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*Research article*

## A maximum-principle approach to the minimisation of a nonlocal dislocation energy<sup>†</sup>

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**Abstract:** In this paper we use an approach based on the maximum principle to characterise the minimiser of a family of nonlocal and anisotropic energies  $I_\alpha$  defined on probability measures in  $\mathbb{R}^2$ . The purely nonlocal term in  $I_\alpha$  is of convolution type, and is isotropic for  $\alpha = 0$  and anisotropic otherwise. The cases  $\alpha = 0$  and  $\alpha = 1$  are special: The first corresponds to Coulombic interactions, and the latter to dislocations. The minimisers of  $I_\alpha$  have been characterised by the same authors in an earlier paper, by exploiting some formal similarities with the Euler equation, and by means of complex-analysis techniques. We here propose a different approach, that we believe can be applied to more general energies.

**Keywords:** nonlocal interaction; potential theory; maximum principle; dislocations; Kirchhoff ellipses

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### 1. Introduction

We consider the family of nonlocal energies

$$I_\alpha(\mu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W_\alpha(x - y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^2} |x|^2 d\mu(x), \quad (1.1)$$

defined on probability measures  $\mu \in \mathcal{P}(\mathbb{R}^2)$ , where the interaction potential  $W_\alpha$  is given by

$$W_\alpha(x) = W_0(x) + \alpha \frac{x_1^2}{|x|^2}, \quad W_0(x) = -\log|x|, \quad (1.2)$$

$x = (x_1, x_2) \in \mathbb{R}^2$ , and  $\alpha \in (-1, 1)$ . The case  $\alpha = 0$  is very classical, and has been studied in a variety of contexts, from random matrices to Coulomb gases, from orthogonal polynomials to Fekete sets in interpolation theory, and for a variety of confining potentials (see, e.g., [11, 15], and the references therein). We note that this is a very special case, as it is the only one for which the energy in (1.1) is isotropic: The potential  $W_0$  is indeed radial, while  $W_\alpha$  is anisotropic whenever  $\alpha \neq 0$ .

Generally speaking, radially of the interactions is a key assumption in most of the mathematical literature on nonlocal energies (see, e.g., [1–6, 10, 17]), and the explicit characterisation, or the derivation of some geometric properties of energy minimisers, has only been done under this assumption. These problems are therefore more challenging in the case of anisotropic interactions, as it is the case of (1.1).

The anisotropic energy (1.1) has been studied in [14] in the case  $\alpha = 1$ , which corresponds to interacting defects in metals, and in [7, 16] for any  $\alpha \in [-1, 1]$ . The main result in these works is the characterisation of the minimiser  $\mu_\alpha$  of  $I_\alpha$ :  $\mu_\alpha \in \mathcal{P}(\mathbb{R}^2)$  is unique, and for  $\alpha \in (-1, 1)$  is of the form

$$\mu_\alpha := \frac{1}{|\Omega_\alpha|} \chi_{\Omega_\alpha}, \quad \Omega_\alpha = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{1-\alpha} + \frac{x_2^2}{1+\alpha} < 1 \right\}. \quad (1.3)$$

More precisely, the minimiser of  $I_\alpha$  is the (normalised) characteristic function of an ellipse for  $\alpha \in (-1, 1)$ , and it converges to a singular, one-dimensional measure (the semi-circle law) for  $\alpha \rightarrow \pm 1$ . This result has been proved in [7] by means of complex-analysis techniques, and in [16] via a more direct proof, based on the explicit computation of the potential  $W_\alpha * \mu_\alpha$  in  $\mathbb{R}^2$ . An extension to the  $n$ -dimensional case has been proved in [8].

In this paper we propose an alternative proof of the characterisation of the minimiser of  $I_\alpha$ , based on a maximum principle for biharmonic functions. We here explain the main idea behind this new approach.

