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On the Solution of Partial Differential Equations Using the Sumudu Transform

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Abstract

In this paper we demonstrate a general procedure for the solution of partial differential equations by using the Sumudu transform. We present some illustrative examples towards this effect.

Keywords: Sumudu transform, Initial-boundary value problems

1 Introduction

The Sumudu transform is an integral transform defined as

$$G(u) = S\{f(t)\} = \frac{1}{u}\int_0^\infty e^{-t/u} f(t)dt,$$

where $f(t)$ is considered to be an element of the set of functions $A = \left\{f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| \leq Me^{|t|/\tau_j}, for\ t \in (-1)^j \times [0,\infty)\right\}$ [1]. Watugala first introduced this integral transform to be able to solve ordinary differential equations in control engineering problems [4]. Many of its elementary properties have been derived, reflecting much parallelism with the results for the Laplace transform [3]. In view of this fact, the Sumudu transform can also be adapted to solve for partial differential equations by mimicking the way Laplace transforms solve partial differential equations [2]. Some of the elementary properties of the Sumudu transform are listed in the table below for quick reference, and can be verified directly from the definition of said integral transform or by reference to [3].

Table 1. Some Sumudu transforms

$\boldsymbol{f(t)}$	$\boldsymbol{G(u) = S(f(t))}$
1	1
t^n	$n!\,u^n$
$t^n e^{at}$	$\frac{n!\,u^n}{(1-au)^{n+1}}$
$\sin at$	$\frac{au}{1+a^2u^2}$
$\cos at$	$\frac{1}{1+a^2u^2}$
$\sinh at$	$\frac{au}{1-a^2u^2}$
$\cosh at$	$\frac{1}{1-a^2u^2}$
$f(at)$	$G(au)$
$e^{ct}f(at)$	$\frac{1}{1-cu}G\left(\frac{au}{1-cu}\right)$
$H(t-a)$	$e^{-a/u}$
$H(t-a)f(t-a)$	$e^{-\frac{a}{u}}G(u)$
$af(t)+bg(t)$	$aS(f(t))+bS(g(t))$
$f'(t)$	$\frac{G(u)-f(0)}{u}$
$f^{(n)}(t)$	$\frac{G(u)}{u^n}-\frac{f(0)}{u^n}-\cdots-\frac{f^{(n-1)}(0)}{u}$
$\int_0^t f(\varphi)d\varphi$	$uG(u)$
$tf'(t)$	$u\frac{dG(u)}{du}$

2 Preliminary Results

The additional statements required to illustrate the technique of solving PDEs by the Sumudu transform will now be established with the following results.

Theorem 1.

i. $S\{u_t(x,t)\} = \frac{1}{u}G(x,u) - \frac{1}{u}u(x,0).$

ii. $S\{u_{tt}(x,t)\} = \frac{1}{u^2}G(x,u) - \frac{1}{u^2}u(x,0) - \frac{1}{u}u_t(x,0)$

Proof. Part (i) follows from the fact that $S\{f'(t)\} = \frac{G(u)-f(0)}{u}$ if $G(u) = S\{f(t)\}$ (indicated at Table 1), with the variable x treated as a parameter.
For part (ii), from the definition we know that

$$S\{u_{tt}(x,t)\} = \frac{1}{u}\int_0^\infty e^{-t/u}u_{tt}(x,t)dt = \frac{1}{u}[-u_t(x,0) + S\{u_t(x,t)\}]$$
$$= \frac{1}{u}\left\{\frac{1}{u}G(x,u) - \frac{1}{u}u(x,0) - u_t(x,0)\right\}$$

by integration by parts, and the result follows immediately.

Theorem 2.
i. $S\{u_x(x,t)\} = G_x(x,u)$.
ii. $S\{u_{xx}(x,t)\} = G_{xx}(x,u)$.

Proof. The two statements follow from integrating the transform integral with respect to the parameter x once and twice, respectively.

Theorem 3. If $G(u) = S\{f(t)\}$ then $S\{tf(t)\} = u^2G'(u) + uG(u)$.

Proof. $G(u) = S\{f(t)\} = \frac{1}{u}\int_0^\infty e^{-t/u}f(t)dt$

$$\Rightarrow G'(u) = -\frac{1}{u^2}\int_0^\infty e^{-t/u}f(t)dt + \frac{1}{u}\int_0^\infty \frac{t}{u^2}e^{-t/u}f(t)dt$$
$$\Rightarrow G'(u) = -\frac{1}{u}G(u) + \frac{1}{u^2}S\{tf(t)\}.$$

The result immediately follows after solving for $S\{tf(t)\}$ in the last equation.

