contained in the stabiliser of a standard tree of Φ ; or
- (iii) H acts properly on a Euclidean plane isometrically embedded in Φ .

The stabilisers of the vertices of Φ , appearing in (i), are cyclic or conjugates of the dihedral Artin groups A_{st} (see Section 2), which are virtually $\mathbb{Z} \times F$ for a free group F , see, for example, [6, Lemma 4.3(i)]. The stabilisers of the standard trees of Φ , appearing in (ii), are also $\mathbb{Z} \times F$ [9, Lemma 4.5] and were described more explicitly in [9, Rm 4.6].

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In the second step of our classification, we will list all $H \cong \mathbb{Z}^2$ satisfying (iii). Note that the statement might seem daunting, but in fact it arises in a straightforward way from reading off the labels of the Euclidean planes obtained in the course of the proof.

We use the following notation. Let S be an alphabet. If $s \in S$, then s^\bullet denotes the language (that is, set of words) of form s^n for $n \in \mathbb{Z} - \{0\}$. We treat a letter $s \in S^\pm$ as a language consisting of a single word. If $\mathcal{L}, \mathcal{L}'$ are languages, then $\mathcal{L}\mathcal{L}'$ denotes the language of words of the form ww' where $w \in \mathcal{L}, w' \in \mathcal{L}'$. If \mathcal{L} is a language, then \mathcal{L}^* denotes the union of the languages \mathcal{L}^n for $n \geq 1$.

THEOREM B. *Let A_S be a two-dimensional Artin group. Suppose that $\mathbb{Z}^2 \subset A_S$ acts properly on a Euclidean plane isometrically embedded in Φ . Then \mathbb{Z}^2 is conjugated into:*

- (a) $\langle w, w' \rangle$, where $w \in A_T, w' \in A_{T'}$ and $m_{tt'} = 2$ for all $t \in T, t' \in T'$, for some disjoint $T, T' \subset S$, or \mathbb{Z}^2 is conjugated into one of the following, where $s, t, r \in S$;
- (b) $\langle strstr, w \rangle$, where $w \in (t^\bullet str)^*$ and $m_{st} = m_{tr} = m_{sr} = 3$;
- (c) $\langle strt, w \rangle$, where $w \in (r^\bullet t^{-1} s^\bullet t)^*$ and $m_{st} = m_{tr} = 4, m_{sr} = 2$;
- (d) $\langle stsrtr, w \rangle$, where $w \in (t^\bullet str)^*$ and $m_{st} = m_{tr} = 4, m_{sr} = 2$;
- (e) $\langle ststsrtr, w \rangle$, where $w \in (t^\bullet ststr)^*$ and $m_{st} = 6, m_{tr} = 3, m_{sr} = 2$;
- (f) $\langle tststr, w \rangle$, where $w \in (s^\bullet tstrt^{-1})^*$ and $m_{st} = 6, m_{tr} = 3, m_{sr} = 2$.

It is easy to check directly that the above groups are indeed abelian. Since A_S is torsion-free [4, Thm B], the only other subgroups of A_S that are virtually \mathbb{Z}^2 are isomorphic to the fundamental group of the Klein bottle. They can be also classified, see Remark 5.8.

In the proof of Theorem B, we will describe in detail the Euclidean planes in Φ stabilised by $\mathbb{Z}^2 \subset A_S$. Huang and Osajda established properties of arbitrary quasiflats in the Cayley complex of A_S , and one can find similarities between our results and [7, § 5.1–5.2 and Proposition 8.3].

Organisation of the article. In Section 2, we describe the modified Deligne complex Φ of Charney and Davis and we prove Theorem A. In Section 3, we prove a lemma on dihedral Artin groups fitting in the framework of [1]. In Section 4, we introduce a *polarisation* method for studying Euclidean planes in Φ . We finish with the classification of admissible polarisations and the proof of Theorem B in Section 5.

2. Modified Deligne complex

Let A_S be a two-dimensional Artin group. For $s, t \in S$ satisfying $m_{st} < \infty$, let A_{st} be the *dihedral* Artin group $A_{S'}$ with $S' = \{s, t\}$ and exponent m_{st} . For $s \in S$, let $A_s = \mathbb{Z}$.

Let K be the following simplicial complex. The vertices of K correspond to subsets $T \subseteq S$ satisfying $|T| \leq 2$ and, in the case where $|T| = 2$ with $T = \{s, t\}$, satisfying $m_{st} < \infty$. We call T the *type* of its corresponding vertex. Vertices of types T, T' are connected by an edge of K , if we have $T \subsetneq T'$ or vice versa. Similarly, three vertices span a triangle of K , if they have types $\emptyset, \{s\}, \{s, t\}$ for some $s, t \in S$.

We define a simple complex of groups \mathcal{K} over K as follows (see [3, § II.12] for background). The vertex groups are trivial, A_s , or A_{st} , when the vertex is of type $\emptyset, \{s\}, \{s, t\}$, respectively. For an edge joining a vertex of type $\{s\}$ to a vertex of type $\{s, t\}$, its edge group is A_s ; all other edge groups and all triangle groups are trivial. All inclusion maps are the obvious ones. It follows directly from the definitions that A_S is the fundamental group of \mathcal{K} .

