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Strong convergence of a Verlet integrator for the semi-linear stochastic wave equation

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Abstract

The full discretization of the semi-linear stochastic wave equation is considered. The discontinuous Galerkin finite element method is used in space and analyzed in a semigroup framework, and an explicit stochastic position Verlet scheme is used for the temporal approximation. We study the stability under a CFL condition and prove optimal strong convergence rates of the fully discrete scheme. Numerical experiments illustrate our theoretical results. Further, we analyze and bound the expected energy and numerically show excellent agreement with the energy of the exact solution.

Key words. semi-linear stochastic wave equation, stochastic Verlet integration, strong convergence, discontinuous Galerkin finite element method, stability, energy conservation

AMS subject classifications. 60H15, 60H35, 65C30, 65C20, 65M60

1 Introduction

We study the semi-linear stochastic wave equation driven by additive noise

$$\begin{aligned} d\dot{u} &= \nabla \cdot (\mathbf{D}\nabla u)dt + f(u)dt + dW && \text{in } \mathcal{D} \times (0, T), \\ u &= 0 && \text{on } \partial\mathcal{D} \times (0, T), \\ u(\cdot, 0) &= u_0, \dot{u}(\cdot, 0) = v_0 && \text{in } \mathcal{D}, \end{aligned} \tag{1.1}$$

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where $\mathcal{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded convex polytopal domain with boundary $\partial\mathcal{D}$ and $\dot{u} = \partial_t u$ stands for the time derivative, f is a globally Lipschitz nonlinear function, and $\{W(t)\}_{t \geq 0}$ is a Q -Wiener process with respect to a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$. We give our assumptions on the noise and f in detail in Section 2. The initial data u_0 and v_0 are \mathcal{F}_0 -measurable variables. We assume that $\mathbf{D} = (d_{ij}(x))_{1 \leq i, j \leq d}$ is a symmetric uniformly positive definite matrix that satisfies the bounds

$$0 < d_{\min} \leq \zeta^T \mathbf{D}(x) \zeta \leq d_{\max} < \infty, \quad \text{for all } \zeta \in \mathbb{R}^d, |\zeta| = 1, x \in \mathcal{D}. \quad (1.2)$$

The stochastic wave equation is of fundamental importance in various applications. For instance, the motion of a strand of DNA floating in a liquid [15]; the dynamics of the primary current density vector field within the grey matter of the human brain [20]; or the vibration of a string under the action of stochastic forces [37]. Our motivating example arises from the sound propagation in the sea, critical for marine reserves and conservation of species [18, 27]. In particular we note that in the marine environment the noise is typically only over a finite range of frequencies and also that often complex computational domains are of interest where sound waves may interact with the shore. The complex geometry motivates the use of the dG method in space and we pay attention not only to space-time rough noise forcing but also to more regular noise in space.

One advantage the dG method offers over the standard continuous finite element methods is that the mass matrix has a block diagonal structure; it can therefore be inverted at a very low computational cost. Hence, the dG method leads to efficient time integration when combined with an explicit time-stepping scheme. In the present paper, we propose such full discretization of (1.1) by applying the dG finite element method in space [2] and a stochastic extension of the explicit position Verlet time-stepping method in time [4, 5]. The position Verlet scheme is a variant of the Störmer-Verlet/leapfrog method and it shares their important geometric properties such as symplecticity. We refer the reader to [24] for a review on the Störmer-Verlet/leapfrog integrators. Stochastic extensions of the position Verlet or Störmer-Verlet/leapfrog time-stepping schemes have been used for second-order ordinary stochastic differential equations, see e.g. [7, 8, 19, 21] and references therein.

In recent years, strong approximations of stochastic wave equations have been studied by many authors [1, 9, 10, 13, 31, 38–42, 44]. We first comment on work analyzing the spatial discretization of stochastic wave equations. Strong convergence estimates for the continuous finite element approximation of the linear stochastic wave equation and of (1.1) with multiplicative noise were derived in [31] and [1], respectively. In [39], [42], and [44] the spectral Galerkin method for one-dimensional semi-linear stochastic wave equations driven by additive noise was used in space and strong convergence rates were proved. In addition, in [41] the stochastic wave equation with Lipschitz continuous nonlinearity and multiplicative noise is discretized by the spectral method in space. Further, the recent preprint [13] studies the spectral Galerkin approximation of the stochastic wave equation with polynomial nonlinearity in \mathbb{R}^d , $d = 1, 2, 3$, and analyzes

strong convergence in $L^p(\Omega)$. Finite difference spatial discretizations for one-dimensional stochastic wave equations were employed in [10, 38, 40]. Secondly we comment on time stepping schemes for stochastic wave equations. Stochastic trigonometric methods have been used for the temporal approximation of various types of stochastic wave equations, see e.g. [1, 9, 10, 41] and improved convergence rates were derived in [42] by using linear functionals of the noise as in [28]. Strong convergence rates for the Störmer-Verlet/leapfrog time-stepping scheme applied to the one-dimensional stochastic wave equation and one-step I -stable time-stepping methods can be found in [40] and [30], respectively. Finally, the recent preprint [13] analyzes the strong convergence of a splitting average vector field time-stepping method.

Concerning the dG finite element approximation of parabolic stochastic partial differential equations (SPDEs) we refer the reader to [33, 34]. In [33] the exact solution was represented in terms of Green's functions and strong convergence estimates were derived for dG approximation to the linear Cahn-Hilliard equation. In the recent publication [34] a local dG method is analyzed for nonlinear parabolic SPDEs with multiplicative noise and strong spatial convergence rates are derived.

Our main aim here is to prove strong convergence to the mild solution of the stochastic position Verlet method (SVM) with a dG discretization in space applied to (1.1), by introducing a discrete norm, under a CFL condition, which is preserved by the time integrator. We note that the same analytical tool could be used to analyze the stability of the Störmer-Verlet/leapfrog method. To the best of our knowledge, the semigroup approach to the dG formulation that we adopt here has not been considered elsewhere. The martingale approach in [40] for the Störmer-Verlet/leapfrog scheme applies only to the one-dimensional case, while our analysis enables us to obtain optimal error bounds for both the displacement and the velocity in multiple dimensions.

The paper is organized as follows. We introduce some notation, introduce our assumptions on the noise and the nonlinearity, and rewrite (1.1) as a first order system in Section 2. In Section 3 we discuss the existence and the uniqueness of the dG finite element approximation of the stochastic equation (1.1) and we extend the results of [1] on the strong convergence estimates for the spatial discretization of our problem. Stability and strong convergence analysis of SVM is considered in Section 4. In Section 5 we state energy results of the full discretization of our problem and in Section 6 we present numerical experiments in order to demonstrate the theory and the efficiency of our discretization.

2 Preliminaries and notation

Let U and H be separable Hilbert spaces with norms $\|\cdot\|_U$ and $\|\cdot\|_H$, respectively. We denote the space of linear operators from U to H by $\mathcal{L}(U, H)$, and we let

$\mathcal{L}_2(U, H)$ be the set of Hilbert-Schmidt operators with norm

$$\|R\|_{\mathcal{L}_2(U, H)} := \left(\sum_{k=1}^{\infty} \|Re_k\|_H^2 \right)^{1/2},$$

where $\{e_k\}_{k=1}^{\infty}$ is an arbitrary orthonormal basis of U . If $H = U$, then we write $\mathcal{L}(U) = \mathcal{L}_2(U, U)$ and $\text{HS} = \mathcal{L}_2(U, U)$. Let $L_2(\Omega, H)$ be the space of H -valued square integrable random variables with norm

$$\|v\|_{L_2(\Omega, H)} := \mathbb{E} [\|v\|_H^2]^{1/2}.$$

Let $Q \in \mathcal{L}(U)$ be a self-adjoint, positive semidefinite operator. The driving stochastic process $W(t)$ in (1.1) is a U -valued Q -Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and has the orthogonal expansion [36, Section 10.2]

$$W(t) = \sum_{j=1}^{\infty} q_j^{1/2} \beta_j(t) \psi_j, \quad (2.1)$$

where $\{\psi_j\}_{j=1}^{\infty}$ are orthonormal eigenfunctions of Q with corresponding eigenvalues $q_j \geq 0$ and $\{\beta_j(t)\}_{j=1}^{\infty}$ are real-valued mutually independent standard Brownian motions. Suppose that $\{\Phi(s)\}_{0 \leq s \leq t} \in \mathcal{L}(U, H)$ and that

$$\int_0^t \|\Phi(s)Q^{1/2}\|_{\text{HS}} ds < \infty,$$

then the stochastic integral $\int_0^t \Phi(s) dW(s)$ is well defined in $L_2(\Omega, H)$ and we have Itô's isometry, see [36, Section 10.2],

$$\left\| \int_0^t \Phi(s) dW(s) \right\|_{L_2(\Omega, H)} = \int_0^t \|\Phi(s)Q^{1/2}\|_{\mathcal{L}_2(U, H)} ds. \quad (2.2)$$

Let us introduce the spaces and norms that we use to describe the spatial regularity of functions. Let $\Lambda = -\nabla \cdot (\mathbf{D}\nabla u)$ denote the linear operator $\Lambda: D(\Lambda) \rightarrow L_2(\mathcal{D})$ with $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ and let $L_2(\mathcal{D})$ have the usual inner product (\cdot, \cdot) and norm $\|\cdot\|$. Further, we define the following spaces

$$\dot{H}^\alpha = D(\Lambda^{\alpha/2}), \quad \|v\|_\alpha = \|\Lambda^{\alpha/2}v\| = \left(\sum_{j=0}^{\infty} \lambda_j^\alpha (v, \phi_j)^2 \right)^{1/2}, \quad \alpha \in \mathbb{R}, v \in \dot{H}^\alpha,$$

where $\{(\lambda_j, \phi_j)\}_{j=1}^{\infty}$ are the eigenpairs of Λ with orthonormal eigenvectors. We also introduce a product Hilbert space with accompanying norm

$$\mathcal{H}^\alpha = \dot{H}^\alpha \times \dot{H}^{\alpha-1}, \quad \|v\|_\alpha^2 = \|v_1\|_\alpha^2 + \|v_2\|_{\alpha-1}^2, \quad \alpha \in \mathbb{R}, v \in \mathcal{H}^\alpha.$$

To study (1.1) as an abstract stochastic differential equation on the Hilbert space \mathcal{H}^1 , we use the notation $u_1 := u$ and $u_2 := \dot{u}_1 = \dot{u}$, and rewrite (1.1) as follows

$$dU(t) = AU(t)dt + F(U(t))dt + BdW, \quad U(0) = U_0, \quad t \in (0, T), \quad (2.3)$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix}, \quad F(U(t)) = \begin{bmatrix} 0 \\ f(u_1(t)) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \quad (2.4)$$

The operator A with $D(A) = (H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})) \times H_0^1(\mathcal{D})$ is the generator of a strongly continuous semigroup (C_0 -semigroup) $E(t) = e^{tA}$ on $H_0^1 \times L_2(\mathcal{D})$ and

$$E(t) = e^{tA} = \begin{bmatrix} C(t) & \Lambda^{-1/2}S(t) \\ -\Lambda^{1/2}S(t) & C(t) \end{bmatrix}, \quad (2.5)$$

where $C(t) = \cos(t\Lambda^{1/2})$ and $S(t) = \sin(t\Lambda^{1/2})$. For more detail we refer the reader to [14, Appendix A] or [6, Chapter 10.3].

We require that the Q -Wiener process $W(t)$ in (2.1) satisfies the following assumption.

Assumption 2.1. *The Q -Wiener process $W(t)$ (2.1) takes values in $\dot{H}^{\beta-1}$ for a fixed $\beta \geq 0$, i.e., $\mathbb{E}[\|W(t)\|_{\beta-1}^2] < \infty$.*

Assumption 2.1 is equivalent to requiring that $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\text{HS}}^2 < \infty$ for a fixed $\beta \geq 0$ since

$$\mathbb{E}[\|W(t)\|_{\beta-1}^2] = \sum_{j=1}^{\infty} \lambda_j^{\beta-1} q_j \mathbb{E}[\beta_j(t)^2] = t \sum_{j=1}^{\infty} \lambda_j^{\beta-1} q_j = t \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\text{HS}}^2. \quad (2.6)$$

Hence, $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\text{HS}}^2 < \infty$ if and only if $\mathbb{E}[\|W(t)\|_{\beta-1}^2] < \infty$ for any finite t .