Since the energy  $I_\alpha$  can be shown to be strictly convex on a class of measures that is relevant for the minimisation, the (unique) minimiser of  $I_\alpha$  is completely characterised by two conditions, called the Euler-Lagrange conditions. Namely the minimality of the measure  $\mu_\alpha$  in (1.3) for  $I_\alpha$  is equivalent to

$$(W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} = C_\alpha \quad \text{for every } x \in \Omega_\alpha, \quad (1.4)$$

$$(W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} \geq C_\alpha \quad \text{for every } x \in \mathbb{R}^2, \quad (1.5)$$

for some constant  $C_\alpha > 0$ . Conditions (1.4)–(1.5) essentially say that the function  $f_\alpha$  defined as  $f_\alpha(x) := (W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2}$  is ‘minimal’ on  $\Omega_\alpha$ . So, intuitively, if  $f_\alpha$  were harmonic outside  $\Omega_\alpha$  and satisfied the stationarity condition (1.4), then (1.5) would follow from the maximum principle for harmonic functions applied in (a bounded subset of)  $\mathbb{R}^2 \setminus \Omega_\alpha$ , since  $f_\alpha$  blows up at infinity. The function  $f_\alpha$ , however, is not harmonic outside  $\Omega_\alpha$ , and therefore this heuristic argument cannot be applied directly. It is in fact biharmonic, which is an obstacle in the application of the maximum principle.

The idea is then to construct an auxiliary function  $g_\alpha$ , harmonic outside  $\Omega_\alpha$ , and to do so in such a careful and clever way that the application of the standard maximum principle for harmonic functions to  $g_\alpha$  gives, as a welcomed byproduct, the unilateral condition (1.5) for  $f_\alpha$ . The idea for this construction is taken from the work [9], where the author formulates several variants of the maximum principle that are valid for biharmonic functions.

## 2. Characterisation of the minimiser of $I_\alpha$ via the maximum principle

We recall that, as proved in [14, Section 2] and [7, Proposition 2.1],  $I_\alpha$  is strictly convex on the class of measures with compact support and finite interaction energy for  $\alpha \in [-1, 1]$ , and hence has a unique minimiser in  $\mathcal{P}(\mathbb{R}^2)$ . Moreover, the minimiser has compact support and finite energy.

We now characterise the minimiser of the energy, for  $\alpha \in (0, 1)$ . Note that considering only positive values of  $\alpha$  is not restrictive, since changing sign to  $\alpha$  corresponds to swapping  $x_1$  and  $x_2$  (up to a constant in the energy), due to the zero-homogeneity of the energy. Hence the minimiser of  $I_{\bar{\alpha}}$  for  $\bar{\alpha} \in (-1, 0)$  can be obtained from the minimiser of  $I_{-\bar{\alpha}}$  by means of a rotation of  $\frac{\pi}{2}$ .

**Theorem 2.1.** *Let  $0 \leq \alpha < 1$ . The measure*

$$\mu_\alpha := \frac{1}{\sqrt{1-\alpha^2}\pi} \chi_{\Omega(\sqrt{1-\alpha}, \sqrt{1+\alpha})}, \tag{2.1}$$

where

$$\Omega(\sqrt{1-\alpha}, \sqrt{1+\alpha}) := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{1-\alpha} + \frac{x_2^2}{1+\alpha} < 1 \right\},$$

is the unique minimiser of the functional  $I_\alpha$  among probability measures  $\mathcal{P}(\mathbb{R}^2)$ , and satisfies the Euler-Lagrange conditions

$$(W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} = C_\alpha \quad \text{for every } x \in \Omega(\sqrt{1-\alpha}, \sqrt{1+\alpha}), \tag{2.2}$$

$$(W_\alpha * \mu_\alpha)(x) + \frac{|x|^2}{2} \geq C_\alpha \quad \text{for every } x \in \mathbb{R}^2, \tag{2.3}$$

with

$$C_\alpha = I_\alpha(\mu_\alpha) - \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 d\mu_\alpha(x) = \frac{1}{2} - \log\left(\frac{\sqrt{1-\alpha} + \sqrt{1+\alpha}}{2}\right) + \alpha \frac{\sqrt{1-\alpha}}{\sqrt{1-\alpha} + \sqrt{1+\alpha}}.$$

**Remark 2.2.** The Euler-Lagrange conditions (2.2)–(2.3) are in general only a necessary condition for minimality (see [15, Theorem 3.1], [14]), namely any minimiser  $\mu$  of  $I_\alpha$  must satisfy them. Due to strict convexity of the energy  $I_\alpha$ , they are also sufficient in our case. In other words, they are in fact equivalent to minimality for  $\alpha \in (-1, 1)$ .

Our new proof consists of two main steps. In Section 2.1 we focus on (2.2): We first compute *explicitly* the convolution of the potential  $W_\alpha$  with the characteristic function of a general ellipse on points within the ellipse. Then, we use this explicit expression to show that there exists a unique ellipse for which (2.2) is satisfied. In Section 2.2 we show that the unique ellipse satisfying condition (2.2) also satisfies (2.3), and consequently is the only minimiser of the energy  $I_\alpha$ . The approach we use to prove that (2.3) is satisfied is based on the maximum principle.