We equip each triangle of K with the *Moussong metric* of a Euclidean triangle of angles $\frac{\pi}{2m_{st}}, \frac{\pi}{2}, \frac{(m_{st}-1)\pi}{2m_{st}}$ at the vertices of types $\{s, t\}, \{s\}, \emptyset$, respectively. As explained in [9, § 3], the

local developments of \mathcal{K} are CAT(0) and hence \mathcal{K} is strictly developable and its development Φ exists and is CAT(0). See [4] for a detailed proof. We call Φ with the Moussong metric the *modified Deligne complex*. In particular, all A_s and A_{st} with $m_{st} < \infty$ map injectively into A_S (which follows also from [8, Theorem 4.13]). Vertices of Φ inherit types from the types of the vertices of K .

Let $r \in S$ and let T be the fixed-point set in Φ of r . Note that since A_S acts on Φ without inversions, T is a subcomplex of Φ . Since the stabilisers of the triangles of Φ are trivial, we have that T is a graph. Since Φ is CAT(0), T is convex and thus it is a tree. Thus we call a *standard tree* the fixed-point set in Φ of a conjugate of a generator $r \in S$ of A_S .

REMARK 2.1 [9, Remark 4.4]. Each edge of Φ belongs to at most one standard tree.

Proof of Theorem A. Let $\Gamma \subset H$ be a finite index normal subgroup isomorphic to \mathbb{Z}^2 . By [2], Γ acts on Φ by semi-simple isometries. Let $\text{Min}(\Gamma) = \bigcap_{\gamma \in \Gamma} \text{Min}(\gamma)$, where $\text{Min}(\gamma)$ is the Minset of γ in Φ . By a variant of the Flat Torus Theorem not requiring properness [3, Theorem II.7.20(1)], $\text{Min}(\Gamma)$ is nonempty. By [3, Theorem II.7.20(4)], we have that H stabilises $\text{Min}(\Gamma)$.

Suppose first that each element of Γ fixes a point of Φ . Then Γ acts trivially on $\text{Min}(\Gamma)$. By the fixed-point theorem [3, Theorem II.2.8(1)], the finite group H/Γ fixes a point of $\text{Min}(\Gamma)$, and since the action is without inversions, we can take this point to be a vertex as required in (i).

Secondly, suppose that Γ has both an element γ that fixes a point of Φ and an element that is loxodromic. Then $\text{Min}(\Gamma)$ is not a single point, so it contains an edge e . Since $\text{Min}(\Gamma) \subset \text{Fix}(\gamma)$, we have that γ fixes e . Thus γ is a conjugate of an element of S and so $\text{Min}(\Gamma)$ is contained in a standard tree T . For any $h \in H$, we have that the intersection $h(T) \cap T$ contains $\text{Min}(\Gamma) \supset e$ and thus by Remark 2.1 we have $h \in \text{Stab}(T)$, as required in (ii).

Finally, suppose that all elements of Γ are loxodromic. By [3, Theorem II.7.20(1,4)], we have $\text{Min}(\Gamma) = Y \times \mathbb{R}^n$ with H preserving the product structure and Γ acting trivially on Y . As before H/Γ fixes a point of Y and so H stabilises \mathbb{R}^n isometrically embedded in Φ . By [3, Theorem II.7.20(2)], we have $n \leq 2$, but since H acts by simplicial isometries, we have $n = 2$ and the action is proper, as required in (iii). \square

3. Girth lemma

LEMMA 3.1. Let $S = \{s, t\}$ with $m_{st} \geq 3$. A word with $2m$ syllables (that is, of form $s^{i_1}t^{j_1} \dots s^{i_m}t^{j_m}$ with all $i_k, j_k \in \mathbb{Z} - \{0\}$) is trivial in A_S if and only if up to interchanging s with t , and a cyclic permutation, it is of the form:

- $s^k \underbrace{t \dots s}_{m-1} t^{-k} \underbrace{s^{-1} \dots t^{-1}}_{m-1}$ for m odd;
- $s^k \underbrace{ts \dots t}_{m-1} s^{-k} \underbrace{t^{-1} s^{-1} \dots t^{-1}}_{m-1}$ for m even,

where $k \in \mathbb{Z} - \{0\}$.

Proof. The ‘if’ part follows immediately from Figure 1. We prove the ‘only if’ part by induction on the size of any reduced (van Kampen) diagram M of the word w in question, where we prove the stronger assertion that, up to interchanging s with t , M is as in Figure 1.

We use the vocabulary from [1], where the 2-cells of M are called *regions* and the *interior degree* $i(D)$ of a region is the number of interior edges of ∂D (after forgetting vertices of valence 2). For example, the two extreme regions in Figure 1 have interior degree 1. A region D is a

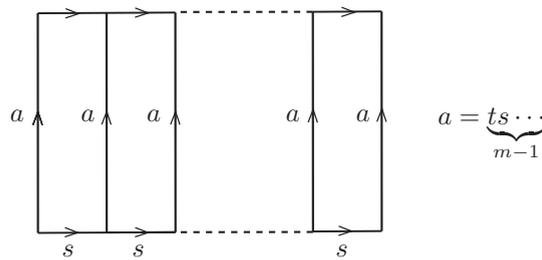


FIGURE 1. The diagram M . The top arrows are either all labelled s or all labelled t , depending on the parity of m .

simple boundary region if $\partial D \cap \partial M$ is nonempty, and $M - \bar{D}$ is connected. For example, the two extreme regions in Figure 1 are simple boundary regions, but the remaining ones are not. A *singleton strip* is a simple boundary region with $i(D) \leq 1$. A *compound strip* is a subdiagram R of M consisting of regions D_1, \dots, D_n , with $n \geq 2$, with $D_{k-1} \cap D_k$ a single interior edge of R (after forgetting vertices of valence 2), satisfying $i(D_1) = i(D_n) = 2, i(D_k) = 3$ for $1 < k < n$ and $M - R$ connected.

