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**SUPPLEMENTARY MATERIALS: MULTISCALE ANALYSIS OF
NUTRIENT UPTAKE BY PLANT ROOTS WITH SPARSE
DISTRIBUTION OF ROOT HAIRS: NONSTANDARD SCALING ***

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SM1. Parameter values. The scaling in the boundary conditions on Γ^ε should be interpreted in terms of the experimental values for nutrient uptake rates by root hairs for different plant types. Considering the nondimensionalization of dimensional Michaelis-Menten boundary condition

$$-D\nabla u \cdot \hat{\mathbf{n}} = \frac{F_h u}{K_h + u}$$

via $x = R\tilde{x}$, $t = R^2\tilde{t}/D$, $u = K_h\tilde{u}$ gives

$$(SM1.1) \quad -\tilde{\nabla}\tilde{u} \cdot \mathbf{n} = \frac{F_h R}{K_h D} \frac{\tilde{u}}{1 + \tilde{u}} = \frac{\varepsilon r_h R^2 F_h}{a_\varepsilon K_h l^2 D} \frac{\tilde{u}}{1 + \tilde{u}} = \frac{\varepsilon}{a_\varepsilon} \tilde{\alpha} \frac{\tilde{u}}{1 + \tilde{u}},$$

where r_h denotes the dimensional hair radius, l denotes the dimensional inter-hair distance and $\tilde{\alpha} = (r_h R^2 F_h)/(K_h l^2 D)$. Considering the range of phosphate uptake parameters F_h and K_h as reviewed in [SM1], and $D = 10^{-5} \text{ cm}^2 \text{ s}^{-1}$ [SM2], as well as $R = 1 \text{ cm}$, $l = 0.01 \text{ cm}$ and $r_h \sim 10^{-4} \text{ cm}$, we conclude that $\tilde{\alpha} = 10$ for wheat, while $\tilde{\alpha} = 1$ arises when modelling sulphur and magnesium uptake by maize [SM3].

SM2. Derivation of macroscopic equations for nonlinear boundary conditions on root hair surfaces.

SM2.1. Case $\varepsilon \ln(1/a_\varepsilon) = O(1)$. Following the same procedure as in Section 3.2.1 of the main text, we obtain the same equations as in (3.29), but with different boundary conditions for u_2^I , u_3^I , and u_4^I , namely

$$(SM2.1) \quad \begin{aligned} D_u \nabla_z u_2^I \cdot \hat{\mathbf{n}} &= -\kappa g(u_0^I) \quad \text{on } \partial B_1, & D_u \nabla_z u_3^I \cdot \hat{\mathbf{n}} &= -\kappa g'(u_0^I) u_1^I \quad \text{on } \partial B_1, \\ D_u \nabla_z u_4^I \cdot \hat{\mathbf{n}} &= -\kappa [g'(u_0^I) u_2^I + \frac{1}{2} g''(u_0^I) (u_1^I)^2] \quad \text{on } \partial B_1. \end{aligned}$$

Hence the corresponding solutions are

$$\begin{aligned} u_j^I(t, x, z) &= u_j^I(t, x), \quad j = 0, 1, & u_2^I(t, x, z) &= (\kappa/D_u) g(u_0^I) \ln(\|z\|) + U_2^I(t, x), \\ u_3^I(t, x, z) &= (\kappa/D_u) g'(u_0^I) u_1^I \ln(\|z\|) + U_3^I(t, x), \\ u_4^I(t, x, z) &= (\kappa/D_u) [g'(u_0^I) U_2^I(t, x) + \frac{1}{2} g''(u_0^I) (u_1^I)^2] \ln(\|z\|) + U_4^I(t, x). \end{aligned}$$

Then by matching inner approximation u_2^I and outer approximation u_2^O we obtain for u_2^O equation (3.32) with $g(u_0^I)$ instead of u_0^I and for u_0^O equation (3.33) with $g(u_0^I)$

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instead of u_0^I . We also obtain the same matching condition (3.45). Hence we obtain an effective equation

$$(SM2.2) \quad \partial_t u_0 = \nabla_x \cdot (D_u \nabla_x u_0) - 2\pi\kappa g(u_0) \chi_{\Omega_L} \quad \text{in } \Omega, t > 0,$$

Case $\varepsilon^2 \ln(1/a_\varepsilon) = O(1)$. Applying the formal asymptotic expansion ansatz (3.25) in multiscale problem (2.1)–(2.3), (2.6), (2.7) again yields (3.53), equipped here with the modified boundary condition

$$\begin{aligned} & \left(e^{\lambda/\varepsilon^2} \varepsilon^{-1} D_u \nabla_z + D_u \nabla_x \right) (u_0 + \varepsilon u_1 + \dots) \cdot \hat{\mathbf{n}} = -\varepsilon \kappa e^{\lambda/\varepsilon^2} g(u_0 + \varepsilon u_1 + \dots) \\ & = -\varepsilon \kappa e^{\lambda/\varepsilon^2} \left[g(u_0) + \varepsilon g'(u_0) u_1 + \varepsilon^2 g'(u_0) u_2 + \varepsilon^2 \frac{1}{2} g''(u_0) u_1^2 + \dots \right] \quad \text{on } \Omega_L \times \partial B_1. \end{aligned}$$

