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PROJECT VALUATION: PRICE FORECASTS BOUND TO DISCOUNT RATES

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Abstract

For their appraisals, most companies use discount rates that account for timing and riskiness of projects. Yet, especially for commodity projects, discounting future cash flows is generally at odds with the assumptions in a company's hurdle rate. With a multitude of technical and market uncertainties, inconsistent assessments lead to biased valuations and poor investment decisions. In this paper, we consider price forecasts and discount rates in an integrated framework. We calibrate the risk premiums in a two-factor stochastic price process with a CAPM-based discount rate. Together with the analysts' long-term prices forecasts, the suggested method improves consistency in valuation and decision making.

Keywords: Risk Premiums; Commodity Price Uncertainty; Project Appraisal; CAPM; Risk-neutral; Application in Petroleum Industry

1. Introduction

In a discussion paper published by the International Valuations Standard Council, participants from mining and petroleum industries commented on how they forecast prices and select discount rates[1]. Most responses revealed a tendency to use market prices and the company's Weighted Average Cost of Capital (WACC) for valuation of capital investments. However, the paper also showed disparity in how to apply the method.

Some respondents used prices of forward contracts in the commodity market and adjusted them with analysts' long-term views. Others relied entirely on price forecasts based on macroeconomic fundamentals. Most respondents agreed that the discount rate should reflect the riskiness of the investment and, in addition, pointed out that risk-adjustment of their project outcomes is more practical than using an all-purpose discount rate. Overall, the responses were illuminating. They showed the industry's concern for key valuation parameters: price forecasts and discount rates. In this paper, we further discuss that this is a single problem of determining risk premiums.

Project appraisals are about expressing uncertainty[2] in terms of value. Traditionally, we discount expected cash flows with a discount rate that reflects both the time value of money and risks. Here, using for example the Capital Asset Pricing Model (CAPM), we calculate a stock's risk and return tradeoff. Then, considering a specific project and its sources of financing, we assess the opportunity cost of capital by further adjusting the discount rate. This approach considers both market preferences and specific project features to assess value. An alternative approach to valuation (common in the real options literature, e.g. Cox et al., 1985, Smith and Nau, 1995) would be to calculate the certainty-equivalent cash flows from market perspective and then discount them with the risk-free rate. Overall,

[1] The discussion papers are available at https://www.ivsc.org/standards/international-valuation-standards
[2] In valuation, "uncertainty" refers to lack of knowledge about the outcome of events. In uncertain projects, there are opportunities to create value by changing course and capitalizing on the desired outcomes or to protect against the downsides. Thus, uncertainty could both destroy or create value.

we either risk-adjust the cash flows or the discount rate. If done consistently, both approaches to valuation yield identical results.

This reveals an important connection; the risk-adjustment of the cash flows or the discount rate account for identical risks. While a projects' risk is an aggregate of multiple factors, we could still make a useful distinction: We assume risk factors are either private or public uncertainties[3]. If we separately present the private uncertainties in our decision model, then, the remaining risk in a project's cash flows should be from public risk—mostly due to the uncertainty in prices, exchange rates, or interest rates. This means, the risk-adjustment of the discount rate (or cash flows) should be only for the public risk of a project.

In other words, forecasting prices and determining the discount rate are two sides of the same coin. They are means to represent public risk in valuations. According to general equilibrium theory (e.g. Debreu, 1959), this is a single problem of estimating prices (or values) for assets in a market with uncertain future states. Even with uncertain supply, demand, and complex interactions between the markets, all valuations are within the same general equilibrium and should converge. In practice, when analysts predict future prices, they should also consult those who assess the discount rate.

In this paper, we develop a joint method of estimating the expected future prices of commodities—price forecasts that include a risk premium. Compared with risk-neutral price forecasts that have no compensation for risks, the expected future prices are slightly higher, reflecting an offset for their riskiness. Here, we use crude oil derivatives that hedge price risk to estimate a risk-adjusted outlook. We then combine these estimates with the risk and return trade-off of petroleum assets within the financial stock markets. These complementary views of the market outlook lead to our estimates of the risk premiums.

Our goal is to contribute to the practice of project appraisal. Yet, any valuation model relies on a model of investment decision—an abstraction of the reality reflecting only key features. To streamline our analysis, we use the two-factor process in Schwartz and Smith (2000) to describe prices and estimate cash flows. We further apply our model to an investment decision in petroleum exploration. Our discussions are however general and apply to other contexts and a variety of price models.[4]

For our price model, we use an implied method of parameter calibration (introduced in Schwartz and Smith, 2000, and implemented by Jafarizadeh and Bratvold, 2012) that uses current market information. Here, the information about commodity derivative contracts (futures and options on futures) reflect the risk-neutral outlook. As this calibration relies on hedged prices, it is natural to assume that we need more information to estimate the risk premiums. Hamilton and Wu (2014) used the information about interaction of hedgers and arbitrageurs in crude oil markets, and Cortazar et al. (2015) and Hahn et al (2018) utilized CAPM-like asset pricing model to estimate risk premiums. More recently, Cortazar et al (2019) incorporated information from analysts' forecasts into estimation of risk premiums. In this paper, we estimate the risk premiums using information from the financial markets—the risk and return tradeoff in a company's stock as reflected in CAPM framework.

The CAPM beta of a stock only partly reflects the price risk. It also reflects other obscuring effects like financial leverage and growth options. But if we remove these effects, then we end up with a market-

[3] Technical risks mostly refer to possibilities specific to a project, like encountering unexpectedly large (or small) hydrocarbon reservoir. On the other hand, general events such as unexpectedly high (or low) oil prices, are public risks. Smith and Nau (1995) further define those risks that could be hedged with market instruments as market risks and those that cannot be hedged as private risk. In this paper, we follow their stronger description.

[4] Our discussions apply to project appraisals involving commodities, especially those traded on futures exchanges. Admittedly, not all commodities have the breadth and depth of the crude oil exchanges. Still, not all investments are as long spanning and global as petroleum exploration and development projects. While most price forecasts extrapolate beyond longest maturity futures contracts, in general they need to fit the valuations.

implied measure of price risk premiums. This "project beta" would be a measure of project's risks from unhedged prices. Following Bernardo et al. (2007 and 2012) and Da et al. (2012), we use external information about leverage and real options potentials of a firm to refine our CAPM estimate of project beta. We then use this measure to assess price risk premiums by comparing the valuation results with those of the risk-neutral approach. The two valuations converge if we consistently account for risks.

In the next section, we discuss the price model and the implied parameter estimation using the market information. We further discuss that, like the Kalman filter method of parameter estimation on historical prices, this method is also incapable of accurately estimating price risk-premiums. Next in section 3, we estimate project discount rate using the added information about a company's market returns. This approach based on CAPM, quantifies the effect of market risk by separating the effect of leverage and growth options. Section 4 applies the method to an oil and gas example and section 5 concludes.

2. Modelling Uncertain Prices

2.1 Price Forecasts in Valuations

A good decision model requires consistent valuation of project outcomes. In corporate decisions, we estimate the economic worth of the outcomes on a scale of "expected net present value". Each outcome could have cash flows with various levels of risk or occurring at differing times, hence, any valuation model should explicitly account for risk and time.