Let R be a strip of M with boundary labelled by rb , where r labels $\partial R \cap \partial M$ and so $w = rw'$. Assume also that R shares no regions with some other strip (such a pair of strips exists by [1, Lemma 2]). By [1, Lemma 5], we have that the syllable lengths satisfy $\|r\| \geq \|b\| + 2$ and so by [1, Lemma 6], we have $\|r\| \geq m + 1$, hence $\|w'\| \leq m + 1$. In fact, since the outside boundary of the other strip has also syllable length $\geq m + 1$, we have $\|r\| = m + 1$, and hence $\|b\| = m - 1$. Let M' be the diagram with boundary labelled by $b^{-1}w'$ obtained from M by removing R . By the induction hypothesis, M' is as in Figure 1. If R is a singleton strip, then there is only one way of gluing R to M' to obtain $\|w\| = 2m$ and it is as in Figure 1. If R is a compound strip, then by the induction hypothesis R is also as in Figure 1. Moreover, since all regions of R share exactly one edge with M' , up to interchanging s with t , and/or b with b^{-1} , we have $b = s^k \underbrace{ts \cdots}_{m-2}$ or $b = \underbrace{\cdots st}_{m-2} s^k$, where $k \geq 2$. Since $m > 2$, we have that b cannot be a subword of the boundary word of M' , unless M' is a mirror copy of R , contradiction. \square

REMARK 3.2. Let $S = \{s, t\}$ with $m_{st} = 2$. A word with four syllables is trivial in $A_S = \mathbb{Z}^2$ if and only if up to interchanging s with t it is of the form $s^k t^l s^{-k} t^{-l}$, where $k, l \in \mathbb{Z} - \{0\}$.

4. Polarisation

DEFINITION 4.1. Let F be a Euclidean plane isometrically embedded in Φ . Then for each vertex v in F of type $\{s, t\}$ there are exactly $4m_{st}$ triangles in F incident to v . We assemble them into regular $2m_{st}$ -gons, and call this complex the *tiling of F* . We say that a cell of this tiling has type T if its barycentre in Φ has type T .

For a Coxeter group $W = W_{S'}$, let Σ denote its *Davis complex*, that is, the complex obtained from the standard Cayley graph by adding k -cells corresponding to cosets of finite $\langle T \rangle$ for $T \subset S'$ of size k . For example, for W the triangle Coxeter group with exponents $\{3, 3, 3\}$, the complex Σ is the tiling of the Euclidean plane by regular hexagons.

LEMMA 4.2. Let F be a Euclidean plane isometrically embedded in Φ . Then the tiling of F is either the standard square tiling, or the one of the Davis complex Σ for W , where W is the triangle Coxeter group with exponents $\{3, 3, 3\}, \{2, 4, 4\}$ or $\{2, 3, 6\}$.

Proof. The 2-cells of the tiling are regular polygons with even numbers of sides, hence their angles lie in $[\frac{\pi}{2}, \pi)$. If there is a vertex v of F incident to four 2-cells, then all these 2-cells are squares. Consequently, any vertex of F adjacent to v is incident to at least two squares, and thus to exactly four squares. Then, since the 1-skeleton of F is connected, the tiling of F is the standard square tiling.

If v is incident to three 2-cells, which are $2m, 2m', 2m''$ -gons, then since $\frac{1}{m} + \frac{1}{m'} + \frac{1}{m''} = 1$, we have $\{m, m', m''\} = \{3, 3, 3\}, \{2, 4, 4\}$ or $\{2, 3, 6\}$. Moreover, a vertex u of F adjacent to v is incident to two of these three 2-cells, and this implies that the third 2-cell incident to u has the same size as the one incident to v . This determines uniquely the tiling of F as the one of Σ . □

Henceforth, let Σ be the Davis complex for W , where W is the triangle Coxeter group with exponents $\{3, 3, 3\}, \{2, 4, 4\}$ or $\{2, 3, 6\}$.

LEMMA 4.3. *Suppose Σ is the tiling of a Euclidean plane isometrically embedded in Φ . Then the natural action of W on Σ preserves the edge types coming from Φ .*

In particular, $W = W_T$ for some $T \subset S$ with $|T| = 3$.

Proof. Chose a vertex v of Σ , and let $\{s\}, \{t\}, \{r\}$ be the types of edges incident to v . Let u be a vertex of Σ adjacent to v , say along an edge e of type $\{r\}$. Hence the 2-cells in Σ incident to e have types $\{s, r\}$ and $\{t, r\}$. Consequently, the types of the remaining two edges incident to u are also $\{s\}$ and $\{t\}$, and in such a way that the reflection of Σ interchanging the endpoints of e preserves the types of these edges. This determines uniquely the types of the edges of Σ , and guarantees that they are preserved by W . □

DEFINITION 4.4. A *polarisation* of Σ is a choice of a longest diagonal $l(\sigma)$ in each 2-cell σ of Σ . A polarisation is *admissible* if every vertex of Σ belongs to exactly one $l(\sigma)$.

