Limit theorems and structural properties for the Cat-and-Mouse Markov chain and its generalisations

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LIMIT THEOREMS AND STRUCTURAL PROPERTIES FOR THE CAT-AND-MOUSE MARKOV CHAIN AND ITS GENERALISATIONS

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Abstract

We revisit the so-called Cat-and-Mouse Markov chain, studied earlier by Litvak and Robert (2012). This is a 2-dimensional Markov chain on the lattice $\mathbb{Z}^2$, where the first component is a simple random walk and the second component changes when the components meet. We obtain new results for two generalisations of the model. Firstly, in the 2-dimensional case we consider general jump distributions for the components and obtain a scaling limit for the second component (the mouse). When we let the first component (the cat) to be again a simple random walk, we further generalise the jump distribution of the second component. Secondly, we consider chains of three and more dimensions, where we investigate structural properties of the model and find a limiting law for the last component.

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1. Introduction

We analyse the dynamics of a stochastic process with dependent coordinates, commonly referred to as the Cat-and-Mouse (CM) Markov chain (MC), and of its generalisations. Let $\mathcal{S}$ be a directed graph. Let $\{(C_n, M_n)\}_{n=0}^{\infty}$ denote the CM MC on $\mathcal{S}^2$, defined as follows. We assume that $\{C_n\}_{n=0}^{\infty}$, the location of the cat, is a MC on $\mathcal{S}$ with transition matrix $P = (p(x,y)), x, y \in \mathcal{S})$. The second coordinate, the location of the mouse, $\{M_n\}_{n=0}^{\infty}$ has the following dynamics:

- If $M_n \neq C_n$, then $M_{n+1} = M_n$,
- If $M_n = C_n$, then, conditionally on M_n, the random variable M_{n+1} has distribution $(p(M_n, y), y \in \mathcal{S})$ and is independent of C_{n+1}.

In our model the cat is trying to catch the mouse. The mouse is usually in hiding and not moving but, if the cat hits the same location of the graph, the mouse jumps. The cat does not notice where the mouse jumps to, so it proceeds independently.

CM MC is an example of models called Cat-and-Mouse games. CM games are common in game theory. We refer to Coppersmith *et al.* (1993), where the authors showed that a CM game is at the core of many on-line algorithms and, in particular, may be used in settings considered by Manasse *et al.* (1990) and Borodin *et al.* (1992). Some special cases of CM games on the plane have been studied by Baeza-Yates *et al.* (1993). Two examples of CM games have been discussed in Aldous and Fill (2002) in the context of reversible MCs.

There are many related models in applied probability where time evolution of the process may be represented as a multi-component Markov chain where one of the components has independent dynamics and forms a Markov chain itself (for example Gamarnik (2004), Gamarnik and Squillante (2005), Borst *et al.* (2008), Foss *et al.* (2012)). Typically such dependence is modelled using Markov modulation. In this paper we consider the case where the first component is a random walk. Thus, our model can be viewed as a random-walk-modulated random walk. We consider null-recurrent and transient cases where we find proper scaling for the components.

We are mainly motivated by the results of the paper by Litvak and Robert (2012) where the authors analyse scaling properties of the (non-Markovian) sequence $\{M_n\}_{n=0}^{\infty}$ for a specific transition matrix P when $\mathcal{S}$ is either $\mathbb{Z}, \mathbb{Z}^2$ or $\mathbb{Z}^+$. Although it deals with a relatively simple case of the CM MC, it clearly illustrates a certain phenomenon related to this type of Markov modulation. This led us to question to what extent such a phenomenon holds, for what kind of distributions of jumps, and for which types of Markov modulation.

In this paper we analyse the case $\mathcal{S} = \mathbb{Z}$. Henceforth, we will take the transition matrix P to satisfy $p(x, x+1) = p(x, x-1) = 1/2$. It was proven in Theorem 3 of Litvak and Robert (2012) that the convergence in distribution

$$\left\{ \frac{1}{\sqrt[4]{n}} M_{[nt]}, t \geq 0 \right\} \Rightarrow \{B_1(L_{B_2}(t)), t \geq 0\}\,, \text{ as } n \to \infty,$$

holds, where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions on $\mathbb{R}$ and $L_{B_2}(t)$ is the local time process of $B_2(t)$ at 0.

This result looks natural, since the mouse, observed at the meeting times with the cat, is a simple random walk. The time intervals between meeting times are independent and identically distributed. They have the same distribution as the time needed for the cat (also a simple random walk) to get from 1 to 0, which has a regularly varying tail with parameter 1/2 (see, e.g., Spitzer (1964)). Thus, the scaling for the location of the mouse is $\sqrt[4]{n} = (n^{1/2})^{1/2}$. Local time $L_{B_2}(t)$ can be interpreted as the scaled duration of time the cat and the mouse spend together.

In this paper we show that similar behaviour holds when jump-size distributions of both components have zero mean and finite variance. The behaviour slightly changes when we introduce heavier tails for the jump-size distribution of the mouse. For this case we develop more general approach based on the work of Jurlewicz *et al.* (2012). In parallel, we introduce additional components while applying an analogue of the aforementioned Markov modulation. Here, through analysis of dynamical structural properties, we show the similar phenomenon for additional components.

More specifically, we provide two generalisations of the CM MC introduced above. The first generalisation relates to the jump distribution of the mouse. Given $C_n = M_n$, random variable $M_{n+1} - M_n$ has a general distribution which has a finite first moment and belongs to the strict domain of attraction of a stable law with index $\alpha \in [1, 2]$, with a normalising function $\{b(n)\}_{n=1}^{\infty}$ (note that distributions with a finite second moment belong to the domain of attraction of a normal distribution). We find a weak limit of $\{b^{-1}(\sqrt{n})M_{[nt]}\}_{t\geq 0}$, as $n \to \infty$. This model is more challenging than the classical setting because, when the mouse jumps, the value of this jump and the time until the next jump may be dependent. Also, if the jump distribution of the mouse has infinite second moment, we can not use classical results such as Theorem 5.1 from Kasahara (1984). Next, we consider the case where both components have general distributions with finite second moments. Here our results take into account the approach developed by Uchiyama (2011a).

In the second generalisation we add more components (we will refer to the objects whose dynamics these components describe as "agents") to the system, with keeping the chain "hierarchy". For instance, adding one extra agent (we refer to it as the dog), acting on the cat the same way as the cat acts on the mouse, slows down the cat and, therefore, also the mouse. We are interested in the effect of this on the scaling properties of the process. Recursive addition of further agents will slow down the mouse further. For the system with three agents we investigate the dynamical structural properties and find a weak limit of $\{n^{-1/8}M_{[nt]}\}_{t\geq 0}$, as $n \to \infty$. The system regenerates when all the agents are at the same point. Therefore, if we find the tail asymptotics of the time intervals between these events, we can split the process into i.i.d. cycles and use classical results (for example, Kasahara (1984)).

For the systems with an arbitrary finite number of agents, we provide a relatively simple result on the weak convergence, for fixed $t > 0$. In this case, the path analysis becomes quite difficult and we have not yet found the asymptotics of the time intervals between regeneration points. Nevertheless, we transform the number of jumps for any agent and use induction and the result from Dobrushin (1955).

Note that we are interested in limit theorems for Markov modulated Markov chains with described dependencies and distributions. We would like to mention that, for such type of models, stability problems were studied in, e.g., [15]–[17] and large deviations problems in, e.g., [18]–[23].

The paper is structured as follows. In Section 2 we define our models and formulate our results. In Sections 3 and 4 we analyse the trajectories of the CM MC and *Dog-and-Cat-and-Mouse* (DCM) MC respectively. This analysis gives the main idea of the proof of our result on scaling properties of DCM MC (Theorem 3). In Section 5.1, we prove our results on scaling properties of general CM MC (Theorem 1 and Theorem 2). We shift the time of our process and use characteristic functions to show that the conditions of Theorem 3.1 from Jurlewicz *et al.* (2012) hold. In Section 5.2, we prove Theorem 3. We approximate the dynamics of the mouse by considering only the values of the process at times when all agents are at the same point of the integer line and then use Theorem 5.1 from Kasahara (1984) to obtain the result. In the Section 5.3, we prove our result on scaling properties for the system with an arbitrary finite number of agents. We approximate the component $X^{(N)}$ by the component $X^{(N-1)}$, slowed down by an independent renewal process.

The Appendix includes definitions and proofs of supplementary results. In Appendix A, we define weak convergence of stochastic processes. In Appendix B we provide auxiliary results on randomly stopped sums with positive summands having regularly varying tail distribution and infinite mean.

Throughout the paper we use the following conventions and notations. For two ultimately positive functions $f(t)$ and $g(t)$ we write $f(t) \sim g(t)$ if $\lim_{t\to\infty} f(t)/g(t) = 1$. For any event A, its *indicator function* $I[A]$ is a random variable that takes value 1 if the event occurs, and value 0, otherwise. Finally we use the following abbreviations: CM – Cat-and-Mouse, DCM – Dog-and-Cat-and-Mouse, MC – Markov chain, i.i.d. – independent and identically distributed, r.v. – random variable, a.s. – almost surely, w.p. – with probability.

2. Models and results

In this section we recall the CM MC on the integers and introduce several of its generalisations.

2.1. “Standard” Cat-and-Mouse Markov chain on $\mathbb{Z}$ $(C \to M)$

Let $\xi = \pm 1$ w.p. $1/2$. Let $\{\xi_n^{(1)}\}_{n=1}^{\infty}$ and $\{\xi_n^{(2)}\}_{n=1}^{\infty}$ be two mutually independent sequences of independent copies of ξ. We define the dynamics of CM MC (C_n, M_n) as follows:

$$C_n = C_{n-1} + \xi_n^{(1)},$$

$$M_n = M_{n-1} + \begin{cases} 0, & \text{if } C_{n-1} \neq M_{n-1}, \\ \xi_n^{(2)}, & \text{if } C_{n-1} = M_{n-1}, \end{cases}$$

for $n \geq 1$.

Let $D[[0,\infty),\mathbb{R}]$ denote the space of all right continuous functions on $[0,\infty)$ having left limits (RCLL, or càdlàg functions).

Let $M(nt) = M_{[nt]}$, $t \geq 0$, be a continuous-time stochastic process taking values M_k, $k \geq 0$, for $t \in [k/n,(k+1)/n)$. Clearly, it is piecewise constant and its trajectories belong to $D[[0,\infty),\mathbb{R}]$.

Litvak and Robert (2012) proved weak convergence

$$\left\{ \frac{1}{\sqrt[4]{n}} M(nt), t \geq 0 \right\} \overset{\mathcal{D}}{\Rightarrow} \left\{ B_1(L_{B_2}(t)), t \geq 0 \right\}, \text{ as } n \to \infty \tag{1}$$

(see Appendix A for definitions), where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions on $\mathbb{R}$ and $L_{B_2}(t)$ is the local time process of $B_2(t)$ at 0.