### 2.1. The Euler-Lagrange condition (2.2)

We start by fixing some notation. For  $0 < a < b$  we denote with

$$\Omega(a, b) := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1 \right\}$$

the domain enclosed by an ellipse of semi-axes  $a$  and  $b$ . We also set

$$\mu_{a,b} := \frac{1}{\pi ab} \chi_{\Omega(a,b)} \quad (2.4)$$

for the normalised characteristic function of the ellipse. We observe that, since we focus on the case  $\alpha > 0$ , it is sufficient to consider  $a < b$ ; the case  $a > b$  corresponds to  $\alpha < 0$  and is completely analogous.

#### 2.1.1. The potential inside an ellipse

In this section we compute the potential  $(W_\alpha * \mu_{a,b})(x)$ , for  $x \in \Omega(a, b)$ . We write

$$(W_\alpha * \mu_{a,b})(x) = \Phi_{a,b}(x) + \alpha \Psi_{a,b}(x), \quad \text{with } \Psi_{a,b}(x) := \int_{\Omega(a,b)} \frac{(x_1 - y_1)^2}{|x - y|^2} dy. \quad (2.5)$$

The *explicit* expression of  $\Phi_{a,b} = W_0 * \mu_{a,b}$ , namely of the logarithmic potential for any ellipse  $\Omega(a, b)$ , is well-known in the whole of  $\mathbb{R}^2$  (see, e.g., [12], [13, Section 159]) and is given by

$$\Phi_{a,b}(x) = \begin{cases} -\frac{bx_1^2 + ax_2^2}{ab(a+b)} - \log\left(\frac{a+b}{2}\right) + \frac{1}{2} & \text{if } x \in \Omega(a, b), \\ -\xi - \frac{1}{2}e^{-2\xi} \cos(2\eta) - \log\frac{c}{2} & \text{if } x \in \mathbb{R}^2 \setminus \Omega(a, b), \end{cases} \quad (2.6)$$

where, for  $a < b$ ,  $c = \sqrt{b^2 - a^2}$  and

$$\begin{cases} x_1 = c \sinh \xi \sin \eta \\ x_2 = c \cosh \xi \cos \eta \end{cases} \quad \text{with } \xi > 0, \quad 0 \leq \eta < 2\pi.$$

We now focus on the computation of the function  $\Psi_{a,b}$  defined in (2.5), namely of the convolution of the anisotropic term of  $W_\alpha$  with  $\mu_{a,b}$ . We write

$$\Psi_{a,b} = H * \mu_{a,b}, \quad \text{with } H(x) := \frac{x_1^2}{|x|^2}.$$

It is easy to see that, since  $\partial_{x_1}(x_1 W_0) = W_0 - H$ ,

$$\Delta H = -2\partial_{x_1}^2 W_0 - 2\pi\delta_0 \quad (2.7)$$

in the sense of distributions. Hence, by (2.4), (2.6) and (2.7) we deduce that

$$\Delta \Psi_{a,b}(x) = -2\partial_{x_1}^2 \Phi_{a,b}(x) - \frac{2}{ab} = \frac{2(b-a)}{ab(a+b)} \quad \text{for } x \in \Omega(a, b), \quad (2.8)$$

namely the Laplacian of  $\Psi_{a,b}$  is constant in  $\Omega(a, b)$ . We now compute  $\nabla\Psi_{a,b}$  on  $\partial\Omega(a, b)$ . The idea is to derive an overdetermined boundary value problem satisfied by  $\Psi_{a,b}$  (namely the elliptic equation (2.8) in  $\Omega(a, b)$  coupled with the value of the gradient on  $\partial\Omega(a, b)$ ); at that point, if we can guess a solution of the boundary value problem, by unique continuation, we can then determine the potential  $\Psi_{a,b}$  in  $\Omega(a, b)$ , up to a constant.