To account for differences in time, we discount cash flows using a risk-free rate. To account for differences in risk, we could either increase the discount rate (a risk premium) or reduce the cash flow estimates (a risk discount). In traditional corporate finance, we risk-adjust the discount rate. We discount riskier cash flows with a higher discount rate. As Modigliani and Miller (1958) show, we accept those projects that their aggregate future cash flows yield greater than WACC—their Weighted Average Cost of Capital. In the alternative risk-neutral approach, instead of adjusting the discount rate we adjust cash flows for risk. For commodity assets, such an adjustment is straightforward; we risk-adjust the cash flows by using risk-adjusted prices. Market instruments that hedge price risk reflect this risk-free view of the prices. This approach to valuation is common the real options literature—applied e.g. in Brennan and Schwartz (1985)—and formulated in Black (1988) for investments in general.

The two approaches to valuation differ in where they account for risk. With consistent assumptions, they should however yield identical values. As Freitas and Brandão (2010) discuss, a joint implementation of the two approaches leads to estimates of the risk premiums. Figure 1 shows this relationship in, e.g., a petroleum production project. Yet in practice, the approaches seem divergent mainly because they rely on dissimilar sources of information. The traditional corporate finance approach uses information from financial markets—a company's stock beta—while the risk-neutral approach uses information from commodity derivative markets—the instruments that hedge price risks, e.g. forwards, futures and options on futures contracts. However, in general equilibrium, any consistent method of valuation should give identical values.

In an aggregate framework, we could jointly estimate price forecasts and discount rates by combining information from financial and commodity markets. Assume we intend to assess a single cash flow, selling Q barrels of crude oil at time T in the future. We use expected future price $\mathrm{E}(S_T)$ to estimate future sales and because the prices are uncertain, we discount this risky cash flow with a rate that accounts for risk. The value of sales today, V_0, is

$$V_0 = \frac{\mathrm{E}(S_T) \times Q}{(1 + \mathrm{r} + \varepsilon)^T} \tag{1}$$

Here, r is the risk-free rate and ε is the premium representing cash flow risk.

In an alternative approach, we could use commodity derivative contracts to estimate future sales. The price of a futures contract for delivery at T is $F_{0,T}$. The contract guarantees a price for Q barrels, therefore the value of this riskless cash flow today is

$$V_0 = \frac{F_{0,T} \times Q}{(1+\mathrm{r})^T} \tag{2}$$

By substituting V_0 in equation (1) we get

$$\frac{\mathrm{E}(S_T)}{(1+\mathrm{r}+\varepsilon)^T} = \frac{F_{0,T}}{(1+r)^T} \tag{3}$$

In other words, the forward price at time T discounted at the risk–free rate is the present value of one barrel of crude oil sold at time T. In addition, the expected price forecast reflects the same risk as the premium in the discount rate.

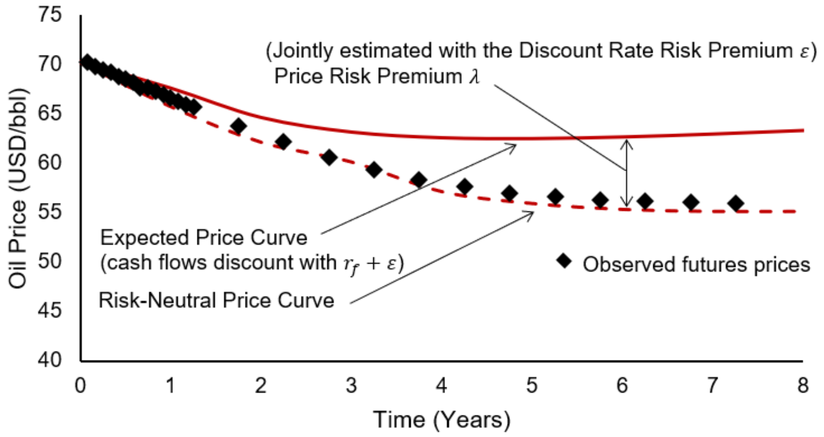

Figure 1—Using either risk neutral or WACC-based valuations, a consistent estimation of risk premiums leads to identical project values. Here, $\boldsymbol{\lambda}$ (price risk premium) and $\boldsymbol{\varepsilon}$ (discount rate's risk premium) are jointly estimated.

While the above relationships are general, the estimates of risk premiums would be project– and model–specific, depending on project length and on models describing price dynamics. In this paper, we use the two-factor price model in Schwartz and Smith (2000) to discuss the nature of the problem. The model is versatile yet simple and could reduce to single-factor price models. We further apply this price model to a petroleum exploration project to estimate risk-premiums.

In the following sections, we first use futures and options on futures contracts to calibrate the price process. This will lead to a risk neutral view of the expected prices like the dashed curve in figure 1. We then use financial information in the CAPM framework to estimate the risk-adjusted discount rate. This leads to an estimate of ε, the risk premium within the discount rate. Finally, in converging the two valuations, we estimate the price risk premium λ and generate price forecasts (the solid curve in figure 1).

2.2 Dynamic Price Model

Price movements are uncertain. Yet unlike stock prices' moves that follow a random walk, commodity prices are to some extent predictable. They follow the equilibrium between supply and demand. When, for example, oil prices constantly increase (decrease) over successive periods, we expect the prices in the next period to fall (rise). In other words, we expect *mean reversion* in commodity prices. Economic arguments admit these expectations: when prices are above an equilibrium level, production that was otherwise too expensive would come online, increasing the aggregate supply, and driving the prices down. When prices are below the equilibrium, expensive production would go offline and fewer barrels of oil in the market drives the prices up towards the equilibrium.[5]

In practice, myriads of factors influence the supply-demand balance. Most price moves are short-lived, but some have a permanent effect. Studies of the term structure of commodity futures, e.g. Schwartz (1997), also confirms this distinction. The market participants expect most price shocks to fade away but some movements to be permanent. In summary, amongst many factors influencing the prices and their complex relationships, we could model two effects: transient factors that follow a mean-reverting process and permanent factors that follow a random walk. Such a two-factor price model is consistent with the economic arguments and the observations in the market. In this paper, we use Schwartz and Smith's (2000) two-factor model. It is a simple, yet useful, model for valuation and communication of results.[6]

We denote S_t as spot price at time t, where $\ln S_t = \xi_t + \chi_t$. The short-term and long-term factors, χ_t and ξ_t, follow stochastic processes

$$d\chi_t = -\kappa\chi_t dt + \sigma_\chi dz_\chi \tag{4}$$

$$d\xi_t = \mu dt + \sigma_\xi dz_\xi \tag{5}$$

Here, κ is the coefficient of mean-reversion, σ_ξ and σ_χ are the volatilities for the long-term and short-term factors, and μ is the drift rate for the long-term factor. In addition, dz_χ and dz_ξ, the increments of the Brownian motion process, are correlated with $dz_\chi dz_\xi = \rho_{\xi\chi} dt$.