2.2. Cat-and-Mouse model with a general jump distribution of the mouse ($C \to M$)

In this Subsection we introduce our results for CM MC with more general distributions of r.v.'s $\xi_n^{(1)}$ and $\xi_n^{(2)}$. We start with the same distribution of $\xi_n^{(1)}$ and generalise distribution of $\xi_n^{(2)}$. Thus, the cat is a simple random walk and the mouse is a general random walk. Then we also generalise the distribution of $\xi_n^{(1)}$, however we need certain restrictions on the mouse (finite second moments).

2.2.1 We continue to assume that the dynamics of the cat is described by a simple random walk on $\mathbb{Z}$. Let $\xi = \pm 1$ w.p. $1/2$. Let $C_0 = 0$, $C_n = C_{n-1} + \xi_n^{(1)}$, where $\xi, \xi_1^{(1)}, \xi_2^{(1)}, \ldots$ are i.i.d r.v.'s.

Let $M_0 = 0$, $M_n = M_{n-1} + \xi_n^{(2)} I[C_{n-1} = M_{n-1}]$ where $\{\xi_n^{(2)}\}_{n=1}^{\infty}$ are i.i.d r.v.'s independent of $\{\xi_n^{(1)}\}_{n=1}^{\infty}$. Assume

$$\mu = \mathbb{E}\xi_1^{(2)} \text{ is finite} \tag{2}$$

and there exist a function $b(c) > 0$, $c \geq 0$, and a r.v. $A^{(2)}$ having a stable distribution with index $\alpha \in [1,2]$ such that

$$\frac{\sum_{k=1}^{n}(\xi_k^{(2)} - \mu)}{b(n)} \Rightarrow A^{(2)}, \text{ as } n \to \infty. \tag{3}$$

Define

$$\tau(0) = 0 \text{ and } \tau(n) = \inf\{m > \tau(n-1) : \ C_m = M_m\}, \text{ for } n \geq 1. \tag{4}$$

Given (2), we show that the tail-distribution of $\tau(1)$ is regularly varying with index $1/2$. As a consequence of this result, there exists a r.v. $D^{(2)}$ having a stable distribution with index $1/2$ such that

$$\frac{\tau(n)}{n^2} \Rightarrow D^{(2)}, \text{ as } n \to \infty. \tag{5}$$

In the proof of Theorem 1 we will show that, in fact, there is a joint convergence

$$\left(\frac{\sum_{k=1}^{n}(\xi_k^{(2)} - \mu)}{b(n)}, \ \frac{\tau(n)}{n^2} \right) \Rightarrow (A^{(2)}, D^{(2)}), \text{ as } n \to \infty, \tag{6}$$

where the r.v.'s on the right-hand side are independent. Further, let $\{(A^{(2)}(t), D^{(2)}(t))\}_{t\geq 0}$ denote a stochastic process with independent increments such that $(A^{(2)}(1), D^{(2)}(1))$ has the same distribution as $(A^{(2)}, D^{(2)})$, or *Lévy process* generated by $(A^{(2)}, D^{(2)})$. Let $E^{(2)}(s) = \inf\{t \geq 0 : \ D^{(2)}(t) > s\}$.

Theorem 1. *Assume that (2) and (3) hold. Then*

- *if* $\mu = 0$, *we have*

$$\left\{ \frac{M(nt)}{b(\sqrt{n})}, \ t \geq 0 \right\} \overset{\mathcal{D}}{\Rightarrow} \{A^{(2)}(E^{(2)}(t)), t \geq 0\}, \ \textit{as } n \to \infty, \tag{7}$$

- *if* $\mu \neq 0$, *we have*

$$\left\{ \frac{M(nt)}{\sqrt{n}}, \ t \geq 0 \right\} \overset{\mathcal{D}}{\Rightarrow} \{\mu E^{(2)}(t), t \geq 0\}, \ \textit{as } n \to \infty. \tag{8}$$

2.2.2 Assume now that both $\xi_1^{(1)}$ and $\xi_1^{(2)}$ have general distributions on the integer lattice. The main difference for the mouse is that we need to assume finite second moment for $\xi_1^{(2)}$. The core of our result is the fact that changing simple random walk to a general random walk does not change the scaling if we assume aperiodicity and finite second moments for the increments.

Theorem 2. *Assume that* $\mathbb{E}\xi^{(1)} = 0$, $\mathbf{Var}\xi_1^{(1)} < \infty$ *and* $\xi_1^{(1)}$ *has an aperiodic distribution. Assume* $\mathbf{Var}\xi_1^{(2)} < \infty$ *and, therefore, (3) holds with* $b(n) = \sqrt{n\mathbf{Var}\xi_1^{(2)}}$ *and a standard normal r.v.* $A^{(2)}$. *Then the statements* (7) *and* (8) *of Theorem* ***1*** *continue to hold, with* $b(\sqrt{n}) = n^{1/4}\sqrt{\mathbf{Var}\xi_1^{(2)}}$ *in* (7).

2.3. Dog-and-Cat-and-Mouse model $(D \to C \to M)$

Let $\xi = \pm 1$ w.p. 1/2. Let $\{\xi_n^{(1)}\}_{n=1}^\infty$, $\{\xi_n^{(2)}\}_{n=1}^\infty$ and $\{\xi_n^{(3)}\}_{n=1}^\infty$ be mutually independent sequences of independent copies of ξ. We can define the dynamics of DCM MC $\{(D_n, C_n, M_n)_n\}_{n=1}^\infty$ as follows:

$$D_n = D_{n-1} + \xi_n^{(1)},$$

$$C_n = C_{n-1} + \begin{cases} 0, & \text{if } D_{n-1} \neq C_{n-1}, \\ \xi_n^{(2)}, & \text{if } D_{n-1} = C_{n-1}, \end{cases}$$

$$M_n = M_{n-1} + \begin{cases} 0, & \text{if } C_{n-1} \neq M_{n-1}, \\ \xi_n^{(3)}, & \text{if } C_{n-1} = M_{n-1}, \end{cases}$$

for $n \geq 1$.

Let $T^{(3)}(0) = 0$ and $T^{(3)}(k) = \min\{n > T^{(3)}(k-1) : \ D_n = C_n = M_n\}$, for $k \geq 1$. We show that the tail-distribution of $T^{(3)}(1)$ is regularly varying with index 1/4. Further, we show that there exists a positive r.v. $D^{(3)}$ with a stable distribution and Laplace transform $\exp(-s^{1/4})$ such that

$$\frac{T^{(3)}(k)}{2^9 k^4} \Rightarrow D^{(3)}, \text{ as } k \to \infty. \tag{9}$$

Let $\{D^{(3)}(t)\}_{t \geq 0}$ be a Lévy process generated by $D^{(3)}$ and $E^{(3)}(s) = \inf\{t \geq 0 : D^{(3)}(t) > s\}$.

Theorem 3. *We have* $\mathbb{E}M(T^{(3)}(1)) = 0$, $\sigma^2 = \mathbf{Var}M(T^{(3)}(1)) = 2$, *and*

$$\left\{\frac{M(nt)}{2^{-9/8}n^{1/8}\sigma}, t \geq 0\right\} \overset{\mathcal{D}}{\Rightarrow} \{B(E^{(3)}(t)), t \geq 0\}, \text{ as } n \to \infty, \tag{10}$$

where $B(t)$ *is a standard Brownian motion, independent of* $E^{(3)}(t)$.

2.4. Linear hierarchical chains ($X^{(1)} \to X^{(2)} \to \ldots \to X^{(N)}$) of length N

In this Subsection we consider a generalisation of the CM MC to the case of N dimensions. Due to the complexity of sample paths for $N > 3$, we have not yet proved the analogue of (5) and (9). Thus, we prove the convergence for every fixed $t > 0$ instead of the weak convergence of the processes.

Let $\xi = \pm 1$ w.p. $1/2$. Let $\{\{\xi_n^{(j)}\}_{n=1}^{\infty}\}_{j=1}^{N}$ be mutually independent sequences of independent copies of ξ. Assume $X_0^{(1)} = \ldots = X_0^{(N)} = 0$. Then MC $(X_n^{(1)}, \ldots, X_n^{(N)})$ is defined as follows:

$$X_n^{(1)} = X_{n-1}^{(1)} + \xi_n^{(1)},$$

$$X_n^{(j)} = X_{n-1}^{(j)} + \begin{cases} 0, & \text{if } X_{n-1}^{(j-1)} \neq X_{n-1}^{(j)}, \\ \xi_n^{(j)}, & \text{if } X_{n-1}^{(j-1)} = X_{n-1}^{(j)}, \end{cases}$$

for $j \in \{2, \ldots, N\}$ and for $n \geq 1$.

For the next result we need the following distribution : let G_α be a one-sided stable distribution satisfying the condition $x^\alpha(1 - G_\alpha(x)) \to (2-\alpha)/\alpha$, as $x \to \infty$.

Theorem 4. *Let $\{\zeta_i\}_{i=1}^{\infty}$ be i.i.d. r.v.'s satisfying $\mathbb{P}\{\zeta_i \geq y\} = G_{1/2}(9/y^2)$. Let ψ be an r.v. with standard normal distribution independent of $\{\zeta_i\}_{i=1}^{\infty}$. Then, for any fixed $t > 0$, we have*

$$\frac{X_{[nt]}^{(N)}}{n^{1/2^N}} \Rightarrow t^{1/2^N} \psi \prod_{i=1}^{N} \sqrt{\sqrt{\frac{\pi}{2}}\zeta_i}, \text{ as } n \to \infty.$$

3. Trajectories of the "standard" Cat-and-Mouse model

Here we revisit the "standard" CM model and highlight a number of properties that are of use in the analysis of the DCM model.

We assume that $C_0 = M_0 = 0$. Let $V_n = |C_n - M_n|$, for $n \geq 0$. Then we can write $M_{n+1} = M_n + \xi_{n+1}^{(2)} I[V_n = 0]$, for $n \geq 1$. Note that $V_{n+1} = |C_{n+1} - M_{n+1}| = |C_n - M_n + \xi_{n+1}^{(1)} - \xi_{n+1}^{(2)} I[V_n = 0]|$. We can further observe that

$$V_{n+1} = \begin{cases} |\xi_{n+1}^{(1)} - \xi_{n+1}^{(2)}| \stackrel{d}{=} 1 + \xi_{n+1}^{(1)}, \text{ if } V_n = 0, \\ |C_n - M_n + \xi_{n+1}^{(1)}| \stackrel{d}{=} V_n + \xi_{n+1}^{(1)}, \text{ if } V_n \neq 0 \end{cases}$$

Thus, V_n forms a MC. Let $p_i(j) = \mathbb{P}\{V_{n+1} = j | V_n = i\}$, for $i, j \geq 0$. Note that $p_0(j) = p_1(j)$ for any j.