To this aim, we compute the gradient of  $\Psi_{a,b}$  on  $\partial\Omega(a, b)$ . Let  $x \in \partial\Omega(a, b)$ ; integration by parts gives

$$\nabla\Psi_{a,b}(x) = \frac{1}{\pi ab} \int_{\Omega(a,b)} \nabla_x H(x-y) dy = -\frac{1}{\pi ab} \int_{\partial\Omega(a,b)} H(x-y) \nu(y) d\mathcal{H}^1(y),$$

where  $\nu$  is the outward unit normal. By rewriting  $x = x(\varphi) = (a \cos \varphi, b \sin \varphi)$ , for some  $\varphi \in [-\pi, \pi)$ , and by parametrising  $\partial\Omega(a, b)$  via  $y = y(\theta) = (a \cos \theta, b \sin \theta)$ , with  $\theta \in [-\pi, \pi)$ , we derive

$$\nabla\Psi_{a,b}(x(\varphi)) = -\frac{1}{\pi ab} \int_{-\pi}^{\pi} \frac{a^2(\cos \varphi - \cos \theta)^2}{a^2(\cos \varphi - \cos \theta)^2 + b^2(\sin \varphi - \sin \theta)^2} (b \cos \theta, a \sin \theta) d\theta. \quad (2.9)$$

Using the trigonometric identities

$$\begin{aligned} \cos \varphi - \cos \theta &= 2 \sin\left(\frac{\theta - \varphi}{2}\right) \sin\left(\frac{\theta + \varphi}{2}\right) \\ \sin \varphi - \sin \theta &= -2 \sin\left(\frac{\theta - \varphi}{2}\right) \cos\left(\frac{\theta + \varphi}{2}\right) \end{aligned}$$

in (2.9), we obtain, by means of elementary manipulations,

$$\begin{aligned} \nabla\Psi_{a,b}(x(\varphi)) &= -\frac{1}{\pi ab} \int_{-\pi}^{\pi} \frac{a^2 \sin^2\left(\frac{\theta + \varphi}{2}\right)}{a^2 \sin^2\left(\frac{\theta + \varphi}{2}\right) + b^2 \cos^2\left(\frac{\theta + \varphi}{2}\right)} (b \cos \theta, a \sin \theta) d\theta \\ &= -\frac{1}{\pi ab} \int_{-\pi}^{\pi} \frac{a^2(1 - \cos(\theta + \varphi))}{a^2 + b^2 + (b^2 - a^2) \cos(\theta + \varphi)} (b \cos \theta, a \sin \theta) d\theta \\ &= -\frac{1}{\pi ab} (b \cos \varphi, -a \sin \varphi) \int_{-\pi}^{\pi} \frac{a^2(1 - \cos(\theta + \varphi)) \cos(\theta + \varphi)}{a^2 + b^2 + c^2 \cos(\theta + \varphi)} d\theta \end{aligned} \quad (2.10)$$

$$- \frac{1}{\pi ab} (b \sin \varphi, a \cos \varphi) \int_{-\pi}^{\pi} \frac{a^2(1 - \cos(\theta + \varphi)) \sin(\theta + \varphi)}{a^2 + b^2 + c^2 \cos(\theta + \varphi)} d\theta. \quad (2.11)$$

To simplify the previous expression, we note for (2.11) that

$$\int_{-\pi}^{\pi} \frac{a^2(1 - \cos(\theta + \varphi)) \sin(\theta + \varphi)}{a^2 + b^2 + c^2 \cos(\theta + \varphi)} d\theta = \int_{-\pi}^{\pi} \frac{a^2(1 - \cos \theta) \sin \theta}{a^2 + b^2 + c^2 \cos \theta} d\theta = 0,$$

since the integrand in the last integral is an odd function. On the other hand, by [15, Lemma IV.1.15] one has

$$\int_{-\pi}^{\pi} \frac{1 - \cos \theta}{a^2 + b^2 + c^2 \cos \theta} d\theta = 2\pi \frac{b - a}{ac^2},$$

and so the integral in (2.10) reduces to

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{a^2(1 - \cos(\theta + \varphi)) \cos(\theta + \varphi)}{a^2 + b^2 + c^2 \cos(\theta + \varphi)} d\theta &= \int_{-\pi}^{\pi} \frac{a^2(1 - \cos \theta) \cos \theta}{a^2 + b^2 + c^2 \cos \theta} d\theta \\ &= \frac{a^2}{c^2} \int_{-\pi}^{\pi} (1 - \cos \theta) d\theta - \frac{a^2(a^2 + b^2)}{c^2} \int_{-\pi}^{\pi} \frac{1 - \cos \theta}{a^2 + b^2 + c^2 \cos \theta} d\theta \\ &= 2\pi \frac{a^2}{c^2} - 2\pi \frac{a}{c^4} (b - a)(a^2 + b^2) = -2\pi \frac{ab}{(a + b)^2}. \end{aligned}$$