We denote $F_{0,T}$ as the current price of a futures contract for delivery at T. Schwartz and Smith (2000) show that

$$\begin{aligned}\ln F_{0,T} = {} & e^{-\kappa T}\chi_0 + \xi_0 + \left(\mu - \lambda_\xi\right)T - (1 - e^{-\kappa T})\frac{\lambda_\chi}{\kappa} \\ & + ½\left((1 - e^{-2\kappa T})\frac{\sigma_\chi^2}{\kappa} + \sigma_\xi^2 T + 2(1 - e^{-\kappa T})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right)\end{aligned} \tag{6}$$

Here, $\lambda_\chi$and λ_ξ are respectively the short-term and long-term risk premiums. Under the risk-neutral measure, these premiums are zero and the expected spot prices at T will be equal to the futures price for delivery at T. The instantaneous variance of $\ln F_{0,T}$ is

$$Var\left(\ln F_{0,T}\right) = e^{-2\kappa T}\sigma_\chi^2 + \sigma_\xi^2 + 2e^{-\kappa T}\rho_{\chi\xi}\sigma_\chi\sigma_\xi \tag{7}$$

In other words, the volatility of the futures prices does not depend on the risk premiums.

The model fits in with the general market observations about the term structure of futures prices and their volatilities.[7] For example, long-maturity futures contracts reflect long-term equilibrium price

[5] Mean reversion is central to understanding commodity prices; yet in practice, different commodities manifest distinct levels of mean-reversion. We expect commodities with inflexible production and excessive cost of storage to exhibit more mean reversion. Empirical studies also confirm this mean reverting behavior in many commodities including metals (Fama and French, 1988) and crude oil (Bessembinder et al, 1995 and Pindyck, 1999).

[6] Gibson and Schwartz (1990) developed the first two-factor commodity price model. It was later reinterpreted in e.g., Baker, Mayfield, and Parsons (1998) and Schwartz and Smith (2000). While the notion of two or more factors describing price dynamics is appealing (and have also led to other price models), in this paper we use the model in Schwartz and Smith (2000) for its simplicity and usefulness.

[7] In practice, owning a physical inventory of a commodity has added benefits compared with owning a derivative contract. Physical prices reflect this "convenience yield". The Schwartz and Smith (2000) price model does not explicitly formulate the convenience yield but is equivalent to the stochastic convenience yield model developed in Gibson and Schwartz (1990). In this context, we estimate risk premiums that reflect the convenience yield.

levels. Increasing T in equation (6) also weakens the influence of the short-term factor but keeps the long-term effect. As another example, the term structure of futures volatilities in the market is generally declining. In equation (7), futures variance also declines with increasing T.

We could use market information to calibrate parameters of the price model. Using futures and options on futures with varying maturities, we simply try to choose model parameters that best fit the market observations. In this approach, we use futures and implied volatility of options on the futures to support our parameter calibration. Compared to the other commonly used method of calibration—using historical futures in a Kalman filter—our implied method is simpler and easier to implement.

The first step in implied parameter calibration is to construct a forward curve, forward prices with increasing maturities, using equation (6) and compare it with observed futures prices from the market. For added support, we can also a construct the volatility term curve using equation (7) and compare it with implied option volatilities from the market. These option volatilities are not directly observable, but we could use the procedure below to obtain them from European options on futures.

The values of a European call option on a futures contract with expiry at T, denoted by c_T, and strike price K, is

$$c_T = e^{-rT}\left(F_{0,T}N(d) - KN\left(d - \sigma_\varphi(T)\right)\right) \tag{8}$$

And the value of a put option, p_T, is

$$p_T = e^{-rT}\left(KN\left(\sigma_\varphi(T) - d\right) - F_{0,T}N(d)\right) \tag{9}$$

Where $\sigma_\varphi(T)$ is the volatility of the futures contract (the annualized volatility would be $\sigma_\varphi(T)/\sqrt{T}$), r is the risk-free discount rate, and $d = \frac{\ln F/K}{\sigma_\varphi(T)} + ½\sigma_\varphi(T)$. In addition, $N(d)$ is the cumulative probability for the standard normal distribution. We assume the options expire at the same time as their underlying futures.

By observing option prices in the markets and "reversing" equations (8) and (9) we could calculate the implied volatility of futures prices. In other words, the implied volatility is the volatility which, when used in equations (8) or (9), returns a theoretical value equal to the observed price of the option. Because theoretical inverse functions are difficult, we developed computer code in the accompanying spreadsheet to perform the "goal seek" operation.

We further use an optimization model that variates model parameters until the curves generated from equations (6) and (7) fit the market observations. We show the details of this parameter estimation procedure in Appendix A. Figure 2 shows the result, were the sum of squared differences between model curves (dashed lines) with the market observations is set to minimum. These fitted curves lead to the parameters in table 1.

Table 1 Parameters of the two-factor price process calibrated to market data

χ_0	0.300
ξ_0	3.960
κ	0.700
σ_χ	0.500
μ	-0.026
σ_ξ	0.200
ρ	0.192

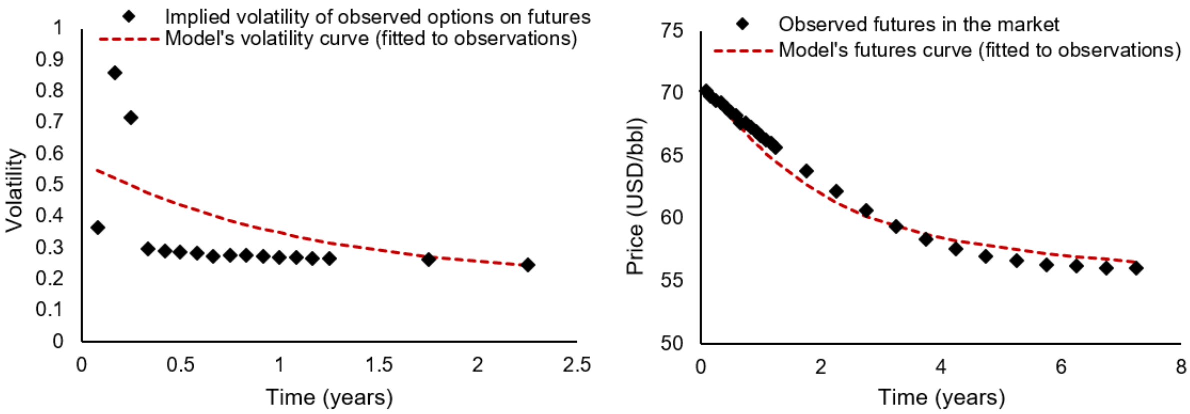

Figure 2 fitting the model's curves to market data

Our software implementation of this parameter estimation technique uses Solver in MS Excel® and is simple and efficient. Schwartz and Smith (2000) suggest weighting the errors to improve the fit, and Jafarizadeh and Bratvold (2012) suggest a sequential approach of first fixing some parameters to parts of the curves and then determining the rest. Our optimization model enables both these provisions with a stepwise automated optimization and user-defined error weights.

