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S. Kokou Dadzie, and M. H. Lakshminarayana Reddy

Recasting Navier-Stokes Equations: Shock Wave Structure Description

S Kokou Dadzie[a)] and M. H. Lakshminarayana Reddy[b)]

School of Engineering and Physical Sciences, Heriot-Watt University, Edinburgh EH14 4AS, Scotland, UK

[a)]*Corresponding author: k.dadzie@hw.ac.uk*
[b)]*Electronic mail: l.mh@hw.ac.uk*

Abstract. Classical Navier-Stokes equations are known to be inadequate in describing some flows in both the compressible and incompressible configurations. A ubiquitous simple example of the failure is the description of shock wave structure. A new compressible hydrodynamic set of equations is here developed using a velocity transformation technique similar in nature to Lorentz transformation and termed "re-casted Navier-Stokes equations". We found that results with the re-casted Navier-Stokes equations fit better the experimental data.

INTRODUCTION

Fluid mechanics is one of the oldest field of science and still widely researched due to its broad spread of applications in many industries. The development of basic fundamental dynamic laws such as Newton's law of motion and Newton's law of viscosity had culminated in the current form of the Navier-Stokes equations and these equations are still widely accepted as the universal basis of modelling fluid motion [1]. Classical Navier-Stokes equations are known to be inadequate in describing some compressible flows accurately [2]. The failure may be tied up to some of the basic assumptions made while deriving them. Improving the range of applicability of the Navier-Stokes equations beyond their limits has been and still is a critical area of research.

One of the best-known example for simple and highly non-equilibrium flow phenomena where Navier-Stokes equations fails is the prediction of a normal shock wave profile. A normal shock wave is a disturbance propagating between a supersonic fluid and a subsonic fluid. In other words, a normal shock involves a transition between a uniform upstream flow and a uniform downstream flow [3, 4]. So we can treat the shock wave as an interface of finite thickness between two different equilibrium states of a gas. Shock waves arise at explosions, detonations, supersonic movements of bodies, and so on. The shock structure problem has been studied extensively numerically, theoretically and experimentally since and serves as a standard benchmark problem for testing the capability (validity) of extended hydrodynamic flow models [5].

In the current work, we propose a new methodology to finding alternatives to Navier-Stokes equations. It involves transforming the fluid velocity field variable within the basic standard fluid flow equations with an appropriately selected *change of variable*. Shock wave profile predictions in Argon gas is then considered to demonstrate the difference in physics between the transformed and the original Navier-Stokes equations. We show a better prediction of the experimental data by the transformed equations (i.e., a fluid flow model where the mass conservation has a diffusive component and the moment equation has a Korteweg type shear stress).

Theory: Re-casted Navier-Stokes Equations

We start with the classical Navier-Stokes-Fourier equations which are a differential form of three classical conservation laws, namely, mass, momentum and energy conservation laws (closed using Newton's and Fourier's laws representing the shear stress and the heat flux) that govern the motion of a fluid. In an Eulerian reference frame they are given by:

mass balance/continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho U] = 0, \tag{1}$$

momentum balance equation

$$\frac{\partial \rho U}{\partial t} + \nabla \cdot [\rho U \otimes U] + \nabla \cdot [p\boldsymbol{I} + \boldsymbol{\Pi}^{(NS)}] = 0, \tag{2}$$

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energy balance equation

$$\frac{\partial}{\partial t}[\frac{1}{2}\rho U^2 + \rho \mathbf{e}_{in}] + \nabla\cdot[\frac{1}{2}\rho U^2 U + \rho \mathbf{e}_{in} U] + \nabla\cdot[(p\boldsymbol{I} + \boldsymbol{\Pi}^{(NS)})\cdot U] + \nabla\cdot\boldsymbol{q}^{(NS)} = 0, \tag{3}$$

where ρ is the mass-density of the fluid, U is the flow mass velocity, p is the hydrostatic pressure and $\mathbf{e}_{in}$ is the specific internal energy of the fluid whose expression is given by, $\mathbf{e}_{in} = p/(\rho(\gamma-1))$ with γ being the isentropic constant. Further, $\boldsymbol{\Pi}^{(NS)}$ and $\boldsymbol{q}^{(NS)}$ represents the shear stress tensor and the heat flux vector, respectively, and $\boldsymbol{I}$ is the identity matrix. All these hydrodynamic fields are functions of time t and spatial variable $\boldsymbol{X}$. Here $\nabla\cdot$ denotes the usual spatial divergence operator. The constitutive relations for the shear stress $\boldsymbol{\Pi}^{(NS)}$ and the heat flux vector $\boldsymbol{q}^{(NS)}$ are given as:

$$\boldsymbol{\Pi}^{(NS)} = -2\mu\left[\frac{1}{2}(\nabla U + \widetilde{\nabla U}) - \frac{1}{3}\boldsymbol{I}\,(\nabla\cdot U)\right] = -2\mu\,\overline{\nabla U}^{\circ}, \quad \boldsymbol{q}^{(NS)} = -\kappa\nabla T, \tag{4}$$

where ∇U is the spatial gradient of U and $\widetilde{\nabla U}$ is the transpose of ∇U. Coefficients μ and κ are the dynamic viscosity and the heat conductivity, respectively.

It is trivial to observe in this system that continuity equation (1) does not contain a diffusion term, whereas the momentum and energy equations do. In the meantime, this system can be shown to satisfy all required mechanical properties and also associates with a second law / entropy equation. It is also important to note that in constitutive equation (4), no complex contributions from effects such as fluid dilation, temperature gradient, fluid vorticity, etc. to the shear stress are described. It is this basic form of the classical fluid flow equations that are traditionally shown to satisfy all known thermo-mechanical properties [6]. Rather than solving directly the system (1) - (4), first, we perform a change of variable within the entire system:

$$U = U_v - \kappa_m \nabla \ln\rho = U_v - \frac{\kappa_m}{\rho}\nabla\rho, \tag{5}$$

where κ_m is the molecular diffusivity co-efficient. Equation (5) is a relation between the fluid mass velocity and the fluid volume velocity, U_v, which originates from *volume diffusion hydrodynamic theory* [7, 8, 9]. It has also been derived using a stochastic variational method [10]. Substituting (5) into (1) - (4), after some laborious transformation technique one arrives at the following system and hereafter termed as re-casted Navier-Stokes system:

re-casted mass balance equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot[\rho\,U_v - \kappa_m\nabla\rho] = 0, \tag{6}$$

re-casted momentum balance equation

$$\frac{\partial}{\partial t}[\rho\,U_v - \kappa_m\nabla\rho] + \nabla\cdot\left[\rho\,U_v\otimes U_v + p\boldsymbol{I} + \boldsymbol{\Pi}_v^{(RNS)}\right] = 0, \tag{7}$$

re-casted energy balance equation

$$\begin{aligned}\frac{\partial}{\partial t}&\left[\frac{1}{2}\rho\,U_v^2 + \rho\,\mathbf{e}_{in} - \kappa_m\,(\rho\,U_v\cdot\nabla\ln\rho) + \frac{1}{2}\kappa_m^2\,(\nabla\rho\cdot\nabla\ln\rho)\right]\\ &+\nabla\cdot\left[\frac{1}{2}\rho\,U_v^2\,U_v + \rho\,\mathbf{e}_{in}\,U_v\right] + \nabla\cdot\left[(p\boldsymbol{I} + \boldsymbol{\Pi}_v)\cdot U_v - \kappa_m\boldsymbol{\Pi}_v\cdot\nabla\ln\rho\right]\\ &+\nabla\cdot\left[\boldsymbol{q}_v^{(RNS)} + \kappa_m\,\mathcal{N}_{v_1} + \kappa_m^2\,\mathcal{N}_{v_2} + \kappa_m^3\,\mathcal{N}_{v_3}\right] = 0,\end{aligned} \tag{8}$$

where the constitutive relations for the new shear stress and the new heat flux vector are given by:

$$\boldsymbol{\Pi}_v^{(RNS)} = \boldsymbol{\Pi}_v + \frac{\kappa_m^2}{\rho}\nabla\rho\otimes\nabla\rho - \kappa_m\,U_v\otimes\nabla\rho - \kappa_m\nabla\rho\otimes U_v, \tag{9}$$

$$\boldsymbol{q}_v^{(RNS)} = \boldsymbol{q}^{(NS)} - \kappa_m\,\rho\,\mathbf{e}_{in}\nabla\ln\rho - \kappa_m\,p\boldsymbol{I}\cdot\nabla\ln\rho, \tag{10}$$

with

$$\boldsymbol{\Pi}_v = -2\mu\overline{\nabla U_v}^{\circ} + 2\,\mu\,\kappa_m\widetilde{\boldsymbol{D}}\ln\rho - \frac{2\mu}{3}\kappa_m\Delta\ln\rho\,\boldsymbol{I}, \tag{11}$$