In the case of inner solutions, for u_0^I and u_1^I we have the same equations and boundary conditions as in (3.29) and for u_2^I, u_3^I , and u_4^I we obtain the same equations as in (3.29) but with different boundary conditions

$$(SM2.3) \quad \begin{aligned} D_u \nabla_z u_2^I \cdot \hat{\mathbf{n}} &= -\kappa g(u_0^I) && \text{on } \partial B_1, \\ D_u \nabla_z u_3^I \cdot \hat{\mathbf{n}} &= -\kappa g'(u_0^I) u_1^I && \text{on } \partial B_1, \\ D_u \nabla_z u_4^I \cdot \hat{\mathbf{n}} &= -\kappa \left[g'(u_0^I) u_2^I + \frac{1}{2} g''(u_0^I) (u_1^I)^2 \right] && \text{on } \partial B_1. \end{aligned}$$

Hence the inner approximation reads

$$(SM2.4) \quad \begin{aligned} u_\varepsilon^I(t, x) &= u_0^I(t, x) + \varepsilon u_1^I(t, x) + \varepsilon^2 U_2^I(t, x) + \varepsilon^2 (\kappa/D_u) g(u_0^I) \ln(\|z\|) \\ &+ \varepsilon^3 \left[(\kappa/D_u) g'(u_0^I) u_1^I \ln(\|z\|) + U_3^I(t, x) \right] \\ &+ \varepsilon^4 \left[\frac{\kappa}{D_u} (g'(u_0^I) U_2^I + \frac{1}{2} g''(u_0^I) (u_1^I)^2) \ln(\|z\|) + U_4^I(t, x) \right] + \dots \end{aligned}$$

Then in terms of outer variables y the inner approximation u_ε^I has the form

$$\begin{aligned} u_\varepsilon^I &= \left(u_0^I + \lambda \frac{\kappa}{D_u} g(u_0^I) \right) + \varepsilon \left(u_1^I + \lambda \frac{\kappa}{D_u} g'(u_0^I) u_1^I \right) \\ &+ \varepsilon^2 \left[U_2^I + \frac{\kappa}{D_u} g(u_0^I) \ln(\|y\|) + \lambda \frac{\kappa}{D_u} (g'(u_0^I) U_2^I + \frac{1}{2} g''(u_0^I) (u_1^I)^2) \right] + \dots \end{aligned}$$

In the same way as in Subsection 3.2.2, for the outer approximation we obtain

$$u_\varepsilon^O(t, x) = u_0^O(t, x) + \varepsilon u_1^O(t, x) + \varepsilon^2 \left(U_2^O(t, x) + 2\pi(\kappa/D_u) g(u_0^I(t, x)) \psi(y) \right) + \dots$$

Then the matching condition for inner and outer solutions for zero order terms implies

$$(SM2.5) \quad u_0^O(t, x) = u_0^I(t, x) + \lambda(\kappa/D_u) g(u_0^I(t, x)),$$

and the macroscopic equation for $u_0(t, x) = u_0^O(t, x)$ reads

$$(SM2.6) \quad \partial_t u_0 = \nabla_x \cdot (D_u \nabla_x u_0) - 2\pi\kappa g(h(u_0)) \chi_{\Omega_L} \quad \text{in } \Omega, t > 0,$$

where $h = h(u_0)$ is the solution of $u_0 = h + \lambda(\kappa/D_u) g(h)$.

Adopting the Michaelis-Menten boundary condition (2.4), condition (SM2.5) can be rewritten as a quadratic equation

$$(SM2.7) \quad (u_0^I)^2 + u_0^I (\lambda(\kappa/D_u) + 1 - u_0^O) - u_0^O = 0,$$

with unique non-negative solution

$$u_0^I = \frac{1}{2} \left[\sqrt{(u_0^O - \lambda(\kappa/D_u) - 1)^2 + 4u_0^O} + u_0^O - \lambda \frac{\kappa}{D_u} - 1 \right],$$

and the effective equation (SM2.6) thus becomes (3.64).

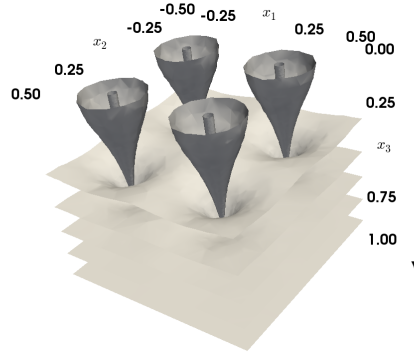


Fig. SM1: Isosurfaces of nutrient concentration support the intuition that with the chosen boundary conditions, the (steady-state) solution has the same behavior in every periodicity cell ($a_\varepsilon = 0.01$, $\varepsilon = 0.5$). The arrow points in the direction of increasing x_3 (i.e. away from the root surface located at $x_3 = 0$).

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