We also assume that the function $f : L_2(\mathcal{D}) \rightarrow L_2(\mathcal{D})$ satisfies

$$\begin{aligned} \|f(u) - f(v)\| &\leq C\|u - v\|, & \text{for } u, v \in L_2(\mathcal{D}), \\ \|f(u)\|_{\gamma} &\leq C(1 + \|u\|_{\gamma}), & \text{for } u \in \dot{H}^{\gamma} \text{ and } \gamma \geq 0. \end{aligned} \quad (2.7)$$

The existence and uniqueness of the mild solution of the stochastic wave equation (1.1) is discussed in [14, Theorem 7.4] and the spatial Sobolev regularity of the solution is discussed in [41, Proposition 3.1].

Theorem 2.2. *Assume that the function f satisfies (2.7) and that $\|U_0\|_{L_2(\Omega, \mathcal{H}^{\beta})} < \infty$, for some $\beta \geq 0$. Then under Assumption 2.1 the stochastic wave equation (2.3) has a unique mild solution, given by*

$$U(t) = E(t)U_0 + \int_0^t E(t-s)F(U(s))ds + \int_0^t E(t-s)BdW(s). \quad (2.8)$$

Additionally, there exists a constant C depending on T and $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\text{HS}}$ such that

$$\|U(t)\|_{L_2(\Omega, \mathcal{H}^{\beta})} \leq C(\|U_0\|_{L_2(\Omega, \mathcal{H}^{\beta})} + 1), \quad 0 \leq t \leq T. \quad (2.9)$$

3 Spatial semi-discretization

In this section, we discretize (1.1) by using the interior penalty dG finite element method [2] in space and provide existence and uniqueness of the dG semi-discrete formulation of (2.3) in a semigroup framework. For an introduction to dG finite element methods we refer to [3, 16, 26]. For a review on dG finite element methods in the context of wave equations see [11]. Furthermore, we derive strong error estimates for the spatial discretization of problem (2.3).

3.1 Discontinuous Galerkin method and semigroup approach

In order to discretize problem (1.1) in space, we consider a family of regular and quasi-uniform meshes \mathcal{T}_h parametrized by the mesh-width $h > 0$. Each \mathcal{T}_h partitions the domain \mathcal{D} into elements \mathfrak{T} , where we denote by $h_{\mathfrak{T}}$ the diameter of the element \mathfrak{T} and $h = \max_{\mathfrak{T}} h_{\mathfrak{T}}$ the mesh-width. We assume that the elements \mathfrak{T} are triangles or parallelograms in two space dimensions, and tetrahedra or parallelepipeds in three dimensions, respectively. We denote by $\mathfrak{F}_h = \cup_{\mathfrak{T} \in \mathcal{T}_h} \partial \mathfrak{T}$ the set of all faces. This is split into boundary $\mathfrak{F}_h^B = \mathfrak{F}_h \cap \partial \mathcal{D}$ and interior faces $\mathfrak{F}_h^I = \mathfrak{F}_h \setminus \mathfrak{F}_h^B$. Let $\mathfrak{T}^+, \mathfrak{T}^-$ be two elements sharing an interior face $\mathfrak{F} \in \mathfrak{F}_h^I$ with respective outward normal unit vectors \mathbf{n}^+ and \mathbf{n}^- . Denoting by u^\pm the trace of $u : \mathcal{D} \rightarrow \mathbb{R}$ taken from within \mathfrak{T}^\pm , we define the average of u over $\mathfrak{F} \in \mathfrak{F}_h^I$ by

$$\{u\} = \frac{1}{2}(u^+ + u^-).$$

Similarly, the jump of u over $\mathfrak{F} \in \mathfrak{F}_h^I$ is given by

$$[[u]] = u^+ \mathbf{n}^+ + u^- \mathbf{n}^-.$$

For a boundary face $\mathfrak{F} \in \mathfrak{F}_h^B$, we set $\{u\} = u$ and $[[u]] = u\mathbf{n}$, where \mathbf{n} denotes the unit outward normal vector on $\partial \mathcal{D}$.

We now define the discontinuous polynomial space

$$V_h = \{u \in L_2(\mathcal{D}) : u|_{\mathfrak{T}} \in \mathcal{P}^p(\mathfrak{T}), \mathfrak{T} \in \mathcal{T}_h\}, \quad (3.1)$$

where $\mathcal{P}^p(\mathfrak{T})$ denotes the polynomials of (total) degree less or equal to $p \geq 1$.

The dG semi-discrete formulation of (1.1) is given by: find $u_h(t) \in V_h$ such that

$$(d\dot{u}_h, v) + B_h(u_h, v) dt = (P_h f(u_h), v) dt + (P_h dW, v) \quad \forall v \in V_h, t \in (0, T), \quad (3.2)$$

$$u_h(\cdot, 0) = u_{h,0}, \dot{u}_h(\cdot, 0) = v_{h,0}, \quad (3.3)$$

where (\cdot, \cdot) is the $L_2(\mathcal{D})$ inner product, $P_h : L_2(\mathcal{D}) \rightarrow V_h$ the L_2 -projection onto V_h , $u_{h,0}, v_{h,0} \in V_h$ projections of initial data to be determined later, and B_h the

symmetric interior penalty discrete bilinear form

$$\begin{aligned}
B_h(u, v) &= \sum_{\mathfrak{T} \in \mathcal{T}_h} \int_{\mathfrak{T}} \mathbf{D}\nabla u \cdot \nabla v \, dx - \sum_{\mathfrak{F} \in \mathfrak{F}_h} \int_{\mathfrak{F}} \{\mathbf{D}\nabla u\} \cdot \llbracket v \rrbracket \, ds \\
&\quad - \sum_{\mathfrak{F} \in \mathfrak{F}_h} \int_{\mathfrak{F}} \llbracket u \rrbracket \cdot \{\mathbf{D}\nabla v\} \, ds + \sigma_0 \sum_{\mathfrak{F} \in \mathfrak{F}_h} \int_{\mathfrak{F}} h_{\mathfrak{F}}^{-1} \mathbf{D} \llbracket u \rrbracket \cdot \llbracket v \rrbracket \, ds,
\end{aligned} \tag{3.4}$$

where $h_{\mathfrak{F}}$ is the diameter of the face \mathfrak{F} . The interior penalty stabilization parameter $\sigma_0 > 0$ has to be chosen sufficiently large but independent of the mesh size. The last three terms in (3.4) correspond to jump and flux terms at the faces and they vanish when $u, v \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. The third term in (3.4) makes the bilinear form symmetric and the last term ensures coercivity of the bilinear form, see Lemma 3.1.

The bilinear form $B_h(\cdot, \cdot)$ defines a discrete linear operator $\Lambda_h : V_h \rightarrow V_h$

$$(\Lambda_h v_h, w) = B_h(v_h, w), \quad \forall w \in V_h.$$

This in turn gives a discrete analogue of the norm $\|\cdot\|_{\alpha}$

$$\|v_h\|_{h, \alpha} := \|\Lambda_h^{\alpha/2} v_h\| = \left(\sum_{j=1}^{N_h} \lambda_{h,j}^{\alpha} (v_h, \phi_{h,j})^2 \right)^{1/2}, \quad v_h \in \dot{H}_h^{\alpha}, \alpha \in \mathbb{R},$$

where $\{\phi_{h,j}\}_{j=1}^{N_h}$, $N_h = \dim V_h$, are the orthonormal eigenvectors of Λ_h with corresponding eigenvalues $\lambda_{h,j} \geq 0$. Note that since Λ_h is a symmetric, positive definite operator, the fractional power is well-defined. We also introduce discrete variants of $\|\cdot\|_{\alpha}$ and \mathcal{H}^{α}

$$\mathcal{H}_h^{\alpha} = V_h \times V_h, \quad \|\llbracket [u_1, u_2]^T \rrbracket\|_{h, \alpha}^2 = \|u_1\|_{h, \alpha}^2 + \|u_2\|_{h, \alpha-1}^2, \quad [u_1, u_2]^T \in V_h \times V_h.$$

We now introduce the broken norm as in [23]

$$\|u\|_* := \left(\sum_{\mathfrak{T} \in \mathcal{T}_h} \|\nabla u\|_{L_2(\mathfrak{T})}^2 + \sum_{\mathfrak{T} \in \mathcal{T}_h} h_{\mathfrak{T}}^2 \|\Delta u\|_{L_2(\mathfrak{T})}^2 + \sum_{\mathfrak{F} \in \mathfrak{F}_h} h_{\mathfrak{F}}^{-1} \|\llbracket u \rrbracket\|_{L_2(\mathfrak{F})}^2 \right)^{1/2}. \tag{3.5}$$

The bilinear form B_h in (3.4) is coercive and continuous in the norm (3.5), see [2, 3].

Lemma 3.1. *For large enough $\sigma_0 > 0$ there exists a constant $C_A > 0$, dependent on σ_0, d_{\max} , and d_{\min} , and independent of the mesh size, such that*

$$|B_h(u, v)| \leq C_A \|u\|_* \|v\|_*, \quad \forall u, v \in \dot{H}^2(\mathcal{D}) + V_h, \tag{3.6}$$

and

$$B_h(u, u) \geq \frac{1}{2} \|u\|_*^2, \quad \forall u \in V_h. \tag{3.7}$$

Consequently, we have the following norm equivalence

$$\frac{1}{2} \|u\|_*^2 \leq \|u\|_{h,1}^2 \leq C_A \|u\|_*^2, \quad \forall u \in V_h. \tag{3.8}$$

We also need the following spectral estimate (Lemma 3.3 in [23]).

Lemma 3.2. *For $u \in V_h + \dot{H}^2(\mathcal{D})$, it holds*

$$B_h(u, u) \leq C_s h^{-2} \|u\|^2, \quad (3.9)$$

where $C_s > 0$, is a constant independent of the mesh size, and depends on σ_0, d_{\max} , and the polynomial degree p .

Remark 3.3. *By Lemma 3.2 we obtain the following bound for the eigenvalues of the discrete operator Λ_h*

$$\lambda_{h,j} \leq C_s h^{-2}, \quad j = 1, \dots, N_h, \quad (3.10)$$

since $(\Lambda_h \phi_{h,j}, \phi_{h,j}) = B_h(\phi_{h,j}, \phi_{h,j}) = \lambda_{h,j}(\phi_{h,j}, \phi_{h,j})$. Additionally, we deduce the inverse estimate for any $u \in V_h$

$$\|u\|_{h,\alpha} = \|\Lambda_h^{\alpha/2} u\| = \left(\sum_{j=1}^{N_h} \lambda_{h,j}^\alpha(u, \phi_{h,j})^2 \right)^{1/2} \leq \sqrt{C_s} h^{-1} \|u\|_{h,\alpha-1}. \quad (3.11)$$

The dG semi-discrete analogue of the first order formulation (2.3) is: find $U_h = [u_{h,1}, u_{h,2}]^T \in V_h \times V_h$ such that

$$\begin{aligned} dU_h(t) &= A_h U_h(t) dt + F(U_h(t)) dt + B P_h dW, \quad t \in (0, T), \\ U_h(\cdot, 0) &= U_{h,0}, \end{aligned} \quad (3.12)$$

where

$$A_h = \begin{bmatrix} 0 & I \\ -\Lambda_h & 0 \end{bmatrix}, \quad U_{h,0} = \begin{bmatrix} u_{h,0} \\ v_{h,0} \end{bmatrix}, \quad F(U_h(t)) = \begin{bmatrix} 0 \\ u_{h,1}(t) \end{bmatrix}, \quad (3.13)$$

and B is as in (2.4). In order to ensure existence and uniqueness of problem (3.12), we first need to show that the discrete operator $A_h : V_h \times V_h \rightarrow V_h \times V_h$ satisfies the hypothesis of the Hille-Yosida Theorem [17, Theorem 3.5], i.e., generates a strongly continuous contraction semigroup on $V_h \times V_h$.