This leads to the simplified expression for the gradient of  $\Psi_{a,b}$  on  $\partial\Omega(a, b)$

$$\nabla \Psi_{a,b}(x(\varphi)) = \frac{2}{(a + b)^2} (b \cos \varphi, -a \sin \varphi).$$

Since

$$(b \cos \varphi, -a \sin \varphi) = \frac{1}{ab} (b^2 x_1, -a^2 x_2),$$

we deduce that

$$\nabla \Psi_{a,b}(x) = \frac{2}{ab(a + b)^2} (b^2 x_1, -a^2 x_2) \quad \text{for } x \in \partial\Omega(a, b). \quad (2.12)$$

Combining (2.8) and (2.12), by unique continuation, we deduce that there exists a constant  $c_{a,b} \in \mathbb{R}$  such that

$$\Psi_{a,b}(x) = \frac{b^2 x_1^2 - a^2 x_2^2}{ab(a + b)^2} + c_{a,b} \quad \text{for } x \in \Omega(a, b). \quad (2.13)$$

We can also compute the constant  $c_{a,b}$  in (2.13): indeed,

$$\begin{aligned} c_{a,b} &= \Psi_{a,b}(0) = \frac{1}{\pi ab} \int_{\Omega(a,b)} H(y) dy = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_0^1 \frac{a^2 \cos^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \rho d\rho d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{a^2(1 + \cos \theta)}{a^2 + b^2 - c^2 \cos \theta} d\theta = \frac{a}{a + b}, \end{aligned}$$

where in the last equality we applied again [15, Lemma IV.1.15]. In conclusion,

$$\Psi_{a,b}(x) = \frac{b^2 x_1^2 - a^2 x_2^2}{ab(a + b)^2} + \frac{a}{a + b} \quad \text{for } x \in \Omega(a, b). \quad (2.14)$$

### 2.1.2. The Euler-Lagrange condition (2.2)

We now show that for every  $\alpha \in (0, 1)$  there exists a unique pair  $(a, b) \in \mathbb{R}^2$ , with  $0 < a < b$ , such that the potential  $W_\alpha * \mu_{a,b}$  satisfies the first Euler-Lagrange condition, i.e.,

$$(W_\alpha * \mu_{a,b})(x) + \frac{|x|^2}{2} = C_\alpha(a, b) \quad \text{for every } x \in \Omega(a, b), \quad (2.15)$$

for some constant  $C_\alpha(a, b)$ . By (2.6) and (2.14) we have that

$$(W_\alpha * \mu_{a,b})(x) = -\frac{bx_1^2 + ax_2^2}{ab(a + b)} + \alpha \frac{b^2 x_1^2 - a^2 x_2^2}{ab(a + b)^2} - \log\left(\frac{a + b}{2}\right) + \frac{1}{2} + \alpha \frac{a}{a + b} \quad (2.16)$$

for every  $x \in \Omega(a, b)$ . Therefore,  $W_\alpha * \mu_{a,b}$  satisfies (2.15) if and only if

$$\begin{cases} -\frac{1}{a(a+b)} + \alpha \frac{b}{a(a+b)^2} = -\frac{1}{2}, \\ -\frac{1}{b(a+b)} - \alpha \frac{a}{b(a+b)^2} = -\frac{1}{2}. \end{cases} \quad (2.17)$$

Multiplying the first equation by  $a$ , the second equation by  $b$ , and taking the difference yield

$$c^2 = b^2 - a^2 = 2\alpha. \quad (2.18)$$

Subtracting the two equations in (2.17) we deduce that

$$\alpha(a^2 + b^2) - c^2 = 0. \quad (2.19)$$

It is immediate to see that the unique solution to (2.18)–(2.19), and hence to (2.17), is given by the pair  $a = \sqrt{1 - \alpha}$  and  $b = \sqrt{1 + \alpha}$ . Hence the measure  $\mu_\alpha$  defined as in (2.1) is a solution of (2.15), and in fact of (2.2).

