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# FROM INDIVIDUAL-BASED MECHANICAL MODELS OF MULTICELLULAR SYSTEMS TO FREE-BOUNDARY PROBLEMS

TOMMASO LORENZI, PHILIP J. MURRAY, MARIYA PTASHNYK

UNIVERSITY OF ST ANDREWS, UK  
UNIVERSITY OF DUNDEE, UK  
HERIOT-WATT UNIVERSITY, EDINBURGH, UK

ABSTRACT. In this paper we present an individual-based mechanical model that describes the dynamics of two contiguous cell populations with different proliferative and mechanical characteristics. An off-lattice modelling approach is considered whereby: (i) every cell is identified by the position of its centre; (ii) mechanical interactions between cells are described via generic nonlinear force laws; and (iii) cell proliferation is contact inhibited. We formally show that the continuum counterpart of this discrete model is given by a free-boundary problem for the cell densities. The results of the derivation demonstrate how the parameters of continuum mechanical models of multicellular systems can be related to biophysical cell properties. We prove an existence result for the free-boundary problem and construct travelling-wave solutions. Numerical simulations are performed in the case where the cellular interaction forces are described by the celebrated Johnson-Kendall-Roberts model of elastic contact, which has been previously used to model cell-cell interactions. The results obtained indicate excellent agreement between the simulation results for the individual-based model, the numerical solutions of the corresponding free-boundary problem and the travelling-wave analysis.

## 1. INTRODUCTION

Continuum mechanical models of multicellular systems have been increasingly used to achieve a deeper understanding of the underpinnings of tissue development, wound healing and tumour growth [1, 2, 3, 6, 8, 10, 13, 15, 26, 32, 40, 44, 46, 52, 53]. These models are formulated in terms of nonlinear partial differential equations for cell densities (or cell volume fractions) and, as such, are amenable to both numerical and analytical approaches that enable insight to be gained into the biological system under study. From a mathematical perspective, over the past few years particular attention has been paid to the existence of travelling-wave solutions with composite shapes [5, 12, 31, 43, 48] and to the convergence to free-boundary problems in the asymptotic limit whereby cells are represented as an incompressible fluid [7, 30, 33, 41, 42].

Whilst continuum mechanical models of multicellular systems are usually defined on the basis of tissue-scale phenomenological considerations, off-lattice individual-based models enable representation of cell mechanics at the level of individual cells [18, 49]. However, as the numerical exploration of such individual-based models requires large computational times for biologically relevant cell numbers and the models are not analytically tractable, it is desirable to derive continuum models in an appropriate limit [4, 9, 17, 18, 25, 28, 29, 34, 35, 36, 37, 47]. Although mechanical interactions between interfacing cell populations with different characteristics arise in many biological contexts (*e.g.* tumour growth, development), relatively little prior work has explored the connection between off-lattice individual-based models and continuum models in such situations.

In this paper we propose an individual-based mechanical model for the dynamics of two contiguous cell populations with different proliferative and mechanical characteristics. In our model: (i) every cell is identified by the position of its centre; (ii) mechanical interactions between cells are described via generic nonlinear force laws; and (iii) cell proliferation is contact inhibited. Formally deriving a continuum counterpart of the discrete model, we obtain a free-boundary problem with nonstandard transmission conditions that governs the dynamics of the cell densities. Our derivation extends a previous method developed for the case of a single cell population [36, 37].

To prove an existence result for the free-boundary problem, a novel extension of methods previously developed for related free-boundary problems [21, 22, 23, 51] is required due to the specific structure

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tl47@st-andrews.ac.uk, p.murray@dundee.ac.uk, m.ptashnyk@hw.ac.uk

of our boundary and transmission conditions. In particular, the jump in the density and in the flux across the moving interface between the two cell populations, along with the fact that there are no conditions of Dirichlet type, prevent us from using existing ideas which are partially based on continuity arguments [22, 23] and from applying the enthalpy method [16, 51].

Moreover, building on a recently presented method for a related system of nonlinear partial differential equations [12, 31], we also construct travelling-wave solutions for the free-boundary problem. In this respect, the novelty of our work lies in the fact that we consider a free-boundary problem subject to biomechanical transmission conditions which are different from those considered in [12, 31]. This requires a different approach when studying the properties of the solution at the interface between the two cell populations and introduces a significant difference in the qualitative properties of the travelling wave.

Numerical simulations are performed in the case where the cellular interaction forces are described by the celebrated Johnson-Kendall-Roberts (JKR) model of elastic contact [27], which has been shown to be experimentally accurate in some cases [14], and has been previously used to approximate mechanical interactions between cells [19, 20]. The results obtained support the findings of the travelling-wave analysis, and demonstrate excellent agreement between the individual-based model and the corresponding free-boundary problem.

The paper is organised as follows. In Section 2, we present our individual-based mechanical model and formally derive the corresponding free-boundary problem. In Section 3, we prove the existence result for the free-boundary problem. In Section 4, we develop the travelling-wave analysis. In Section 5, we compare simulation results for the individual-based model and numerical solutions of the free-boundary problem. Section 6 concludes the paper and provides a brief overview of possible research perspectives.

## 2. FORMULATION OF THE INDIVIDUAL-BASED MODEL AND DERIVATION OF THE CORRESPONDING FREE-BOUNDARY PROBLEM

**2.1. Formulation of the individual-based model.** We consider a one-dimensional multicellular system that consists of two populations of cells that are arranged along the real line  $\mathbb{R}$  and characterised by different proliferative and mechanical properties. We label the two cell populations by the letters  $A$  and  $B$  and make the assumption that, during the considered time interval, the cells in population  $A$  can proliferate, whereas the cells in population  $B$  cannot. We denote the number of cells in population  $B$  by  $M > 0$ . Moreover, at time  $\tau \geq 0$  we let the function  $m(\tau)$  represent the number of cells in population  $A$  and compute the total number of cells inside the system as  $n(\tau) = m(\tau) + M$ .

We adopt a discrete off-lattice modelling approach whereby every cell is identified by the position of its centre [49]. Building upon the ideas presented in [36, 37], we model the two cell populations as a chain of masses and springs with the masses corresponding to the cell centres, and assume the cell order to be fixed. We label each cell by an index  $i = 1, \dots, n(\tau)$  and describe the position of the  $i^{\text{th}}$  cell's centre at time  $\tau$  by means of the function  $r_i(\tau)$ . Without loss of generality, we let the cells of population  $A$  be on the left of the cells of population  $B$ .

We assume that the centre of the first cell of population  $A$  is pinned at a point  $s_0 \in \mathbb{R}$ , i.e.

$$(2.1) \quad r_1(\tau) = s_0, \quad \text{for all } \tau \geq 0.$$

We describe the effect of cell proliferation and mechanical interactions between cells on the dynamics of the multicellular system using the modelling strategies and the assumptions described hereafter. Mathematical modelling of cell proliferation. We assume that cell proliferation is contact dependent such that the proliferation rate  $g$  of the  $j^{\text{th}}$  cell in population  $A$  depends on the position of neighbouring cells, *i.e.*  $g \equiv g(r_j(\tau) - r_{j-1}(\tau))$  with  $j = 2, \dots, m(\tau) - 1$ . Hence in a time interval  $\Delta\tau$  the proliferation of cells in population  $A$  will result in a reindexing  $\Delta i$  of the  $i^{\text{th}}$  cell by an amount

$$(2.2) \quad (\Delta i)_i = \left[ \sum_{j=2}^i g(r_j - r_{j-1}) \Delta\tau \right] \quad \text{for } i = 2, \dots, m - 1$$

such that

$$(2.3) \quad r_i(\tau + \Delta\tau) = r_{i - (\Delta i)_i}(\tau) \quad \text{for } i = 2, \dots, m - 1,$$

where  $m \equiv m(\tau)$ .