Let

$$U^{(2)}(0) = 0 \text{ and } U^{(2)}(k) = \min\{n > U^{(2)}(k-1) : \ V_n \in \{0,1\}\}. \tag{11}$$

Since $p_0(j) = p_1(j)$ for any j, we have that r.v.'s $\{U^{(2)}(k) - U^{(2)}(k-1)\}_{k=1}^{\infty}$ are i.i.d. and r.v. $(U^{(2)}(k) - U^{(2)}(k-1))$ does not depend on $V_{U^{(2)}(k-1)}$, for $k \geq 1$. From the Markov property we have

$$V_{U^{(2)}(k)+1} \stackrel{d}{=} 1 + \xi_1^{(1)} = \begin{cases} 0, & \text{w.p. } \frac{1}{2}, \\ 2, & \text{w.p. } \frac{1}{2}. \end{cases} \tag{12}$$

Thus, after each time-instant $U^{(2)}(k)$ the cat and the mouse jump with equal probabilities either to the same point or to two different points distant by 2. In the latter case, $V_{U^{(2)}(k+1)} = 1$, since the cat's jumps are 1 or -1. For the cat, let $\tau_m^{(1)} = \min\{n : \sum_{k=1}^{n} \xi_k^{(1)} = m\}$ denote the hitting time of the state m. Then

$$U^{(2)}(1) \stackrel{d}{=} 1 + \begin{cases} 0, & \text{w.p. } \frac{1}{2}, \\ \tau_1^{(1)}, & \text{w.p. } \frac{1}{2}. \end{cases} \tag{13}$$

The tail asymptotics for $\tau_1^{(1)}$ are known: $\mathbb{P}\{\tau_1^{(1)} > n\} \sim \sqrt{2/(\pi n)}$, as $n \to \infty$ (see, e.g., Section III.2 in Feller (1971a) for related result). Since $\tau_1^{(1)}$ has a distribution with a regularly varying tail, for any $m = 2, 3, \ldots$ we have

$$\mathbb{P}\{\tau_m^{(1)} > n\} \sim m\mathbb{P}\{\tau_1^{(1)} > n\} \sim \sqrt{2m^2/(\pi n)}, \text{ as } n \to \infty.$$

4. Trajectories in the Dog-and-Cat-and-Mouse model

In this Section we look at the structural properties of the DCM MC on $\mathbb{Z}$. Let us describe the main idea of the analysis which may be of independent interest as, we believe, it may be applied to other multi-component MCs.

Let $\{T^{(3)}(n)\}_{n=0}^{\infty}$ be the meeting time-instants, when all the agents meet at a certain point of $\mathbb{Z}$, and $\{J_k\}_{k=1}^{\infty} = \{T^{(3)}(k) - T^{(3)}(k-1)\}_{k=1}^{\infty}$ be the times between such events. Let $M_{T^{(3)}(n)}$, $n = 0, 1, \ldots$, be the locations of the mouse (and, therefore, the common location of the agents) at the embedded epochs $T^{(3)}(n)$ and $\{Y_k\}_{k=1}^{\infty} = \{M_{T^{(3)}(k)} - M_{T^{(3)}(k-1)}\}_{k=1}^{\infty}$ the corresponding jump sizes between the embedded epochs. Due to time homogeneity, random vectors $\{(Y_k, J_k)\}$ are i.i.d..

Let $N(t) = \max\{n : \ T^{(3)}(n) = \sum_{k=1}^{n} J_k \leq t\}$, for $t \geq 0$. Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} Y_k$. We show that the statement of Theorem 3 holds if we swap M_n with a continuous-time process

$$\widetilde{M}(t) = S_{N(t)} = \sum_{k=1}^{N(t)} Y_k, \text{ for } t \geq 0. \tag{14}$$

The process $\widetilde{M}(t)$ is a so-called *coupled continuous-time random walk* (see Becker-Kern *et al.* (2004)) and we use Theorem 5.1 from Kasahara (1984) to obtain its scaling properties.

4.1. Distribution of r.v. $J_1^{(3)}$

We assume that $D_0 = C_0 = M_0 = 0$. Let $V_n = (V_{n1}, V_{n2}) = (|D_n - C_n|, |C_n - M_n|)$. Then we can write

$$(D_{n+1}, C_{n+1}, M_{n+1}) = (D_n + \xi_{n+1}^{(1)}, C_n + \xi_{n+1}^{(2)} I[V_{n1} = 0], M_n + \xi_{n+1}^{(3)} I[V_{n2} = 0]).$$

Note further that

$$V_{n+1} \stackrel{d}{=} \begin{cases} (1 + \xi_{n+1}^{(1)}, 1 + \xi_{n+1}^{(2)}), \text{ if } V_{n1} = V_{n2} = 0, \\ (1 + \xi_{n+1}^{(1)}, V_{n2} + \xi_{n+1}^{(2)}), \text{ if } V_{n1} = 0 \text{ and } V_{n2} \neq 0, \\ (V_{n1} + \xi_{n+1}^{(1)}, 1), \text{ if } V_{n1} \neq 0 \text{ and } V_{n2} = 0, \\ (V_{n1} + \xi_{n+1}^{(1)}, V_{n2}), \text{ if } V_{n1} \neq 0 \text{ and } V_{n2} \neq 0. \end{cases} \tag{15}$$

Thus, $\{V_n\}_{n=0}^{\infty}$ is a MC. Let $p_{ij}(k,l) = \mathbb{P}\{V_{n+1} = (k,l) | V_n = (i,j)\}$, for $i,j,k,l \geq 0$. Note that $p_{00}(k,l) = p_{01}(k,l)$ for any k, l.

Let

$$U^{(3)}(0) = 0 \text{ and } U^{(3)}(k) = \min\{n > U^{(3)}(k-1) : \ V_n \in \{(0,0),(0,1)\}\}. \tag{16}$$

Since $p_{00}(m,l) = p_{01}(m,l)$ for any m, l, we have that r.v.'s $\{U^{(3)}(k) - U^{(3)}(k-1)\}_{k=1}^{\infty}$ are i.i.d. and r.v. $(U^{(3)}(k) - U^{(3)}(k-1))$ does not depend on $V_{U^{(3)}(k-1)}$, for $k \geq 1$.

In other words, the auxiliary states are $D_n = C_n = M_n \pm 1$. To find the resulting asymptotics we need the asymptotics of $U^{(3)}(1)$ and the relation between time-instants $T^{(3)}(1)$ and $\{U^{(3)}(k)\}_{k=1}^{\infty}$.

Lemma 1. *Let $V_0 \in \{(0,0),(0,1)\}$. Then we have*

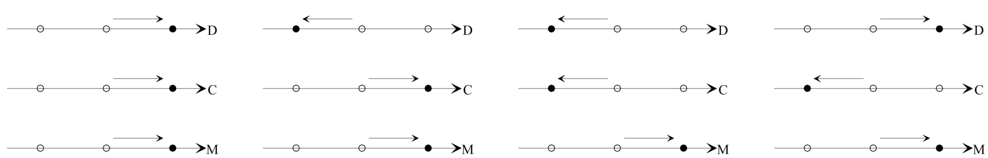
$$\mathbb{P}\{U^{(3)}(1) > n\} \sim \frac{2^{1/4}}{\Gamma(3/4)n^{1/4}}, \ \textit{as } n \to \infty.$$

Further, $U^{(3)}(1) = 1$ *iff* $V_{U^{(3)}(1)} = (0,0)$.

Proof. Let $V_0 = (0,0)$. It is apparent from the first line of equation (15) that

$$\mathbb{P}\{V_1 = (0,0)\} = \mathbb{P}\{V_1 = (2,0)\} = \mathbb{P}\{V_1 = (0,2)\} = \mathbb{P}\{V_1 = (2,2)\} = \frac{1}{4}. \tag{17}$$

Since $p_{00}(k,l) = p_{01}(k,l)$, r.v. V_1 has the same distribution given $V_0 = (0,1)$.



(a) $V_1 = (0,0)$. (b) $V_1 = (2,0)$. (c) $V_1 = (0,2)$. (d) $V_1 = (2,2)$.

Figure 1: The positioning after the first jump.

Let $V_1 = (0,2)$ (Figure 1c). From the second and the fourth lines of equation (15) we know that $|V_{(k+1)2} - V_{k2}| \in \{0,1\}$, given $V_{k2} \neq 0$. Therefore V_{k2} arrives at 1, before hitting 0 and $V_{U^{(3)}(1)} = (0,1)$. Let $\tau, \tau_1, \tau_2, \ldots$ be independent copies of $\tau_1^{(1)}$. Then $U^{(3)}(1)$ has the same distribution as $\sum_{k=1}^{\tau} \tau_k$ and we have that

$$\mathbb{P}\{\sum_{k=1}^{\tau} \tau_k > n\} \sim n^{-1/4} \frac{\Gamma^{1/2}(1/2)\Gamma(1/2)}{\Gamma(3/4)} \sqrt{\sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}} = \frac{2^{3/4}}{\Gamma(3/4)n^{1/4}},$$

as $n \to \infty$ (see Appendix B).

Let $V_1 = (2,2)$ (Figure 1d). From the fourth line of equation (15), V_{k2} remains at 2 (the cat and the mouse do not move) until V_{k1} reaches 0. This happens after a time which has the same distribution as $\tau_2^{(1)} = \min\{n > 0 : \sum_{k=1}^{n} \xi_k^{(1)} = 2\}$. Thus, we travel from $(2,2)$ to $(0,2)$ while never hitting $(0,0)$. We also know that the tail distribution of the travel time is $\mathbb{P}\{\tau_2^{(1)} > n\} \sim \sqrt{8/\pi n}$, as $n \to \infty$. Therefore, we travel from $(2,2)$ to $(0,2)$ much faster than from $(0,2)$ to $(0,1)$ and, given $V_1 = (2,2)$, we again have $\mathbb{P}\{U^{(3)}(1) > n\} \sim \mathbb{P}\{\sum_{k=1}^{\tau} \tau_k > n\}$, as $n \to \infty$.

Finally, let $V_1 = (2,0)$ (Figure 1b). From the third line of equation (15) we have $V_2 \stackrel{d}{=} (2+\xi_2^{(1)},1)$ and $V_{U^{(3)}(1)} = (0,1)$, where $\mathbb{P}\{U^{(3)}(1) > n\} \sim \sqrt{8/\pi n}$, as $n \to \infty$.

Thus,

$$\mathbb{P}\{U^{(3)}(1) > n\} \sim \frac{1}{2}\mathbb{P}\{\sum_{k=1}^{\tau} \tau_k > n\} \sim \frac{2^{1/4}}{\Gamma(3/4)n^{1/4}}, \text{ as } n \to \infty. \tag{18}$$

□

Thus, we get the relation between time-instants $T^{(3)}(1)$ and $\{U^{(3)}(k)\}_{k=1}^{\infty}$. Each time we are at the auxiliary state we have a probability $1/4$ to jump into the state $D_n = C_n = M_n$ independent of anything else. Using Lemma 1 and the results of Section 1.5 from Borovkov & Borovkov (2008) we get the following result.