DEFINITION 4.5. Suppose $\mathbb{Z}^2 \subset A_S$ acts properly and cocompactly on $\Sigma \subset \Phi$. For an edge e of type $\{s\}$ in Σ , its vertices correspond to elements $g, gs^k \in A_S$ for $k > 0$. We direct e from g to gs^k . By Lemma 3.1, the boundary of each 2-cell σ is subdivided into two directed paths joining two opposite vertices. The *induced polarisation* of Σ assigns to each σ the longest diagonal $l(\sigma)$ joining these two vertices.

LEMMA 4.6. *An induced polarisation is admissible.*

Proof. Step 1. For each vertex v of Σ , there is at most one $l(\sigma)$ containing v .

Indeed, suppose that we have $v \in l(\sigma), l(\tau)$. Without loss of generality suppose that the edge $e = \sigma \cap \tau$ is directed from v . Then the other two edges incident to v are also directed from v . We will now prove by induction on the distance from v that each edge of Σ is oriented from its vertex closer to v to its vertex farther from v in the 1-skeleton Σ^1 (they cannot be at equal distance since Σ^1 is bipartite).

For the induction step, suppose we have already proved the induction hypothesis for all edges closer to v than an edge uu' , where u' is closer to v than u . Let u'' be the first vertex on a geodesic from u' to v in Σ^1 . Let σ be the 2-cell containing the path $uu'u''$. By [10, Theorems 2.10 and 2.16], σ has two opposite vertices u_0 closest to v and u_{\max} farthest from v . By the induction hypothesis, the edge $u'u''$ is oriented from u'' to u' , and both edges of σ incident to u_0 are oriented from u_0 . Thus if the edge uu' was oriented to u' , we would have that u' is opposite to u_0 , so $u' = u_{\max}$, contradiction. This finishes the induction step.

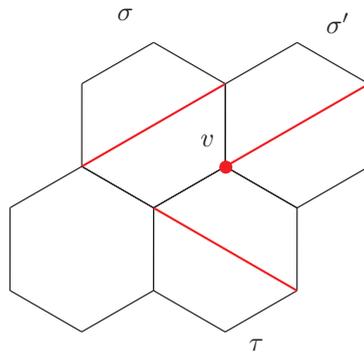


FIGURE 2 (colour online).

As a consequence, v is the unique vertex of Σ with all edges incident to v oriented from v . This contradicts the cocompactness of the action of \mathbb{Z}^2 on Σ and proves Step 1.

Step 2. For each v , there is at least one $l(\sigma)$ containing v .

Among the edges incident to v there are at least two edges directed from v or at least two edges directed to v . The 2-cell σ containing such two edges satisfies $l(\sigma) \ni v$. □

5. Classification

PROPOSITION 5.1. *Let Σ be the Davis complex for W the triangle Coxeter group with exponents $\{3, 3, 3\}$ with an admissible polarisation l . Then there is an edge e such that each hexagon γ of Σ satisfies*

- ♣ *the diagonal $l(\gamma)$ has endpoints on edges of γ parallel to e .*

Note that if the conclusion of Proposition 5.1 holds, then the translation ρ mapping one hexagon containing e to the other preserves l .

REMARK 5.2. It is easy to prove the converse, that is, that if each $l(\gamma)$ has endpoints on edges of γ parallel to e , and if l is ρ -invariant, then l is admissible. This can be used to classify all admissible polarisations, but we will not need it.

To prove Proposition 5.1, we need the following reduction.

LEMMA 5.3. *Let e be an edge and ρ a translation mapping one hexagon containing e to the other. If ♣ holds for all hexagons γ in one ρ -orbit, then it holds for all γ .*

Proof. Suppose that ♣ holds for all hexagons γ in the ρ -orbit of a hexagon σ . Let τ be a hexagon adjacent to two of them, say to σ and $\rho(\sigma)$. Let $v = \sigma \cap \rho(\sigma) \cap \tau$. Since ♣ holds for $\gamma = \sigma$ and $\gamma = \rho(\sigma)$, by the admissibility of l , v belongs to one of $l(\sigma), l(\rho(\sigma))$. Thus $v \notin l(\tau)$ and hence ♣ holds for $\gamma = \tau$. Proceeding inductively, by the connectivity of Σ , we obtain ♣ for all γ . □

Proof of Proposition 5.1. Case 1. There are adjacent hexagons σ, τ with nonparallel $l(\sigma), l(\tau)$.

Let $f = \sigma \cap \tau$. Without loss of generality $l(\sigma) \cap f = \emptyset, l(\tau) \cap f \neq \emptyset$. Let v be the vertex of f outside $l(\tau)$. By the admissibility of l , v is contained in $l(\sigma')$ for the third hexagon σ' incident to v . Hence ♣ holds for $e = \sigma \cap \sigma'$ and $\gamma = \sigma, \sigma', \tau$ (see Figure 2). Let ρ be the translation

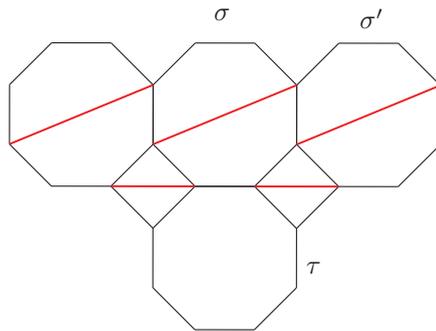


FIGURE 3 (colour online).

mapping σ to σ' . Replacing the pair σ, τ with τ, σ' and repeating inductively the argument shows that \clubsuit holds for $\gamma = \rho^n(\sigma), \rho^n(\tau)$ for all $n > 0$ (note that e gets replaced by parallel edges in this procedure).