Mathematical modelling of mechanical interactions between cells. We make the assumption that mechanical interactions between nearest neighbour cells depend on the distance between their centres. We denote the force exerted on the  $i^{\text{th}}$  cell of population  $l$  by its left and right neighbours by  $F_l(r_i - r_{i-1})$  and  $F_l(r_{i+1} - r_i)$ , respectively, and introduce the parameter  $\eta_l > 0$  to model the damping coefficient of cells in population  $l$ , where  $l = A, B$ . With this notation and neglecting cell-cell friction, the dynamics of the positions of the cell centres are described via the following system of differential equations

$$(2.4) \quad \begin{aligned} \frac{dr_i}{d\tau} &= \frac{1}{\eta_A} (F_A(r_i - r_{i-1}) - F_A(r_{i+1} - r_i)), \quad i = 2, \dots, m-1, \\ \frac{dr_i}{d\tau} &= \frac{1}{\eta_B} (F_B(r_i - r_{i-1}) - F_B(r_{i+1} - r_i)), \quad i = m+1, \dots, n-1, \end{aligned}$$

where  $m \equiv m(\tau)$  and  $n \equiv n(\tau)$ . We complete system (2.4) with the following differential equations

$$(2.5) \quad \begin{aligned} \frac{dr_1}{d\tau} &= 0, \\ \frac{dr_m}{d\tau} &= \frac{1}{\eta_A} F_A(r_m - r_{m-1}) - \frac{1}{\eta_B} F_B(r_{m+1} - r_m), \\ \frac{dr_n}{d\tau} &= \frac{1}{\eta_B} F_B(r_n - r_{n-1}). \end{aligned}$$

**2.2. Derivation of the corresponding free-boundary problem.** In order to formally derive a continuum version of our individual-based mechanical model (2.4) and (2.5), considering the scenario where the number of cells in both populations is large, we introduce the continuous variable  $y \in \mathbb{R}$  so that, for some  $\delta > 0$  sufficiently small,

$$r_i(\tau) = r(\tau, y_i) \quad \text{with} \quad y_i = i \delta,$$

and

$$r_{i\pm 1}(\tau) = r(\tau, y_{i\pm 1}) = r(\tau, y_i \pm \delta), \quad r_{i-(\Delta i)_i}(\tau) = r(\tau, y_{i-(\Delta i)_i}) = r(\tau, y_i - (\Delta i)_i \delta).$$

Moreover, we use the notation

$$(2.6) \quad r(\tau, y_1) = s_0, \quad r(\tau, y_m) = s_1(\tau), \quad r(\tau, y_n) = s_2(\tau), \quad \text{for } \tau > 0.$$

We assume the function  $r(\tau, y)$  to be continuously differentiable with respect to the variable  $\tau$  and twice continuously differentiable with respect to the variable  $y$ . Under these assumptions, letting  $\Delta\tau$  and  $\delta$  be sufficiently small, and using the Taylor expansions

$$\begin{aligned} r(\tau + \Delta\tau, y_i) &= r(\tau, y_i) + \frac{\partial r(\tau, y_i)}{\partial \tau} \Delta\tau + o(\Delta\tau), \\ r(\tau, y_i - (\Delta i)_i \delta) &= r(\tau, y_i) - \frac{\partial r(\tau, y_i)}{\partial y} (\Delta i)_i \delta + o(\delta), \end{aligned}$$

along with the approximation

$$\left[ \sum_{j=2}^i g(r_j - r_{j-1}) \Delta\tau \right] \approx \int_{y_1}^{y_i} g \left( \frac{\partial r(\tau, y')}{\partial y'} \delta \right) dy' \frac{\Delta\tau}{\delta},$$

from (2.3) we obtain

$$(2.7) \quad \frac{\partial r(\tau, y_i)}{\partial \tau} \approx - \left( \int_{y_1}^{y_i} g \left( \frac{\partial r(\tau, y')}{\partial y'} \delta \right) dy' \right) \frac{\partial r(\tau, y_i)}{\partial y} \quad \text{for } i = 2, \dots, m-1.$$

Moreover, using the Taylor expansions

$$\begin{aligned} r(\tau, y_i + \delta) &= r(\tau, y_i) + \frac{\partial r(\tau, y_i)}{\partial y} \delta + \frac{1}{2} \frac{\partial^2 r(\tau, y_i)}{\partial y^2} \delta^2 + o(\delta^2), \\ r(\tau, y_i - \delta) &= r(\tau, y_i) - \frac{\partial r(\tau, y_i)}{\partial y} \delta + \frac{1}{2} \frac{\partial^2 r(\tau, y_i)}{\partial y^2} \delta^2 + o(\delta^2), \end{aligned}$$

and making the additional assumption that the functions  $F_A$  and  $F_B$  are twice continuously differentiable, we approximate the force terms in (2.4) for  $i = 2, \dots, n-1$  as

$$(2.8) \quad F_l(r_i - r_{i-1}) - F_l(r_{i+1} - r_i) \approx -F_l' \left( \frac{\partial r}{\partial y} \delta \right) \frac{\partial^2 r}{\partial y^2} \delta^2, \quad l = A, B.$$

Using the approximations (2.7) and (2.8), and combining proliferation (2.3) and mechanical interaction (2.4) processes, we obtain

$$(2.9) \quad \frac{\partial r}{\partial \tau} = -\frac{1}{\eta_A} F'_A \left( \frac{\partial r}{\partial y} \delta \right) \frac{\partial^2 r}{\partial y^2} \delta^2 - \left( \int_{y_1}^y g \left( \frac{\partial r}{\partial y'} \delta \right) dy' \right) \frac{\partial r}{\partial y} \quad \text{for } y \in (y_1, y_m),$$

and

$$(2.10) \quad \frac{\partial r}{\partial \tau} = -\frac{1}{\eta_B} F'_B \left( \frac{\partial r}{\partial y} \delta \right) \frac{\partial^2 r}{\partial y^2} \delta^2 \quad \text{for } y \in (y_m, y_n).$$