3. Estimating risk premiums

3.1 Price risk premiums and Discount Rate

The calibrated model describes the risk-neutral behavior of prices. We fitted the model parameters to the futures prices and their implied volatility. In other words, the model describes the expected prices as if they were equal to futures prices.

$$\mathrm{E}^*(S_t) = F_{0,t} \tag{10}$$

This also means that the risk premiums λ_χ and λ_ξ are set to zero. If we exogenously determine λ_χ and λ_ξ, we could use the equations below to transform the risk neutral process into the true process, as implied by equation (6)

$$\mu = \mu^* + \lambda_\xi \tag{11}$$

$$\xi_0 = \xi_0^* + \frac{\lambda_\chi}{\kappa} \tag{12}$$

And the short-term factor is $\chi_0 = \ln(S_0) - \xi_0$.

With earlier discussions about the relationship between expected and risk-neutral prices, we replace the terms of equations (6) and (10-12) in equation (3). We will have

$$\frac{\exp\left(e^{-\kappa T}\chi_0 + \xi_0 + B(T)\right)}{(1 + \mathrm{r} + \varepsilon)^T} = \frac{\exp\left(e^{-\kappa T}\chi_0 + \xi_0 + A(T)\right)}{(1 + r)^T} \tag{13}$$

Where

$$B(T) = \mu T + ½\left((1 - e^{-2\kappa T})\frac{\sigma_\chi^2}{\kappa} + \sigma_\xi^2 T + 2(1 - e^{-\kappa T})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right)$$

$$A(T) = (\mu - \lambda_\xi)T - (1 - e^{-\kappa T})\frac{\lambda_\chi}{\kappa} + ½\left((1 - e^{-2\kappa T})\frac{\sigma_\chi^2}{\kappa} + \sigma_\xi^2 T + 2(1 - e^{-\kappa T})\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\right)$$

This shows the relationship between risk adjustments of price forecast and the discount rate. Yet, the derivation that led to equation (13) assumed we sell Q barrels of oil in the future; meaning, we have a single cash flow purely affected by price risk. In practice, project appraisals are more sophisticated. There are other risks in a project's cash flows; the discount rate may reflect uncertainty in costs, tariffs, or exchange rates. In addition, projects cash flows reflect the embedded managerial and growth options. The price risk is arguably a significant portion of the total systematic risk of a project. With these complexities, we need a more general version of equation (13) for effective estimation of risk premiums. The premiums would be project specific and will generally depend on a project's length—an issue discussed in Myers and Turnbull (1977) and more recently in e.g. Hahn and Dyer (2018).

We assume that we can separate the effect of various sources of uncertainty in a project and account for each of them individually. In this ideal environment, we discount project cash flows (that are merely affected by price risk) using project WACC $(r + \varepsilon)$. The value of this project, V_0, is the discounted sum of all its future cash flows and it should remain the same whether we use risk-neutral or project WACC approach

$$V_0 = \sum_{t=0}^{T} \frac{\mathrm{E}(S_t) \times Q_t - C_t - D_t}{(1 + r + \varepsilon)^t} = \sum_{t=0}^{T} \frac{F_{0,t} \times Q_t - C_t - D_t}{(1 + r)^t} \tag{14}$$

Here, costs, C_t, production, Q_t, and deductions, D_t, $(t = 0, \ldots, T)$ are considered certain. The project-specific version of equation (13) becomes

$$\sum_{t=0}^{T} \frac{\exp\left(e^{-\kappa T}\chi_0 + \xi_0 + B(t)\right) \times Q_t - C_t - D_t}{(1 + r + \varepsilon)^t} = \sum_{t=0}^{T} \frac{\exp\left(e^{-\kappa T}\chi_0 + \xi_0 + A(t)\right) \times Q_t - C_t - D_t}{(1 + r)^t} \tag{15}$$

Where $A(t)$ and $B(t)$ are the same as in equation (13).

With equation (15), we have an explicit relationship between price risk premiums λ_χ and λ_ξ and the discounting risk premium ε. This estimation technique is simple and easy, but the question really is: how do we estimate ε so that it only stands for price uncertainty?

3.2 Estimating Project Risk and the Premium ε

To estimate a project's discount rate $r + \varepsilon$, we expand the approach in Jafarizadeh (2019). We first consider the CAPM interpretation of the risk premium. Assuming the expected market return is μ_m then the expected return on the firm's stock is $r + \beta(\mu_m - r)$ where the equity beta, β, measures the co-movement of firm's stock price with the market. The premium $\beta(\mu_m - r)$ reflects for the systematic risk of investing in the firm's stock.

However, the risk premium from CAPM reflects the aggregate corporate uncertainty. It reflects the risk of uncertain prices together with other obscuring effects, including the execution of embedded real options, financial leverage, and macro-economic factors. Although not an exact method, we use the following procedure to strip away other effects from the CAPM beta until we are left with a measure of price risk. Jafarizadeh and Bratvold (2019) discuss a detailed description of this approach.

First, there is financial leverage. Companies use debt to magnify their investing capabilities. Debt usually has a lower cost, but the commitment to pay the interests escalates the risk (and beta) of company's stock. In the end, the shares of a leveraged company are riskier even though the nature of its projects stays the same.

Using Modigliani and Miller (1958) arguments, we account for the effect of financial leverage in beta. We distinguish business risk from the total equity risk. Here, we calculate asset beta (unlevered beta, β_{Asset}) by compensating for the effect of leverage. The asset beta is then a measure of riskiness in a company's assets:

$$\beta_{\text{Asset}} = \frac{\beta}{\left(1 + (1 - Tax\ Rate)\frac{D}{E}\right)} \tag{16}$$

Where D and E are respectively the market values for company's debt and equity. Assuming a corporate tax rate of 35%, Figure 3 shows the equity and asset beta in our sample of US domiciled petroleum companies[8].

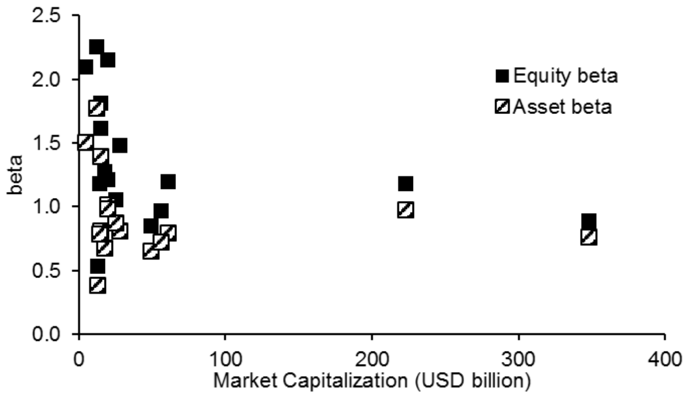

Figure 3 asset beta for our sample of upstream petroleum firms (Jafarizadeh and Bratvold, 2019)

Second, even though we compensate for the effect of financial leverage in β_{Asset}, there are still great variations in the companies' risk profiles. Their asset betas still represent the mix of their assets, some of which have significant growth options. If we use a company's β_{Asset} in project valuations, we are discounting cash flows with a rate that reflects both the projects' systematic risk and its growth options. If we explicitly model these options in a decision tree, using β_{Asset} for discounting means we are double counting the effect of the options.