Proposition 3.4. *The discrete operator $A_h : V_h \times V_h \rightarrow V_h \times V_h$ in (3.13) generates a strongly continuous contraction semigroup on $V_h \times V_h$.*

Proof. The proof of the proposition follows from [6, Chapter 10.3]. \square

Similarly to (2.5), the C_0 -semigroup $E_h(t)$ generated by the discrete operator A_h is given by

$$E_h(t) = e^{tA_h} = \begin{bmatrix} C_h(t) & \Lambda_h^{-1/2} S_h(t) \\ -\Lambda_h^{1/2} S_h(t) & C_h(t) \end{bmatrix}, \quad (3.14)$$

where $C_h(t) = \cos(t\Lambda_h^{1/2})$ and $S_h(t) = \sin(t\Lambda_h^{1/2})$.

Similarly to the continuous case, see Theorem 2.2, we have the existence of the mild solution to the semi-discrete system.

Lemma 3.5. *Assume that $W(t)$ satisfies Assumption 2.1 and that f satisfies (2.7). Also let $\|U_{h,0}\|_{L_2(\Omega, \mathcal{H}_h^\beta)} < \infty$, then the dG formulation (3.12) has a unique mild solution given by*

$$U_h(t) = E_h(t)U_{h,0} + \int_0^t E_h(t-s)P_h F(U_h(s))ds + \int_0^t E_h(t-s)BP_h dW(s). \quad (3.15)$$

Further, there exists a constant $C := C(T, \|U_{h,0}\|_{L_2(\Omega, \mathcal{H}_h^\beta)}, \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\text{HS}})$, and independent of h such that

$$\|U_h(t)\|_{L_2(\Omega, \mathcal{H}_h^\beta)} \leq C, \quad 0 \leq t \leq T. \quad (3.16)$$

Proof. The proof follows from [1, Proposition 3] by using that $\|v\|_{h,\beta} = \|\Lambda_h^{\beta/2}v\|$. \square

3.2 Strong convergence in space

In this subsection, we prove strong convergence of the dG approximation of the stochastic wave equation (3.12), first with respect to the broken norm (3.5) and then with respect to the L_2 -norm. To analyze the strong convergence of the spatial approximation (3.12), we need to derive error estimates for the sine and cosine operators as in Corollary 4.2 in [31]. Before we state strong error estimates for the semi-discrete dG formulation (3.12) we derive optimal error bounds for the deterministic homogeneous wave equation.

Deterministic homogeneous wave equation. We now look at the deterministic homogeneous wave equation

$$\begin{aligned} d\dot{u} + \Lambda u dt &= 0 && \text{in } \mathcal{D} \times (0, T), \\ u &= 0 && \text{on } \partial\mathcal{D} \times (0, T), \\ u(\cdot, 0) &= u_0, \dot{u}(\cdot, 0) = v_0 && \text{in } \mathcal{D}. \end{aligned} \quad (3.17)$$

The dG semi-discrete formulation of (3.17) is: find $u_h(t) \in V_h$ such that

$$\begin{aligned} (d\dot{u}_h, v) + B_h(u_h, v) dt &= 0 \quad \forall v \in V_h, t \in (0, T), \\ u_h(\cdot, 0) &= u_{h,0}, \dot{u}_h(\cdot, 0) = v_{h,0}, \quad u_{h,0}, v_{h,0} \in V_h. \end{aligned} \quad (3.18)$$

We recall some useful results for the dG finite element method. For $u \in \dot{H}^2(\mathcal{D})$, the Galerkin projection $\Pi_h u \in V_h$ is defined as follows

$$B_h(\Pi_h u - u, v) = 0, \quad v \in V_h. \quad (3.19)$$

Since $\partial_t^i(\pi_I u) = \pi_I(\partial_t^i u)$, $i = 0, \dots, 2$, where π_I can be chosen as P_h or Π_h , we have the following error bound in the L_2 -norm

$$\|\partial_t^i(u - \pi_I u)\| \leq Ch^{p+1}\|\partial_t^i u\|_{p+1}, \quad \partial_t^i u \in \dot{H}^{p+1}, p \geq 1, \quad (3.20)$$

where we recall that p is the (local) polynomial degree of the discrete space V_h . The error estimate for the Galerkin projection in the broken norm (3.5) is

$$\|u - \Pi_h u\|_* \leq Ch^p \|u\|_{p+1}, \quad u \in \dot{H}^{p+1}, p \geq 1. \quad (3.21)$$

Estimates (3.20) and (3.21) can be found in Lemma 4.1 in [23].

Theorem 3.6. *Let the exact solution u of (3.17) satisfy*

$$u, \dot{u}, \ddot{u} \in L^\infty([0, T]; \dot{H}^{p+1}(\mathcal{D})),$$

for $p \geq 1$, and u_h be the dG approximation obtained by (3.18). Setting $e(t) = u(t) - u_h(t)$, $t \in [0, T]$, we have for a constant $C > 0$, independent of the mesh size h ,

$$\|e(t)\|_* \leq C \{ \|u_0 - \Pi_h u_0\|_* + \|v_0 - \Pi_h v_0\| \} + h^p \left\{ \|u\|_{p+1} + \int_0^t \|\ddot{u}(s)\|_p ds \right\}, \quad (3.22)$$

$$\|\dot{e}(t)\| \leq C \{ \|u_0 - \Pi_h u_0\|_{h,1} + \|v_0 - \Pi_h v_0\| \} + h^{p+1} \left\{ \|\dot{u}\|_{p+1} + \int_0^t \|\ddot{u}(s)\|_{p+1} ds \right\}, \quad (3.23)$$

$$\begin{aligned} \|e(t)\| &\leq C \{ \|u_0 - \Pi_h u_0\| + \|v_0 - P_h v_0\|_{h,-1} \} \\ &\quad + h^{p+1} \left\{ \|u(s)\|_{p+1} + \int_0^t \|\dot{u}(s)\|_{p+1} ds \right\}. \end{aligned} \quad (3.24)$$

Proof. We set as in [32]

$$e = u - \Pi_h u + \Pi_h u - u_h = \rho + \theta. \quad (3.25)$$

Then, using the Galerkin projection (3.19), the error satisfies

$$(\ddot{\theta}, v) + B_h(\theta, v) = -(\ddot{\rho}, v) \quad \forall v \in V_h. \quad (3.26)$$

Choosing $v = \dot{\theta}$ and using (3.7), we conclude in the standard way that, see [32, Theorem 13.1],

$$\|\dot{\theta}(t)\| + \|\theta(t)\|_{h,1} \leq C \left\{ \|\theta(0)\|_{h,1} + \|\dot{\theta}(0)\| + \int_0^t \|\ddot{\rho}(s)\| ds \right\}. \quad (3.27)$$

By the triangle inequality, the norm equivalence (3.8), and estimates (3.20) for $\pi_I = \Pi_h$, and (3.21), we conclude (3.22) and (3.23).

Although, [31] uses continuous polynomials, the proof of estimate (3.24), follows along the same lines as [31, Theorem 4.1] by rewriting the problem in a first-order formulation and using estimates (3.20) for the Galerkin projection (3.19). \square

Remark 3.7. In [22] optimal convergence rates are derived for the displacement with respect to the L_2 -norm and with respect to the broken norm (3.5) for (3.17). A bound for the velocity in the L_2 -norm follows from [22, Theorem 4.1], but is not optimal. Theorem 3.6 provides optimal error estimates for both the displacement and the velocity with respect to the L_2 -norm.

In the following lemma we state error estimates for the sine and cosine operators.

Lemma 3.8. Denote $U_0 = [u_0, v_0]^T$ and let

$$\begin{aligned} G_h(t)U_0 &= (C_h(t)\Pi_h - C(t))u_0 + \Lambda_h^{-1/2}S_h(t)v_{h,0} - \Lambda^{-1/2}S(t)v_0, \\ \dot{G}_h(t)U_0 &= -\left(\Lambda_h^{1/2}S_h(t)\Pi_h - \Lambda^{1/2}S(t)\right)u_0 + (C_h(t)\Pi_h - C(t))v_0. \end{aligned} \quad (3.28)$$

Then there exists a constant $C > 0$ independent of the mesh size h such that:

i. If $v_{h,0} = \Pi_h v_0$, then

$$\|G_h(t)U_0\|_* \leq C(1+t)h^{\frac{p}{p+1}(\beta-1)} \|U_0\|_\beta, \quad t \in [0, T], \quad \beta \in [1, p+2]. \quad (3.29)$$

ii. If $v_{h,0} = P_h v_0$, then

$$\|G_h(t)U_0\| \leq C(1+t)h^{\frac{p+1}{p+2}\beta} \|U_0\|_\beta, \quad t \in [0, T], \quad \beta \in [0, p+2]. \quad (3.30)$$

iii. If $v_{h,0} = \Pi_h v_0$, then

$$\|\dot{G}_h(t)U_0\| \leq C(1+t)h^{\frac{p+1}{p+2}(\beta-1)} \|U_0\|_\beta, \quad t \in [0, T], \quad \beta \in [1, p+3]. \quad (3.31)$$

Proof. In [31], problem (3.18) is discretized in space by using piecewise continuous polynomials and error bounds are derived for the G_h and \dot{G}_h operators in terms of initial conditions in Corollary 4.2. The proof of the above lemma follows from [31], employing the estimates in Theorem 3.6 and by the norm equivalence (3.8). \square

As a corollary, we obtain the following error estimates.

Theorem 3.9. Assume that f satisfies (2.7). Denote $U_0 = [u_0, v_0]^T$, and let $U = [u_1, u_2]^T$ and $U_h = [u_{h,1}, u_{h,2}]^T$ be given by (2.8) and (3.15), respectively. Choosing $u_{h,0} = \Pi_h u_0$, we have, for $t \in [0, T]$ the following estimates:

i. If $v_{h,0} = P_h v_0$ and $W(t)$ satisfies Assumption 2.1, for some $\beta \geq 0$, then

$$\|u_{h,1}(t) - u_1(t)\|_{L_2(\Omega, L_2(\mathcal{D}))} \leq Ch^{\min(\frac{p+1}{p+2}\beta, p+1)}. \quad (3.32)$$

ii. If $v_{h,0} = \Pi_h v_0$ and $W(t)$ satisfies Assumption 2.1 for some $\beta \geq 1$, then

$$\|u_{h,1}(t) - u_1(t)\|_{L_2(\Omega, \dot{H}_h^1)} \leq Ch^{\min(\frac{p}{p+1}(\beta-1), p+1)}. \quad (3.33)$$

iii. If $v_{h,0} = \Pi_h v_0$ and $W(t)$ satisfies Assumption 2.1 for some $\beta \geq 1$, then

$$\|u_{h,2}(t) - u_2(t)\|_{L_2(\Omega, L_2(\mathcal{D}))} \leq Ch^{\min(\frac{p+1}{p+2}(\beta-1), p+1)}. \quad (3.34)$$

The constant C depends on $t, \|U_0\|_{L_2(\Omega, \mathcal{H}^\beta)}, \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\text{HS}}$, and is independent of h .

Proof. Estimates (3.32) and (3.34) are proved as in [1, Theorem 4] where piecewise continuous polynomials are used for the spatial discretization. The proof of estimate (3.33) follows along the same lines with the proof of (3.32). \square

4 The stochastic position Verlet method

We now consider the full discretization of the stochastic wave equation (1.1). Our goal is to prove optimal convergence of the strong error for the full discretization of problem (1.1). Let $\tau > 0$ be the time step size, so that $t_n = n\tau$, $n = 1, 2, \dots, N$, and $T = N\tau$. Letting $X^n = [X_1^n, X_2^n]^T$ be the numerical approximation of $U_h(t_n)$ in (3.12), the stochastic position Verlet (SVM) scheme is

$$\begin{aligned} X_1^{n-1/2} &= X_1^{n-1} + \frac{\tau}{2} X_2^{n-1}, \\ X_2^n &= X_2^{n-1} - \tau \Lambda_h X_1^{n-1/2} + \tau P_h f(X_1^{n-1/2}) + P_h \Delta W^n, \\ X_1^n &= X_1^{n-1/2} + \frac{\tau}{2} X_2^n, \end{aligned} \quad (4.1)$$

where $X^0 = U_{h,0}$ and $\Delta W^n = W(t_n) - W(t_{n-1})$.