Proposition 1. *Let* $\nu = \inf\{k \geq 1 : \ U^{(3)}(k) - U^{(3)}(k-1) = 1\}$. *Then* ν *has a geometric distribution with parameter* $1/4$ *and*

$$\mathbb{P}\{J_1^{(3)} > n\} = \mathbb{P}\{U^{(3)}(\nu) > n\} \sim 4\mathbb{P}\{U^{(3)}(1) > n\}, \text{ as } n \to \infty, \tag{19}$$

and therefore there exists a positive r.v. $D^{(3)}$ *with a stable distribution and Laplace transform* $\exp(-s^{1/4})$ *such that*

$$\frac{T^{(3)}(n)}{2^9 n^4} = \frac{\sum_{k=1}^{n} J_k^{(3)}}{2^9 n^4} \Rightarrow D^{(3)}, \text{ as } n \to \infty. \tag{20}$$

4.2. Distribution of r.v. $Y_1^{(3)}$

In the previous Subsection we analysed the time our process spends between auxiliary states. In this Subsection we analyse the total jumps of the mouse between the states (it can have either zero jumps, one jump, or two jumps).

Let $\{Z_k\}_{k=0}^{\infty}$ be an auxiliary Markov chain which satisfies $Z_k = M_{U^{(3)}(k)} - C_{U^{(3)}(k)} \in \{-1,0,1\}$. As before, at the times $n = 0$ and $n = U^{(3)}(\nu) = T^{(3)}(1)$ we have $D_n = C_n = M_n$ and, therefore, $Z_0 = Z_\nu = 0$. For any $k \in \{1, \nu - 1\}$ we have $Z_k = \pm 1$.

Let

$$\gamma_k^{(3)} = M_{U^{(3)}(k)} - M_{U^{(3)}(k-1)} \text{ for } k \geq 1. \tag{21}$$

These r.v.'s depend on each other through the auxiliary Markov chain $\{Z_k\}_{k=0}^{\infty}$. If we condition on the values of $\{Z_k\}_{k=0}^{\infty}$, r.v.'s $\{\gamma_k^{(3)}\}_{k=1}^{\infty}$ become independent. We represent the total jump as follows:

$$Y_1^{(3)} = M_{T^{(3)}(1)} = \sum_{k=1}^{\nu} \gamma_k^{(3)}.$$

We calculate its first and second moment and comment on a general power moment.

Since $D_0 = C_0 = M_0 = 0$, r.v. ν equals one if and only if $Z_1 = 0$ and $\gamma_1^{(3)} = \pm 1$. Additionally, we have

$$\mathbb{P}\{\gamma_1^{(3)} = \pm 1,\ Z_1 = 0 \mid Z_0 = 0\} = \mathbb{P}\{D_1 = C_1 = M_1 = \pm 1\} = \frac{1}{8}. \tag{22}$$

Another case of exactly one jump is when the cat and the mouse jump in different directions. Here we have

$$\mathbb{P}\{\gamma_1^{(3)} = \pm 1,\ Z_1 = \pm 1 \mid Z_0 = 0\} = \mathbb{P}\{C_1 = \mp 1, M_1 = \pm 1\} = \frac{1}{4}. \tag{23}$$

Further, the mouse can have two jumps in the same direction w.p.

$$\mathbb{P}\{\gamma_1^{(3)} = \pm 2,\ Z_1 = \pm 1 \mid Z_0 = 0\} = \mathbb{P}\{D_1 = \mp 1, C_1 = M_1 = \pm 1, M_2 = \pm 2\} = \frac{1}{16}, \tag{24}$$

or two jumps in the opposite directions w.p.

$$\mathbb{P}\{\gamma_1^{(3)} = 0,\ Z_1 = \pm 1 \mid Z_0 = 0\} = \mathbb{P}\{D_1 = \pm 1, C_1 = M_1 = \mp 1, M_2 = 0\} = \frac{1}{16}, \tag{25}$$

Thus, given $Z_0 = 0$ we have

$$\mathbb{P}\{Z_1 = 0\} = \frac{1}{4} \text{ and } \mathbb{P}\{Z_1 = \pm 1\} = \frac{3}{8}, \tag{26}$$

$$\mathbb{P}\{\gamma_1^{(3)} = 0\} = \frac{1}{8},\ \mathbb{P}\{\gamma_1^{(3)} = \pm 1\} = \frac{3}{8} \text{ and } \mathbb{P}\{\gamma_1^{(3)} = \pm 2\} = \frac{1}{16}. \tag{27}$$

From the obtained we get that $\mathbb{E}\gamma_1^{(3)} = 0$ and $\mathbf{Var}\gamma_1^{(3)} = 5/4$.

To analyse the distribution of $\gamma_k^{(3)}$, $k \geq 2$, and not overcomplicate the indices we assume that $D_0 = C_0 = 0$ and $M_0 = 1$, so $Z_0 = 1$. The case $Z_0 = -1$ is analogous by the symmetry. Then the mouse can make either zero jumps or exactly one before the next auxiliary state. In the case of zero jumps we have

$$\mathbb{P}\{\gamma_1^{(3)} = \pm 0,\ Z_1 = 0 \mid Z_0 = 1\} = \mathbb{P}\{D_1 = C_1 = 1\} = \frac{1}{4}, \tag{28}$$

$$\mathbb{P}\{\gamma_1^{(3)} = \pm 0,\ Z_1 = 1 \mid Z_0 = 1\} = \mathbb{P}\{C_1 = -1\} = \frac{1}{2}. \tag{29}$$

For the case of exactly one jump we have

$$\mathbb{P}\{\gamma_1^{(3)} = 1,\ Z_1 = 1 \mid Z_0 = 1\} = \mathbb{P}\{D_1 = -1, C_1 = 1, M_2 = 2\} = \frac{1}{8}, \tag{30}$$

$$\mathbb{P}\{\gamma_1^{(3)} = -1,\ Z_1 = -1 \mid Z_0 = 1\} = \mathbb{P}\{D_1 = -1, C_1 = 1, M_2 = 0\} = \frac{1}{8}. \tag{31}$$

Thus, given $Z_0 = 1$ we have

$$\mathbb{P}\{Z_1 = 0\} = \frac{1}{4},\ \mathbb{P}\{Z_1 = 1\} = \frac{5}{8} \text{ and } \mathbb{P}\{Z_1 = -1\} = \frac{1}{8}, \tag{32}$$

$$\mathbb{P}\{\gamma_1^{(3)} = 0\} = \frac{3}{4} \text{ and } \mathbb{P}\{\gamma_1^{(3)} = \pm 1\} = \frac{1}{8}. \tag{33}$$

From the obtained we get that $\mathbb{E}\left(\gamma_1^{(3)} | Z_0 = 1\right) = 0$ and $\mathbf{Var}\left(\gamma_1^{(3)} | Z_0 = 1\right) = 1/4$.

Let us return to the case $D_0 = C_0 = M_0 = 0$. Combining the results (27) and (33) we get

$$\mathbb{E}Y_1^{(3)} = \mathbb{E}\sum_{k=1}^{\nu} \gamma_k^{(3)} = \mathbb{E}\sum_{k=1}^{\infty}\left(\gamma_k^{(3)} I[\nu \geq k]\right) = \mathbb{E}\gamma_1^{(3)} + \sum_{k=2}^{\infty}\mathbb{E}\left(\gamma_k^{(3)} I[\nu \geq k]\right)$$
$$= \sum_{k=2}^{\infty}\mathbb{E}\left(\gamma_k^{(3)} I[Z_{k-1} = 1] + \gamma_k^{(3)} I[Z_{k-1} = -1]\right) = 0. \tag{34}$$

In the similar manner we transform the second moment:

$$\mathbb{E}(Y_1^{(3)})^2 = \sum_{k=1}^{\infty}\mathbb{E}\left((\gamma_k^{(3)})^2 I[\nu \geq k]\right) + 2\sum_{k=1}^{\infty}\sum_{m=k+1}^{\infty}\mathbb{E}\left(\gamma_k^{(3)}\gamma_m^{(3)} I[\nu \geq m]\right). \tag{35}$$

If we fix the values of $\{Z_k\}_{k=0}^{\infty}$ the r.v.'s $\gamma_k^{(3)}$ and $\gamma_m^{(3)}$ become independent. Combined with the fact that $\mathbb{E}\left(\gamma_m^{(3)} | Z_{m-1} = \pm 1\right) = 0$, we get that the second sum in (35) equals zero. Now we use the obtained conditioned second moments and the fact that the r.v. ν has a geometric distribution with parameter $1/4$. We obtain

$$\mathbb{E}(Y_1^{(3)})^2 = \mathbb{E}(\gamma_1^{(3)})^2 + \sum_{k=2}^{\infty}\mathbb{E}\left((\gamma_k^{(3)})^2 |\ Z_{k-1} = \pm 1\right)\mathbb{P}\{\nu \geq k\} = \frac{5}{4} + \frac{1}{4}(\mathbb{E}\nu - 1) = 2. \tag{36}$$

Finally, we have $|\gamma_1^{(3)}| \leq 2$ and r.v. ν has a light-tailed distribution. Therefore, r.v. $Y_1^{(3)} = M_{J_1^{(3)}} = \sum_{k=1}^{\nu}\gamma_k^{(3)}$ has a light-tailed distribution. Using that and a symmetry argument, we get the following result.

Proposition 2. *We have* $\mathbb{E}Y_1^{(3)} = 0$, $\mathbf{Var}Y_1^{(3)} = 2$ *and* $\mathbb{E}(Y_1^{(3)})^m < \infty$, *for any* $m \geq 3$.

5. Proofs of main results

In this section we provide proofs of our main results.

5.1. Proof of Theorem 1

For $i = 1, 2$, let $S_0^{(i)} = 0$ and $S_n^{(i)} = \sum_{k=1}^n \xi_k^{(i)}$, for $n \geq 1$. Let

$$\tau(0) = 0 \text{ and } \tau(n) = \inf\{m > \tau(n-1) : \; S_m^{(1)} = S_n^{(2)}\}, \text{ for } n \geq 1. \tag{37}$$

Since $\{\xi_k^{(1)}\}_{k=1}^\infty$ and $\{\xi_k^{(2)}\}_{k=1}^\infty$ are independent sequences of i.i.d. r.v.'s, we have that $\tau(n) - \tau(n-1) \overset{d}{=} \tau(1)$, for $n \geq 1$.