Similarly, mechanical interaction processes (2.5) for  $i = 1$ ,  $i = m$  and  $i = n$  yield, respectively,

$$(2.11) \quad \begin{aligned} \frac{\partial r}{\partial \tau} &= 0 && \text{at } y = y_1, \\ \frac{\partial r}{\partial \tau} &= \frac{1}{\eta_A} \left[ F_A \left( \frac{\partial r}{\partial y} \delta \right) - \frac{1}{2} F'_A \left( \frac{\partial r}{\partial y} \delta \right) \frac{\partial^2 r}{\partial y^2} \delta^2 \right] \\ &\quad - \frac{1}{\eta_B} \left[ F_B \left( \frac{\partial r}{\partial y} \delta \right) + \frac{1}{2} F'_B \left( \frac{\partial r}{\partial y} \delta \right) \frac{\partial^2 r}{\partial y^2} \delta^2 \right] && \text{at } y = y_m, \end{aligned}$$

and

$$(2.12) \quad \frac{\partial r}{\partial \tau} = \frac{1}{\eta_B} \left[ F_B \left( \frac{\partial r}{\partial y} \delta \right) - \frac{1}{2} F'_B \left( \frac{\partial r}{\partial y} \delta \right) \frac{\partial^2 r}{\partial y^2} \delta^2 \right] \quad \text{at } y = y_n.$$

Based on the ideas presented in [36, 37] and according to the considerations given in Remark 2.1, we define the cell number densities of populations  $A$  and  $B$  as

$$(2.13) \quad \rho_A(\tau, y) = \left( \frac{\partial r}{\partial y} \delta \right)^{-1} \quad \text{for } y \in [y_1, y_m], \quad \rho_B(\tau, y) = \left( \frac{\partial r}{\partial y} \delta \right)^{-1} \quad \text{for } y \in [y_m, y_n].$$

**Remark 2.1.** *The definitions of the cell densities given by (2.13) are based on the observation that, at any time  $\tau$ , the quotient of the number of cells in a generic interval  $[r_i, r_j]$ , with  $j > i$ , and the length of the interval is*

$$\frac{j - i}{r_j(\tau) - r_i(\tau)} = \frac{j - i}{r(\tau, y_j) - r(\tau, y_i)}.$$

*From the above relation, choosing  $j = i + 1$  and using the fact that  $\delta$  is small, we obtain the following approximate expression for the cell density*

$$(2.14) \quad \rho_l(\tau, y_i) = \frac{1}{r_{i+1}(\tau) - r_i(\tau)} \approx \frac{1}{\delta} \frac{1}{\frac{\partial r(\tau, y_i)}{\partial y}}.$$

The change of coordinates  $(\tau, y) \mapsto (t, r)$ , with  $t = \tau$  and  $r = r(\tau, y)$  yields [36]

$$\frac{\partial r}{\partial \tau} = -\frac{\partial r}{\partial y} \frac{\partial y}{\partial t} = -\frac{1}{\delta} \frac{1}{\rho_l} \frac{\partial y}{\partial t}.$$

Substituting this relation along with the expressions

$$\begin{aligned} \frac{\partial r}{\partial y} &= \frac{1}{\delta} \frac{1}{\rho_l}, & \frac{\partial^2 r}{\partial y^2} &= \frac{1}{\delta} \frac{\partial}{\partial r} \left( \frac{1}{\rho_l} \right) \frac{\partial r}{\partial y} = -\frac{1}{\delta^2} \frac{1}{\rho_l^3} \frac{\partial \rho_l}{\partial r}, \\ \left( \int_{y_1}^y g \left( \frac{\partial r}{\partial y'} \delta \right) dy' \right) \frac{\partial r}{\partial y} &= \left( \int_{s_0}^r g(1/\rho_A) \rho_A dr' \right) \frac{1}{\rho_A} \end{aligned}$$

into equations (2.9) and (2.10) yields

$$(2.15) \quad \frac{1}{\delta} \frac{1}{\rho_A} \frac{\partial y}{\partial t} = -\frac{F'_A(1/\rho_A)}{\eta_A \rho_A^3} \frac{\partial \rho_A}{\partial r} + \int_{s_0}^r g(1/\rho_A) \rho_A dr' \frac{1}{\rho_A}$$

and

$$(2.16) \quad \frac{1}{\delta} \frac{1}{\rho_B} \frac{\partial y}{\partial t} = -\frac{F'_B(1/\rho_B)}{\eta_B \rho_B^3} \frac{\partial \rho_B}{\partial r} + \int_{s_0}^{s_1} g(1/\rho_A) \rho_A dr' \frac{1}{\rho_B}.$$

Multiplying equations (2.15) and (2.16) by  $\rho_A$  and  $\rho_B$ , respectively, differentiating with respect to  $r$ , using the fact that

$$\frac{\partial}{\partial r} \left( \frac{1}{\delta} \frac{\partial y}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{1}{\delta} \frac{\partial y}{\partial r} \right) = \frac{\partial}{\partial t} \left( \frac{\partial r}{\partial y} \delta \right)^{-1} = \frac{\partial \rho_l}{\partial t},$$

$$\frac{\partial}{\partial r} \int_{s_0}^r g(1/\rho_A) \rho_A dr' = G(\rho_A) \rho_A \quad \text{and} \quad \frac{\partial}{\partial r} \int_{s_0}^{s_1} g(1/\rho_A) \rho_A dr' = 0,$$

with the growth rate  $G$  of population  $A$  defined as

$$(2.17) \quad G(\rho_A) = g(1/\rho_A),$$

and renaming  $r$  to  $x$ , we obtain the following equations for the cell densities  $\rho_A(t, x)$  and  $\rho_B(t, x)$

$$(2.18) \quad \partial_t \rho_A = \partial_x (D_A(\rho_A) \partial_x \rho_A) + G(\rho_A) \rho_A \quad \text{for } x \in (s_0, s_1(t)), \quad t > 0,$$

$$(2.19) \quad \partial_t \rho_B = \partial_x (D_B(\rho_B) \partial_x \rho_B) \quad \text{for } x \in (s_1(t), s_2(t)), \quad t > 0,$$

where

$$(2.20) \quad D_l(\rho_l) = -\frac{F_l'(1/\rho_l)}{\eta_l \rho_l^2} \quad \text{for } l = A, B.$$

Similarly, the evolution equations for the positions of the free boundaries  $s_1(t)$  and  $s_2(t)$  are obtained from equations (2.11) and (2.12), respectively, yielding

$$(2.21) \quad \begin{aligned} \frac{ds_1}{dt} &= \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A}\right) - \frac{1}{\eta_B} F_B\left(\frac{1}{\rho_B}\right) - \frac{1}{2} \left[ \frac{D_A(\rho_A)}{\rho_A} \partial_x \rho_A + \frac{D_B(\rho_B)}{\rho_B} \partial_x \rho_B \right] \quad \text{at } x = s_1(t), \\ \frac{ds_2}{dt} &= \frac{1}{\eta_B} F_B\left(\frac{1}{\rho_B}\right) - \frac{1}{2} \frac{D_B(\rho_B)}{\rho_B} \partial_x \rho_B \quad \text{at } x = s_2(t). \end{aligned}$$

In order to obtain the boundary conditions (*i.e.* the conditions at  $s_0$  and  $s_2(t)$ ) and the transmission conditions (*i.e.* the conditions at  $s_1(t)$ ), that are needed to complete the problem, we consider the mass balance equations

$$\begin{aligned} \int_{s_0}^{s_1(t)} G(\rho_A) \rho_A dx &= \frac{d}{dt} \left( \int_{s_0}^{s_1(t)} \rho_A dx + \int_{s_1(t)}^{s_2(t)} \rho_B dx \right), \\ \int_{s_0}^{s_1(t)} G(\rho_A) \rho_A dx &= \frac{d}{dt} \int_{s_0}^{s_1(t)} \rho_A dx. \end{aligned}$$