Furthermore, by the admissibility of l , since $l(\rho^{-1}(\tau))$ is disjoint from $l(\sigma)$ and $l(\tau)$, it leaves us only one choice for $l(\rho^{-1}(\tau))$, and it satisfies \clubsuit for $\gamma = \rho^{-1}(\tau)$. Replacing the pair σ, τ with $\rho^{-1}(\tau), \sigma$ and repeating inductively the argument shows that \clubsuit holds for $\gamma = \rho^{-n}(\sigma), \rho^{-n}(\tau)$ for all $n > 0$. It remains to apply Lemma 5.3.

Case 2. All the $l(\sigma)$ are parallel.

In this case, it suffices to take any edge e intersecting some $l(\sigma)$. □

PROPOSITION 5.4. *Let Σ be the Davis complex for W the triangle Coxeter group with exponents $\{2, 4, 4\}$ with an admissible polarisation l . Then there is an edge e such that each octagon γ of Σ satisfies:*

\diamond *the diagonal $l(\gamma)$ has endpoints on edges of γ parallel to e .*

An edge e of Σ lies either in two octagons σ, σ' or there is a square with two parallel edges e, e' in octagons σ, σ' . The translation of Σ mapping σ to σ' is called an e -translation. Note that if the conclusion of Proposition 5.4 holds, then an e -translation preserves l .

LEMMA 5.5. *Let e be an edge and ρ an e -translation. If \diamond holds for all octagons γ in one ρ -orbit, then it holds for all γ .*

Proof. Suppose that \diamond holds for all octagons γ in the ρ -orbit of an octagon σ . We can assume $e \subset \sigma$. Suppose first that e lies in another octagon σ' . Then let τ be an octagon outside the ρ -orbit of σ adjacent to some $\rho^k(\sigma)$, say σ . Let $\square, \rho(\square)$ be the two squares adjacent to both σ and τ (see Figure 3). By the admissibility of l , we have that $l(\square), l(\rho(\square))$ contain the two vertices of $\sigma \cap \tau$. Consequently, $l(\tau)$ intersects the edge $\tau \cap \rho(\tau)$, and so $\gamma = \tau$ satisfies \diamond . It is easy to extend this to all the octagons γ .

It remains to consider the case where there is a square with two parallel edges e, e' in octagons σ, σ' . Let τ be an octagon adjacent to two of them, say to σ and $\rho(\sigma)$. Let $v = e \cap \tau, x = e' \cap \tau$. Since \diamond holds for $\gamma = \sigma, \sigma'$, each of v, x lies in one of $l(\sigma), l(\square), l(\sigma')$. Thus by the admissibility of l , we have $v, x \notin l(\tau)$. Any of the two remaining choices for $l(\tau)$ satisfy \diamond for $\gamma = \tau$. It is again easy to extend this to all the octagons γ . □

Proof of Proposition 5.4. Note that we fall in one of the following two cases.

Case 1. There is an edge f in octagons σ, τ with $l(\sigma) \cap f = \emptyset, l(\tau) \cap f \neq \emptyset$.

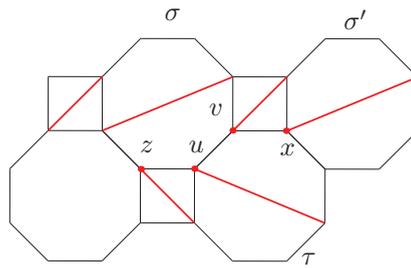


FIGURE 4 (colour online).

Let v be the vertex of f distinct from $u = l(\tau) \cap f$. By the admissibility of l , the vertex v is contained in $l(\square)$ for the square \square incident to v . Let x be the vertex in $\tau \cap \square$ distinct from v , and let σ' be the octagon incident to x distinct from τ . By the admissibility of l , the vertex x is contained in $l(\sigma')$. Hence \diamond holds for $e = \sigma \cap \square$ and $\gamma = \sigma', \tau$ (see Figure 4). Let ρ be the translation mapping σ to σ' .

Now let \blacksquare be the square incident to u , and let z be the vertex in $\sigma \cap \blacksquare$ distinct from u . Note that by the admissibility of l , we have $z \in l(\blacksquare)$, and consequently $\rho^{-1}(x) \in l(\sigma)$ (so \diamond holds for $\gamma = \sigma$) and $\rho^{-1}(v) \in l(\rho^{-1}(\square))$. Hence $l(\rho^{-1}(\tau))$ cannot contain neither z , nor $\rho^{-1}(v)$, nor $\rho^{-1}(x)$. There is only one remaining choice for $l(\rho^{-1}(\tau))$, and it satisfies \diamond for $\gamma = \rho^{-1}(\tau)$.