Corollary. If $G(u(x,t)) = S\{u(x,t)\}$ then $S(tu(x,t)) = u^2\frac{d}{du}G(x,u) + uG(x,u)$.

3 Demonstration of the Technique

In this section we proceed to illustrate how the Sumudu transform can be utilized to solve certain partial differential equations.

Example 1. Solve the differential equation with the following initial and boundary conditions.

$$u_x + u_t = x^2, \qquad x > 0, t > 0$$
$$u(0,t) = 0, \qquad t > 0$$
$$u(x,0) = 0, \qquad x > 0$$

Taking the Sumudu transforms of both sides, we obtain

$S\{u_x(x,t)\} + S\{u_t(x,t)\} = S\{x^2\} = x^2,$ or
$G_x(x,u) + \frac{1}{u}\big(G(x,u) - u(x,0)\big) = x^2$
$\Leftrightarrow G_x(x,u) + \frac{1}{u}G(x,u) = x^2.$

This is a first-order linear ordinary differential equation in x, and can be solved by determining the integrating factor, which is $e^{\int \frac{dx}{u}} = e^{\frac{x}{u}}$. This implies that

$$e^{\frac{x}{u}}G_x(x,u) + \frac{1}{u}e^{\frac{x}{u}}G(x,u) = x^2 e^{\frac{x}{u}},$$
$$\Rightarrow \frac{d}{dx}\left\{e^{\frac{x}{u}}G(x,u)\right\} = x^2 e^{\frac{x}{u}} \ .$$
$$\Rightarrow e^{\frac{x}{u}}G(x,u) = \int x^2 e^{\frac{x}{u}}\,dx = ux^2 e^{\frac{x}{u}} - 2u^2 x e^{\frac{x}{u}} + 2u^3 e^{\frac{x}{u}} + C,$$

after performing integration by parts twice on the indefinite integral.

$$\Leftrightarrow G(x,u) = ux^2 - 2u^2 x + 2u^3 + Ce^{-\frac{x}{u}}.$$

Now the boundary condition

$u(0,t) = 0 \ \Rightarrow G(0,u) = 0 \ \Rightarrow 2u^3 + C = 0$, or $C = -2u^3$.
$$\Rightarrow G(x,u) = ux^2 - 2u^2 x + 2u^3 - 2u^3 e^{-\frac{x}{u}}.$$

By directly applying the inverse Sumudu transform, we obtain the solution
$$u(x,t) = tx^2 - t^2 x + \frac{t^3}{3} - \frac{1}{3}H(t-x)(t-x)^3.$$

Example 2. Solve the specific case of the Telegraph Equation [5], having the following initial and boundary conditions:

$$2u_t + 3u_{tt} = 3u_{xx} + 2(x^2 - 9x - t + 3)$$
$$u(x,0) = x^3 + x, \quad u_t(x,0) = x^2$$
$$u(0,t) = t^2, \quad u_x(0,t) = 1$$

We apply the Sumudu transform to both sides of the differential equation.

$$2S\{u_t\} + 3S\{u_{tt}\} = 3S\{u_{xx}\} + 2S\{x^2\} - 18S\{x\} - 2S\{t\} + 6S\{1\}.$$
$$\Rightarrow \frac{2}{u}G(x,u) - \frac{2}{u}u(x,0) + \frac{3}{u^2}G(x,u) - \frac{3}{u^2}u(x,0) - \frac{3}{u}u_t(x,0)$$
$$= 3G_{xx}(x,u) + 2x^2 - 18x - 2u + 6,$$

from **Theorems 1** and **2**.

$$\Leftrightarrow \frac{2u+3}{u^2}G(x,u) - \frac{2}{u}(x^3+x) - \frac{3}{u^2}(x^3+x) - \frac{3}{u}x^2$$
$$= 3G_{xx}(x,u) + 2x^2 - 18x - 2u + 6$$
$$\Leftrightarrow \frac{2u+3}{u^2}G(x,u) - \frac{2u+3}{u}(x^3+x) - \frac{3}{u}x^2$$
$$= 3G_{xx}(x,u) + 2x^2 - 18x - 2u + 6$$

A rearrangement of terms leads to the following equation:

$$3G_{xx}(x,u) - \frac{2u+3}{u^2}G(x,u)$$
$$= -\frac{2u+3}{u^2}x^3 - \left(\frac{2u+3}{u}\right)x^2 + \left(18 - \frac{2u+3}{u^2}\right)x + 2u - 6$$

This can be treated as a second-order differential equation in x, with complementary solution given by

$$y_c = c_1 e^{\frac{\sqrt{2u+3}}{\sqrt{3}u}x} + c_2 e^{-\frac{\sqrt{2u+3}}{\sqrt{3}u}x}.$$