Using the fact that  $\frac{ds_0}{dt} = 0$ , together with equations (2.18) and (2.19), yields

$$\begin{aligned} \int_{s_0}^{s_1(t)} G(\rho_A) \rho_A dx &= \int_{s_0}^{s_1(t)} \partial_t \rho_A dx + \rho_A(t, s_1) \frac{ds_1}{dt} - \rho_A(t, s_0) \frac{ds_0}{dt} \\ &\quad + \int_{s_1(t)}^{s_2(t)} \partial_t \rho_B dx + \rho_B(t, s_2) \frac{ds_2}{dt} - \rho_B(t, s_1) \frac{ds_1}{dt} \\ &= \int_{s_0}^{s_1(t)} G(\rho_A) \rho_A dx + D_A(\rho_A) \partial_x \rho_A \Big|_{x=s_1} - D_A(\rho_A) \partial_x \rho_A \Big|_{x=s_0} + \rho_A(t, s_1) \frac{ds_1}{dt} \\ &\quad + D_B(\rho_B) \partial_x \rho_B \Big|_{x=s_2} - D_B(\rho_B) \partial_x \rho_B \Big|_{x=s_1} + \rho_B(t, s_2) \frac{ds_2}{dt} - \rho_B(t, s_1) \frac{ds_1}{dt} \end{aligned}$$

and

$$\begin{aligned} \int_{s_0}^{s_1(t)} G(\rho_A) \rho_A dx &= \int_{s_0}^{s_1(t)} \partial_t \rho_A dx + \rho_A(t, s_1) \frac{ds_1}{dt} - \rho_A(t, s_0) \frac{ds_0}{dt} = \rho_A(t, s_1) \frac{ds_1}{dt} \\ &\quad + \int_{s_0}^{s_1(t)} G(\rho_A) \rho_A dx + D_A(\rho_A) \partial_x \rho_A \Big|_{x=s_1} - D_A(\rho_A) \partial_x \rho_A \Big|_{x=s_0}. \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \frac{ds_1}{dt} (\rho_A(t, s_1) - \rho_B(t, s_1)) + (D_A(\rho_A) \partial_x \rho_A - D_B(\rho_B) \partial_x \rho_B) \Big|_{x=s_1} \\ &\quad + \frac{ds_2}{dt} \rho_B(t, s_2) + D_B(\rho_B) \partial_x \rho_B \Big|_{x=s_2} - D_A(\rho_A) \partial_x \rho_A \Big|_{x=s_0} \end{aligned}$$

and

$$0 = \frac{ds_1}{dt} \rho_A(t, s_1) + D_A(\rho_A) \partial_x \rho_A \Big|_{x=s_1} - D_A(\rho_A) \partial_x \rho_A \Big|_{x=s_0}.$$

The above equations along with equations (2.21) give

$$\begin{aligned}
(2.22) \quad & D_A(\rho_A)\partial_x\rho_A = 0 && \text{at } x = s_0, \\
& \frac{1}{\eta_A}F_A\left(\frac{1}{\rho_A}\right) = \frac{1}{\eta_B}F_B\left(\frac{1}{\rho_B}\right) && \text{at } x = s_1(t), \\
& \frac{ds_1}{dt}\rho_A = -D_A(\rho_A)\partial_x\rho_A && \text{at } x = s_1(t), \\
& \frac{ds_1}{dt}(\rho_A - \rho_B) = -(D_A(\rho_A)\partial_x\rho_A - D_B(\rho_B)\partial_x\rho_B) && \text{at } x = s_1(t), \\
& \frac{ds_2}{dt} = \frac{1}{\eta_B}F_B\left(\frac{1}{\rho_B}\right) - \frac{1}{2}\frac{1}{\rho_B}D_B(\rho_B)\partial_x\rho_B && \text{at } x = s_2(t), \\
& \frac{ds_2}{dt}\rho_B = -D_B(\rho_B)\partial_x\rho_B && \text{at } x = s_2(t).
\end{aligned}$$

We complement the free-boundary problem (2.18), (2.19) and (2.22) with the following initial conditions for the moving boundaries  $s_1(t)$  and  $s_2(t)$ , and the cell densities  $\rho_A(t, x)$  and  $\rho_B(t, x)$ :

$$\begin{aligned}
(2.23) \quad & s_1(0) = s_1^*, & s_2(0) = s_2^*, \\
& \rho_A(0, x) = \rho_A^0(x) & \text{for } x \in (s_0, s_1^*), \\
& \rho_B(0, x) = \rho_B^0(x) & \text{for } x \in (s_1^*, s_2^*),
\end{aligned}$$

where  $s_0 < s_1^* < s_2^*$ . Letting  $\rho_l^{\text{eq}} > 0$  denote the equilibrium cell density (*i.e.* the density below which intercellular forces are zero) and  $\rho^M > \rho_A^{\text{eq}}$  a critical cell density above which cells stop dividing due to contact inhibition, throughout the rest of the paper we will make the following assumptions:

$$(2.24) \quad \rho_A^0(x) \geq \rho_A^{\text{eq}} \text{ for all } x \in (s_0, s_1^*), \quad \rho_B^0(x) \geq \rho_B^{\text{eq}} \text{ for all } x \in (s_1^*, s_2^*),$$

$$(2.25) \quad F_l(1/\rho_l) = 0, \quad F_l'(1/\rho_l) = 0 \text{ for } \rho_l \leq \rho_l^{\text{eq}}, \quad F_l(1/\rho_l) > 0, \quad F_l'(1/\rho_l) < 0 \text{ for } \rho_l > \rho_l^{\text{eq}},$$

where  $l = A, B$ , and

$$(2.26) \quad g(\cdot) > 0 \text{ in } (1/\rho^M, \infty), \quad g(\cdot) = 0 \text{ in } (0, 1/\rho^M], \quad g'(\cdot) < 0 \text{ in } [1/\rho^M, \infty).$$

Assumptions (2.25) and (2.26), together with notations (2.17) and (2.20), imply that the nonlinear diffusion coefficient  $D_l(\rho_l)$  and the growth rate  $G(\rho_A)$  are such that

$$(2.27) \quad D_l(\rho_l) = 0 \text{ for } \rho_l \leq \rho_l^{\text{eq}}, \quad D_l(\rho_l) > 0 \text{ for } \rho_l > \rho_l^{\text{eq}}, \quad l = A, B,$$

and

$$(2.28) \quad G(\cdot) > 0 \text{ in } (0, \rho^M), \quad G(\cdot) = 0 \text{ in } [\rho^M, \infty), \quad G'(\cdot) < 0 \text{ in } (0, \rho^M].$$

### 3. AN EXISTENCE RESULT FOR THE FREE-BOUNDARY PROBLEM

Due to the specific structure of our boundary and transmission conditions, the existing well-posedness results for one-dimensional free-boundary problems, such as those presented in [21, 22, 23, 51], are not directly applicable to our problem. Therefore, in this section we prove an existence result for the free-boundary problem (2.18), (2.19), (2.22) and (2.23).