We can explain the above point using a simple example: a major oil and gas company considers buying the rights to an exploration tract. Investing in this tract is a real option. If they drill and discover oil, then they consider the expected mid-sized development project, otherwise they pay the drilling cost and walk away. Here, success in drilling is pivotal. Its chance overshadows the uncertainty in the ensuing development project. However, the company's β_{Asset} reflects the systematic risk of existing projects *and* the expectations from real options in all other tracts they currently consider. If they use β_{Asset} to evaluate this tract, they over-discount the cash flows. In other words, the discount rate already reflects the expectation for drilling success or failure in all existing opportunities, including this tract.

In general, the equity returns of companies reflect the risk of existing operations together with embedded real options; these include the option to delay investments, contract/expand operations, and value of information. Furthermore, the options, at least from CAPM point of view, are riskier than the

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[8] We selected a group of sixteen upstream petroleum companies from COMPUSTAT that is large enough for this method, and in addition, domiciled in the United States so that the data does not have additional systematic risks such as variations in exchange rate. Although this group is not ideal, we could still draw coherent conclusions about project WACC based on the publicly available information on the companies' portfolio of upstream projects.

existing projects (as observed by Berk, Green, Naik, 1999, Dechow, Sloan, and Soliman, 2004, and Da, Guo, Jagannathan, 2012). If we could separate the effect of options from the systematic risk of projects, the valuations would become straightforward.

We assume the value of all firm's assets A is composed of the value of its projects P and their embedded growth options G. Using the arguments in Bernardo, Chowdhry, and Goyal (2007) we will have

$$A = P + G \tag{17}$$

The firm's β_{Asset} would be the weighted average of β_{Project} and β_{Option}, with weights corresponding to the ratio of P and G to the total value.

$$\beta_{\text{Asset}} = \frac{P}{A}\,\beta_{\text{Project}} + \left(1 - \frac{P}{A}\right)\,\beta_{\text{Option}} \tag{18}$$

This separates the risk effects of projects from the embedded options. Some authors, including Smith and Watts (1992) and Chen, Novy–Marx, Zhang (2010), argue that proxies such as book–to–market ratio or return on asset (ROA) can be useful in explaining the share of growth options in a firm's value. In other words, even without the knowledge of projects and operations, proxies could provide a rough measure of $\frac{P}{A}$ in equation (18). We assume "share book-to-price" ratio commonly reported by financial services estimates $\frac{P}{A}$. The "book" refers to the accounting valuation of a firm, while "price" is the market perception of the value. As accounting conventions do not recognize intangible assets and growth options (while market returns do) this ratio could reveal the potentials of value creation from embedded real options.

In addition, we assume that β_{Project} is the same for all firms in the upstream petroleum industry (following Bernardo, Chowdhry, and Goyal, 2007). This is a convenient assumption and allows us to determine the parameters of equation (18), but at the same time sacrifices some realism by implying that a project (stripped of its options) has the same risk across all companies within the sector. In other words, we assume the variability in the asset betas is only caused by their embedded real options. Some may argue that there are cross–company variations in project risk, e.g. mature and declining fields are less risky compared to those in their ramp-up period. However, we believe that these differences are insignificant compared to the benefits of better decision making via an aggregate project beta.

With these assumptions, we could use information about β_{Asset} and book-to-market ratio in a regression analysis that estimates the parameters of equation (18). However, as this regression-based method is problematic (Bernardo, Chowdhry, and Goyal, 2007), we split the data into two portfolios based on their market-to-book values. These portfolios represent the (equally weighted) means of β_{Asset} and market-to-book values of the stocks in them. A straight line that connects these two points yields the intercept and slope coefficients for equation (18).

Inherent variabilities aside, we observe that asset beta (and project beta) is slightly below one for most petroleum companies; as if upstream petroleum development projects, irrespective of their ownership, have similar level of risk across the sector. With these similarities, perhaps it is useful to set up standard valuation procedures using a single industry beta. Our analysis concludes $\beta_{\text{Project}} = 0.8$ for upstream petroleum valuations. Assuming the risk-free rate $r = 2\%$ and return on market $\mu_m = 7\%$, this leads to project risk premium $\varepsilon = \beta_{\text{Project}}(\mu_m - r) = 0.8 \times (7\% - 2\%) = 4\%$ for projects with all equity financing.

4. Application and Discussion

4.1 A Petroleum Exploration Example

In this section, we apply our method to the appraisal of a petroleum exploration project. Here, drilling a wildcat well could yield nothing (a dry hole) or lead to a discovery worth much the same as its estimated cash flows. We first estimate expected project cash flows using Bloomberg's crude oil price forecast and then discount them with 9% hurdle rate. These assumptions are comparable to common industry valuations. We then compare this valuation with that of our method—using a price forecast bound with the discount rate. We conclude by illustrating and discussing where the two methodologies potentially diverge.

Assume a company holds the drilling rights to an exploration tract. It costs USD 10 million to drill a wildcat well and the chance of success is estimated to be 30%. If successful, they expect a discovery that (after a year of development) produces crude oil for eight years. The net worth of the discovery would be the sum of all its discounted cash flows, reflecting all costs and revenue from selling oil.

The company also has the option of selling the drilling rights to a third party at a price of USD 5 million for the right to drill plus with a USD 5 million bonus if the buyer discovers oil. Which course of action, drilling or selling, is best? The answer depends primarily on the net present value of the development project. Figure 4 shows the decision tree model.

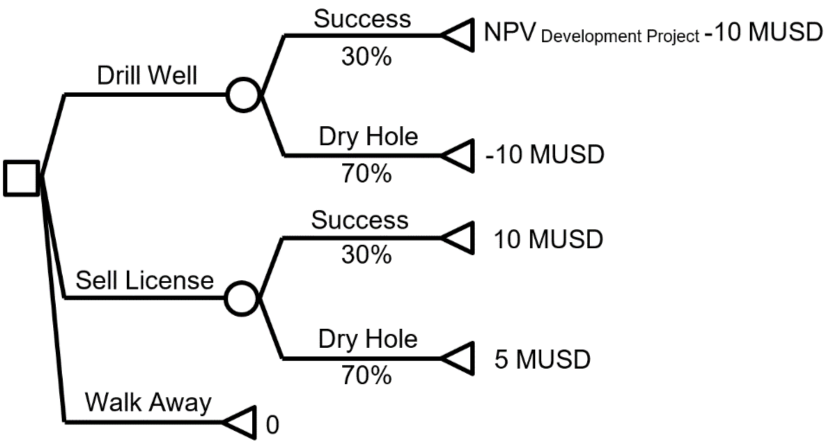

Figure 4 the decision tree model of the example

The value of the development project depends on its expected production and costs. However, even with similar cost and production forecasts, valuations could still differ—a biased view of price risk could lead to inconsistent valuations. In table 2, we show that, Depending on whether we estimate the cash flows using the corporate planning price and discount with 9% hurdle rate, or using forward prices and discount with 2% risk-free rate, the result would vary. In this example, the difference in NPVs is USD 8.6 million. However, as shown in the bottom panel of Table 2, our method of jointly estimating price and discounting risk premiums results in the same value as in the risk-neutral valuation.[9]

Table 2 cash flow profile and valuation of the development project

Traditional Valuation (Using industry price forecasts and discounting with corporate hurdle rate 9%)

Year	0	1	2	3	4	5	6	7	8
Industry Price Forecasts (USD/bbl)	71.0	69.0	68.0	68.0	68.0	68.0	68.0	68.0	68.0
Production (1000 bbl)		600	500	420	360	320	300	290	290
Cost (Million USD)	70	5	5	5	5	5	5	5	10

[9] Note that in table 2 the futures prices (middle panel) and the prices from the fitted risk-neutral curve (lower panel) are not necessarily the same. The curve of the two-factor process describes the futures prices well, but still leaves some movements unexplained.