## 2.2. The Euler-Lagrange condition (2.3)

In this section we show that for every  $\alpha \in (0, 1)$

$$(W_\alpha * \mu_\alpha)(x) + \frac{1}{2}|x|^2 \geq C_\alpha \quad \text{for every } x \in \mathbb{R}^2 \setminus \Omega(\sqrt{1 - \alpha}, \sqrt{1 + \alpha}), \quad (2.20)$$

where  $\mu_\alpha$  is defined as in (2.1) and, from (2.16),

$$C_\alpha = -\log\left(\frac{\sqrt{1 - \alpha} + \sqrt{1 + \alpha}}{2}\right) + \frac{1}{2} + \alpha \frac{\sqrt{1 - \alpha}}{\sqrt{1 - \alpha} + \sqrt{1 + \alpha}}.$$

Let now  $\alpha \in (0, 1)$ ; for simplicity of notation we set

$$\Omega_\alpha := \Omega(\sqrt{1 - \alpha}, \sqrt{1 + \alpha})$$

and

$$f_\alpha(x) := (W_\alpha * \mu_\alpha)(x) + \frac{1}{2}|x|^2 \quad \text{for every } x \in \mathbb{R}^2. \quad (2.21)$$

It is easy to see that  $f_\alpha \in C^1(\mathbb{R}^2)$  and  $f_\alpha \in C^\infty(\mathbb{R}^2 \setminus \overline{\Omega_\alpha})$ . We also recall that in Section 2.1 we have proved that

$$f_\alpha(x) = C_\alpha \quad \text{for every } x \in \overline{\Omega_\alpha}. \quad (2.22)$$

Moreover, by (2.7), we have that

$$\Delta f_\alpha = \Delta(W_0 * \mu_\alpha) + \alpha \Delta(H * \mu_\alpha) + 2 = -2\alpha \partial_{x_1}^2 (W_0 * \mu_\alpha) + 2 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega_\alpha}, \quad (2.23)$$

hence,

$$\Delta^2 f_\alpha = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega_\alpha}. \quad (2.24)$$



Let now  $x^0 \in \mathbb{R}^2 \setminus \overline{\Omega}_\alpha$ . We write  $x^0$  as  $x^0 = y^0 + tv$ , where  $y^0 \in \partial\Omega_\alpha$ ,  $t > 0$ , and  $v$  denotes the external unit normal to  $\partial\Omega_\alpha$  at  $y^0$ . In view of (2.22), the second Euler-Lagrange condition (2.20) is proved if we show that

$$\partial_\nu f_\alpha(x^0) \geq 0. \quad (2.25)$$

We prove (2.25) by means of a subtle use of the maximum principle applied to an auxiliary, harmonic function, see [9, Theorem 4]. Let  $R > 0$  be a large enough parameter that will be chosen later, such that  $x^0 \in B_R(0) \setminus \overline{\Omega}_\alpha$ . We consider the auxiliary function  $g_\alpha : \overline{B_R(0)} \setminus \Omega_\alpha \rightarrow \mathbb{R}$  defined by

$$g_\alpha(x) := \partial_\nu f_\alpha(x) - \frac{1}{2}(x - x^0) \cdot \nu \Delta f_\alpha(x) \quad (2.26)$$

for every  $x \in \overline{B_R(0)} \setminus \Omega_\alpha$ . From (2.24) it follows that

$$\Delta g_\alpha = \Delta(\partial_\nu f_\alpha) - \frac{1}{2}(x - x^0) \cdot \nu \Delta^2 f_\alpha - \nu \cdot \nabla(\Delta f_\alpha) = 0 \quad \text{in } B_R(0) \setminus \overline{\Omega}_\alpha,$$

in other words,  $g_\alpha$  is harmonic in  $B_R(0) \setminus \overline{\Omega}_\alpha$ . Therefore, by the maximum principle we deduce that

$$g_\alpha(x^0) \geq \min\{g_\alpha(x) : x \in \partial B_R(0) \cup \partial\Omega_\alpha\}. \quad (2.27)$$

Note that the value of  $g_\alpha$  on  $\partial\Omega_\alpha$  is intended as a limit from  $\mathbb{R}^2 \setminus \overline{\Omega}_\alpha$ .

We claim that the function in the right-hand side of (2.27) is nonnegative for large enough  $R$ . This claim clearly implies (2.25), since  $g_\alpha(x^0) = \partial_\nu f_\alpha(x^0)$ .