Let $\eta(t) = \max\{k \geq 0 : \; \tau(k) \leq t\}$ for $t \geq 0$. Define a continuous-time process $M'(t)$ by

$$M'(t) = 0 \text{ for } t \in [0,1) \text{ and } M'(t) = S^{(2)}_{\eta(t-1)+1} = \sum_{k=1}^{\eta(t-1)+1} \xi_k^{(2)}, \text{ for } t \geq 1. \tag{38}$$

It is straightforward to verify that $\{M'(n), \; n \geq 0\} \overset{d}{=} \{M(n), \; n \geq 0\}$. In the rest of the section we will omit the dash and simply write $M(t)$. The process $\{\widehat{M}(t)\}_{t\geq 0} = \{M'(t+1)\}_{t\geq 0}$ is a so-called *oracle continuous-time random walk* (see, e.g., Jurlewicz *et al.* (2012)). We need the following proposition.

Proposition 3. *We have*

$$d_{\mathcal{J}_1,\infty}\left(\left\{\frac{M(ct)}{b(\sqrt{c})}, t \geq 0\right\}, \left\{\frac{M(ct+1)}{b(\sqrt{c})}, t \geq 0\right\}\right) \overset{a.s.}{\to} 0, \;\; \text{as } c \to \infty. \tag{39}$$

Proof. This result follows from properties of the Skorokhod topology and the fact that $b(c) \to \infty$, as $c \to \infty$. First, we change time interval $[0, \infty)$ to $[0, T]$ with an arbitrary finite T. Second, we introduce an appropriate function λ_c (see Appendix A) such that $M(\lambda(ct))$ and $\widehat{M}(ct + 1)$ differ only near 0 and T. Then the distance between processes on time interval $[0, T]$ can be bounded by a r.v. which has the same distribution as $\max(\xi_1^{(2)}, \xi_2^{(2)})/b(c)$. Since this bound converges to 0 a.s., we get the result. $\square$

First, consider the case $\mathbb{E}\xi_1^{(2)} = 0$. We want to show that

$$\left(\frac{S_n^{(2)}}{b(n)}, \frac{\tau(n)}{n^2}\right) \Rightarrow (A^{(2)}, D^{(2)}), \text{ as } n \to \infty. \tag{40}$$

Given that, we will show that the first part of Theorem **1** follows from the next proposition.

Proposition 4. *(Theorem 3.1, Jurlewicz et al. (2012)) Assume (40) holds. Then*

$$\left\{\frac{\widehat{M}(ct)}{b(\sqrt{c})}, t \geq 0\right\} = \left\{\frac{M(ct+1)}{b(\sqrt{c})}, t \geq 0\right\} \overset{\mathcal{D}}{\Rightarrow} \left\{A^{(2)}(E^{(2)}(t)), t \geq 0\right\}, \ \textit{as } c \to \infty. \quad (41)$$

A similar result was proven in Theorem 3.6 from Henry and Straka (2011).

We will now show that relation (40) holds and that r.v.'s $A^{(2)}$ and $D^{(2)}$ are independent, which means

$$\mathbb{E}\exp\left(i\left(\lambda_1 \frac{S_n^{(2)}}{b(n)} + \lambda_2 \frac{\tau(n)}{n^2}\right)\right) = \mathbb{E}\exp\left(i\left(\lambda_1 \frac{\xi_1^{(2)}}{b(n)} + \lambda_2 \frac{\tau(1)}{n^2}\right)\right)^n$$
$$= \left(1 + \frac{f_1(\lambda_1) + f_2(\lambda_2)}{n} + o\left(\frac{1}{n}\right)\right)^n, \quad (42)$$

as $n \to \infty$, for some functions f_1 and f_2. Indeed, convergence of characteristic functions is equivalent to weak convergence of r.v.'s and for independence of r.v.'s it is sufficient to verify that the characteristic function of the sum is equal to the product of respective characteristic functions. Since the right-hand side of (42) converges to $\exp(f_1(\lambda_1))\exp(f_2(\lambda_2))$, it will prove the convergence and the independence of the limits $A^{(2)}$ and $D^{(2)}$.

3.1 We start with the case of Theorem **1**. From condition (3) we have a weak convergence of $S_n^{(2)}/b(n)$ to a r.v. $A^{(2)}$. Again, this is equivalent to convergence of characteristic functions. Thus, (3) implies

$$\mathbb{E}\exp\left(i\lambda_1 \frac{\sum_{k=1}^n \xi_k^{(2)}}{b(n)}\right) = \left[\mathbb{E}\exp\left(i\lambda_1 \frac{\xi_1^{(2)}}{b(n)}\right)\right]^n \to \mathbb{E}\exp\left(i\lambda_1 A^{(2)}\right), \text{ as } n \to \infty. \quad (43)$$

Additionally, if $B^n(n) \to z$, as $n \to \infty$, then $n \log B(n) \to \log z$, which leads to $\log B(n) \sim n^{-1}\log z$. Finally, such relation leads to $B(n) \sim 1 + n^{-1}\log z$, as $n \to \infty$. Thus, we have the following

$$\mathbb{E}\exp\left(i\lambda_1 \frac{\xi_1^{(2)}}{b(n)}\right) \sim 1 + \frac{l_1(\lambda_1)}{n}, \text{ as } n \to \infty, \quad (44)$$

where $l_1(\lambda) = \log \mathbb{E}\exp(i\lambda A^{(2)})$, the logarithmic characteristic function of $A^{(2)}$.

Let $\{\tau_k^{(1)}\}_{k=1}^\infty$ be independent copies of τ, the time needed for the simple random walk to hit 0 if it starts from 1, independent of $\{\xi_n^{(2)}\}_{n=1}^\infty$. Then we have the following relation for $\tau(1)$:

$$\tau(1) \stackrel{d}{=} I[\xi_n^{(2)} \neq 0] \sum_{k=1}^{|\xi_n^{(2)}|} \tau_k^{(1)} + I[\xi_n^{(2)} = 0](1 + \tau_1^{(1)}). \tag{45}$$

Since $\mathbb{P}\{\tau > n\} \sim \sqrt{2/(\pi n)}$, as $n \to \infty$, we have

$$\mathbb{P}\{\tau(1) > n\} \sim (\mathbb{E}|\xi_1^{(2)}| + \mathbb{P}\{\xi_1^{(2)} = 0\})\mathbb{P}\{\tau > n\} \tag{46}$$

and there exists a r.v. $D^{(2)}$ having a stable distribution with index 1/2 such that

$$\frac{\tau(n)}{n^2} \Rightarrow D^{(2)}, \text{ as } n \to \infty. \tag{47}$$

Using the same argument as for (44) we get

$$\mathbb{E}\exp\left(i\lambda_2 \frac{\tau}{n^2}\right) \sim 1 + \frac{l_2(\lambda_2)}{n}, \text{ as } n \to \infty, \tag{48}$$

where $\lambda_2(\lambda) = \log \mathbb{E}\exp(i\lambda D^{(2)}/(\mathbb{E}|\xi_1^{(2)}| + \mathbb{P}\{\xi_1^{(2)} = 0\}))$, the logarithmic characteristic function of $D^{(2)}/(\mathbb{E}|\xi_1^{(2)}| + \mathbb{P}\{\xi_1^{(2)} = 0\})$. Then we use the total probability formula to get the following:

$$\begin{aligned}
&\mathbb{E}\exp\left(i\left[\lambda_1 \frac{\xi_1^{(2)}}{b(n)} + \lambda_2 \frac{\tau(1)}{n^2}\right]\right) \\
&\quad = \sum_{-\infty}^{\infty} \exp\left(i\lambda_1 \frac{k}{b(n)}\right) \mathbb{P}\{\xi_1^{(2)} = k\} \mathbb{E}\exp\left(i\lambda_2 \frac{\tau(1)}{n^2} |\xi_1^{(2)} = k\right) \\
&\quad = \mathbb{P}\{\xi_1^{(2)} = 0\} \mathbb{E}\exp\left(i\lambda_2 \frac{1+\tau}{n^2}\right) + \\
&\quad + \sum_{k \neq 0} \exp\left(i\lambda_1 \frac{k}{b(n)}\right) \mathbb{P}\{\xi_1^{(2)} = k\} \left(\mathbb{E}\exp\left(i\lambda_2 \frac{\tau}{n^2}\right)\right)^{|k|}.
\end{aligned}$$

In order to transform the last sum in the last equation we use (48) and get the following, for any $m > 0$:

$$\begin{aligned}
\left(\mathbb{E}\exp\left(i\lambda_2 \frac{\tau}{n^2}\right)\right)^m &= \left(1 + \frac{l_2(\lambda_2)}{n} + o\left(\frac{1}{n}\right)\right)^m \\
&= \exp\left(m \ln(1 + \frac{l_2(\lambda_2) + o(1)}{n})\right) \\
&= \exp\left(\frac{m l_2(\lambda_2)}{n}(1 + o(1))\right) \\
&= 1 + \frac{m l_2(\lambda_2)(1 + o(1))}{n} + \frac{1}{n^2} \sum_{j=2}^{\infty} \frac{(m l_2(\lambda_2)(1 + o(1)))^j}{n^{j-2} j!},
\end{aligned}$$

as $n \to \infty$. Now we use the fact that if $\sum_{-\infty}^{\infty} A_n = \sum_{-\infty}^{\infty} B_n + \sum_{-\infty}^{\infty} C_n$ and if series $\sum_{-\infty}^{\infty} A_n$ and $\sum_{-\infty}^{\infty} B_n$ converge, then $\sum_{-\infty}^{\infty} C_n$ converges too. We have

$$\begin{aligned}
&\sum_{k\neq 0} \exp\left(i\lambda_1 \frac{k}{b(n)}\right) \mathbb{P}\{\xi_1^{(2)} = k\} \left(\mathbb{E}\exp\left(i\lambda_2 \frac{\tau}{n^2}\right)\right)^{|k|} \\
&\quad = \left(\mathbb{E}\exp\left(i\lambda_1 \frac{\xi_1^{(2)}}{b(n)}\right) - \mathbb{P}\{\xi_1^{(2)} = 0\}\right) \\
&\quad + \frac{l_2(\lambda_2)}{n} \sum_{k\neq 0} |k| \exp\left(i\lambda_1 \frac{k}{b(n)}\right) \mathbb{P}\{\xi_1^{(2)} = k\} + o\left(\frac{1}{n}\right) \\
&\quad = \left(\mathbb{E}\exp\left(i\lambda_1 \frac{\xi_1^{(2)}}{b(n)}\right) - \mathbb{P}\{\xi_1^{(2)} = 0\}\right) + \mathbb{E}|\xi_1^{(2)}| \frac{l_2(\lambda_2)}{n} + o\left(\frac{1}{n}\right),
\end{aligned}$$

as $n \to \infty$. Using (44) and (48), we have

$$\begin{aligned}
&\mathbb{E}\exp\left(i\left[\lambda_1 \frac{\xi_1^{(2)}}{b(n)} + \lambda_2 \frac{\tau(1)}{n^2}\right]\right) \\
&\qquad\qquad = 1 + \frac{l_1(\lambda_1)}{n} + (\mathbb{E}|\xi_1^{(2)}| + \mathbb{P}\{\xi_1^{(2)} = 0\}) \frac{l_2(\lambda_2)}{n} + o\left(\frac{1}{n}\right), \quad (49)
\end{aligned}$$

as $n \to \infty$. We have proved that equation (42) holds with $f_1(\lambda_1) = l_1(\lambda_1)$ and $f_2(\lambda_2) = (\mathbb{E}|\xi_1^{(2)}| + \mathbb{P}\{\xi_1^{(2)} = 0\}) l_2(\lambda_2)$. Therefore, equation (40) holds and we can use Propositions **3** and **4** to prove the first part of Theorem **1**.