Cash Flow (Million USD)	-70	36	29	24	19	17	15	15	10
NPV (with Hurdle Rate)	52.8								

Risk-Neutral Valuation (Using futures prices and discounting with risk-free rate 2%)

Year	0	1	2	3	4	5	6	7	8
Futures Prices (USD/bbl)	71.0	66.6	63.0	61.0	58.0	56.8	56.2	56.0	56.0
Production (1000 bbl)		600	500	420	360	320	300	290	290
Cost (Million USD)	70	5	5	5	5	5	5	5	10
Risk-free Cash Flow (Million USD)	-70	35.0	26.5	20.6	15.9	13.2	11.9	11.2	6.2
NPV (Risk Neutral)	61.4								

Joint Valuation (Using expected prices and discounting with project WACC rate 5%)

Year	0	1	2	3	4	5	6	7	8
Fitted Risk Neutral Curve (USD/bbl)	71.0	65.6	62.0	59.8	58.5	57.7	57.1	56.6	56.2
Expected Price Curve (USD/bbl)	71.0	67.2	64.7	63.8	63.8	64.2	65.0	65.8	66.8
Production (1000 bbl)		600	500	420	360	320	300	290	290
Cost (Million USD)	70	5	5	5	5	5	5	5	10
Expected Cash Flow (Million USD)	-70	35.3	27.4	21.8	18.0	15.6	14.5	14.1	9.4
NPV (with Industry Beta)	61.4								

Here, in our before-tax valuations, we assume the company has 50% debt in its capital structure at 4% interest, and that the market risk premium is 5%. We use the project beta of 0.8 leading to project WACC at 5%. This is lower than the hurdle rates most oil and gas companies use.

To calculate price risk premiums, we use the above discount rate. If $r + \varepsilon = 5\%$ and $r = 2\%$, then we could replace $\varepsilon = 3\%$ in equation (15) and numerically estimate the price risk premiums λ_ξ and λ_χ. The result is the expected price curve in table 2. In Figure 5 we compare this expected price curve with the observed futures prices and their fitted risk-neutral curve. The difference between expected and risk-neutral prices reflects the effect of the price risk premium.

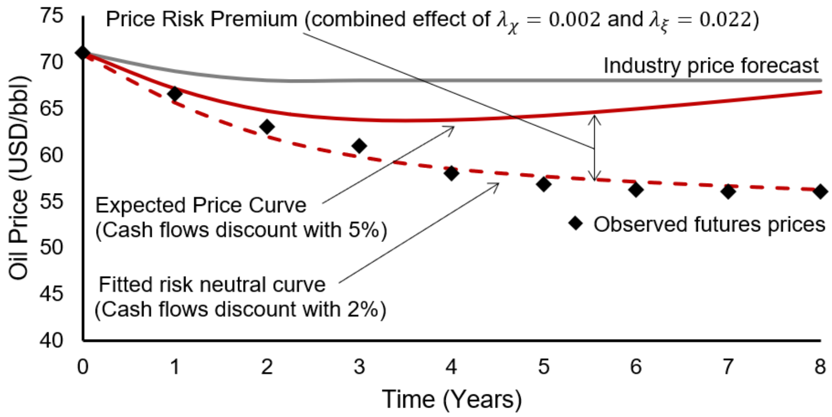

Figure 5 Forward curve and an estimate of expected spot prices through numerical estimation of risk premiums

In our example, different value estimates lead to different courses of action. As shown in Figure 6, the company would sell the drilling rights following a traditional valuation but would drill the prospect had they implemented the risk neutral (or our joint estimation) scheme. To maximize shareholder value, the company should make decisions based on consistent value measures. In this case, they should not sell but drill the prospect themselves.

Many real-world projects have non-linear payoffs. The value from a specific course of action is usually measured against other alternatives with different payoff and risk profiles. In our example, we compare the value of the development project (subjected to price uncertainty over several years) to the comparatively risk-free payoff from selling drilling rights. Like our example, in real-world projects the key to good decision making is the consistent assessment of relative values.

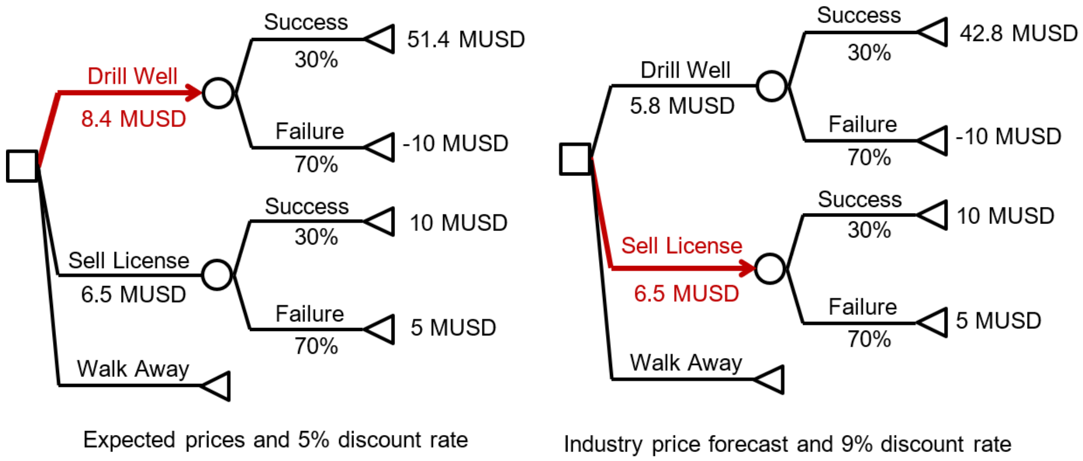

Figure 6 Optimal decisions when proposed valuation method is used (left) and when corporate planning prices and 9% discount rate used (right)

In our joint estimation approach, we use expected prices to calculate production revenues in this project as well as other projects with similar risk levels. With these cash flows, discounting at 5% rate should result in the same NPV as in the risk-neutral valuation method. In other words, whenever forward prices are available for the duration of a project, we could use the project WACC and estimate the price risk premiums. For periods where forward prices are available, this makes price forecasting straightforward. In the next section, we discuss other aspects, including projects with expected cash flows further in the future than the maturity of existing futures contracts.