4.1 Stability of the scheme

To study the stability of the above scheme, we rewrite system (4.1) as follows

$$X^n = M X^{n-1} + \tau D P_h f(X_1^{n-1/2}) + D P_h \Delta W^n, \quad (4.2)$$

where $D = [\frac{\tau}{2} I, I]^T$, and

$$M(\tau) = \begin{bmatrix} I - \frac{\tau^2}{2} \Lambda_h & \tau I - \frac{\tau^3}{4} \Lambda_h \\ -\tau \Lambda_h & I - \frac{\tau^2}{2} \Lambda_h \end{bmatrix}. \quad (4.3)$$

By recursion, the approximate solution X^n can be written as

$$X^n = M^n X^0 + \sum_{j=1}^n M^{n-j} \tau D P_h f(X_1^{j-1/2}) + \sum_{j=1}^n M^{n-j} D P_h \Delta W^j. \quad (4.4)$$

We study the stability of the scheme under a CFL condition [12]. More specifically, we assume that the mesh size h and the time step τ satisfy the following restriction

$$\tau < C_{\text{CFL}} h, \quad (4.5)$$

where $C_{\text{CFL}} < \frac{2}{\sqrt{C_S}}$ and C_S is the constant from Lemma 3.2.

4.1.1 Discrete norm

Under the CFL condition (4.5), we introduce the following inner product for $v = [v_1, v_2]^T$, $w = [w_1, w_2]^T \in \mathcal{H}_h^\alpha$

$$\langle v, w \rangle_{m, \alpha} = (V^{-1}w)^* \begin{bmatrix} \Lambda_h^{\alpha-1} & 0 \\ 0 & \Lambda_h^{\alpha-1} \end{bmatrix} V^{-1}v, \quad \alpha \in \mathbb{R}, \quad (4.6)$$

where $(V^{-1}w)^*$ is the conjugate transpose, and

$$V^{-1} = \begin{bmatrix} i\Lambda_h^{1/2}(I - \frac{\tau^2}{4}\Lambda_h)^{-1/2}/2 & I/2 \\ -i\Lambda_h^{1/2}(I - \frac{\tau^2}{4}\Lambda_h)^{-1/2}/2 & I/2 \end{bmatrix}.$$

The inner product $\langle \cdot, \cdot \rangle_{m, \alpha}$ defines the discrete norm

$$\|v\|_{m, \alpha}^2 = \langle v, v \rangle_{m, \alpha} \quad \alpha \in \mathbb{R}, \quad v = [v_1, v_2]^T \in \mathcal{H}_h^\alpha. \quad (4.7)$$

A key result for the stability analysis of our scheme is showing that the time integrator M (4.3) preserves the $\|\cdot\|_{m, \alpha}$ norm. To do so, we consider the following spectral decomposition of the matrix M in (4.3).

Lemma 4.1. *Under the CFL condition (4.5) we have that*

$$M = VDV^{-1}$$

where $V = \begin{bmatrix} m_+ & m_- \end{bmatrix}$ and $D = \text{diag}(\mu_+, \mu_-)$ with

$$\mu_\pm = I - \frac{\tau^2 \Lambda_h}{2} \pm i\tau \Lambda_h^{1/2} (I - \tau^2 \Lambda_h / 4)^{1/2}, \quad (4.8)$$

and

$$m_\pm = \begin{bmatrix} \pm i\Lambda_h^{-1/2} (I - \frac{\tau^2}{4}\Lambda_h)^{1/2} \\ I \end{bmatrix}. \quad (4.9)$$

Further, $\mu_+^* \mu_+ = I$ and $\mu_-^* \mu_- = I$, where μ_\pm^* is the adjoint with respect to the L^2 -inner product, respectively.

Proof. The expressions (4.8) and (4.9) can be verified by direct computation. We first note that μ_+^* is given by

$$\mu_+^* = I - \frac{\tau^2 \Lambda_h}{2} - i\tau \Lambda_h^{1/2} (I - \tau^2 \Lambda_h / 4)^{1/2} = \mu_-,$$

since Λ_h is a real symmetric operator. Similarly, $\mu_-^* = \mu_+$. Then it is not difficult to see that $\mu_+^* \mu_+ = \mu_- \mu_+ = I$ and $\mu_-^* \mu_- = \mu_+ \mu_- = I$. \square

Lemma 4.2. *Let $v = [v_1, v_2]^T \in \mathcal{H}_h^\alpha$, under the CFL condition (4.5),*

$$\|Mv\|_{m, \alpha} = \|v\|_{m, \alpha}, \quad \alpha \in \mathbb{R}. \quad (4.10)$$

Proof. Using the spectral decomposition of M in Lemma 4.1 and the definition of the $\|\cdot\|_{m,\alpha}$ norm in (4.7), we have $v = [v_1, v_2]^T \in \mathcal{H}_h^\alpha$

$$\begin{aligned} \|Mv\|_{m,\alpha}^2 &= \|VDV^{-1}v\|_{m,\alpha}^2 = (V^{-1}VDV^{-1}v)^* \begin{bmatrix} \Lambda_h^{\alpha-1} & 0 \\ 0 & \Lambda_h^{\alpha-1} \end{bmatrix} V^{-1}VDV^{-1}v \\ &= (DV^{-1}v)^* \begin{bmatrix} \Lambda_h^{\alpha-1} & 0 \\ 0 & \Lambda_h^{\alpha-1} \end{bmatrix} DV^{-1}v = (V^{-1}v)^* D^* \begin{bmatrix} \Lambda_h^{\alpha-1} & 0 \\ 0 & \Lambda_h^{\alpha-1} \end{bmatrix} DV^{-1}v \\ &= v^* (V^{-1})^* \begin{bmatrix} \Lambda_h^{\alpha-1} & 0 \\ 0 & \Lambda_h^{\alpha-1} \end{bmatrix} V^{-1}v = \|v\|_{m,\alpha}^2, \end{aligned}$$

since μ_\pm and $\Lambda_h^{\alpha-1}$ commute; recall $D = \text{diag}(\mu_+, \mu_-)$, and

$$D^*D = \begin{bmatrix} \mu_+^* & 0 \\ 0 & \mu_-^* \end{bmatrix} \begin{bmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{bmatrix} = \begin{bmatrix} \mu_+^* \mu_+ & 0 \\ 0 & \mu_-^* \mu_- \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

□

Next, we prove that the norms $\|\cdot\|_{h,\alpha}$ and $\|\cdot\|_{m,\alpha}$ are equivalent.

Lemma 4.3. *Under the CFL condition (4.5), there exist constants $C_1, C_2 > 0$, independent of the mesh size h and the time step τ , such that*

$$C_1 \|v\|_{h,\alpha}^2 \leq \|v\|_{m,\alpha}^2 \leq C_2 \|v\|_{h,\alpha}^2 \quad \alpha \in \mathbb{R}, v \in \mathcal{H}_h^\alpha. \quad (4.11)$$

Proof. By the definition of the norm $\|\cdot\|_{m,\alpha}$ in (4.7) we obtain

$$\|v\|_{m,\alpha}^2 = \frac{1}{2} \|\Lambda_h^{\alpha/2} (I - \frac{\tau^2}{4} \Lambda_h)^{-1/2} v_1\|^2 + \frac{1}{2} \|\Lambda_h^{(\alpha-1)/2} v_2\|^2$$

Thus, we get under the CFL condition (4.5)

$$\begin{aligned} \|v\|_{m,\alpha}^2 &\leq \frac{1}{2} \|(I - \frac{\tau^2}{4} \Lambda_h)^{-1/2}\|_{\mathcal{L}(V_h)}^2 \|\Lambda_h^{\alpha/2} v_1\|^2 + \frac{1}{2} \|\Lambda_h^{(\alpha-1)/2} v_2\|^2 \\ &\leq \max\{C/2, 1/2\} \|v\|_{h,\alpha}^2 = C_2 \|v\|_{h,\alpha}^2. \end{aligned} \quad (4.12)$$

We also have under the CFL condition (4.5)

$$\begin{aligned} \|v\|_{h,\alpha}^2 &= \|\Lambda_h^{\alpha/2} v_1\|^2 + \|\Lambda_h^{(\alpha-1)/2} v_2\|^2 \\ &\leq \|(I - \frac{\tau^2}{4} \Lambda_h)^{1/2}\|_{\mathcal{L}(V_h)}^2 \|\Lambda_h^{\alpha/2} (I - \frac{\tau^2}{4} \Lambda_h)^{-1/2} v_1\|^2 + \|\Lambda_h^{(\alpha-1)/2} v_2\|^2 \\ &\leq C \|\Lambda_h^{\alpha/2} (I - \frac{\tau^2}{4} \Lambda_h)^{-1/2} v_1\|^2 + \|\Lambda_h^{(\alpha-1)/2} v_2\|^2 \leq C_1 \|v\|_{m,\alpha}^2. \end{aligned} \quad (4.13)$$

Estimates (4.12) and (4.13) complete the proof of (4.11). □

We also want to analyse the stability SVM applied to the linear analogue of (1.1). The linear stochastic wave equation is given by

$$d\dot{u} = -\Lambda u dt + dW, \quad (4.14)$$

with initial conditions as in (1.1), i.e., $u(\cdot, 0) = u_0$, $\dot{u}(\cdot, 0) = v_0$. Following notation from Section 3.1, the dG approximation to the linear stochastic wave equation is given with (3.12) for $f \equiv 0$, i.e., find $U_h(t) = [u_{h,1}, u_{h,2}]^T \in V_h \times V_h$ such that

$$\begin{aligned} dU_h(t) &= A_h U_h(t) dt + B P_h dW, \quad t \in (0, T), \\ U_h(\cdot, 0) &= U_{h,0}. \end{aligned}$$

Let $Y^n = [Y_1^n, Y_2^n]^T$ be the temporal approximation to the above problem. Then Y^n is given by (4.4) for $f = 0$, i.e.,

$$Y^n = M^n Y^0 + \sum_{j=1}^n M^{n-j} D P_h \Delta W^j, \quad (4.15)$$

where $Y^0 = X^0 = U_{h,0}$.