We can now argue exactly as in the proof of Proposition 5.1 that \diamond holds for $\gamma = \rho^n(\sigma), \rho^n(\tau)$ for all $n \in \mathbb{Z}$. It then remains to apply Lemma 5.5.

Case 2. For each edge e in octagons σ, σ' with $l(\sigma) \cap e \neq \emptyset$, we have $l(\sigma') \cap e \neq \emptyset$.

Let σ be any octagon and e an edge contained in another octagon σ' and intersecting $l(\sigma)$. Let ρ be the translation mapping σ to σ' . One can show inductively that \diamond holds for octagons $\gamma = \rho^n(\sigma)$ for all $n \in \mathbb{Z}$. It then remains to apply Lemma 5.5. \square

Note that for l satisfying \diamond for all octagons γ , the values of l on octagons determine its values on squares.

PROPOSITION 5.6. *Let Σ be the Davis complex for W the triangle Coxeter group with exponents $\{2, 3, 6\}$ with an admissible polarisation l . Then there is an edge e such that each 12-gon γ of Σ satisfies:*

\heartsuit *the diagonal $l(\gamma)$ has endpoints on edges of γ parallel to e .*

Let e be an edge. An e -translation is the translation of Σ mapping σ to σ' in the two following configurations. In the first configuration, we have a square with two parallel edges e, e' in 12-gons σ, σ' . In the second configuration, we have four parallel edges e, e', e'', e''' such that e', e'' lie in a square, e, e' in a hexagon ϕ and e'', e''' in another hexagon, and we consider 12-gons $\sigma \supset e, \sigma' \supset e'''$. Again, if the conclusion of Proposition 5.6 holds, then an e -translation preserves l . To see this in the configuration with hexagons, it suffices to observe that $l(\phi)$ (and similarly for the other hexagon) is not parallel to e : otherwise $l(\phi)$ would intersect $l(\square)$ for \square the square containing $e \cap l(\sigma)$.

LEMMA 5.7. *Let e be an edge and ρ an e -translation. If \heartsuit holds for all 12-gons γ in one ρ -orbit, then it holds for all γ .*

The proof is easy, it goes along the same lines as the proofs of Lemmas 5.3 and 5.5 and we omit it.

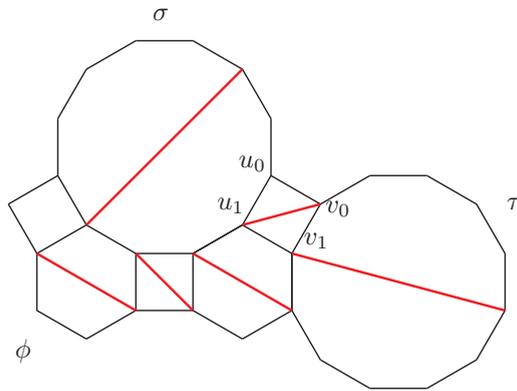


FIGURE 5 (colour online).

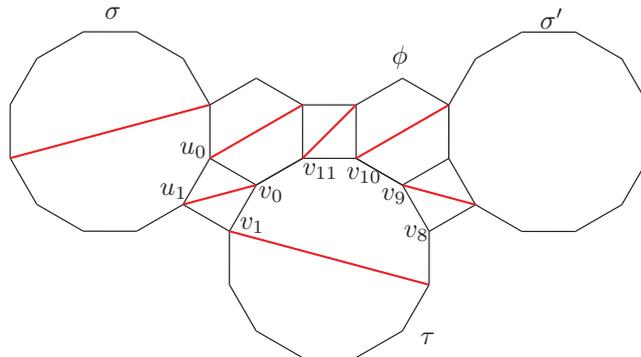


FIGURE 6 (colour online).

Proof of Proposition 5.6. We adopt the convention that if we label the vertices of an edge in a 12-gon σ by v_0v_1 , then all the other vertices of σ get cyclically labelled by $v_2 \cdots v_{11}$.

Let τ be a 12-gon and suppose that $l(\tau)$ contains a vertex v_1 of an edge $v_0v_1 \subset \tau$ for a square $\square = v_0v_1u_1u_0$. Let σ be the 12-gon containing u_0u_1 . Then $l(\square) = u_1v_0$ and furthermore l assigns to the hexagon and square containing u_1u_2, u_2u_3 , respectively, the longest diagonal containing u_2, u_3 , respectively. Thus the only three remaining options for $l(\sigma)$ are the diagonals u_0u_6, u_5u_{11} and u_4u_{10} . Thus we fall in one of the following three cases.

Case 0. There is such a τ with $l(\sigma) = u_4u_{10}$.

Let ϕ be the hexagon containing u_3u_4 . Then $l(\phi)$ is parallel to u_3u_4 and there is no admissible choice for l in the square containing u_4u_5 (see Figure 5). This is a contradiction.

Case 1. There is such a τ with $l(\sigma) = u_5u_{11}$.

It is easy to see that l agrees with Figure 6 on the hexagon containing v_0v_{11} and the square containing $v_{11}v_{10}$. Thus the only 2-cell ϕ with $l(\phi)$ containing v_{10} may be (and is) the hexagon containing $v_{10}v_9$. Consequently the only 2-cell \blacksquare with $l(\blacksquare)$ containing v_9 may be (and is) the square containing v_9v_8 . Denote by σ' the 12-gon adjacent to both ϕ and \blacksquare at the vertex $x \neq v_9$. Then x may lie (and lies) only in $l(\sigma')$. Denote by ρ the translation mapping σ to σ' .