**Assumption 3.1.** *We make the following assumptions on the force terms  $F_A$  and  $F_B$ , the diffusion coefficients  $D_A$  and  $D_B$ , the growth rate  $G$ , and the initial conditions  $\rho_A^0$  and  $\rho_B^0$ .*

- (i) *The force terms  $F_l \in H^1(0, \infty) \cap C^3(\rho_l^{\text{eq}}, \infty)$ , with  $l = A, B$ , and satisfy (2.25).*
- (ii) *The diffusion coefficients  $D_l \in C^2(\rho_l^{\text{eq}}, \infty) \cap L^\infty(0, R)$ , for any  $R > 0$ , with  $l = A, B$ , are defined by (2.20), satisfy assumptions (2.27), and  $D_l(\xi) \geq d_l > 0$  for  $\xi > \rho_l^{\text{eq}}$ .*
- (iii) *The growth rate  $G \in C^3(\mathbb{R})$  and satisfies assumptions (2.28).*
- (iv) *The initial conditions  $\rho_A^0, \rho_B^0 \in C_0^3(\mathbb{R})$  and satisfy assumptions (2.24).*

Throughout this section we use the notation

$$\Omega_A(t) = (s_0, s_1(t)) \quad \text{and} \quad \Omega_B(t) = (s_1(t), s_2(t)),$$

for  $t \in [0, T]$ , with  $T > 0$ , and consider solutions in the sense specified by the following definition.

**Definition 3.2.** A solution of the free-boundary problem (2.18), (2.19), (2.22), (2.23) is given by functions  $s_1, s_2 \in W^{1,3}(0, T)$  and  $\rho_l \in H^1(0, T; L^2(\Omega_l(t))) \cap L^2(0, T; H^2(\Omega_l(t)))$ , with  $\rho_l \geq \rho_l^{\text{eq}}$  and  $\rho_l \in L^\infty(0, T; L^\infty(\Omega_l(t)))$  for  $l = A, B$ , that satisfy equations (2.18) and (2.19), the following boundary and transmission conditions

$$(3.1) \quad \begin{aligned} \partial_x F_A\left(\frac{1}{\rho_A}\right) &= 0 && \text{at } x = s_0, \\ \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A}\right) &= \frac{1}{\eta_B} F_B\left(\frac{1}{\rho_B}\right) && \text{at } x = s_1(t), \\ \frac{\partial_x F_A(1/\rho_A)}{\eta_A \rho_A} &= \frac{\partial_x F_B(1/\rho_B)}{\eta_B \rho_B} && \text{at } x = s_1(t), \\ \frac{\partial_x F_B(1/\rho_B)}{\eta_B \rho_B} &= -\frac{2}{\eta_B} F_B\left(\frac{1}{\rho_B}\right) && \text{at } x = s_2(t), \end{aligned}$$

the equations for the free boundaries

$$(3.2) \quad \begin{aligned} \frac{ds_1}{dt} &= -\frac{\partial_x F_A(1/\rho_A)}{\eta_A \rho_A} && \text{at } x = s_1(t), \\ \frac{ds_2}{dt} &= \frac{2}{\eta_B} F_B\left(\frac{1}{\rho_B}\right) && \text{at } x = s_2(t), \end{aligned}$$

and the initial conditions (2.23).

**Theorem 3.3.** Under Assumptions 3.1 there exists a solution of the free-boundary problem (2.18), (2.19), (2.22) and (2.23).

*Proof.* In order to prove the existence of a solution of the free-boundary problem (2.18), (2.19), (2.22) we consider iterations over successive time intervals and use a fixed point argument. In particular, we first show the existence of a solution on a time interval  $[0, T_1]$  such that

$$(3.3) \quad |s_1(t) - s_1^*| \leq \frac{s_1^* - s_0}{8} \quad \text{and} \quad |s_2(t) - s_2^*| \leq \frac{s_2^* - s_1^*}{8}, \quad \text{for } t \in [0, T_1].$$

Subsequently, the boundedness of  $s_1'$  and  $s_2'$ , shown at the end of the proof, will allow iteration over successive time intervals in order to obtain an existence result for  $t \in (0, T]$ .

We begin by making the change of variables

$$(3.4) \quad (t, x) \mapsto (t, y), \quad \text{with} \quad x = y + \zeta(y)(s_1(t) - s_1^*) + \xi(y)(s_2(t) - s_2^*),$$

with  $\zeta, \xi \in C_0^2(\mathbb{R})$  such that  $\zeta(y) = 1$  for  $|y - s_1^*| < \alpha$  and  $\zeta(y) = 0$  for  $|y - s_1^*| > 2\alpha$ , while  $\xi(y) = 1$  for  $|y - s_2^*| < \alpha$  and  $\xi(y) = 0$  for  $|y - s_2^*| > 2\alpha$ , where  $\alpha = \min\{(s_1^* - s_0)/4, (s_2^* - s_1^*)/4\}$ . The change of variables (3.4) transforms the time-dependent domains  $\Omega_A(t) = (s_0, s_1(t))$  and  $\Omega_B(t) = (s_1(t), s_2(t))$  into the fixed intervals  $\Omega_A^* = (s_0, s_1^*)$  and  $\Omega_B^* = (s_1^*, s_2^*)$ , respectively. A similar change of variables was considered in [21]. Notice that, for  $s_1(t)$  and  $s_2(t)$  satisfying conditions (3.3), such a change of variables defines a diffeomorphism from  $[0, +\infty)$  into  $[0, +\infty)$ . Hence we obtain

$$\rho_l(t, x) = \rho_l(t, y + \zeta(y)(s_1(t) - s_1^*) + \xi(y)(s_2(t) - s_2^*)) = w_l(t, y) \quad \text{for } l = A, B,$$

where  $w_A$  and  $w_B$  satisfy the reaction-diffusion-convection equations

$$(3.5) \quad \begin{aligned} J_A(s_1) \partial_t w_A - \partial_y (R_A^2(s_1) D_A(w_A) \partial_y w_A) - Q_A(s_1') \partial_y w_A - J_A(s_1) G_A(w_A) &= 0, \\ J_B(s_1, s_2) \partial_t w_B - \partial_y (R_B^2(s_1, s_2) D_B(w_B) \partial_y w_B) - Q_B(s_1', s_2') \partial_y w_B &= 0, \end{aligned}$$

complemented with the nonlinear transmission and boundary conditions

$$(3.6) \quad \begin{aligned} \partial_y F_A(w_A) &= 0 && \text{at } y = s_0, \\ F_A(w_A) &= F_B(w_B) && \text{at } y = s_1^*, \\ \frac{\partial_y F_A(w_A)}{w_A} &= \frac{\partial_y F_B(w_B)}{w_B} && \text{at } y = s_1^*, \\ \frac{\partial_y F_B(w_B)}{w_B} &= -2F_B(w_B) && \text{at } y = s_2^*, \end{aligned}$$



and the equations for the velocities  $s'_1$  and  $s'_2$

$$(3.7) \quad \begin{aligned} \frac{ds_1}{dt} w_A &= -\partial_y F_A(w_A) \quad \text{at } y = s_1^*, \\ \frac{ds_2}{dt} w_B &= -\partial_y F_B(w_B) \quad \text{at } y = s_2^*. \end{aligned}$$

Notice that for ease of notation we denote

$$F_l(w_l) \equiv \frac{1}{\eta_l} F_l(1/w_l) \quad \text{and} \quad F'_l(w_l) \equiv -\frac{1}{\eta_l} \frac{F'_l(1/w_l)}{w_l^2}, \quad l = A, B,$$

$G_A(w_A) = G(w_A)w_A$ , and the functions  $D_l(w_l)$  are defined in terms of  $F_l(w_l)$  according to (2.20).