Remark 4.4. *Throughout this section, the constant C denotes a generic positive constant that may vary from line to line and is independent of h and τ .*

In order to prove the stability of the temporal approximation of the linear and nonlinear problem, we need the following relation between Λ_h and Λ for a constant C , see proof of Theorem 4.4 in [29],

$$\|\Lambda_h^{-\delta} P_h \Lambda^\delta\|_{\mathcal{L}(L_2(\mathcal{D}))} \leq C, \quad \delta \in [0, 1/2]. \quad (4.16)$$

Lemma 4.5. *Assume that $\|Y^0\|_{L_2(\Omega, \mathcal{H}_h^\beta)} < \infty$ and that $\|X^0\|_{L_2(\Omega, \mathcal{H}_h^\beta)} < \infty$. Let $W(t)$ satisfy Assumption 2.1 for some $\beta \geq 0$ and let f satisfy (2.7), then, under the CFL condition (4.5), there exists a constant C , independent of h and τ , such that*

$$\|Y^n\|_{L_2(\Omega, \mathcal{H}_h^\beta)} \leq C \left(\|Y^0\|_{L_2(\Omega, \mathcal{H}_h^\beta)} + t_n^{1/2} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\text{HS}} \right) \quad (4.17)$$

and

$$\|X^n\|_{L_2(\Omega, \mathcal{H}_h^\beta)} \leq C \exp(C t_n) \left(\|X^0\|_{L_2(\Omega, \mathcal{H}_h^\beta)} + t_n^{1/2} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\text{HS}} + t_n \right). \quad (4.18)$$

Proof. Writing the increments ΔW^n as $\Delta W^n = \int_{t_{n-1}}^{t_n} dW(s)$, we have for (4.15)

by using Itô's isometry (2.2) and Lemma 4.2

$$\begin{aligned}
\mathbb{E} \left[\|Y^n\|_{m,\beta}^2 \right] &= \mathbb{E} \left[\|M^n Y^0\|_{m,\beta}^2 \right] + \mathbb{E} \left[\left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} M^{n-j} DP_h dW(s) \right\|_{m,\beta}^2 \right] \\
&= \mathbb{E} \left[\|Y^0\|_{m,\beta}^2 \right] + \sum_{j=1}^n (t_j - t_{j-1}) \sum_{k=0}^{\infty} \left\| DP_h Q^{1/2} e_k \right\|_{m,\beta}^2 \\
&= \mathbb{E} \left[\|Y^0\|_{m,\beta}^2 \right] + t_n \sum_{k=0}^{\infty} \left\| DP_h Q^{1/2} e_k \right\|_{m,\beta}^2.
\end{aligned}$$

By the norm equivalence (4.11), it follows that

$$\mathbb{E} \left[\| \|Y^n\|_{h,\beta}^2 \right] \leq \frac{C_2}{C_1} \left(\mathbb{E} \left[\| \|X^0\|_{h,0}^2 \right] + t_n \sum_{k=0}^{\infty} \left\| \|DP_h Q^{1/2} e_k\|_{h,\beta} \right\|^2 \right).$$

Letting $I = \sum_{k=0}^{\infty} \left\| \|DP_h Q^{1/2} e_k\|_{h,\beta} \right\|^2$, employing the definition of the $\|\cdot\|_{h,\beta}$ norm, and by the inverse estimate (3.11) we obtain

$$I \leq \frac{\tau^2 C_s}{4h^2} \|\Lambda_h^{(\beta-1)/2} P_h Q^{1/2}\|_{\text{HS}}^2 + \|\Lambda_h^{(\beta-1)/2} P_h Q^{1/2}\|_{\text{HS}}^2.$$

Using the CFL condition (4.5) and (4.16) for $\beta/2 \in [-1, 2]$, we finally get

$$I \leq 2 \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\text{HS}}^2.$$

The above estimate completes the proof of bound (4.17).

To prove estimate (4.18), we use (4.4) and (4.10) to get

$$\begin{aligned}
\mathbb{E}[\|X^n\|_{m,\beta}^2] &\leq 3\mathbb{E}[\|X^0\|_{m,\beta}^2] + 3\mathbb{E} \left[\left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} DP_h dW(s) \right\|_{m,\beta}^2 \right] \\
&\quad + 3n\tau^2 \sum_{j=1}^n \mathbb{E} \left[\|DP_h f(X_1^{j-1/2})\|_{m,\beta}^2 \right].
\end{aligned}$$

Using (4.11) and noting that the second term is bounded as above, we obtain

$$\begin{aligned}
\mathbb{E}[\| \|X^n\|_{h,\beta}^2] &\leq C \left(\mathbb{E} \left[\| \|X^0\|_{h,\beta}^2 \right] + t_n \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\text{HS}}^2 \right. \\
&\quad \left. + n\tau^2 \sum_{j=1}^n \mathbb{E} \left[\left\| \|DP_h f(X_1^{j-1/2})\|_{h,\beta} \right\|^2 \right] \right). \tag{4.19}
\end{aligned}$$

Letting $II = n\tau^2 \sum_{j=1}^n \mathbb{E} \left[\left\| \|DP_h f(X_1^{j-1/2})\|_{h,\beta} \right\|^2 \right]$, we have by the definition of

the norm $\|\cdot\|_{h,\beta}$ and by noting that $\|\Lambda_h^{-1/2}\|_{\mathcal{L}(L_2(\mathcal{D}))}^2 \leq C$,

$$\begin{aligned} II &= n\tau^2 \sum_{j=1}^n \mathbb{E} \left[\frac{\tau^2}{4} \|P_h f(X_1^{j-1/2})\|_{h,\beta}^2 + \|P_h f(X_1^{j-1/2})\|_{h,\beta-1}^2 \right] \\ &\leq (C + \frac{\tau^2}{4}) t_n \tau \sum_{j=1}^n \mathbb{E} \left[\|P_h f(X_1^{j-1/2})\|_{h,\beta}^2 \right]. \end{aligned}$$

Using (2.7) and triangle inequality, we obtain

$$\begin{aligned} II &\leq (C + \frac{\tau^2}{4}) t_n \tau \sum_{j=1}^n \mathbb{E} \left[1 + \|X_1^{j-1/2}\|_{h,\beta}^2 \right] \\ &\leq (C + \frac{\tau^2}{4}) t_n \tau \sum_{j=1}^n \left(1 + 2\mathbb{E} \left[\|X_1^{j-1}\|_{h,\beta}^2 + \frac{\tau^2}{4} \|X_2^{j-1}\|_{h,\beta}^2 \right] \right). \end{aligned}$$

By the inverse estimate (3.11) and the CFL condition (4.5), we deduce for II

$$\begin{aligned} II &\leq Ct_n \tau \sum_{j=1}^n \left(1 + 2\mathbb{E} \left[\|X_1^{j-1}\|_{h,\beta}^2 + \frac{\tau^2 C_s}{4h^2} \|X_2^{j-1}\|_{h,\beta-1}^2 \right] \right) \\ &\leq Ct_n \tau \sum_{j=1}^n \left(1 + 2\mathbb{E} \left[\|X_1^{j-1}\|_{h,\beta}^2 + \|X_2^{j-1}\|_{h,\beta-1}^2 \right] \right) \\ &= Ct_n \tau \sum_{j=1}^n \left(1 + 2\mathbb{E} \left[\|X^{j-1}\|_{h,\beta}^2 \right] \right). \end{aligned}$$

Using the above estimate, we obtain for (4.19)

$$\begin{aligned} \mathbb{E}[\|X^n\|_{h,\beta}^2] &\leq C \left(\mathbb{E}[\|X^0\|_{h,\beta}^2] + t_n \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\text{HS}}^2 \right. \\ &\quad \left. + t_n^2 + 2t_n \tau \sum_{j=1}^n \left(\mathbb{E}[\|X^{j-1}\|_{h,\beta}^2] \right) \right). \end{aligned}$$

The discrete version of Gronwall's inequality applied to the above inequality and taking square roots, gives

$$\|X^n\|_{L_2(\Omega, \mathcal{H}_h^\beta)} \leq C \exp(Ct_n) \left(\|X^0\|_{L_2(\Omega, \mathcal{H}_h^\beta)} + t_n^{1/2} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\text{HS}} + t_n \right).$$

The above bound completes the proof of (4.18). \square

4.2 Strong convergence analysis

In this subsection, we derive strong error estimates for the full discretization of (1.1). Before we analyze the strong convergence of the temporal discretization (4.4), we present Hölder continuity of the semi-discrete mild solution (3.15).

Lemma 4.6. *Let $U_h = [u_{h,1}, u_{h,2}]^T$ be the solution to the dG semi-discrete formulation (3.2) given by (3.15). Also let all conditions in Lemma 3.5 be fulfilled, then*

$$\mathbb{E}[\|u_{h,1}(t) - u_{h,1}(s)\|_{h,0}^2] \leq C|t - s|^{2\min(\beta,1)}, \quad (4.20)$$

where the constant $C > 0$ depends on T , $\|U_{h,0}\|_{L_2(\Omega, \mathcal{H}_h^\beta)}$, and $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\text{HS}}$, and is independent of h and τ .

Proof. In [1, Proposition 3] Hölder continuity of the finite element approximation to the stochastic wave equation (1.1) is proved. The proof for the dG approximation follows in the same way using that the discrete operator Λ_h defines the discrete norm $\|v\|_{h,\beta} = \|\Lambda_h^{\beta/2} v\|$. \square

We also need to derive a bound for the difference between the semigroup E_h (3.14) and our time integrator M (4.3).

Lemma 4.7. *Let $\alpha = 0, 1$. The following estimate holds for the error between the semigroup E_h (3.14) and the time integrator M (4.3):*

$$\|(E_h - M)V\|_{h,\alpha} \leq \tau^3 \|V\|_{h,\alpha+3}, \quad V = [v_1, v_2]^T \in \mathcal{H}_h^{\alpha+3}. \quad (4.21)$$

Proof. We first note that

$$(E_h - M)V = \begin{bmatrix} (\cos(\tau\Lambda_h^{1/2}) - (I - \frac{\tau^2\Lambda_h}{2}))v_1 + (\Lambda_h^{-1/2} \sin(\tau\Lambda_h^{1/2}) - (\tau I - \frac{\tau^3\Lambda_h}{4}))v_2 \\ (-\Lambda_h^{1/2} \sin(\tau\Lambda_h^{1/2}) + \tau\Lambda_h)v_1 + (\cos(\tau\Lambda_h^{1/2}) - (I - \frac{\tau^2\Lambda_h}{2}))v_2 \end{bmatrix}.$$

Therefore, we have by the triangle inequality

$$\begin{aligned} \|(E_h - M)V\|_{h,\alpha} &\leq \|(\cos(\tau\Lambda_h^{1/2}) - (I - \frac{\tau^2\Lambda_h}{2}))v_1\|_{h,\alpha} \\ &\quad + \|(\Lambda_h^{-1/2} \sin(\tau\Lambda_h^{1/2}) - (\tau I - \frac{\tau^3\Lambda_h}{4}))v_2\|_{h,\alpha} \\ &\quad + \|(-\Lambda_h^{1/2} \sin(\tau\Lambda_h^{1/2}) + \tau\Lambda_h)v_1\|_{h,\alpha-1} \\ &\quad + \|(\cos(\tau\Lambda_h^{1/2}) - (I - \frac{\tau^2\Lambda_h}{2}))v_2\|_{h,\alpha-1} \\ &= I + II + III + IV. \end{aligned} \quad (4.22)$$

By the definition of the $\|\cdot\|_{h,\alpha}$ norm, we have for I

$$I = \left(\sum_{j=1}^{N_h} \lambda_{h,j}^\alpha |\cos(\tau\sqrt{\lambda_{h,j}}) - 1 + \frac{\tau^2\lambda_{h,j}}{2}|^2 (v_1, \phi_{h,j})^2 \right)^{1/2}.$$

Using Taylor's theorem it holds that $|\cos(\tau\sqrt{\lambda_{h,j}}) - (1 - \frac{\tau^2}{2}\lambda_{h,j})| = \frac{|\sin(\xi)|\tau^3\lambda_{h,j}^{3/2}}{6} \leq \frac{\tau^3\lambda_{h,j}^{3/2}}{6}$, for some $\xi \in (0, \tau\sqrt{\lambda_{h,j}})$, thus we get

$$I \leq \left(\sum_{j=1}^{N_h} \lambda_{h,j}^\alpha \left| \frac{\tau^3\lambda_{h,j}^{3/2}}{6} \right|^2 (v_1, \phi_{h,j})^2 \right)^{1/2} = \frac{\tau^3}{6} \|v_1\|_{h,\alpha+3}. \quad (4.23)$$

We now look at II and again by Taylor's theorem it holds that $|\lambda_{h,j}^{-1/2} \sin(\tau\sqrt{\lambda_{h,j}}) - \tau| \leq \frac{\tau^3 \lambda_{h,j}^2}{6}$, hence we obtain

$$II \leq \|(\Lambda_h^{-1/2} \sin(\tau\Lambda_h^{1/2}) - \tau I)v_2\|_{h,\alpha} + \frac{\tau^3}{4}\|v_2\|_{h,\alpha+2} \leq \frac{5}{12}\tau^3\|v_2\|_{h,\alpha+2}. \quad (4.24)$$

Similarly, we get for III

$$III \leq \left(\sum_{j=1}^{N_h} \lambda_{h,j}^{\alpha-1} (\tau^3 \lambda_{h,j}^2)^2 (v_1, \phi_{h,j})^2 \right)^{1/2} = \frac{\tau^3}{6} \|v_1\|_{h,\alpha+3}. \quad (4.25)$$