It is also easy to see that the 2-cells surrounding σ and $\rho^{-1}(\tau)$ depicted in Figure 7 have $l(\cdot)$ as indicated. This leaves only two choices for $l(\rho^{-1}(\tau))$, where one of them leads to Case 0, and the other satisfies \heartsuit .

Replacing repeatedly σ and τ in the above argument by τ and σ' or by $\rho^{-1}(\tau)$ and σ gives that \heartsuit holds for $\gamma = \rho^n(\sigma), \rho^n(\tau)$ for all $n \in \mathbb{Z}$. It remains to apply Lemma 5.7.

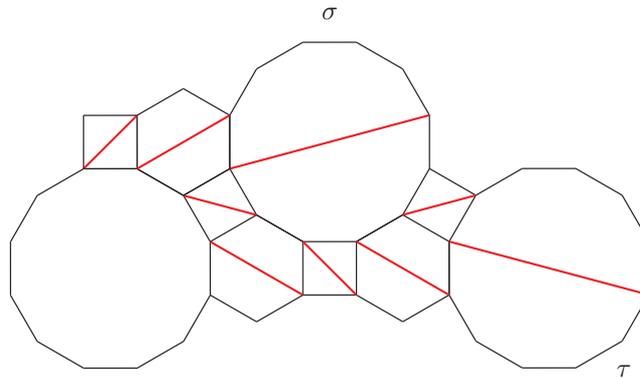


FIGURE 7 (colour online).

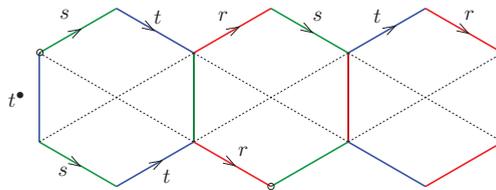


FIGURE 8 (colour online). Dashed lines indicate possible polarisations.

Case 2. There is no such τ as in Case 0 or 1.

It is then easy to see that $e = v_0v_1$ and ρ mapping σ to τ satisfy the hypothesis of Lemma 5.7. □

Note that for l satisfying \heartsuit for all 12-gons γ , the values of l on 12-gons determine its values on squares and hexagons.

We are now ready for the following.

Proof of Theorem B. Let F be a Euclidean plane isometrically embedded in Φ with a proper (and thus cocompact) action of \mathbb{Z}^2 . By Lemma 4.2, the tiling of F is either the standard square tiling, or the one of the Davis complex Σ for W , where W is the triangle Coxeter group with exponents $\{3, 3, 3\}$, $\{2, 4, 4\}$ or $\{2, 3, 6\}$.

First consider the case where the tiling of F is the standard square tiling. We can partition the set of edges into two classes *horizontal* and *vertical* of parallel edges. Let T be the set of types of horizontal edges and T' be the set of types of vertical edges. Since for each square the type of its two horizontal (respectively, vertical) edges is the same, we have $m_{tt'} = 2$ for all $t \in T, t' \in T'$. Moreover, by Remark 3.2, if one of the edges is of form g, gt^k , then the other is of form h, ht^k . Thus, up to a conjugation, the stabiliser of F in A_S is generated by a horizontal translation $w \in A_T$ and a vertical translation $w' \in A_{T'}$. This brings us to Case (a) in Theorem B.

It remains to consider the case where the tiling of F is the one of Σ . Consider its induced polarisation l from Definition 4.5. By Lemma 4.6, l is admissible. By Propositions 5.1, 5.4, and 5.6, there is an edge e such that for each γ a maximal size 2-cell, the diagonal $l(\gamma)$ has endpoints on edges of γ parallel to e , and there is a particular translation ρ in the direction perpendicular to e preserving l .

For an edge f of type $\{s\}$ in Σ , its vertices are of form g, gs^k for $k > 0$, directed from g to gs^k . If $k > 1$, then we call f k -long. By Lemma 3.1 and Remark 3.2, if f is k -long, then so

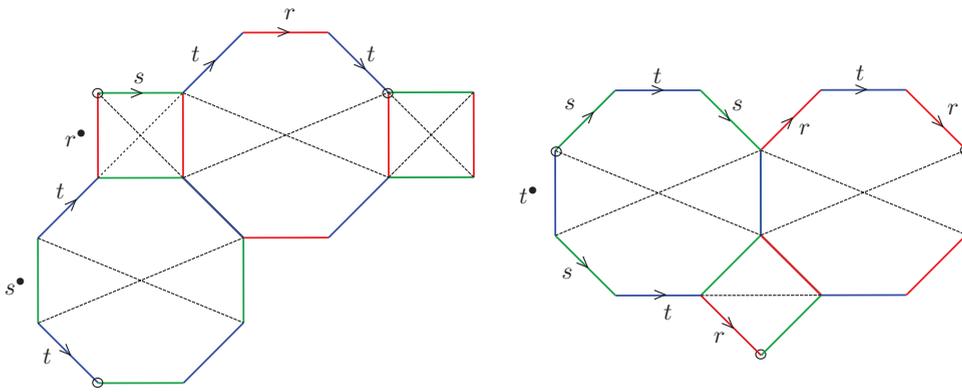


FIGURE 9 (colour online).