For the particular solution consider the polynomial $Ax^3 + Bx^2 + Cx + D$.
After direct substitution and equating coefficients of like terms, we get the following values for $A, B, C,$ and D:

$A = 1$, $B = u$, $C = 1$, and $D = 2u^2$.

$$\Rightarrow G(x,u) = c_1 e^{\frac{\sqrt{2u+3}}{\sqrt{3}u}x} + c_2 e^{-\frac{\sqrt{2u+3}}{\sqrt{3}u}x} + x^3 + ux^2 + x + 2u^2.$$

Now $u(0,t) = t^2 \Rightarrow G(0,u) = 2u^2$, and after substitution into the equation above, $G(0,u) = c_1 + c_2 + 2u^2$.
This implies that $c_1 + c_2 = 0$, or $-c_1 = c_2$.
In addition, the boundary condition $u_x(0,t) = 1$ implies that $G_x(0,u) = 1$.

$$\Rightarrow c_1 \frac{\sqrt{2u+3}}{\sqrt{3}u} - c_2 \frac{\sqrt{2u+3}}{\sqrt{3}u} = 0 \;\Rightarrow\; c_1 = c_2 = 0.$$
$$\Rightarrow G(x,u) = x^3 + ux^2 + x + 2u^2.$$

Applying the inverse Sumudu transform directly gives the solution $u(x,t) = x^3 + tx^2 + x + t^2$.

Example 3. Solve the differential equation with the following initial and boundary conditions.

$$tu_t(x,t) = -tu_{xx}(x.t) + u(x,t)$$
$$0 \le x \le 2\pi, \quad u_x(2\pi,t) = te^t, \quad u_t(x,0) = \sin x$$

Assume that the solution comes in the form of $u(x,t) = f(t)g(x)$.

This then implies that $tf'(t)g(x) = -tf(t)g''(x) + f(t)g(x)$ after direct substitution into the PDE. Applying the Sumudu transform to the terms of the differential equation, we obtain

$$uG'(u)g(x) = -[u^2G'(u) + uG(u)]g''(x) + G(u)g(x).$$
$$\Leftrightarrow [uG'(u) + G(u)]g''(x) + \left[G'(u) - \frac{1}{u}G(u)\right]g(x) = 0$$
$$\Leftrightarrow g''(x) + \frac{G'(u) - \frac{1}{u}G(u)}{uG'(u) + G(u)}g(x) = 0,$$

which is a 2nd – order ODE in terms of x with the coefficient of $g(x)$ considered as a function of the parameter u.

Now $u(x,t) = f(t)g(x) \Rightarrow u_t(x,t) = f'(t)g(x) \Rightarrow u_t(x,0) = f'(0)g(x)$

$$f'(0)g(x) = \sin x,$$
$$\Rightarrow g(x) = \frac{1}{f'(0)}\sin x.$$

Note that $f'(0) \neq 0$, since the function $g(x)$ is not identically zero.
Therefore, $g(x) = \frac{1}{f'(0)}\sin x$ is a solution of the above 2nd – order ODE, and hence, this implies that

$$\frac{G'(u) - \frac{1}{u}G(u)}{uG'(u) + G(u)} = 1$$

or $G'(u) - \frac{1}{u}G(u) = uG'(u) + G(u)$.

$$\Leftrightarrow G'(u) + \frac{u+1}{u(u-1)}G(u) = 0,$$
$$\Leftrightarrow G'(u) + \left(\frac{2}{u-1} - \frac{1}{u}\right)G(u) = 0.$$

The integrating factor for this ODE is $\frac{(u-1)^2}{u}$. From this we get

$$\frac{d}{du}\left(\frac{(u-1)^2}{u}G(u)\right) = 0, \text{ or } \frac{(u-1)^2}{u}G(u) = k,$$

for some constant k. Then we get
$G(u) = k\frac{u}{(1-u)^2}$, which implies that $f(t) = kte^t$, after direct application of the inverse Sumudu transform. This implies that

$$f'(t) = k(t+1)e^t \Rightarrow f'(0) = k$$
$$\Rightarrow f(t) = f'(0)te^t$$
$$\Rightarrow u(x,t) = f(t)g(x) = (f'(0)te^t) \cdot \left(\frac{1}{f'(0)}\sin x\right) = te^t \sin x.$$

To complete the solution, we find that that $u_x(2\pi, t) = te^t \Rightarrow te^t = kte^t \Rightarrow k = f'(0) = 1$. Note that with the assumption of $u(x,t) = f(t)g(x)$, the PDE can also be solved directly without using the Sumudu transform, and can be found to yield the same solution.

4 Conclusion

It was just demonstrated in this paper that the Sumudu transform is capable of solving partial differential equations by taking advantage of properties that are similar to that of the Laplace transform.

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