Finally, we have for IV

$$IV \leq \left(\sum_{j=1}^{N_h} \lambda_{h,j}^{\alpha-1} \left(\frac{\tau^3 \lambda_{h,j}^{3/2}}{6} \right)^2 (v_2, \phi_{h,j})^2 \right)^{1/2} = \frac{\tau^3}{6} \|v_2\|_{h,\alpha+2}. \quad (4.26)$$

Using (4.23), (4.24), (4.25), and (4.26), gives for (4.22)

$$\|(E_h - M)V\|_{h,\alpha} \leq \frac{1}{3}\tau^3\|v_1\|_{h,\alpha+3} + \frac{7}{12}\tau^3\|v_2\|_{h,\alpha+2} \leq \tau^3\|V\|_{h,\alpha+3}.$$

The above bound completes the proof of the lemma. \square

Theorem 4.8. *Let $X^n = [X_1^n, X_2^n]^T$ be the numerical approximation of (3.12) by the stochastic position Verlet method (4.1). Assume that $\|U_{h,0}\|_{L_2(\Omega, \mathcal{H}_h^\beta)} < \infty$ for some $\beta \geq 0$ and that f satisfies (2.7). Then there exists a constant $C > 0$ depending on T , $\|U_{h,0}\|_{L_2(\Omega, \mathcal{H}_h^\beta)}$, and $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\text{HS}}$, but independent of h and τ , under the CFL condition (4.5), such that:*

i. If $W(t)$ satisfies Assumption 2.1 for some $\beta \geq 0$, then

$$\|u_{h,1}(t_n) - X_1^n\|_{L_2(\Omega, \dot{H}_h^0)} \leq C\tau^{\min(\frac{2}{3}\beta, 1)}. \quad (4.27)$$

ii. If $W(t)$ satisfies Assumption 2.1 for some $\beta \geq 1$, then

$$\|u_{h,2}(t_n) - X_2^n\|_{L_2(\Omega, \dot{H}_h^0)} \leq C\tau^{\min(\frac{2}{3}(\beta-1), 1)}. \quad (4.28)$$

Proof. Let $\alpha \in [0, 1]$. By the stability of the semi-discrete mild solution (3.16) at discrete times t_n and the stability of the approximate solution (4.18), and by the triangle inequality, we have

$$\begin{aligned} \|U_h(t_n) - X^n\|_{L_2(\Omega, \mathcal{H}_h^\alpha)}^2 &\leq 2\|U_h(t_n)\|_{L_2(\Omega, \mathcal{H}_h^\alpha)}^2 + 2\|X^n\|_{L_2(\Omega, \mathcal{H}_h^\alpha)}^2 \\ &\leq C \left(\left(\|X^0\|_{L_2(\Omega, \mathcal{H}_h^\alpha)}^2 + 1 \right) + \|\Lambda^{(\alpha-1)/2} Q^{1/2}\|_{\text{HS}}^2 \right). \end{aligned} \quad (4.29)$$

Recall that the mild solution (3.15) is given at the discrete times $t_n = n\tau$ by

$$U_h(t_n) = E_h(\tau)U_h(t_{n-1}) + \int_{t_{n-1}}^{t_n} E_h(t_n - s)F(U_h(s))ds + \int_{t_{n-1}}^{t_n} E_h(t_n - s)BP_h dW(s).$$

Subtracting (4.2) from the above equation, and adding and subtracting $MU_h(t_{n-1})$, we get

$$\begin{aligned} U_h(t_n) - X^n &= (E_h(\tau) - M)U_h(t_{n-1}) + M(U_h(t_{n-1}) - X^{n-1}) \\ &\quad + \int_{t_{n-1}}^{t_n} E_h(t_n - s)P_h F(U_h(s))ds - \tau DP_h f(X_1^{n-1/2}) \\ &\quad + \int_{t_{n-1}}^{t_n} (E_h(t_n - s)B - D)P_h dW(s). \end{aligned}$$

Letting $\text{Err}_U^n = U_h(t_n) - X^n$, $\text{Err}_d^n = (E_h(\tau) - M)U_h(t_n)$, $\text{Err}_{\text{non}}^n = \int_{t_{n-1}}^{t_n} E_h(t_n - s)P_h F(U_h(s))ds - \tau DP_h f(X_1^{n-1/2})$, and $\text{Err}_s^n = \int_{t_{n-1}}^{t_n} (E_h(t_n - s)B - D)P_h dW(s)$, we have

$$\begin{aligned} \text{Err}_U^n &= \text{Err}_d^{n-1} + M\text{Err}_U^{n-1} + \text{Err}_{\text{non}}^n + \text{Err}_s^n \\ &= \sum_{j=1}^n M^{n-j}\text{Err}_d^{j-1} + \sum_{j=1}^n M^{n-j}\text{Err}_{\text{non}}^j + \sum_{j=1}^n M^{n-j}\text{Err}_s^j, \end{aligned}$$

since $\text{Err}_U^0 = U_{h,0} - X^0 = 0$, see (4.4). Using the discrete norm (4.7) and that $\mathbb{E} \left[\langle \text{Err}_s^j, \text{Err}_s^k \rangle_{m,\alpha} \right] = 0$, for $j \neq k$, we obtain

$$\begin{aligned} \mathbb{E}[\|\text{Err}_U^n\|_{m,\alpha}^2] &\leq 3n \sum_{j=1}^n \mathbb{E} \left[\left\| M^{n-j}\text{Err}_d^{j-1} \right\|_{m,\alpha}^2 \right] + 3n \sum_{j=1}^n \mathbb{E} \left[\left\| M^{n-j}\text{Err}_{\text{non}}^j \right\|_{m,\alpha}^2 \right] \\ &\quad + 3 \sum_{j=1}^n \mathbb{E} \left[\left\| M^{n-j}\text{Err}_s^j \right\|_{m,\alpha}^2 \right]. \end{aligned}$$

Employing (4.10) we have for the above equation

$$\mathbb{E}[\|\text{Err}_U^n\|_{m,\alpha}^2] \leq 3 \sum_{j=1}^n \left(n \mathbb{E} \left[\left\| \text{Err}_d^{j-1} \right\|_{m,\alpha}^2 \right] + n \left[\|\text{Err}_{\text{non}}^j\|_{m,\alpha}^2 \right] + \mathbb{E} \left[\|\text{Err}_s^j\|_{m,\alpha}^2 \right] \right).$$

By the norm equivalence (4.11), we have

$$\begin{aligned} \mathbb{E} \left[\left\| \|\text{Err}_U^n\|_{h,\alpha}^2 \right\| \right] &\leq C \sum_{j=1}^n \left(n \mathbb{E} \left[\left\| \|\text{Err}_d^{j-1}\|_{h,\alpha}^2 \right\| \right] + n \mathbb{E} \left[\left\| \|\text{Err}_{\text{non}}^j\|_{h,\alpha}^2 \right\| \right] + \mathbb{E} \left[\left\| \|\text{Err}_s^j\|_{h,\alpha}^2 \right\| \right] \right) \\ &= C (\text{Err}_1 + \text{Err}_2 + \text{Err}_3). \end{aligned} \tag{4.30}$$

Using the estimates (4.21) and (3.16) gives for Err_1

$$\text{Err}_1 \leq \tau^6 n \mathbb{E} \left[\left\| U_h(t_{j-1}) \right\|_{h,\alpha+3}^2 \right] \leq T \tau^5 \sup_{t_{j-1} \in [0,T]} \mathbb{E} \left[\left\| U_h(t_{j-1}) \right\|_{h,\alpha+3}^2 \right] \leq C \tau^5. \quad (4.31)$$

By the definition of the $\|\cdot\|_{h,\alpha}$ norm, we have for Err_2

$$\begin{aligned} \text{Err}_2 &= n \left(\mathbb{E} \left[\left\| \int_{t_{j-1}}^{t_j} \Lambda_h^{-1/2} S_h(t_j - s) P_h f(u_{h,1}(s)) ds - \frac{\tau^2}{2} P_h f(X_1^{j-1/2}) \right\|_{h,\alpha}^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\left\| \int_{t_{j-1}}^{t_j} C_h(t_j - s) P_h f(u_{h,1}(s)) ds - \tau P_h f(X_1^{j-1/2}) \right\|_{h,\alpha-1}^2 \right] \right) \\ &= n (\text{Err}_{[2,1]} + \text{Err}_{[2,2]}). \end{aligned}$$

Using the triangle inequality, Taylor's theorem for $\Lambda_h^{-1/2} S_h(t_j - s)$ up to first order and (2.7) for f , we obtain for $\text{Err}_{[2,1]}$

$$\begin{aligned} \text{Err}_{[2,1]} &\leq 2C \left(\tau \int_{t_{j-1}}^{t_j} |t_j - s|^2 \mathbb{E} [1 + \|u_{h,1}(s)\|_{h,\alpha}^2] ds + \frac{\tau^4}{4} \mathbb{E} \left[1 + \left\| X_1^{j-1/2} \right\|_{h,\alpha}^2 \right] \right) \\ &\leq 2C \left(\tau \int_{t_{j-1}}^{t_j} |t_j - s|^2 \sup_{t \in [0,T]} \mathbb{E} [1 + \|u_{h,1}(t)\|_{h,\alpha}^2] ds \right. \\ &\quad \left. + \frac{\tau^4}{4} \mathbb{E} \left[1 + \left\| X_1^{j-1/2} \right\|_{h,\alpha}^2 \right] \right) \\ &\leq C \tau^4 \left(\mathbb{E} \left[1 + \left\| X^0 \right\|_{h,\alpha}^2 \right] + \|\Lambda^{(\alpha-1)/2} Q^{1/2}\|_{\text{HS}}^2 \right), \end{aligned}$$

by estimates (3.16) and (4.18).

Adding and subtracting $P_h f(u_{h,1}(s)) ds$ and using the triangle inequality gives

$$\begin{aligned} \text{Err}_{[2,2]} &\leq 2 \left(\mathbb{E} \left[\left\| \int_{t_{j-1}}^{t_j} (C_h(t_j - s) - I) P_h f(u_{h,1}(s)) ds \right\|_{h,\alpha-1}^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\left\| \int_{t_{j-1}}^{t_j} P_h f(u_{h,1}(s)) ds - \tau P_h f(X_1^{j-1/2}) \right\|_{h,\alpha-1}^2 \right] \right) \\ &= 2 (\text{Err}_{[2,2]}^1 + \text{Err}_{[2,2]}^2). \end{aligned}$$

Using Taylor's theorem for $|\cos((t_j - s)\sqrt{\lambda_{h,j}}) - 1| \leq (t_j - s)\sqrt{\lambda_{h,j}}$, (2.7) for f and the stability estimate (3.16), we get for $\text{Err}_{[2,2]}^1$

$$\text{Err}_{[2,2]}^1 \leq \tau \int_{t_{j-1}}^{t_j} |t_j - s|^2 ds \sup_{t \in [0,T]} \mathbb{E} [1 + \|u_{h,1}(t)\|_{h,\alpha}^2] \leq C \tau^4.$$

Adding and subtracting $\int_{t_{j-1}}^{t_j} P_h f(u_{h,1}(t_{j-1}) + \frac{\tau}{2}u_{h,2}(t_{j-1}))ds$, we obtain for $\text{Err}_{[2,2]}^2$

$$\begin{aligned} \text{Err}_{[2,2]}^2 &\leq 2\mathbb{E} \left[\tau \int_{t_{j-1}}^{t_j} \left\| P_h \left(f(u_{h,1}(s)) - f \left(u_{h,1}(t_{j-1}) + \frac{\tau}{2}u_{h,2}(t_{j-1}) \right) \right) \right\|_{h,\alpha-1}^2 ds \right] \\ &\quad + 2\mathbb{E} \left[\left\| \tau P_h f \left(u_{h,1}(t_{j-1}) + \frac{\tau}{2}u_{h,2}(t_{j-1}) \right) - \tau P_h f(X_1^{j-1/2}) \right\|_{h,\alpha-1}^2 \right]. \end{aligned}$$