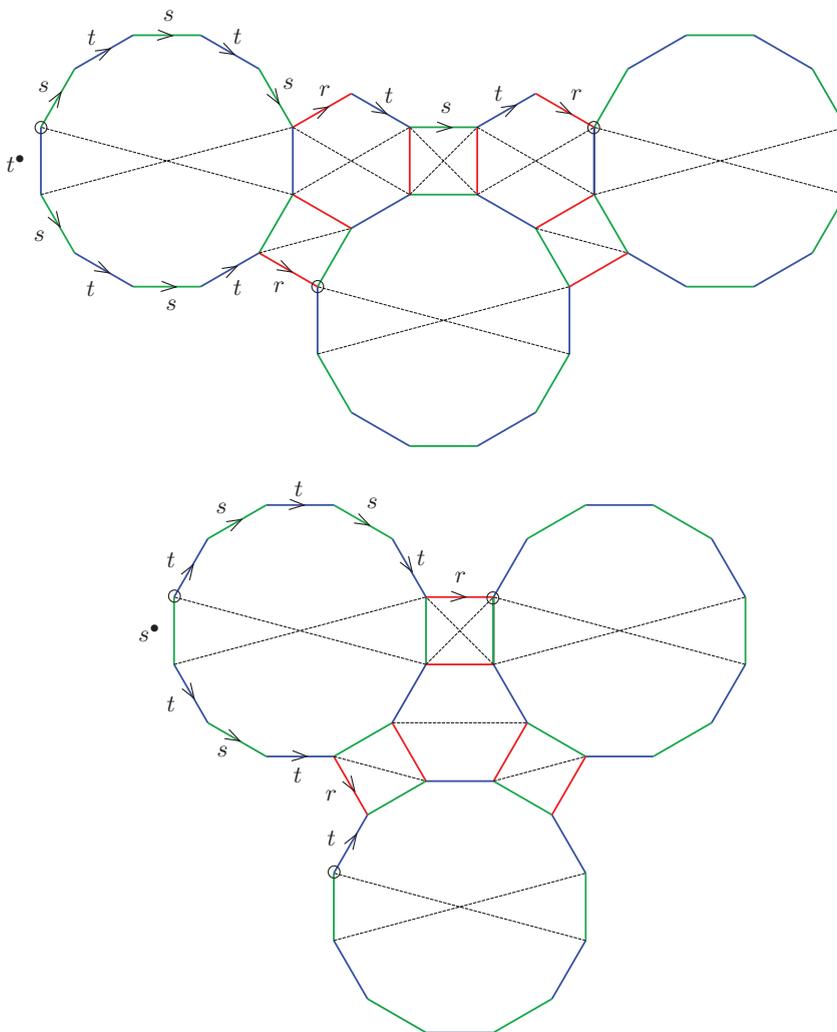


FIGURE 10 (colour online).

is its opposite edge in both of the 2-cells that contain f . Consequently all the edges crossing the bisector of f are k -long. Moreover, all such bisectors are parallel, since otherwise the 2-cell \square where they crossed would have four long edges, so \square would be a square by Lemma 3.1. Analysing l in the 2-cells adjacent to \square leaves then no admissible choice for $l(\square)$.

Furthermore, by Lemmas 3.1, 5.3, 5.5 and 5.7, if f is a long edge, then we can assume that f is parallel to e .

Suppose first that $W = W_T$ is the triangle Coxeter group with exponents $\{3, 3, 3\}$ and $T = \{s, t, r\}$. Let ω be a combinatorial axis for the action of ρ on Σ . Since none of the edges of ω are parallel to e , by the definition of the induced polarisation we see that they are all directed consistently (see Figure 8).

Thus, up to replacing F by its translate and interchanging t with r , the element $strstr$ preserves ω , and coincides on it with ρ^3 . In fact, ρ^3 not only preserves the types of edges, but also by Lemma 3.1 their direction and k -longness. Thus $strstr$ preserves F . The second generator of the type preserving translation group of Σ in W_T is $tstr$. Note that the path representing it in $\Sigma \subset \Phi$ corresponds to a word in $t^\bullet str \subset A_T$. That word depends on whether the second edge of the path is long and on the polarisation. Since \mathbb{Z}^2 acts cocompactly, there is a power of $tstr$ such that its corresponding path in $\Sigma \subset \Phi$ reads off a word in $(t^\bullet str)^* \subset A_T$ that is the other generator of the orientation preserving stabiliser of F in A_S . This brings us to Case (b) in Theorem B. One similarly obtains the characterisations of orientation-preserving stabilisers of F for the two other W (see Figures 9 and 10). \square

REMARK 5.8. Analysing the full stabilisers of F in A_S , one can easily classify also the subgroups of A_S acting properly on Φ isomorphic to the fundamental group of the Klein bottle. For example, suppose that the second generator of the \mathbb{Z}^2 in Case (b) of Theorem B has the form $g = (t^k str)(t^{-k} str)$. Then our \mathbb{Z}^2 is generated by $strstr$ and $g' = g(strstr)^{-1} = t^k sr^{-k} s^{-1}$. Note that str normalises our \mathbb{Z}^2 with $(str)^{-1}g'(str) = (g')^{-1}$. Thus $\langle str, g' \rangle$ is isomorphic to the fundamental group of the Klein bottle. We do not include a full classification, since it is not particularly illuminating.

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