and $\mathscr{N}_{v_i}$ for $i = 1$ to 3 represents additional nonlinear terms not explicitly listed here. In equation (11), $\widetilde{\boldsymbol{D}}$ and Δ denotes the Hessian and Laplacian operators, respectively. The continuum flow system (6) - (10) is a type of mass diffusion set of continuum equations. That is, it contains: (i) a mass diffusion component in the conservation of mass equation, (ii) explicit fluid dialation terms in the momentum stress tensor, and (iii) a non-Fourier heat flux term. Next we solve the new system (6) - (8) for the normal shock wave problem and compare its solutions to that of the classical system (1) - (3) solutions and experimental data to show that the transformed equations may be more appropriate to solve directly for flows involving large density variations/gradients.

THE SHOCK WAVE STRUCTURE PROBLEM IN A MONATOMIC GAS

In the one-dimensional planar stationary shock wave problem all flow variables are function of the spatial coordinate x. We denote the upstream ($x \to -\infty$) and the downstream ($x \to \infty$) conditions of the shock, located at $x = 0$, by subscript 1 and 2, respectively. These upstream and downstream states of the shock are connected by jump conditions, the Rankine-Hugoniot (RH) conditions.

The re-casted Navier-Stokes equations, for the one-dimensional stationary shock flow configuration, are given by:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho\, u_v - \kappa_m \frac{\mathrm{d}\rho}{\mathrm{d}x}\right] = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\rho\, u_v^2 + p + \Pi_v^{(RNS)}\right] = 0, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\rho\, u_v \left(\frac{1}{2}u_v^2 + C_p\, T\right) + \left(\Pi_v - \frac{3}{2}\kappa_m\, u_v \frac{\mathrm{d}\rho}{\mathrm{d}x}\right)\left(u_v - \frac{\kappa_m}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x}\right) - \frac{\kappa_m^3}{2\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}x}\right)^3 + q_v^{(RNS)}\right] = 0, \tag{13}$$

with the only non-zero longitudinal new shear stress $\Pi_v^{(RNS)}$ and the new heat flux vector $q_v^{(RNS)}$ given by:

$$\Pi_v^{(RNS)} = \Pi_v - 2\,\kappa_m\, u_v \frac{\mathrm{d}\rho}{\mathrm{d}x} + \frac{\kappa_m^2}{\rho}\left(\frac{\mathrm{d}\rho}{\mathrm{d}x}\right)^2 \quad \text{with} \quad \Pi_v = -\frac{4}{3}\mu\,\frac{\mathrm{d}u_v}{\mathrm{d}x} + \frac{4}{3}\frac{\mu\,\kappa_m}{\rho}\frac{\mathrm{d}^2\rho}{\mathrm{d}x^2} - \frac{4}{3}\frac{\mu\,\kappa_m}{\rho^2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}x}\right)^2, \tag{14}$$

$$q_v^{(RNS)} = -\kappa\,\frac{\mathrm{d}T}{\mathrm{d}x} - \frac{\gamma}{(\gamma - 1)}\kappa_m\,\frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}x}. \tag{15}$$

The system (12)-(15) can be interpreted in a dimensionless form using dimensionless parameters as defined in reference [11]. The new additional transport coefficient κ_m is assumed related to the viscosity coefficient by $\kappa_m = -\kappa_{m_o}\mu/\rho$, and its dimensionless form given by, $\kappa_m^* = -\kappa_{m_o}\mu^*/\rho^*$ where μ^* and ρ^* are respectively the dimensionless dynamic viscosity and density. Constant coefficient κ_{m_o} is assumed being of the magnitude order of unity. A numerical strategy, namely, finite difference global solution (FDGS) scheme developed with well-posed boundary conditions in the same reference is used to solve the above reduced nonlinear system. The ideal gas equation of state, $p = \rho\boldsymbol{R}T$, where $\boldsymbol{R}$ is the specific gas constant, and the power law temperature dependent viscosity formula with temperature-viscosity exponent equal to $s = 0.75$ are used.