Applying (2.7) for both terms above and the fact that $\|\Lambda_h^{(\alpha-1)/2}u\| \leq \|u\|$, $\alpha \in [0, 1]$ for the first term, gives

$$\begin{aligned} \text{Err}_{[2,2]}^2 &\leq 4C\mathbb{E} \left[\tau \int_{t_{j-1}}^{t_j} \left(\|u_{h,1}(s) - u_{h,1}(t_{j-1})\|^2 + \left\| \frac{\tau}{2}u_{h,2}(t_{j-1}) \right\|^2 \right) ds \right] \\ &\quad + 4C\tau^2\mathbb{E} \left[\left\| u_{h,1}(t_{j-1}) - X_1^{j-1} \right\|_{h,\alpha-1}^2 + \left\| \frac{\tau}{2}u_{h,2}(t_{j-1}) - \frac{\tau}{2}X_2^{j-1} \right\|_{h,\alpha-1}^2 \right]. \end{aligned}$$

By Hölder's continuity (4.20) for the first term in the above inequality, we have

$$\begin{aligned} \text{Err}_{[2,2]}^2 &\leq 4C\tau \int_{t_{j-1}}^{t_j} |s - t_{j-1}|^2 ds + 4C\frac{\tau^3}{4} \int_{t_{j-1}}^{t_j} \sup_{t \in [0, T]} \mathbb{E} \left[\|u_{h,2}(t)\|_{h,\alpha-1}^2 \right] ds \\ &\quad + 4C\tau^2\mathbb{E} \left[\left\| u_{h,1}(t_{j-1}) - X_1^{j-1} \right\|_{h,\alpha-1}^2 + \frac{\tau^2}{4} \left\| u_{h,2}(t_{j-1}) - X_2^{j-1} \right\|_{h,\alpha-1}^2 \right]. \end{aligned}$$

Using the stability estimate (3.16) for the second term, the inverse estimate (3.11) together with the CFL condition (4.5) for the third term, we get for $\text{Err}_{[2,2]}^2$

$$\begin{aligned} \text{Err}_{[2,2]}^2 &\leq C\tau^4 + 4\tau^2\mathbb{E} \left[\left\| u_{h,1}(t_{j-1}) - X_1^{j-1} \right\|_{h,\alpha-1}^2 + \left\| u_{h,2}(t_{j-1}) - X_2^{j-1} \right\|_{h,\alpha-2}^2 \right] \\ &= C\tau^4 + 4C\tau^2\mathbb{E} \left[\left\| \text{Err}_U^{j-1} \right\|_{h,\alpha}^2 \right]. \end{aligned}$$

Combining the estimates for $\text{Err}_{[2,2]}^1$ and $\text{Err}_{[2,2]}^2$, we have for $\text{Err}_{[2,2]}$

$$\text{Err}_{[2,2]} \leq C\tau^4 + 4C\tau^2\mathbb{E} \left[\left\| \text{Err}_U^{j-1} \right\|_{h,\alpha}^2 \right].$$

Then combining this and the estimate for $\text{Err}_{[2,1]}$, we obtain for Err_2

$$\begin{aligned} \text{Err}_2 &\leq C\tau^3 \left(\mathbb{E} \left[1 + \left\| X^0 \right\|_{h,\alpha}^2 \right] + \|\Lambda^{(\alpha-1)/2}Q^{1/2}\|_{\text{HS}}^2 \right. \\ &\quad \left. + 4\tau \sum_{j=1}^n \mathbb{E} \left[\left\| \text{Err}_U^{j-1} \right\|_{h,\alpha}^2 \right] \right). \end{aligned} \tag{4.32}$$

By Itô's isometry (2.2) we have for Err_3

$$\begin{aligned}\text{Err}_3 &= \int_{t_{j-1}}^{t_j} \|\Lambda_h^{\alpha/2}(\Lambda_h^{-1/2}S_h(t_j - s) - \frac{\tau}{2}I)P_hQ^{1/2}\|_{\text{HS}}^2 ds \\ &\quad + \int_{t_{j-1}}^{t_j} \|\Lambda_h^{(\alpha-1)/2}(C_h(t_j - s) - I)P_hQ^{1/2}\|_{\text{HS}}^2 ds.\end{aligned}$$

Using triangle inequality for the first term, and Taylor's theorem for $\Lambda_h^{-1/2}S_h(t_j - s)$ and $C_h(t_j - s)$ up to first order, we get

$$\begin{aligned}\text{Err}_3 &\leq 3 \int_{t_{j-1}}^{t_j} |t_j - s|^2 \|\Lambda_h^{\alpha/2}P_hQ^{1/2}\|_{\text{HS}}^2 ds + 2\frac{\tau^3}{4} \|\Lambda_h^{\alpha/2}P_hQ^{1/2}\|_{\text{HS}} \\ &\leq \tau^3 \|\Lambda_h^{\alpha/2}P_hQ^{1/2}\|_{\text{HS}}^2.\end{aligned}$$

By estimates (4.31), (4.32) and the above estimate for Err_3 , we have for (4.30)

$$\begin{aligned}\mathbb{E} \left[\|\text{Err}_U^n\|_{h,\alpha}^2 \right] &\leq C \left(\tau^4 + \tau^2 \mathbb{E} \left[1 + \|X^0\|_{h,\alpha}^2 \right] \right. \\ &\quad \left. + \tau^2 \|\Lambda^{\alpha/2}Q^{1/2}\|_{\text{HS}}^2 + \tau \sum_{j=1}^n \mathbb{E} \left[\|\text{Err}_U^{j-1}\|_{h,\alpha}^2 \right] \right).\end{aligned}$$

By applying the discrete Gronwall's inequality to the above, we get

$$\mathbb{E} \left[\|\text{Err}_U^n\|_{h,\alpha}^2 \right] \leq C \left(\tau^4 + \tau^2 \mathbb{E} \left[1 + \|X^0\|_{h,\alpha}^2 \right] + \tau^2 \|\Lambda^{\alpha/2}Q^{1/2}\|_{\text{HS}}^2 \right).$$

By interpolation between (4.29) and the above estimate and taking square roots, we have for $\beta \geq \alpha$

$$\begin{aligned}\|\text{Err}_U^n\|_{L_2(\Omega, \mathcal{H}_h^\alpha)} &\leq C \left(\tau^{\frac{2}{3}(\beta-\alpha)} (\|X^0\|_{L_2(\Omega, \mathcal{H}_h^\beta)} + 1) \right. \\ &\quad \left. + \tau^{\min(\beta-\alpha, 1)} \|\Lambda_h^{(\beta-1)/2}Q^{1/2}\|_{\text{HS}} \right).\end{aligned}$$

The proof of estimate (4.27) follows from setting $\alpha = 0$ and the proof of estimate (4.28) follows from setting $\alpha = 1$ in the above bound. \square

We now state the strong convergence rates for the fully discrete stochastic wave equation (1.1).

Theorem 4.9. *Let $U = [u_1, u_2]^T$ and $X = [X_1^n, X_2^n]^T$ be given by (2.8) and (4.4), respectively. Also let the assumptions of Theorems 3.9 and 4.8 be fulfilled. Then the following estimates hold, under the CFL condition (4.5), at discrete times $t_n \in [0, T]$*

$$\|u_1(t_n) - X_1^n\|_{L_2(\Omega, \dot{H}^0)} \leq C(\tau^{\min(\frac{2}{3}\beta, 1)} + h^{\min(\frac{p+1}{p+2}\beta, p+1)}), \quad \beta \geq 0,$$

$$\|u_2(t_n) - X_2^n\|_{L_2(\Omega, \dot{H}^0)} \leq C(\tau^{\min(\frac{2}{3}(\beta-1), 1)} + h^{\min(\frac{p+1}{p+2}(\beta-1), p+1)}), \quad \beta \geq 1.$$

Proof. The proof follows from Theorems 3.9 and 4.8 and triangle inequality. \square

5 Energy conservation

In this section, we state bounds for the energy (or Hamiltonian) of the fully discrete stochastic wave equation (3.12) and the linear analogue of it. We consider a trace-class Q -Wiener process, i.e., $\text{Tr}(Q) = \|Q^{1/2}\|_{\text{HS}}^2 < \infty$ and the nonlinearity $f(u) = -V'(u)$ for a smooth potential V . The ‘‘Hamiltonian’’ function for the dG approximation U_h is defined on $\mathcal{H}_h^1 = \dot{H}_h^1 \times \dot{H}_h^0$ as

$$H(U_h) = \frac{1}{2}\|u_{h,1}\|_*^2 + \frac{1}{2}\|u_{h,2}\|^2 + \int_{\mathcal{D}} V(u_{h,1})dx,$$

where the broken norm $\|\cdot\|_*$ is defined in (3.5). In the following proposition we state the trace formula for the dG semi-discrete exact solution (3.15).

Proposition 5.1. *Let $f(u) = -V'(u)$ for a smooth potential $V: \mathbb{R} \rightarrow \mathbb{R}$, $\text{Tr}(Q) < \infty$, and the Hamiltonian H be defined as above. Then, the dG approximation to the stochastic wave equation (1.1), $U_h(t)$ in (3.15), satisfies the trace formula*

$$\mathbb{E}[H(U_h(t))] = \mathbb{E}[H(U_{h,0})] + \frac{1}{2}t\text{Tr}(P_hQP_h), \quad t \in [0, T]. \quad (5.1)$$

Proof. The proof follows from Proposition 5 in [1] by taking into account that $\|u\|_* = \|\Lambda_h^{1/2}u\|$. \square

We now derive a bound for the Hamiltonian of the temporal approximation to the nonlinear stochastic wave equation.

Theorem 5.2. *Let f and W be as in Proposition 5.1. Also let the assumptions in Lemma 4.5 be fulfilled for $\beta = 1$. The numerical approximation of (3.12) by the stochastic position Verlet method (4.4) satisfies, under the CFL condition (4.5) the following bound for the Hamiltonian H*

$$\mathbb{E}[H(X^n)] \leq \mathbb{E}[H(X^0)] + C \exp(2Ct_n)(t_n\|Q^{1/2}\|_{\text{HS}}^2 + t_n^2), \quad (5.2)$$

for $0 \leq t_n \leq T$ and \hat{C} independent of h, τ , and T .

Proof. We first note that

$$\begin{aligned} \mathbb{E}[H(X^n)] - \mathbb{E}[H(X^0)] &= \frac{1}{2}\mathbb{E}\left[\|X^n\|_{h,1}^2\right] - \frac{1}{2}\mathbb{E}\left[\|X^0\|_{h,1}^2\right] \\ &\quad + \mathbb{E}\left[\int_{\mathcal{D}} (V(X_1^n) - V(X_1^0))dx\right]. \end{aligned} \quad (5.3)$$

The first term in the above inequality is bounded from estimate (4.18) for $\beta = 1$

$$\mathbb{E}\left[\|X^n\|_{h,1}^2\right] \leq C \exp(2Ct_n) \left(t_n\|Q^{1/2}\|_{\text{HS}}^2 + t_n^2\right).$$

For the third term in (5.3), using the mean value theorem we obtain

$$\begin{aligned}\mathbb{E} [\|V(X_1^n) - V(X_1^0)\|_{L_1(\mathcal{D})}] &\leq \mathbb{E} [\|V(X_1^n) - V(X_1^0)\|_{L_2(\mathcal{D})}] \\ &\leq C \left(\mathbb{E} [\|V'(\xi)(X_1^n - X_1^0)\|_{L_2(\mathcal{D})}^2] \right)^{1/2}.\end{aligned}$$

Recalling that $V'(u) = -f(u)$, we have by (2.7) $\|V'(\xi)\| = \|f(\xi)\| \leq C(1 + \|\xi\|)$. Since $\xi \in (X^0, X^n)$, we have by triangle inequality and estimate (4.18) for $\beta = 0$

$$\mathbb{E} [\|V(X_1^n) - V(X_1^0)\|_{L_1(\mathcal{D})}] \leq C \exp(2Ct_n) \left(t_n \|\Lambda^{-1/2} Q^{1/2}\|_{\text{HS}}^2 + t_n^2 \right).$$

The above completes the proof of (5.2). \square

In the case of the linear stochastic wave equation (4.14), the discrete energy of the temporal approximation is given by,

$$\mathcal{E}^n = \frac{1}{2} \|\Lambda_h^{1/2} Y_1^n\|^2 + \frac{1}{2} \|Y_2^n\|^2$$

where we recall that $Y^n = [Y_1^n, Y_2^n]^T$ is the numerical approximation in (4.15). We also introduce a so-called modified energy

$$\mathcal{E}_m^n = \frac{1}{2} \left\| \left(I - \frac{\tau^2}{4} \Lambda_h \right)^{-1/2} \Lambda_h^{1/2} Y_1^n \right\|^2 + \frac{1}{2} \|Y_2^n\|^2 = \|Y^n\|_{m,1}^2.$$

Theorem 5.1 in [9] proves that the expected value of the energy of the exact semi-discrete solution to the linear stochastic equation grows linearly with time t . We observe that the SVM applied to this problem preserves the linear growth of the expected value of the modified energy \mathcal{E}_m^n with the time t .