To show that  $g_\alpha$  is nonnegative on  $\partial B_R(0)$ , we start by rewriting it more explicitly, by using the definition (2.21) of  $f_\alpha$ , as

$$g_\alpha(x) = x^0 \cdot \nu + \partial_\nu(W_\alpha * \mu_\alpha)(x) - \frac{1}{2}(x - x^0) \cdot \nu \Delta(W_\alpha * \mu_\alpha)(x). \quad (2.28)$$

Using the fact that  $|\nabla W_\alpha(x)| \leq (1 + \alpha)/|x|$  and  $|\Delta W_\alpha(x)| \leq 2\alpha/|x|^2$  for every  $x \neq 0$ , one can easily check that

$$\lim_{|x| \rightarrow +\infty} \partial_\nu(W_\alpha * \mu_\alpha)(x) = \lim_{|x| \rightarrow +\infty} (x - x^0) \cdot \nu \Delta(W_\alpha * \mu_\alpha)(x) = 0.$$

From (2.28) we immediately conclude that

$$\lim_{|x| \rightarrow +\infty} g_\alpha(x) = x^0 \cdot \nu > 0,$$

which implies that for  $R$  large enough

$$\min\{g_\alpha(x) : x \in \partial B_R(0)\} > 0.$$

It remains to show that  $\min\{g_\alpha(x) : x \in \partial\Omega_\alpha\} \geq 0$ . To see this note that, since  $f_\alpha \in C^1(\mathbb{R}^2)$  and satisfies (2.22), we have that  $\partial_\nu f_\alpha(x) = 0$  for every  $x \in \partial\Omega_\alpha$ . Hence, from the definition (2.26) of  $g_\alpha$ , we have that

$$g_\alpha(x) = -\frac{1}{2}(x - x^0) \cdot \nu \Delta f_\alpha(x) \quad \text{for } x \in \partial\Omega_\alpha.$$

Moreover,  $(x - x^0) \cdot \nu \leq 0$  for every  $x \in \partial\Omega_\alpha$ , by the convexity of  $\Omega_\alpha$ . Therefore,

$$g_\alpha \geq 0 \quad \text{on } \partial\Omega_\alpha \quad \text{if and only if} \quad \Delta f_\alpha \geq 0 \quad \text{on } \partial\Omega_\alpha.$$

By (2.23) it remains to show that

$$\alpha \lim_{\substack{x \rightarrow \partial \Omega_\alpha \\ x \in \mathbb{R}^2 \setminus \overline{\Omega_\alpha}}} \partial_{x_1}^2 (W_0 * \mu_\alpha)(x) \leq 1. \quad (2.29)$$

To prove (2.29) we use the expression (2.6) of the logarithmic potential of the ellipse  $\Omega_\alpha$  for points  $x \in \mathbb{R}^2 \setminus \overline{\Omega_\alpha}$ . By symmetry it is enough to work in the first quadrant, where it is convenient to use an alternative set of coordinates, namely

$$\begin{cases} z = \sinh \xi \\ \rho = \sin \eta \end{cases} \quad \text{with} \quad \xi > 0, \quad 0 \leq \eta \leq \frac{\pi}{2},$$

which are then related to the Cartesian coordinates by the transformation

$$\begin{cases} x_1 = cz\rho \\ x_2 = c\sqrt{(1+z^2)(1-\rho^2)} \end{cases} \quad \text{with} \quad z > 0, \quad 0 \leq \rho \leq 1.$$

Note that, in the  $(z, \rho)$  coordinates

$$\mathbb{R}^2 \setminus \Omega_\alpha = \left\{ z \geq \frac{a}{c} \right\},$$

and the logarithmic potential in (2.6) outside  $\Omega_\alpha$ , in the first quadrant, becomes

$$(W_0 * \mu_\alpha)(x) = -\log\left(z + \sqrt{z^2 + 1}\right) - \frac{1}{2} \frac{1 - 2\rho^2}{(z + \sqrt{z^2 + 1})^2} - \log \frac{c}{2},$$

for  $0 \leq \rho \leq 1$  and  $z \geq \frac{a}{c}$ . Now we recall that the gradient of the  $(z, \rho)$ -coordinates with respect to the Cartesian coordinates is given by the following formulas:

$$\begin{aligned} \nabla \rho(x) &= \frac{1}{c(z^2 + \rho^2)} \left( z(1 - \rho^2), -\rho \sqrt{(1 + z^2)(1 - \rho^2)} \right), \\ \nabla z(x) &= \frac{1}{c(z^2 + \rho^2)} \left( \rho(1 + z^2), z \sqrt{(1 + z^2)(1 - \rho^2)} \right). \end{aligned}$$