In equations (3.5), since  $\xi(y) = 0$  for  $y < s_2^* - 2\alpha$  and  $s_1(t) < s_2^* - 2\alpha$  for  $t \in [0, T_1]$ , we have

$$\begin{aligned} R_A(s_1) &= \frac{dy}{dx} = \frac{1}{1 + \zeta'(y)(s_1(t) - s_1^*)} && \text{for } s_0 < x < s_1^*, \\ R_B(s_1, s_2) &= \frac{dy}{dx} = \frac{1}{1 + \zeta'(y)(s_1(t) - s_1^*) + \xi'(y)(s_2(t) - s_2^*)} && \text{for } s_1^* < x < s_2^*, \\ J_A(s_1) &= 1 + \zeta'(y)(s_1(t) - s_1^*) && \text{for } s_0 < x < s_1^*, \\ J_B(s_1, s_2) &= 1 + \zeta'(y)(s_1(t) - s_1^*) + \xi'(y)(s_2(t) - s_2^*) && \text{for } s_1^* < x < s_2^*, \\ Q_A(s'_1) &= \zeta(y) s'_1(t), \quad Q_B(s'_1, s'_2) = \zeta(y) s'_1(t) + \xi(y) s'_2(t), \\ \frac{dy}{dt} &= \frac{Q_A(s'_1)}{J_A(s_1)} \quad \text{for } s_0 < x < s_1^*, \quad \frac{dy}{dt} = \frac{Q_B(s'_1, s'_2)}{J_B(s_1, s_2)} && \text{for } s_1^* < x < s_2^*. \end{aligned}$$

The assumptions on  $F_l$  and  $D_l$ , for  $l = A, B$ , ensure that

$$(3.8) \quad \begin{aligned} D_A(w_A)R_A(s_1)\partial_y w_A &= 0 && \text{for } 0 < w_A \leq \rho_A^{\text{eq}}, \\ D_B(w_B)R_B(s_1, s_2)\partial_y w_B &= 0 && \text{for } 0 < w_B \leq \rho_B^{\text{eq}}. \end{aligned}$$

Notice that, without loss of generality, we can focus on the case where  $\rho_l^0 > \rho_l^{\text{eq}}$  for  $l = A, B$ . In fact, if  $\rho_l^0 = \rho_l^{\text{eq}}$  the growth term in the equation for  $w_A$  would result into  $w_A(t, y) > \rho_A^{\text{eq}}$  and  $F_A(w_A) > 0$ , thus ensuring that  $w_B(t, y) > \rho_B^{\text{eq}}$  due to the transmission conditions at  $s_1^*$  and the convection term in the equation for  $w_B$ . In the case where  $\rho_l^0 > \rho_l^{\text{eq}}$  for  $l = A, B$ , using the maximum principle and relations (3.8) we obtain that  $w_l(t, x) > \rho_l^{\text{eq}}$  for  $(t, x) \in (0, T) \times \Omega_l^*$ . Therefore, we conclude that system (3.6), (3.7), or equivalently system (2.18), (2.19), is nondegenerate. Notice also that assuming  $s_j(t) \in H^2(0, T)$  with  $s'_j(t) \geq 0$  for  $t \in [0, T]$  and  $j = 1, 2$ , and considering  $F_l(w_l)$ ,  $\partial_y^2 \partial_t F_l(w_l)$  and  $\partial_t^2 w_l$  as test functions in equations (3.6) and (3.7), one can prove that  $w_l$  is continuous in  $\bar{\Omega}_{l,T}^*$ , while  $\partial_t w_l$  and  $\partial_y^2 F_l(w_l)$  are continuous in  $(0, T) \times \Omega_l^*$  for  $l = A, B$ , which is the regularity required to apply the maximum principle. A similar approach was previously used in the analysis of the porous medium equation [50].

The assumptions on  $F_B$  imply that

$$D_B(w_B)\partial_y w_B = -2w_B F_B(w_B) \leq 0 \quad \text{at } y = s_2^*, \quad t \geq 0,$$

and, applying the maximum principle to the equation for  $w_B$ , we find that  $w_B$  has a minimum at  $s_2^*$  and a maximum at  $s_1^*$ . Hence,  $\partial_y w_B(t, y) \leq 0$  at  $y = s_1^*$  for  $t > 0$  and, therefore,  $\partial_y F_A(w_A) \leq 0$  at  $y = s_1^*$ . Applying the comparison principle and using the fact that  $F'_A(w_A) = 0$  for  $0 < w_A(t, y) \leq \rho_A^{\text{eq}}$ , along with the assumptions on  $G$  and on the initial conditions, we obtain

$$(3.9) \quad \rho_A^{\text{eq}} \leq w_A(t, y) \leq \rho_A^{\text{M}} \quad \text{in } [s_0, s_1^*], \quad t \geq 0,$$

where  $\rho_A^{\text{M}} = \max\{\rho^{\text{M}}, \max_{x \in [s_0, s_1^*]} \rho_A^0(x)\}$ . Moreover, applying the maximum principle to the equation for  $w_B$  and using the assumptions on  $F_B$  and on the initial data, along with the boundedness of  $w_A$  and the transmission conditions at  $y = s_1^*$ , yield

$$(3.10) \quad \rho_B^{\text{eq}} \leq w_B(t, y) \leq \rho_B^{\text{M}} \quad \text{in } [s_1^*, s_2^*], \quad t \geq 0,$$

where  $\rho_B^M = \max\{F_B^{-1}(F_A(\rho_A^M)), \max_{x \in [s_1^*, s_2^*]} \rho_B^0(x)\}$ . Using these results along with the change of variables given by equation (3.4) we conclude that

$$\begin{aligned} \rho_A^{\text{eq}} \leq \rho_A(t, x) \leq \rho_A^M & \quad \text{for } x \in [s_0, s_1(t)], \quad t \geq 0, \\ \rho_B^{\text{eq}} \leq \rho_B(t, x) \leq \rho_B^M & \quad \text{for } x \in [s_1(t), s_2(t)], \quad t \geq 0. \end{aligned}$$