Turn now to the second part and assume $\mathbb{E}\xi^{(2)} = \mu \neq 0$. Then the above arguments are applicable to $\sum_{k=1}^{\eta(t)+1} (\xi_k^{(2)} - \mu)$. Thus, we have shown that the process $\left(\left(\sum_{k=1}^{\eta(nt)+1} (\xi_k^{(2)} - \mu)\right) / b(\sqrt{n}),\ t \geq 0\right)$ weakly converges to the limiting one (see Appendix A for corresponding definitions). Since $\mu < \infty$, we have $b(n) = o(n)$, as $n \to \infty$, and therefore the process

$$\left(\frac{\sum_{k=1}^{\eta(nt)+1} (\xi_k^{(2)} - \mu)}{\sqrt{n}},\ t \geq 0\right) = \left(\frac{\sum_{k=1}^{\eta(nt)+1} (\xi_k^{(2)} - \mu)}{b(\sqrt{n})} \frac{b(\sqrt{n})}{\sqrt{n}},\ t \geq 0\right) \qquad (50)$$

converges to the zero-valued process.

Thus, it follows from the representation

$$\frac{\widehat{M}(nt)}{\sqrt{n}} = \frac{\sum_{k=1}^{\eta(nt)+1} (\xi_k^{(2)} - \mu)}{\sqrt{n}} + \frac{\mu(\eta(nt) + 1)}{\sqrt{n}} \qquad (51)$$

and from the Corollary of Theorem 3.2 from Meerschaert and Scheffler (2004) that

$$\left\{\frac{\widehat{M}(nt)}{\sqrt{n}},\ t \geq 0\right\} \stackrel{\mathcal{D}}{\Rightarrow} \{\mu E^{(2)}(t),\ t \geq 0\},\ \text{as } n \to \infty. \qquad (52)$$

3.2 Turn now to the second case. Under the assumption of the finiteness of second moments we can expand our result to the case where both $\xi^{(1)}$ and $\xi^{(2)}$ have general distributions. Assume now that $\{S_n^{(1)}\}_{n=0}^{\infty} = \{\sum_{k=1}^{n} \xi_k^{(1)}\}_{n=0}^{\infty}$ is an aperiodic random walk with zero-mean and finite-variance-σ_1^2 increments. A theory of general random walks and their hitting times is well developed. Nevertheless, it was challenging for us to find results uniform in terms of the hitting point. From Section **3.3** of Uchiyama (2011a), we have that, uniformly in x,

$$\mathbb{E}\left[\exp\left(it\tau(1)\right) \mid \xi_1^{(2)} = x\right] = 1 - (a^*(x) + e_x(t))(\sigma_1\sqrt{-2it} + o(\sqrt{|t|})), \text{ as } t \to 0, \quad (53)$$

where

$$a^*(x) = 1 + \sum_{n=1}^{\infty} \left(\mathbb{P}\{S_n^{(1)} = 0\} - \mathbb{P}\{S_n^{(1)} = -x\}\right), \quad (54)$$

$$e_x(t) = c_x(t) + is_x(t), \quad (55)$$

$$|c_x(t)| = O\left(x^2\sqrt{|t|}\right), \text{ as } t \to 0, \quad (56)$$

$$s_0(t) = 0 \text{ and } \frac{s_x(t)}{x} = o(1), \text{ as } t \to 0, \text{ uniformly in } x \neq 0. \quad (57)$$

Following steps similar to those used in the previous part we take $t = \lambda_2/n^2$ and, eventually, let n become large. A very important relation here is (56). When we take characteristic function $\mathbb{E}\exp\left(i\left[\lambda_1\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}} + \lambda_2\frac{\tau(1)}{n^2}\right]\right)$ and start to separate it into different summands the relation (56) leads to a summand

$$\sum_{x\in\mathbb{Z}} O\left(\frac{x^2}{n^2}\right)\mathbb{P}\{\xi_1^{(2)} = x\}, \text{ as } n \to \infty, \quad (58)$$

and this is the main reason why we need to assume that $\xi_1^{(2)}$ has a finite second moment.

Assume now that $\mathbb{E}\xi_1^{(2)} = 0$ and $\sigma_2 = \mathbf{Var}\xi_1^{(2)} < \infty$. We have (see, e.g., Proposition 7.2 from Uchiyama (2011b))

$$\sigma_1^2(a^*(x) - I(x = 0)) \sim |x|, \quad \text{as } |x| \to \infty. \quad (59)$$

As a consequence we get $\mathbb{E}a^*(\xi_1^{(2)}) < \infty$. Let $p^{(2)}(x) = \mathbb{P}\{\xi_1^{(2)} = x\}$. Then the total probability formula gives us

$$\mathbb{E}\exp\left(i\left[\lambda_1\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}+\lambda_2\frac{\tau(1)}{n^2}\right]\right)$$
$$=\sum_{x\in\mathbb{Z}}\exp\left(i\lambda_1\left[\frac{x}{\sigma_2\sqrt{n}}\right]\right)\mathbb{E}\left[\exp\left(i\left[\lambda_2\frac{\tau(1)}{n^2}\right]\right)\mid \xi_1^{(2)}=x\right]p^{(2)}(x). \quad (60)$$

Now we use (53)-(57) to get

$$\mathbb{E}\exp\left(i\left[\lambda_1\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}+\lambda_2\frac{\tau(1)}{n^2}\right]\right)=\mathbb{E}\left[\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right]-$$
$$-\frac{\sigma_1\sqrt{-2i\lambda_2}}{n}\mathbb{E}\left[a^*(\xi_1^{(2)})\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right]+$$
$$+O\left(\frac{1}{n^2}\mathbb{E}\left[\left(\xi_1^{(2)}\right)^2\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right]\right)+$$
$$+o\left(\frac{1}{n}\mathbb{E}\left[\xi_1^{(2)}\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right]\right)+o\left(\frac{1}{n}\right),$$

as $n\to\infty$. Next, we use relation (59) and the Taylor expansion for the exponent to get

$$\mathbb{E}\left[a^*(\xi_1^{(2)})\exp\left(i\lambda_1\left[\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}\right]\right)\right]=\mathbb{E}\left[a^*(\xi_1^{(2)})\right]+o(1),\ \text{as } n\to\infty. \quad (61)$$

Since $\mathbb{E}\xi_1^{(2)}=0$ and $\mathbf{Var}\xi_1^{(2)}<\infty$, the Central Limit Theorem holds. Thus, we have the analogue of (44) with l_1 being a logarithmic characteristic function of a r.v. with standard normal distribution. Finally, we get

$$\mathbb{E}\exp\left(i\left[\lambda_1\frac{\xi_1^{(2)}}{\sigma_2\sqrt{n}}+\lambda_2\frac{\tau(1)}{n^2}\right]\right)=1+\frac{l_1(\lambda_1)}{n}-\frac{\sigma_1\sqrt{-2i\lambda_2}}{n}\mathbb{E}\left[a^*(\xi_1^{(2)})\right]+o\left(\frac{1}{n}\right). \quad (62)$$

Thus, we proved equation (42) for this case and the rest of the proof follows the same argument as in the previous case.

5.2. Proof of Theorem 3

Random vectors $\{Y_n^{(3)},J_n^{(3)}\}_{n=1}^\infty$ are i.i.d., where $Y_1^{(3)}=\sum_{k=1}^{\nu}\gamma_k^{(3)}$ and $J_1^{(3)}=T^{(3)}(1)=U^{(3)}(\nu)$. We have

$$\eta(t)=\max\{n>0:\ \sum_{k=1}^{n}J_k^{(3)}\le t\}\ \text{and}\ \widetilde{M}(t)=\sum_{k=1}^{\eta(t)}Y_k^{(3)}. \quad (63)$$

From Propositions **1** and **2** we have

$$\mathbb{E}Y_1^{(3)} = 0,\ \mathbf{Var}Y_1^{(3)} = 2,\ \mathbb{E}(Y_1^{(3)})^m < \infty,\ \text{for } m \geq 2, \text{ and } \mathbb{P}\{J_1^{(3)} > n\} \sim \frac{2^{9/4}}{\Gamma(3/4)n^{1/4}}, \tag{64}$$

as $n \to \infty$. From Theorem 5.1 from Kasahara (1984) we have

$$\left\{ \frac{\widetilde{M}(nt)}{2^{-9/8}n^{1/8}\sqrt{\mathbf{Var}Y_1^{(3)}}}, t \geq 0 \right\} \overset{\mathcal{D}}{\Rightarrow} \{B(E^{(3)}(t)), t \geq 0\},\ \text{as } n \to \infty, \tag{65}$$

where $B(t)$ is a standard Brownian motion, independent of $E^{(3)}(t)$.

We show now that (65) holds with $M(nt)$ in the place of $\widetilde{M}(nt)$. It is sufficient to prove that for any fixed $T > 0$

$$\frac{\max_{1\leq k\leq [nT]} \left\{\widetilde{M}_k - M_k\right\}}{n^{1/8}} \overset{a.s.}{\to} 0,\ \text{as } n \to \infty. \tag{66}$$

In the time interval $(\eta(nt), nT]$ there are no time-instants n when $D(n) = C(n) = M(n)$, however the mouse may have jumps. Nevertheless, the number of this jumps can be bounded by $2\widehat{\nu}$, where r.v. $\widehat{\nu}$ has a geometric distribution with parameter $1/4$.

Let $U_n = \max_{T^{(3)}(n-1)\leq l\leq T^{(3)}(n)} |\widetilde{M}_l - M_l|$, $n \geq 1$. We have

$$\max_{1\leq l\leq J_1^{(3)}} |\widetilde{M}_l - M_l| \overset{d}{\leq} \sum_{k=1}^{\nu} |\xi_k|. \tag{67}$$

Proposition 5. *For any $m \geq 1$ we have $\mathbb{E}U_1^m < \infty$ and $n^{-1/m}\max_{1\leq l\leq n} U_l$ converges to 0 a.s., as $n \to \infty$.*

We have $\eta(nT) \to \infty$ a.s. and there exists a r.v. ζ such that $n^{-1/4}\eta(nT) \Rightarrow \zeta$, as $n \to \infty$ (see, e.g., Section XI.5 in Feller (1971b)). Thus,

$$\begin{aligned}
\frac{|\max_{1\leq k\leq [nT]} \left\{\widetilde{M}_k - M_k\right\}|}{n^{1/8}} &\leq \frac{\max_{1\leq l\leq \eta(nT)} \{U_l\}}{n^{1/8}} + \frac{2\widehat{\nu}}{n^{1/8}} \\
&= \frac{\max_{1\leq l\leq \eta(nT)} \{U_l\}}{\eta^{1/4}(nT)} \left(\frac{\eta(nT)}{n^{1/2}}\right)^{1/4} + \frac{2\widehat{\nu}}{n^{1/8}} \overset{a.s.}{\to} 0.
\end{aligned} \tag{68}$$

This completes the proof of Theorem **3**.