RESULTS AND CONCLUSIONS

Figure 1 present the numerical predictions by the re-casted Navier-Stokes equations for the shock wave structure for Argon gas at different Mach numbers. We assume the constant κ_{m_0} in the molecular mass diffusivity κ_m to be unity in all our present simulations. The normalized density profile defined by, $\rho_N = (\rho - \rho_1)/(\rho_2 - \rho_1)$ is used here as the primary selected property for the prediction comparison as full experimental data exist for this property [2]. In each panel, the dotted black line and the solid red line represents the solutions of Navier-Stokes and re-casted Navier-Stokes equations respectively. The filled blue circles represents the experimental data of Alsmeyer [2]. One observes an excellent agreement between the prediction of the re-casted Navier-Stokes equations and the experimental data at Mach number between 2.05 and 3.8 (see panel (b) and (c) of figure 1). Overall the re-casted Navier-Stokes equations produce better agreement with the experimental data and its solutions clearly deviate from the original equation solutions at all Mach numbers studied here (see panels (a-e) of figure 1). Due to lack of experimental data about temperature profiles, we only compare the normalized temperature profiles, $T_N = (T - T_1)/(T_2 - T_1)$, between the original and the re-casted equations in figure 1(f-h). At both the low and high Mach numbers, differences observed between predictions by the original and the re-casted equations on the temperature profiles are similar to those seen

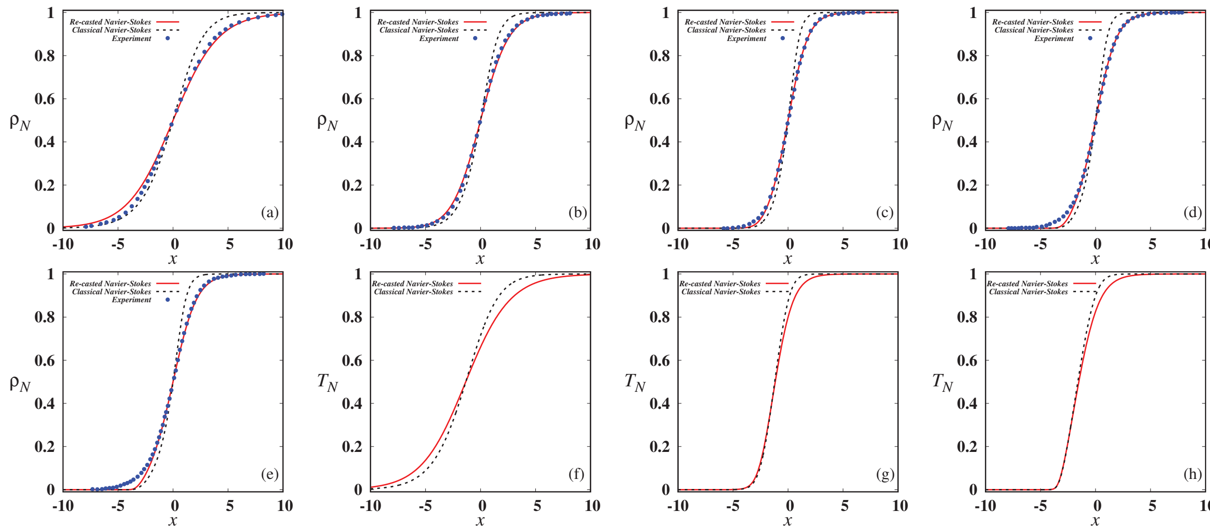

FIGURE 1. (a-e) The variation of normalized density profiles in Ar shock layer, (ρ_N): for (a) $\mathrm{Ma}_1 = 1.55$, (b) $\mathrm{Ma}_1 = 2.05$, (c) $\mathrm{Ma}_1 = 3.8$, (d) $\mathrm{Ma}_1 = 6.5$ and (e) $\mathrm{Ma}_1 = 9$. (f-h) The variation of normalized temperature profiles in Ar shock layer, (T_N): for (f) $\mathrm{Ma}_1 = 1.55$, (g) $\mathrm{Ma}_1 = 3.8$ and (h) $\mathrm{Ma}_1 = 9$.

between the two models previously on the density profiles. That is, taking for example the density profile at the low Mach number in figure 1(a), re-casted Navier-Stokes predicts ρ_N higher than Navier-Stokes at the upstream region and lower than it downstream. Then observing figure 1(f), re-casted Navier-Stokes again predicts T_N higher than Navier-Stokes at the upstream and lower than it downstream. Moving into the high Mach numbers, in figure 1(e), re-casted Navier-Stokes predicts ρ_N close to Navier-Stokes at the upstream, although better than it, and deviates from it clearly downstream. Then observing figure 1(h), re-casted Navier-Stokes again predicts T_N similar to Navier-Stokes at the upstream and deviates from it downstream.

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