If  $w_B$  is nonconstant in  $(s_1^*, s_2^*)$  and  $w_B(t, s_j^*) \neq \rho_B^{\text{eq}}$  for  $j = 1, 2$  and  $t \geq 0$ , the maximum principle yields  $\partial_y w_B(t, s_2^*) < 0$  and  $\partial_y w_B(t, s_1^*) < 0$ . This along with the assumptions on  $F_B$  ensures the monotonicity of the free boundaries  $\{x = s_1(t)\}$  and  $\{x = s_2(t)\}$ , *i.e.*

$$\begin{aligned} \frac{ds_2(t)}{dt} > 0 & \quad \text{if } w_B(t, s_2^*) > \rho_B^{\text{eq}}, & \frac{ds_2(t)}{dt} = 0 & \quad \text{if } w_B(t, s_2^*) = \rho_B^{\text{eq}}, & t \geq 0, \\ \frac{ds_1(t)}{dt} > 0 & \quad \text{if } w_B(t, s_1^*) > \rho_B^{\text{eq}}, & \frac{ds_1(t)}{dt} = 0 & \quad \text{if } w_B(t, s_1^*) = \rho_B^{\text{eq}}, & t \geq 0. \end{aligned}$$

To prove the existence of a solution of problem (3.5)-(3.7) we use a fixed point argument. Let

$$(3.11) \quad s_1^{*,1} = -\frac{1}{\rho_A^0 \eta_A} \partial_x F_A\left(\frac{1}{\rho_A^0(s_1^*)}\right), \quad s_2^{*,1} = -\frac{1}{\rho_B^0 \eta_B} \partial_x F_B\left(\frac{1}{\rho_B^0(s_2^*)}\right),$$

which are both well-defined quantities due to the assumptions on  $F_l$  and  $\rho_l^0$ , for  $l = A, B$ . Moreover, consider

$$\begin{aligned} \mathcal{W}_l &= \{u \in L^6(0, T_1; W^{1,4}(\Omega_l^*)) : \rho_l^{\text{eq}} \leq u(t, x) \leq \rho_l^M \text{ for } (t, x) \in \Omega_{l, T_1}^*, \|\partial_t u\|_{L^2(\Omega_{l, T_1}^*)} \leq \mu\}, \\ \mathcal{W}_s &= \{(s_1, s_2) \in W^{1,3}(0, T_1)^2 : \|s_j' - s_j^{*,1}\|_{L^3(0, T_1)} \leq 1, \text{ for } j = 1, 2\}, \end{aligned}$$

for  $l = A, B$ , some constant  $\mu > 0$ , and  $T_1 > 0$ . Notice that for  $(s_1, s_2) \in \mathcal{W}_s$  we have

$$\sup_{(0, T_1)} |s_j(t) - s_j^*| \leq \int_0^{T_1} \left| \frac{ds_j}{dt} \right| dt \leq T_1^{\frac{2}{3}} \|s_j'(t)\|_{L^3(0, T_1)} \leq T_1^{\frac{2}{3}} (1 + \|s_j^{*,1}\|_{L^3(0, T_1)}), \quad j = 1, 2.$$

Therefore, choosing

$$T_1 = \min \left\{ \left( (s_1^* - s_0)/8 \right)^{\frac{3}{2}} (1 + \|s_1^{*,1}\|_{L^3(0, T_1)})^{-\frac{3}{2}}, \left( (s_2^* - s_1^*)/8 \right)^{\frac{3}{2}} (1 + \|s_2^{*,1}\|_{L^3(0, T_1)})^{-\frac{3}{2}} \right\},$$

we find that  $s_j$  satisfies the conditions (3.3), for  $j = 1, 2$ , and the change of coordinates (3.4) is well-defined for all  $(s_1, s_2) \in \mathcal{W}_s$ .

For some given  $(\tilde{s}_1, \tilde{s}_2) \in \mathcal{W}_s$  and  $\tilde{w}_l \in \mathcal{W}_l$ , with  $l = A, B$ , we first consider the problem given by the following equations for  $w_A$  and  $w_B$

$$(3.12) \quad \begin{aligned} J_A(\tilde{s}_1) \partial_t w_A - \partial_y (R_A^2(\tilde{s}_1) \partial_y F_A(w_A)) - Q_A(\tilde{s}_1') \partial_y w_A &= J_A(\tilde{s}_1) G_A(\tilde{w}_A) & \text{in } \Omega_A^*, t > 0, \\ J_B(\tilde{s}_1, \tilde{s}_2) \partial_t w_B - \partial_y (R_B^2(\tilde{s}_1, \tilde{s}_2) \partial_y F_B(w_B)) - Q_B(\tilde{s}_1', \tilde{s}_2') \partial_y w_B &= 0 & \text{in } \Omega_B^*, t > 0, \\ \partial_y F_A(w_A) &= 0 & \text{at } y = s_0, t > 0, \\ \frac{\partial_y F_A(w_A)}{\tilde{w}_A} = \frac{\partial_y F_B(w_B)}{\tilde{w}_B}, \quad F_A(w_A) = F_B(w_B) & & \text{at } y = s_1^*, t > 0, \\ \frac{\partial_y F_B(w_B)}{\tilde{w}_B} = -2F_B(w_B) & & \text{at } y = s_2^*, t > 0, \\ w_A(0) &= \rho_A^0 & \text{in } (s_0, s_1^*), \\ w_B(0) &= \rho_B^0 & \text{in } (s_1^*, s_2^*). \end{aligned}$$

For  $(\tilde{s}_1, \tilde{s}_2) \in \mathcal{W}_s$  and  $\tilde{w}_l \in L^2(0, T_1; H^1(\Omega_l^*))$ , with  $l = A, B$ , applying the Rothe-Galerkin method and using the a priori estimates obtained by considering  $F_l(w_l)/\tilde{w}_l$  as a test function in the equations for  $w_l$ , we obtain the existence of a weak solution  $F_l(w_l) \in L^2(0, T_1; H^1(\Omega_l^*))$ , with  $\partial_t w_l \in L^2(0, T_1; H^{-1}(\Omega_l^*))$ , of problem (3.12). Notice that for  $\tilde{s}_1, \tilde{s}_2 \in H^2(0, T_1)$ , in the same way as below, we can show that the solutions of (3.12) satisfy the regularity properties required by the maximum principle, and obtain that the solutions are bounded and satisfy (3.9) and (3.10), and the equations in (3.12) are nondegenerate.

To derive a priori estimates for  $\partial_t w_A$  and  $\partial_t w_B$ , we consider  $\phi = \partial_t F_A(w_A)/\tilde{w}_A$  and  $\psi = \partial_t F_B(w_B)/\tilde{w}_B$  as test functions for the equations in problem (3.12). In this way, we obtain