Theorem 5.3. *Let the assumptions in Lemma 4.5 be fulfilled for $\beta = 1$. Then under the CFL condition (4.5), the expected value of the modified energy \mathcal{E}_m^n satisfies*

$$\mathbb{E} [\mathcal{E}_m^n] = \mathbb{E} [\|Y^0\|_{m,1}^2] + t_n \sum_{k=0}^{\infty} \left\| DP_h Q^{1/2} e_k \right\|_{m,1}^2.$$

Further, the expected value of the discrete energy \mathcal{E}^n is bounded, at $t_n = n\tau$, by

$$\mathbb{E} [\mathcal{E}^n] \leq C \left(\mathbb{E} [\|Y^0\|_{h,1}^2] + t_n \|Q^{1/2}\|_{\text{HS}}^2 \right),$$

where C is a constant independent of h , τ and n .

Proof. We have for (4.15) by using the Itô's isometry (2.2) and (4.10)

$$\begin{aligned}\mathbb{E} [\|Y^n\|_{m,1}^2] &= \mathbb{E} [\|M^n Y^0\|_{m,1}^2] + \mathbb{E} \left[\left\| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} M^{n-j} DP_h dW(s) \right\|_{m,1}^2 \right] \\ &= \mathbb{E} [\|Y^0\|_{m,1}^2] + t_n \sum_{k=0}^{\infty} \left\| DP_h Q^{1/2} e_k \right\|_{m,1}^2.\end{aligned}$$

This completes the proof for the bound on \mathcal{E}_m^n .

The proof for the bound on \mathcal{E}^n follows from estimate (4.17) for $\beta = 1$. \square

6 Numerical experiments

We consider the following 1-dimensional Sine-Gordon equation

$$\begin{aligned} d\dot{u} &= \Delta u dt - \sin(u)dt + dW && \text{in } \mathcal{D} \times [0, 1], \\ u &= 0 && \text{on } \partial\mathcal{D} \times [0, 1], \\ u(\cdot, 0) &= 0, \quad \dot{u}(\cdot, 0) = \sin(\pi x) && \text{in } \mathcal{D}, \end{aligned} \tag{6.1}$$

where $\mathcal{D} = (0, 1)$. We approximate the solution of (6.1) with the dG finite element method (3.12) in space. Letting X^n be the numerical approximation of (3.12) at discrete times $t_n = \tau n$, $n = 1, \dots, N$, we consider the following integrators:

- i. The stochastic trigonometric method (STM), see [9],

$$X^n = E_h(\tau)X^{n-1} + E_h(\tau)BP_h f(X_1^{n-1})\tau + E_h(\tau)BP_h\Delta W^n,$$

where $E_h(\tau)$ is the C_0 -semigroup defined in (3.14) and $B = [0 \ I]^T$. The strong convergence rates for the full discretization are $\mathcal{O}(\tau^{\min(\beta, 1)} + h^{\frac{2}{3}\beta})$, see [1, Theorem 4].

- ii. The semi-implicit Euler-Maruyama method (SEM)

$$X^n = X^{n-1} + \tau A_h X^n + BP_h f(X_1^{n-1})\tau + BP_h\Delta W^n.$$

We refer to [25] for the mean-square errors of this scheme applied to stochastic parabolic partial differential equations.

- iii. The stochastic position Verlet method (SVM) considered here.

For our numerical experiments, we set $Q = \Lambda^{-s}$, $s \in \mathbb{R}$, then using the asymptotic behaviour of the eigenvalues of Λ , $\lambda_j \sim j^{2/d}$, where d is the dimension of the domain \mathcal{D} , see [43], we get

$$\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\text{HS}}^2 = \sum_{j=1}^{\infty} \lambda_j^{(\beta-1-s)} \approx \sum_{j=1}^{\infty} j^{\frac{2}{d}(\beta-1-s)}.$$

The above series converges if and only if $\beta < 1 + s - d/2$.

We examine simultaneously the spatial error and the temporal error for the displacement for $s = 0$, which corresponds to space-time white noise, and $s = 1$, which corresponds to correlated noise. We choose different timesteps $\tau_i = 2^{-i}$, $i = 2, \dots, 8$, and different mesh sizes $h_i = 2^{-i}$, $i = 4, \dots, 10$. Furthermore, we take the exact solution to be the dG approximation on a fine mesh with mesh size $h_{\text{exact}} = 2^{-10}$ in space and the time integration is done by STM with time step $\tau_{\text{exact}} = 2^{-12}$. We consider $M = 100$ realisations to compute the expected values. Figures 1 display the strong convergence rates and the efficiency of the

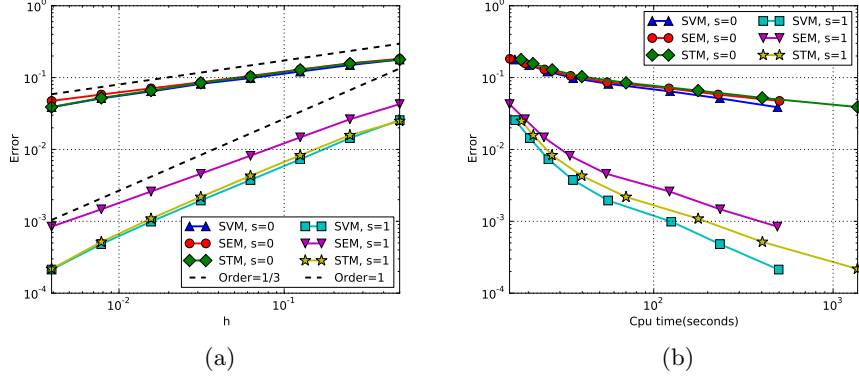


Figure 1: Plots (a) exhibits the spatial rates of convergence of the stochastic position Verlet method (SVM), the stochastic trigonometric method (STM), and the semi-implicit Euler-Maruyama method (SEM) and (b) the efficiency of these time integrators for space-time white noise, i.e., $Q = \Lambda^{-s}$, $s = 0$, and for correlated noise, i.e., $Q = \Lambda^{-s}$, $s = 1$.

above numerical schemes. The spatial mean-square error is defined at final time $T = 1$ as

$$\text{Error} = \left(\frac{1}{M} \sum_{m=1}^M \|u_{h,m}(\cdot, 1) - u_m^{\text{ref}}(\cdot, 1)\|^2 \right)^{1/2} \approx (\mathbb{E} [\|u_h(\cdot, 1) - u^{\text{ref}}(\cdot, 1)\|^2])^{1/2}.$$

We only present the strong numerical error vs. the mesh size h since the spatial convergence rates dominate the convergence rates of the temporal discretization see Theorem 4.9. We observe that the expected convergence rates are confirmed for SVM and STM for both space-time white noise and correlated noise. Further, SVM is the most efficient of the three methods.

In addition, we investigate the spatial error for polynomials of total degree equal to $p = 2$. Recalling estimate (3.32), we have that the convergence rate is $\mathcal{O}(h^{\frac{3}{4}\beta})$ for the displacement. We consider correlated noise by setting $Q = \Lambda^{-1}$. We choose different mesh sizes $h_i = 2^{-i}$, $i = 2, \dots, 7$. We take the exact solution to be the dG approximation on a fine mesh with mesh size $h_{\text{exact}} = 2^{-9}$ in space and the time integration is done by the stochastic Verlet scheme with time step $\tau_{\text{exact}} = 2^{-11}$. Figure 2 exhibits the spatial rate of convergence and the efficiency for polynomials of total degree equal to $p = 1, 2$. Again, we consider $M = 100$ realisations to calculate the expected values. We observe that the second-order polynomials are more accurate and efficient when used for the linear problem.

Finally, we are concerned with the energy results given in Section 5. In order to illustrate the results from Section 5, we set $Q = \Lambda^{-s}$, $s = 1$, and we choose $h = 2^{-5}$. For the stochastic Verlet method we choose the timestep $\tau = 1/6400$. For STM and SEM we choose timestep $\tau = 1/500$. Figure 3 displays the expected value of the Hamiltonian along the numerical solutions

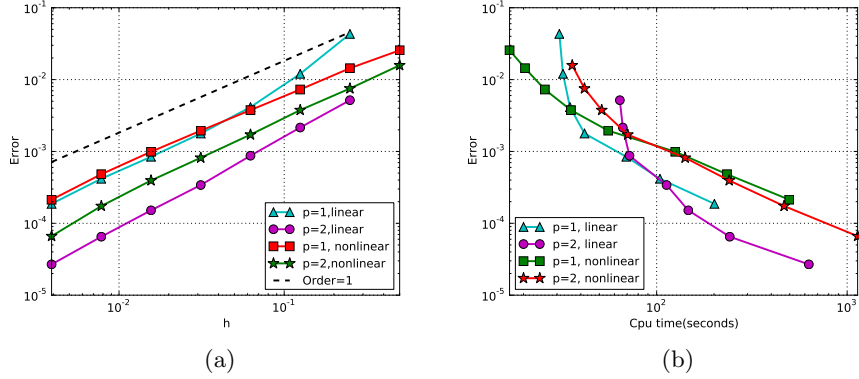


Figure 2: Plots (a) show the strong convergence rates in space and the efficiency for first-order degree polynomials ($p=1$) and (b) second-order degree polynomials ($p=2$) for (6.1) and the linear analogue of it.

of (6.1) and the linear analogue of it over the time interval $[0, 50]$. Further, we take $M = 8000$ samples to approximate the expected values of the energy of the schemes. We observe that SVM reproduces the linear growth of the exact energy, although this is not expected from Theorems 5.2 and 5.3. In case of the linear stochastic wave equation, STM preserves the linear growth of the expected value of the energy, see [9], as Figure 3 (a) verifies. The unsatisfactory behaviour of SEM has also been previously observed when applied the nonlinear stochastic wave equation [1] and the linear analogue of it [9].

All the numerical experiments were performed in Python using the finite element software library Fenics [35].

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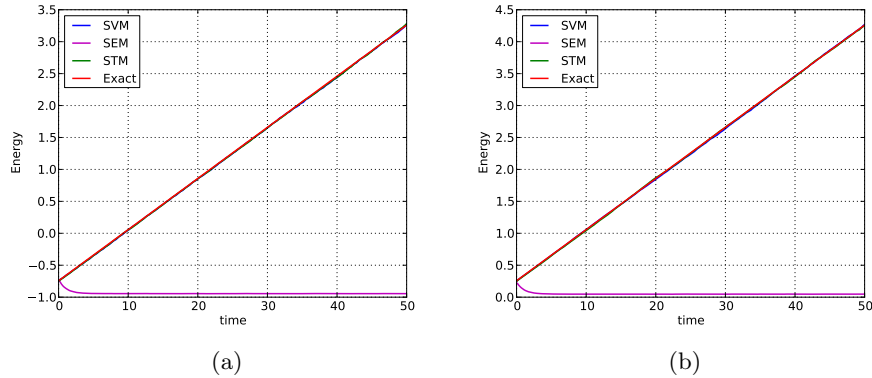


Figure 3: Plot (a) shows the expected values of the Hamiltonian H of (6.1) along the numerical solutions given by the three time stepping schemes presented above. Plot (b) illustrates the expected values of the discrete energy \mathcal{E}^n of the linear analogue of problem (6.1).

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