4.2 Price Forecasts in Practice

Most projects have time horizons beyond the longest maturity of futures contracts. For these projects, there are no futures prices available to assess cash flows, and no risk-neutral valuation to compare the results with. We believe even in the absence of such hedging contracts, project WACC and an extrapolation of the risk-neutral curve could still generate consistent valuations. The price risk premiums would likely be different for projects with different lengths, but they would otherwise lead to general price forecasts applicable to valuations.

To extend price forecasts beyond the breadth of market instruments, we first assume that the fitted risk neutral curve further applies to longer time horizons. In other words, we assume the calibrated parameters of the price process will remain valid beyond the time horizon of the futures contracts. We then estimate the risk neutral value of a longer project using this extended price curve. Finally, we forecast expected prices by applying our procedure of estimating price risk premiums. Figure 7 shows price forecasts that apply to a longer (and more capital-intensive) project. Here, the estimates of the risk premiums are remarkably close to earlier estimates.[10]

[10] In conventional oil and gas projects, larger projects produce longer, but are also more expensive to develop and have a longer lead time. This means their production revenues appear further in the future, reaches a higher plateau, and then more slowly declines. Naturally, our estimates of risk premiums should vary by project. Still, in a simplistically prolonged project (for example, by assuming constant production over longer periods) our approach leads to identical risk premium as in shorter projects.

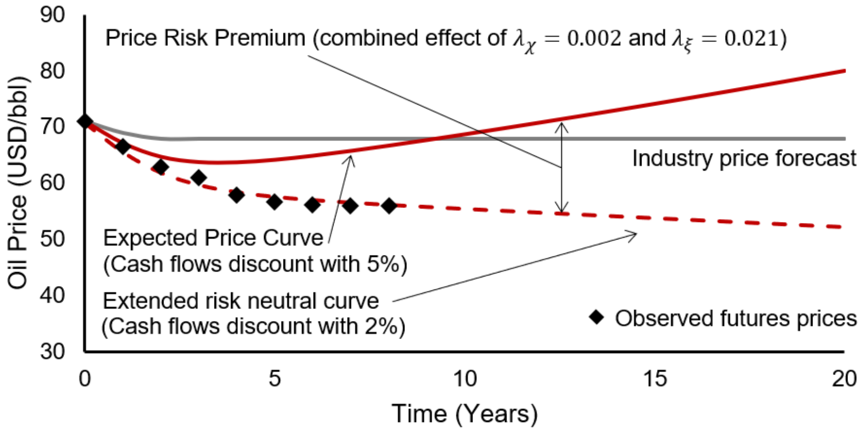

Figure 7—long run price forecasts

In general, practitioners incorporate additional information about risk into their estimates. Industry price forecasts and investment hurdle rates may convey added insights about risk premiums. As Cortazar et al (2018) show in their calibration of three factor price model, combining analysts' forecast with market's futures data improves the consistency of expected price estimates. The result would be price curves that agree with the market in the short-term and extend to long-term by relying on experts' data. In addition, a company's hurdle rate may also account for cash flow estimation biases. For example, Titman and Martin (2011) point out the behavioral biases in estimating project cash flows and that the companies try to compensate for such over-optimism by increasing their hurdle rate.

We believe such external sources of information can also be utilized in our method. We use a version of CAPM that roughly eliminates the effects of leverage and growth options from a company's beta. We then estimate project WACC for a "typical" project of any length and use it to forecast expected prices. While we may still have behavioral effects in our estimates, we believe individual analysis and quantification of various effects contributes to the quality of investment decision making. In addition, the external estimates could either inform the decision makers of the implied discount rate, or if the discount rate is determined, adjust the price forecasts. Overall, price forecasts and discounting are central to any valuation process. Our main contribution is their joint estimation.

5. Conclusions

Valuations are central to corporate decision making. In most projects, the multitude of uncertainties and their long lifespans make it difficult to consistently account for risks and opportunities. Here, we discuss measures of accounting for uncertain commodity prices. If valuations rely on disparate representation of risk in prices and discounting, they will likely confuse the investment decisions resulting in reduced value creation. In this paper, we suggest a consistent valuation framework that jointly estimates the risk premiums in prices and the discount rate. Our approach is consistent, agrees with the risk-neutral approach in the real options literature, and allows for individual assessments of key sources of uncertainty.

In principle, value estimates should be the same whether we account for risk in a project cash flows or in its discount rate. For commodity assets, risk adjusted prices are already available in market hedging instruments, so we use these to estimate project value. We then illustrate that the value should be the same in an approach that uses risk adjusted discount rate. This leads to estimates of price risk premiums.

The estimation of risk premiums depends on the frame of investment decisions. In this paper, we discuss such estimations in the context of a two-factor price model that applies to project appraisals. For a petroleum development project, we assess risk-adjusted prices by calibrating the parameters of the model to the implied information in futures and options on futures contracts. Discounting cash flows with the risk-free rate leads to risk-neutral values. We then compare these values to an approach that uses expected prices instead of risk-adjusted prices and discounts cash flows with a risk-adjusted rate. We estimate such a risk-adjusted discount rate by removing the effects of leverage and growth options from the CAPM measure of systematic risk. Finally, we estimate the price risk-premiums numerically so that the project's value from expected prices converges to the risk-neutral valuation.

Our method is simple and easy to understand and implement. In addition, the optimization model is also versatile enough to accommodate a range of applications. We believe the benefits of this valuation framework—the joint estimation of price forecasts and discount rate—goes beyond its initial applications. For example, the concept of project beta could lead to long-term price forecasts in parallel with analysts' interpretations, serving as an extra source of information and helping analysts calibrate their long-term predictions. In other instances, we could compare implied discount rates from external and independent price forecasts. These insights contribute to planning, investment decision making, and thus, value creation.

The spreadsheet model of our method (including market data and the VBA code) is available in the link below

https://www.dropbox.com/s/sft4emhjgefm0ku/Price_and_Discount%20Rate.xlsm?dl=0

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Appendix A: Empirical Data and the Two-Factor Price Model

In this paper, we use an implied method to calibrate the parameters of the two-factor price model. Here, we estimate parameters of the price model by fitting the futures and option volatility curves to observed market information. The calibrated parameters are the solution to the following minimization problem

$$\begin{aligned}\{\chi_t,\xi_t\} \in \underset{\chi_t,\xi_t}{\operatorname{argmin}} \sum_{i=1}^{N} & w_i\left(\ln \hat{F}_{0,i}(\chi_t,\xi_t,T_i,\mathbf{\Omega}) - \ln F_{0,i}\right)^2 \\ & + w_i\left(Var\left(\ln \hat{F}_{0,i}(\chi_t,\xi_t,T_i,\mathbf{\Omega})\right) - Var\left(\ln F_{0,i}\right)\right)^2 \\ \text{with } \mathbf{\Omega}&: \left\{\mu,\kappa,\sigma_\chi,\sigma_\xi,\rho\right\}\end{aligned} \qquad \text{A-1}$$

Here, $\hat{F}_{0,i}$ is the model's price for futures contract with maturity i, and $F_{0,i}$ is the observed market price at this same maturity. In addition, $Var\left(\ln \hat{F}_{0,i}\right)$ is the model's variance of the futures contract and $Var\left(\ln \hat{F}_{0,i}\right)$ is the implied variance of the options on the futures as observed in the market. We assume there are N market observations across varying maturities. We define importance weights w_i to enable customized fits. Note that we do not estimate the price risk premiums, λ_χ and λ_ξ. Hedging instruments generally do not inform estimation of risk premiums.