5.3. Proof of Theorem 4

Random process $X^{(1)}$ is a simple random walk on $\mathbb{Z}$ and for $j \in [2, N]$ we have

$$\mathbb{P}\{X^{(j)}(n) - X^{(j)}(n-1) = 1 |\ X^{(j)}(n-1) = X^{(j-1)}(n-1)\}$$
$$= \mathbb{P}\{X^{(j)}(n) - X^{(j)}(n-1) = -1 |\ X^{(j)}(n-1) = X^{(j-1)}(n-1)\} = \frac{1}{2}.$$

Let us give a new representation for such process. Let $X^{(1)}(0) = X^{(2)}(0) = \ldots = X^{(N)}(0) = 0$ and let r.v. $T_j(n)$ denote the time when $X^{(j)}$ makes n-th step. Let $T_j(0) = 0$. Note the difference between T_j and $T^{(3)}$. Thus, $\{X^{(j)}(T_j(k))\}_{k=0}^{\infty}$ is a simple random walk on $\mathbb{Z}$ and if $X^{(j)}(n) \neq X^{(j)}(n-1)$ then $n \in \{T_j(k)\}_{k=1}^{\infty}$. Let

$$\xi_k^{(j)} = X^{(j)}(T_j(k)) - X^{(j)}(T_j(k-1)) = X^{(j)}(T_j(k)) - X^{(j)}(T_j(k)-1)$$

for $j \geq 1$ and $k \geq 1$. By definition $\{\{\xi_k^{(j)}\}_{k=0}^{\infty}\}_{j=1}^{N}$ are mutually independent and equal ± 1 w.p. $1/2$.

Since $X^{(1)}$ jumps every time, $T_1(k) = k$ for $k \geq 0$. Let τ be the time that simple random walk goes from point 1 to 0. From Section 3 it is easy to deduce that the time between meeting time-instants of the cat and the mouse has the same distribution as τ. Thus, if we look at the system only at the times $\{T_j(k)\}_{k=0}^{\infty}$ the time between meeting time-instants of $X^{(j)}$ and $X^{(j+1)}$ has the same distribution as τ.

Let us define $\nu_j(n) = \max\{k \geq 0 :\ T_j(k) \leq n\}$, the number of time-instants up to time n when $X^{(j)}$ changed its value. Then we can rewrite the dynamics of the j-th coordinate as

$$X^{(j)}(n) = \sum_{k=1}^{\nu_j(n)} \xi_{T_j(k)}^{(j)}.$$

Our restrictions on the distribution of the increments $\xi_k^{(j)}$, for $k \geq 1$, give us the next important property of our process.

Proposition 6. *Sequences $\{T_j(k)\}_{k=0}^{\infty}$ and $\{\xi_k^{(j)}\}_{k=1}^{\infty}$ are independent for any $j \in \{1, \ldots, N\}$.*

This property comes from the space-symmetry of the model.

For $j = 1$ the result is trivial, since $\nu_1(n) = n$. We show the result for $j = 2$ and then extend it onto $j > 2$. Define

$$^1\tau(0) = 0 \text{ and } ^1\tau(k) = \inf\{n > {}^1\tau(k-1) :\ X^{(1)}(n) = X^{(2)}(n)\}, \text{ for } k \geq 1. \quad (69)$$

One can see that in our model $T_2(k) = 1 + {}^1\tau(k-1)$, for $k \geq 1$. In the time interval $[1, {}^1\tau(1)]$ the second coordinate changes its value only at the time $T_2(1) = 1$. Thus, the time ${}^1\tau(1)$ does not depend on $\xi_k^{(2)}$, for $k \geq 2$. Additionally, the trajectory $\{X^{(1)}(n)\}_{n=0}^{\infty}$ has the same distribution as $\{-X^{(1)}(n)\}_{n=0}^{\infty}$. Thus,

$$\mathbb{P}\{{}^1\tau(1) = n,\ \xi_1^{(2)} = 1\} = \mathbb{P}\{{}^1\tau(1) = n,\ \xi_1^{(2)} = -1\}. \tag{70}$$

As a corollary of the last equation, we get that ${}^1\tau(1)$ has the same distribution as the time that is needed for the simple random walk to hit 0 if it starts from 1. This implies that

$$\mathbb{P}\{{}^1\tau(1) > n\} \sim \sqrt{\frac{2}{\pi n}},\ \text{as } n \to \infty. \tag{71}$$

From the symmetry of our model, it follows further that the sequence $\{{}^1\tau(k)\}_{k=0}^{\infty}$, and subsequently the sequences $\{T_2(k)\}_{k=0}^{\infty}$ and $\{\nu_2(n)\}_{n=1}^{\infty}$, do not depend on $\{\xi_k^{(2)}\}_{k=1}^{\infty}$ (and, therefore, on $\{\xi_k^{(j)}\}_{k\geq 1, j\geq 2}$).

For the analysis of $\{T_j(k)\}_{k=0}^{\infty}$, $j > 2$, we need to define an 'embedded version' of ${}^1\tau(k)$. Let

$${}^j\tau(0) = 0 \text{ and } {}^j\tau(k) = \inf\{m > {}^j\tau(k-1) :\ X^{(j)}(T_j(m)) = X^{(j+1)}(T_j(m))\},\ \text{for } k \geq 1. \tag{72}$$

The process $\{{}^j\tau(k)\}_{k=0}^{\infty}$ counts the number of times that the process $X^{(j)}$ changed its value between the time-instants when $X^{(j)}$ and $X^{(j+1)}$ have the same value. Using the same argument as before, we get that the sequence $\{{}^j\tau(k)\}_{k=0}^{\infty}$ does not depend on $\{\xi_k^{(j+1)}\}_{k=1}^{\infty}$.

The j-th coordinate $X^{(j)}$ changes its value for the k-th time at the time-instant n if and only if up to time $n-1$ processes $X^{(j-1)}$ and $X^{(j)}$ had the same value exactly $k-1$ times (not including $X^{(j-1)}(0) = X^{(j)}(0) = 0$) and the last time was at the time-instant $n-1$ (which also means that at the time-instant $n-1$ the process $X^{(j-1)}$ changes its value). This can be rewritten as

$$T_j(k) = n \ \Leftrightarrow\ n - 1 = T_{j-1}({}^{j-1}\tau(k-1)),\ \text{for } j \geq 2,\ k \geq 1, \tag{73}$$

and thus $T_j(k) = 1 + T_{j-1}({}^{j-1}\tau(k-1))$. Thus, since sequences $\{T_2(k)\}_{k=0}^{\infty}$ and $\{{}^2\tau(k)\}_{k=0}^{\infty}$ do not depend on $\{\xi_k^{(j)}\}_{k\geq 1, j\geq 3}$, the same holds for $\{T_3(k)\}_{k=0}^{\infty}$. Therefore, using the induction, we get that the sequences $\{T_j(k)\}_{k=0}^{\infty}$ and $\{\xi_k^{(j)}\}_{k=1}^{\infty}$ are independent for any $j \geq 1$.

As a corollary of this result we get

$$X^{(j)}(n) = \sum_{k=1}^{\nu_j(n)} \xi^{(j)}_{T_j(k)} \stackrel{d}{=} \sum_{k=1}^{\nu_j(n)} \xi^{(j)}_k. \tag{74}$$

Let ${}^j\eta(n) = \max\{k \geq 0 : \ {}^j\tau(k) \leq n\}$ for $n \geq 0$ and $j \in [1, \dots, N]$. Since the sequence $\{{}^j\tau(k)\}_{k=0}^{\infty}$ depends only on the sequence $\{\xi^{(j)}_k\}_{k=1}^{\infty}$, we have that $\{{}^j\eta(n)\}_{j=1}^{N-1}$ are i.i.d. r.v.'s. For $n \geq 1$ and $j \in \{1, \dots, N\}$ we have

$$\begin{aligned}\nu_j(n) = \max\{k \geq 0 : \ T_j(k) \leq n\} &= \max\{k \geq 1 : \ 1 + T_{j-1}({}^{j-1}\tau(k-1)) \leq n\} \\ &= 1 + \max\{k \geq 0 : \ T_{j-1}({}^{j-1}\tau(k)) \leq n-1\} \\ &= 1 + \max\{k \geq 0 : \ {}^{j-1}\tau(k) \leq \nu_{j-1}(n-1)\} \\ &= 1 + {}^{j-1}\eta(\nu_{j-1}(n-1)).\end{aligned} \tag{75}$$

For $n < N-1$ we can iterate the process and get

$$\begin{aligned}\nu_N(n) &= 1 + {}^{N-1}\eta(1 + {}^{N-2}\eta(\dots(1 + {}^{N-n}\eta(0))\dots)) \\ &= 1 + {}^{N-1}\eta(1 + {}^{N-2}\eta(\dots(1 + {}^{N-n+1}\eta(1))\dots)) \\ &\stackrel{d}{=} 1 + {}^{n-1}\eta(1 + {}^{n-2}\eta(\dots(1 + {}^{1}\eta(1))\dots)) \\ &= \nu_n(n).\end{aligned} \tag{76}$$

For $n \geq N-1$ we have

$$\nu_N(n) = 1 + {}^{N-1}\eta(1 + {}^{N-2}\eta(\dots + {}^1\eta(n-N+1))). \tag{77}$$

We want to construct a process with the same distribution as $\{\nu_N(n)\}_{n=0}^{\infty}$ in a form of $\nu_{N-1}(\varphi(n))$, where process $\{\varphi(n)\}_{n=0}^{\infty}$ is independent of everything else. Define process $\{\eta(n)\}_{n=0}^{\infty} \stackrel{d}{=} \{{}^{N-1}\eta(n)\}_{n=0}^{\infty}$, which is independent of everything else. Then, for $n \geq N-1$, we have

$$\nu_N(n) \stackrel{d}{=} 1 + {}^{N-2}\eta(1 + {}^{N-3}\eta(\dots + \eta(n-N+1))). \tag{78}$$

Using the same formula for $\nu_{N-1}(m)$ with such m that $m-(N-1)+1 = 1 + {}^{N-1}\eta(n - N+1)$, we get

$$\nu_N(n) \stackrel{d}{=} \nu_{N-1}(N-1+\eta(n-N+1)), \text{ for } n \geq N-1. \tag{79}$$

Then, for $n \geq N$ we have $X^{(N)}(n) \stackrel{d}{=} X^{(N-1)}(N-1+\eta(n-N+1))$. There exists a non-degenerate r.v. ζ (see Section XI.5 in Feller (1971b)) such that $\mathbb{P}\{\zeta_i \geq y\} = G_{1/2}(9/y^2)$ and

$$\eta(n)\mathbb{P}\{{}^{N-1}\tau(1) > n\} \Rightarrow \zeta, \text{as } n \to \infty. \tag{80}$$

Therefore, using (71) we get

$$\frac{j-1+\eta(n-j+1)}{\sqrt{n}} = \frac{j-1+\eta(n-j+1)}{\sqrt{n-j+1}} \frac{\sqrt{n-j+1}}{\sqrt{n}} \Rightarrow \sqrt{\frac{\pi}{2}}\zeta, \tag{81}$$

as $n \to \infty$, for $j \geq 1$. We now present a known result that we utilise to prove Theorem **4**.