Then, since

$$\begin{aligned} \partial_z(W_0 * \mu_\alpha) &= \frac{-2(z^2 + \rho^2 + z\sqrt{z^2 + 1})}{(z + \sqrt{z^2 + 1})^2 \sqrt{z^2 + 1}}, \\ \partial_\rho(W_0 * \mu_\alpha) &= \frac{2\rho}{(z + \sqrt{z^2 + 1})^2}, \end{aligned}$$

we have that

$$\partial_{x_1}(W_0 * \mu_\alpha)(x) = -\frac{2}{c} \left( \frac{\rho}{z + \sqrt{z^2 + 1}} \right).$$

After similar computations, we obtain

$$\partial_{x_1}^2(W_0 * \mu_\alpha)(x) = -\frac{2}{c} \partial_{x_1} \left( \frac{\rho}{z + \sqrt{z^2 + 1}} \right) = \frac{2}{c^2} \left( 1 - \frac{z\sqrt{z^2 + 1}}{z^2 + \rho^2} \right).$$

Since the expression at the right-hand side achieves its maximum value at  $\rho = 1$ , we have that for  $x \in \mathbb{R}^2 \setminus \Omega_\alpha$

$$\partial_{x_1}^2 (W_0 * \mu_\alpha)(x) \leq \frac{2}{c^2} \left( 1 - \frac{z}{\sqrt{z^2 + 1}} \right). \tag{2.30}$$

On the other hand,

$$\frac{2\alpha}{c^2} \lim_{z \rightarrow \frac{a}{c}^+} \left( 1 - \frac{z}{\sqrt{z^2 + 1}} \right) = \frac{2\alpha}{c^2} \left( 1 - \frac{a}{b} \right) \leq 1 \tag{2.31}$$

for  $a = \sqrt{1 - \alpha}$  and  $b = \sqrt{1 + \alpha}$  (and  $c^2 = 2\alpha$ ). Inequalities (2.30) and (2.31) prove the claim (2.29).

**Remark 2.3** (The higher-dimensional case). For the case  $n \geq 3$ , one could in principle try to adapt the maximum-principle approach adopted in this section to prove (2.3), where now

$$W_\alpha(x) = W_0(x) + \alpha \frac{x_1^2}{|x|^n}, \quad W_0(x) = \frac{1}{|x|^{n-2}}. \tag{2.32}$$

Let  $\alpha \geq 0$ . Proceeding as in Section 2.2 one can define, for  $n \geq 3$ , an auxiliary function  $g_\alpha : \overline{B_R(0)} \setminus \Omega_\alpha \rightarrow \mathbb{R}$  as

$$g_\alpha(x) := \partial_\nu f_\alpha(x) - \frac{1}{2}(x - x^0) \cdot \nu \Delta f_\alpha(x) - \left( 1 - \frac{n}{2} \right) (x - x_0) \cdot \nu. \tag{2.33}$$

It is easy to see that  $g_\alpha$  is harmonic in  $B_R(0) \setminus \overline{\Omega}_\alpha$  and that, for  $R$  large enough

$$\min\{g_\alpha(x) : x \in \partial B_R(0)\} > 0.$$

To complete the maximum-principle argument, in analogy with the two-dimensional case, it would remain to show that  $\min\{g_\alpha(x) : x \in \partial\Omega_\alpha\} \geq 0$ . Similarly as in (2.29), this condition can be equivalently rewritten as

$$\frac{\alpha}{n - 2} \lim_{\substack{x \rightarrow \partial\Omega_\alpha \\ x \in \mathbb{R}^2 \setminus \overline{\Omega}_\alpha}} \partial_{x_1}^2 (W_0 * \mu_\alpha)(x) \leq 1. \tag{2.34}$$

Using the explicit expression of  $W_0 * \mu_\alpha$  outside  $\Omega_{a,b}$  (see, e.g., [8, Section 3.2.2]), proving (2.34) is equivalent (modulo lengthy computations) to showing that

$$\frac{n\alpha}{2} \left( \frac{2}{ab^{n-1}} - \int_{a^2}^\infty \frac{d\sigma}{\sigma^{3/2}(\sigma + c^2)^{\frac{n-1}{2}}} \right) \leq 1. \tag{2.35}$$

It is however not so immediate to verify whether (2.35) holds true, in particular since in higher dimension the first Euler condition does not determine the semi-axes  $a$  and  $b$  as explicit functions of  $\alpha$ . Moreover, condition (2.34) is not a necessary condition for (2.3), it is only sufficient, and so is (2.35). For these reasons we developed an alternative approach for the higher-dimensional case (see [8]).

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## Conflict of interest

The authors declare that they have no conflict of interest and guarantee the compliance with the Ethics Guidelines of the journal.

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