A software model could estimate the parameters of the two-factor model by solving the following minimization problem.

$$\min_{\{\chi_0,\xi_0,\Omega\}} \sum_{i=1}^{N} w_i\left(\ln \hat{F}_{0,i}(\chi_0,\xi_0,T_i,\mathbf{\Omega}) - \ln F_{0,i}\right)^2 + w_i\left(Var\left(\ln \hat{F}_{0,i}(\chi_0,\xi_0,T_i,\mathbf{\Omega})\right) - Var\left(\ln F_{0,i}\right)\right)^2 \qquad \text{A-2}$$

Here, we calculate model's futures prices $\hat{F}_{0,i}(\chi_0,\xi_0,T_i,\mathbf{\Omega})$ using equation (6) and $Var\left(\ln \hat{F}_{0,i}(\chi_0,\xi_0,T_i,\mathbf{\Omega})\right)$ using equation (7). In addition, we calculate the implied variance of the futures prices, $Var\left(\ln F_{0,i}\right)$ by varying $\sigma_\varphi(T)$ in equations (8) for call options, or equation (9) for put options, so that the price of the option's contract converges the observed price. The accompanying spreadsheet performs this parameter estimation process.

Like any other parameter calibration method, the implied approach described above has its limitations. First, the data from futures and options on futures reflect a risk-free outlook of the prices. They do not hint at estimation of risk premiums. In this regard, the implied approach is no worse than the alternative approach of using Kalman filter on historical prices. For example, Schwartz (1997), Schwartz and Smith (2000), and Cortazar and Naranjo (2006) mention large errors and statistically insignificant estimates for their risk premiums. Cortazar et al. (2015) suggest using external sources of information to estimate these premiums; an approach that we also adopt in this paper. We use information from financial stock markets to estimate the premiums. Second, the two-factor price process is a model (an abstraction) of the real world. The market data holds patterns and white noises not accounted for in this model. We could extend our price model by considering additional factors—e.g. model III in Schwartz (1997), the extended model of Schwartz and Smith (1998), and the multi-factor model in Cortazar and Schwartz (2003)—yet the added effort in modeling should also justify better investment decision making. If more sophisticated price models lead to the same investment decisions, then the added efforts would have been in vain. In this paper, we use the two-factor model as we believe it is a simple and affordable yet subtle enough to support investment decisions.

Still, it would be insightful to discuss the inconsistencies between market observations and our price model. The two-factor model assumes the growth rate of the long-term factor is constant but a careful examination of fitted curve to observed futures prices (figure 2) would imply that the growth rate varies. The observed futures prices slightly oscillate around the model's curve, suggesting a mean reverting growth rate with a half-life of two to three years. Schwartz and Smith (1998) confirm this in their data and suggest a third factor to model changes in the long-term growth rate.

The model's description of volatility also differs from market observations. The general equilibrium model of supply and demand in Carlson, Khokher, and Titman (2007) also generates somewhat different volatility estimates than those of the two-factor model. The two-factor model assumes equal implied volatilities for options with different strike prices but same maturity. In other words, the implied volatility should not change with strike prices. Yet as shown in Figure 7, specially for crude oil, the implied volatility of the observed options contracts changes with strike prices. Options on futures with extremely high or low strike prices imply larger volatilities than those closer to the current price level. In fact, the assumption of normal distribution for log of futures prices, through the expression $N(d)$ in equations (8) and (9), is questionable[11]. Higher probability of extreme prices causes a pattern called "volatility smile". For options with same maturity, the implied volatility increases as the strike prices deviates further from the current price.

Figure 8 shows that in our data, the volatility smile was most discernible in options with the shortest maturities and almost disappears in the long-maturity options. In our parameter estimation, we selected an implied volatility curve out of numerous curves within the volatility surface. Which curve better represent the volatility curve? Some authors suggest at-the-money curves, or those with highest traded volume (Geman, 2005). However, for oil markets, even though the strike price of 100 USD/bbl is highly traded, at current spot price of 75 USD/bbl, it is not at-the-money. For this reason, it is to analysts' discretion which curve to select for parameter estimation.

[11] Jarque and Bera (1987) develop a statistic that assesses the normality of time-series by combining tests for skewness and kurtosis. Recalling that a normally distributed dataset has no skewness and kurtosis, this statistic rejects normality when the test's value is less than the significance level. Applying this test to futures price usually reveals significant skewness at 5% level.

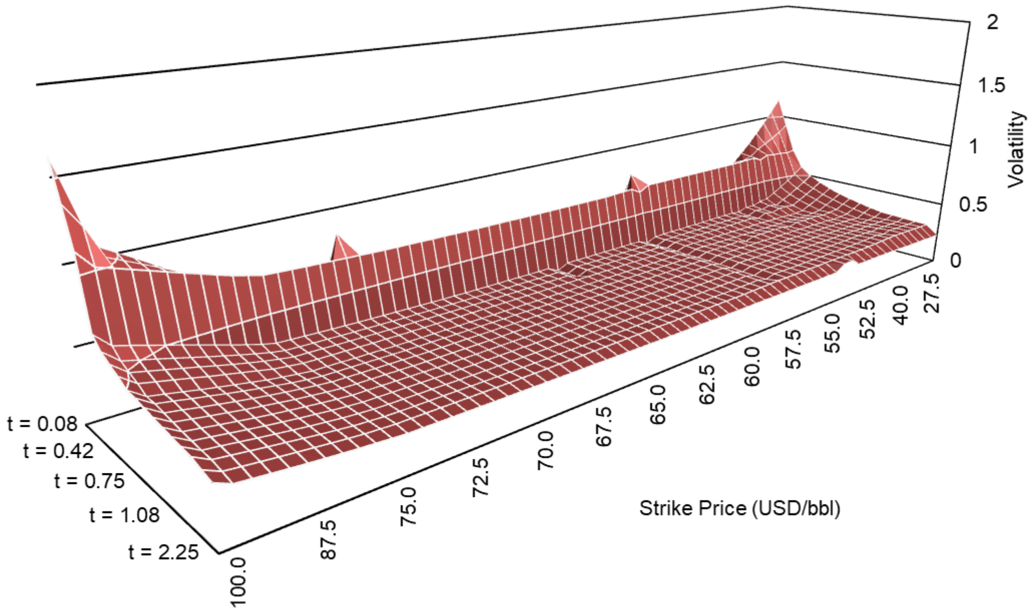

Figure 8 volatility surface (implied volatility across different maturities and strike prices) for European put options on futures, market data retrieved from NYMEX on 01/09/2018

Overall, implied parameter estimation depends on the source data and the importance weights that we assign to each part of the curves. If the futures and the implied volatility curves are sufficiently informative and analysts' sound judgment selects the inputs, then we believe the parameter estimations will be reliable and useful for valuations.