Proposition 7. *(Dobrushin (1955), (v)) Let $Y(t)$ and τ_n be independent sequences of r.v.'s such that*

$$\frac{Y(t)}{bt^\beta} \Rightarrow Y, \text{ as } t \to \infty, \text{ and } \frac{\tau_n}{dn^\delta} \Rightarrow \tau, \text{ as } n \to \infty. \tag{82}$$

Then for independent Y and τ we have

$$\frac{Y(\tau_n)}{bd^\beta n^{\beta\delta}} \Rightarrow Y\tau^\beta, \text{ as } n \to \infty. \tag{83}$$

Indeed, by the Central Limit Theorem $X^{(1)}(n)/\sqrt{n}$ weakly converges to a normally distributed r.v. ψ (we assume that ψ and ζ are independent). Together with (81) and independence of $X^{(1)}(n)$ and $\eta(n)$, this insures that condition (82) holds with $Y(t) = X^{(1)}([t])$, $\tau_n = 1 + \eta(n-1)$ and $\beta = \delta = 1/2$. By the Proposition **7**, we get

$$\frac{X^{(2)}(n)}{n^{1/4}} \stackrel{d}{=} \frac{X^{(1)}(1+\eta(n-1))}{n^{1/4}} \Rightarrow \psi\sqrt{\sqrt{\frac{\pi}{2}}\zeta}, \text{ as } n \to \infty. \tag{84}$$

Let $\{\zeta_j\}_{j=2}^N$ be independent copies of ζ which are independent of ψ. Next, we use the induction argument. For some $j \geq 1$ condition (82) holds with $Y(t) = X^{(j)}([t])$, $\tau_n = j-1+\eta(n-j+1)$, $\beta = 2^{-j}$ and $\delta = 2^{-1}$. By Proposition **7**, we get

$$\frac{X^{(j+1)}(n)}{n^{2^{-(j+1)}}} \stackrel{d}{=} \frac{X^{(j)}(j-1+\eta(n-j+1))}{n^{2^{-(j+1)}}} \Rightarrow \psi\prod_{i=2}^{j+1}\sqrt{\sqrt{\frac{\pi}{2}}\zeta_i}, \text{ as } n \to \infty. \tag{85}$$

This concludes the proof of Theorem **4**.

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Appendix

Appendix A. Weak convergence for processes from $D[[0,\infty),\mathbb{R}]$

To make the paper self-contained we recall definition of $\mathcal{J}_1$-topology (see, e.g., Skorokhod (1956)). Let $D[[0,T],\mathbb{R}]$ denote the space of all right continuous functions on $[0,T]$ having left limits. For any $g \in D[[0,T],\mathbb{R}]$ let $\|g\| = \sup_{t\in[0,T]} |g(t)|$.

Let Λ be the set of increasing continuous functions $\lambda : [0,T] \to [0,T]$, such that $\lambda(0)=0$ and $\lambda(T)=T$. Let λ_{id} denote the identity function. Then

$$d_{\mathcal{J}_1,T}(g_1,g_2) = \inf_{\lambda\in\Lambda} \max(\|g_1\circ\lambda - g_2\|, \|\lambda - \lambda_{id}\|)$$

defines a metric inducing $\mathcal{J}_1$.

On the space $D[[0,\infty),\mathbb{R}]$ the $\mathcal{J}_1$-topology is defined by the metric

$$d_{\mathcal{J}_1,\infty}(g_1,g_2) = \int_0^\infty e^{-t} \min(1, d_{\mathcal{J}_1,t}(g_1,g_2))dt.$$

Convergence $g_n \to g$ in $(D[[0,\infty),\mathbb{R}],\tau)$ means that $d_{\tau,T}(g_n,g) \to 0$ for every continuity point T of g.

Let $\{\{X_n(t)\}_{t\geq 0}\}_{n=1}^\infty$ and $\{X(t)\}_{t\geq 0}$ be stochastic processes with trajectories from $D[[0,\infty),\mathbb{R}]$. We say that weak convergence

$$\{X_n(t)\}_{t\geq 0} \overset{\mathcal{D}}{\Rightarrow} \{X(t)\}_{t\geq 0},$$

holds if

$$\mathbb{E}f(\{X_n(t)\}_{t\geq 0}) \to \mathbb{E}f(\{X(t)\}_{t\geq 0}), \text{ as } n\to\infty,$$

for any continuous and bounded function f on $D[[0,\infty),\mathbb{R}]$ endowed with $\mathcal{J}_1$-topology.

Proposition 8. *Let $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$ be two sequences of stochastic processes with trajectories from $D[[0,\infty),\mathbb{R}]$. Given $d_{\mathcal{J}_1,\infty}(X_n,Y_n) \overset{a.s.}{\to} 0$, we have*

$$X_n - Y_n \overset{\mathcal{D}}{\Rightarrow} 0.$$

Appendix B. Tail asymptotics for randomly stopped sum

Let $\xi_1, \xi_2, \dots$ be positive i.i.d. r.v.'s with a common distribution function F. Let $S_0 = 0$ and $S_k = \xi_1 + \dots \xi_k$, $k \geq 1$. Let τ be a counting r.v. with distribution function G, independent of $\{\xi_k\}_{k=1}^{\infty}$. For a general overview concerning asymptotics of tail-distribution of S_τ see, e.g., Denisov *et al.* (2010) and references therein. The next result follows from Corollary **3** from Foss & Zachary (2003).

Proposition 9. *Assume that $\overline{F}(x) \sim l_1(x)/x^\alpha$, $\alpha \in [0,1)$ and τ has any distribution with $\mathbb{E}\tau < \infty$. Then*

$$\mathbb{P}\{S_\tau > n\} \sim \mathbb{E}\tau \mathbb{P}\{\xi > n\} \text{ as } n \to \infty. \tag{86}$$

The next result we use in Lemma 1 and we prove it using Tauberian theorems.

Proposition 10. *Assume that $\overline{F}(x) \sim l_1(x)/x^\alpha$ and $\overline{G}(x) \sim l_2(x)/x^\beta$, $\alpha, \beta \in (0,1)$. Then*

$$\mathbb{P}\{S_\tau > n\} \sim n^{-\alpha\beta} \frac{\Gamma^\beta(1-\alpha)\Gamma(1-\beta)}{\Gamma(1-\alpha\beta)} l_1^\beta(n)\, l_2\left(\frac{n^\alpha}{\Gamma(1-\alpha) l_1(n)}\right), \text{ as } n \to \infty. \tag{87}$$

Proof. Denote the c.d.f. of S_τ as H. Let

$$\overline{F}(x) = 1 - F(x),\ x \in \mathbb{R}, \tag{88}$$

$$\widehat{F}(\lambda) = \mathbb{E}e^{-\lambda\xi_1} = \int_0^\infty e^{-\lambda x} dF(x),\ \lambda \geq 0. \tag{89}$$

Define $\overline{G}, \widehat{G}, \overline{H}$, and $\widehat{H}$ similarly. We use the following result.

Proposition 11. ***(Corollary 8.1.7, Bingham, Goldie and Teugels (1987))*** *For a constant $\alpha \in [0,1]$, l and for a slowly varying at infinity function, the following are equivalent:*

$$1 - \widehat{F}(\lambda) \sim \lambda^\alpha l\left(\frac{1}{\lambda}\right), \text{ as } \lambda \downarrow 0, \tag{90}$$

$$\begin{cases} \overline{F}(x) \sim \frac{l(x)}{x^\alpha \Gamma(1-\alpha)}, \text{ as } x \to \infty, & \text{if } 0 \leq \alpha < 1, \\ \int_0^x t dF(t) \sim \int_0^x \overline{F}(t) dt \sim l(x), \text{ as } x \to \infty, & \text{if } \alpha = 1. \end{cases} \tag{91}$$

Using this result, we get

$$1-\widehat{F}(\lambda)\sim\lambda^{\alpha}\Gamma(1-\alpha)l_1\left(\frac{1}{\lambda}\right)\text{ and }1-\widehat{G}(\lambda)\sim\lambda^{\beta}\Gamma(1-\beta)l_2\left(\frac{1}{\lambda}\right),\text{ as }\lambda\downarrow 0. \tag{92}$$

Let us analyse $\widehat{H}$:

$$\widehat{H}(\lambda)=\mathbb{E}e^{-\lambda S_\tau}=\sum_{k=1}^{\infty}e^{-\lambda(\xi_1+\ldots+\xi_k)}\mathbb{P}\{\tau=k\}=\mathbb{E}\left(\mathbb{E}e^{-\lambda\xi_1}\right)^{\tau}=\widehat{G}(-\ln\widehat{F}(\lambda)). \tag{93}$$

Since

$$-\ln\widehat{F}(\lambda)=-\ln(1-(1-\widehat{F}(\lambda)))\sim 1-\widehat{F}(\lambda),\text{ as }\lambda\downarrow 0, \tag{94}$$

we have

$$\begin{aligned}1-\widehat{H}(\lambda)&\sim 1-\widehat{G}\left(\lambda^{\alpha}\Gamma(1-\alpha)l_1\left(\frac{1}{\lambda}\right)\right)\\&\sim\lambda^{\alpha\beta}\Gamma^{\beta}(1-\alpha)\Gamma(1-\beta)l_1^{\beta}\left(\frac{1}{\lambda}\right)l_2\left(\frac{1}{\lambda^{\alpha}\Gamma(1-\alpha)l_1\left(\frac{1}{\lambda}\right)}\right),\end{aligned} \tag{95}$$

as $\lambda\downarrow 0$, and finally

$$\overline{H}(x)\sim x^{-\alpha\beta}\frac{\Gamma^{\beta}(1-\alpha)\Gamma(1-\beta)}{\Gamma(1-\alpha\beta)}l_1^{\beta}(x)\,l_2\left(\frac{x^{\alpha}}{\Gamma(1-\alpha)l_1(x)}\right),\text{ as }x\to\infty. \tag{96}$$

□

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