$$\begin{aligned}
(3.13) \quad & \sum_{l=A,B} \left\{ \int_0^\tau \int_{\Omega_l^*} \frac{J_l(\tilde{s}) D_l(w_l)}{\tilde{w}_l} |\partial_t w_l|^2 dy dt + \frac{1}{2} \int_0^\tau \frac{d}{dt} \int_{\Omega_l^*} \frac{R_l(\tilde{s})^2}{\tilde{w}_l} |\partial_y F_l(w_l)|^2 dy dt \right. \\
& \quad - \int_0^\tau \int_{\Omega_l^*} \left[ Q_l(\tilde{s}') \frac{\partial_y F_l(w_l)}{\tilde{w}_l} \partial_t w_l + \frac{R_l(\tilde{s})^2}{\tilde{w}_l^2} \partial_y F_l(w_l) \partial_t F_l(w_l) \partial_y \tilde{w}_l \right] dy dt \\
& \quad \left. + \int_0^\tau \int_{\Omega_l^*} \left[ \frac{1}{2} \frac{R_l^2(\tilde{s})}{\tilde{w}_l^2} |\partial_y F_l(w_l)|^2 \partial_t \tilde{w}_l - \frac{R_l(\tilde{s}) \partial_t R_l(\tilde{s})}{\tilde{w}_l} |\partial_y F_l(w_l)|^2 \right] dy dt \right\} \\
& \quad - \int_0^\tau \int_{\Omega_A^*} J_A(\tilde{s}_1) G_A(\tilde{w}_A) \frac{\partial_t F_A(w_A)}{\tilde{w}_A} dy dt + 2 \int_0^\tau F_B(w_B) \partial_t F_B(w_B) \Big|_{y=s_2^*} dt \\
& \quad = \int_0^\tau \left[ \frac{\partial_y F_A(w_A)}{\tilde{w}_A} \partial_t F_A(w_A) - \frac{\partial_y F_B(w_B)}{\tilde{w}_B} \partial_t F_B(w_B) \right]_{y=s_1^*} dt,
\end{aligned}$$

for  $\tau \in (0, T_1]$ , where  $J_A(\tilde{s}) = J_A(\tilde{s}_1)$ ,  $J_B(\tilde{s}) = J_B(\tilde{s}_1, \tilde{s}_2)$ ,  $R_A(\tilde{s}) = R_A(\tilde{s}_1)$ ,  $R_B(\tilde{s}) = R_B(\tilde{s}_1, \tilde{s}_2)$ ,  $Q_A(\tilde{s}') = Q_A(\tilde{s}'_1)$ , and  $Q_B(\tilde{s}') = Q_B(\tilde{s}'_1, \tilde{s}'_2)$ .

The transmission conditions in problem (3.12) ensure that the integral at  $y = s_1^*$  is equal to zero, while for the integral at  $y = s_2^*$  we have

$$2 \int_0^\tau F_B(w_B) \partial_t F_B(w_B) \Big|_{y=s_2^*} dt = |F(w_B(\tau, s_2^*))|^2 - |F(w_B(0, s_2^*))|^2.$$

From the equation for  $w_A$  in problem (3.12) we obtain

$$\begin{aligned}
\|R_A^2 \partial_y^2 F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 & \leq \|2R_A \partial_y R_A \partial_y F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 + \|J_A(\tilde{s}_1) \partial_t w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 \\
& \quad + \|Q_A(\tilde{s}'_1) \partial_y w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 + \|J_A(\tilde{s}_1) G_A(\tilde{w}_A)\|_{L^2(\Omega_{A,\tau}^*)}^2.
\end{aligned}$$

Using the definition of  $Q_A$  and Hölder inequality, the third term on the right-hand side is estimated as

$$\|Q_A(\tilde{s}'_1) \partial_y w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 \leq C_1 \int_0^\tau |\tilde{s}'_1|^2 \|\partial_y w_A\|_{L^2(\Omega_A^*)}^2 dt \leq C_\delta \int_0^\tau |\tilde{s}'_1|^3 dt + \delta \int_0^\tau \|\partial_y w_A\|_{L^2(\Omega_A^*)}^6 dt,$$

for any fixed  $\delta > 0$ . The assumptions on  $G_A$  and  $F_A$ , the boundedness of  $J_A(\tilde{s}_1)$ ,  $R_A$ , and  $\partial_y R_A$ , and the fact that  $R_A^2(\tilde{s}_1) \geq 4/9$  and  $F_A'(w_A) = D_A(w_A) \geq d_A > 0$  for  $w_A > \rho_A^{\text{eq}}$ , imply

$$\begin{aligned}
\|\partial_y^2 F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 & \leq C_1 \left[ \|\partial_t w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 + \|\partial_y F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 \right] + C_\delta \|\tilde{s}'_1\|_{L^3(0,\tau)}^3 \\
& \quad + \delta \int_0^\tau \|\partial_y F_A(w_A)\|_{L^2(\Omega_A^*)}^6 dt + C_2 \tau,
\end{aligned}$$

for  $\tau \in (0, T_1]$ . Notice that for  $s'_1 \in L^\infty(0, T_1)$  we would have the  $L^2$ -norm of  $\partial_y F_A(w_A)$  on the right-hand side of the last inequality. A similar inequality for  $\|\partial_y^2 F_B(w_B)\|_{L^2(\Omega_{B,\tau}^*)}^2$  follows from the equation for  $w_B$  in problem (3.12). The Gagliardo-Nirenberg inequality gives

$$(3.14) \quad \|\partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^6 \leq C \left( \|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_l^*)}^2 \|F_l(w_l)\|_{L^\infty(\Omega_l^*)}^4 + \|F_l(w_l)\|_{L^2(\Omega_l^*)}^6 \right),$$

and, using the fact that  $w_l$  is uniformly bounded and choosing  $\delta > 0$  sufficiently small, we obtain

$$(3.15) \quad \|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 \leq C_1 \|\partial_t w_l\|_{L^2(\Omega_{l,\tau}^*)}^2 + C_2 \|\partial_y F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 + C_\delta \|\tilde{s}'\|_{L^3(0,\tau)}^3 + C_3 \tau,$$

for  $\tau \in (0, T_1]$ , where  $|\tilde{s}'| = |\tilde{s}'_1|$  if  $l = A$  and  $|\tilde{s}'| = |\tilde{s}'_1| + |\tilde{s}'_2|$  if  $l = B$ . In a similar way, we also obtain the following pointwise in the time variable estimate

$$(3.16) \quad \|\partial_y^2 F_l(w_l(t))\|_{L^2(\Omega_l^*)}^2 \leq C_1 \|\partial_t w_l(t)\|_{L^2(\Omega_l^*)}^2 + C_2 \|\partial_y F_l(w_l(t))\|_{L^2(\Omega_l^*)}^2 + C_\delta |\tilde{s}'(t)|^3 + C_3$$

for a.e.  $t \in [0, T_1]$ . Additionally, using the Gagliardo-Nirenberg inequality we have

$$\begin{aligned}
(3.17) \quad & \|\partial_y F_l(w_l)\|_{L^4(\Omega_l^*)}^4 \leq C \left( \|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_l^*)}^2 \|F_l(w_l)\|_{L^\infty(\Omega_l^*)}^2 + \|F_l(w_l)\|_{L^2(\Omega_l^*)}^4 \right), \\
& \|\partial_y F_l(w_l)\|_{L^4(\Omega_l^*)}^4 \leq C \left( \|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_l^*)} \|\partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^3 + \|F_l(w_l)\|_{L^2(\Omega_l^*)}^4 \right).
\end{aligned}$$

We shall estimate each term in equation (3.13) separately. Using estimates (3.14) and (3.15) yields

$$\begin{aligned} \int_0^\tau \int_{\Omega_l^*} \left| Q_l(\tilde{s}') \frac{\partial_y F_l(w_l)}{\tilde{w}_l} \partial_t w_l \right| dy dt &\leq \delta \int_0^\tau \left[ \|\partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^6 + \|\partial_t w_l\|_{L^2(\Omega_l^*)}^2 \right] dt \\ &+ C_\delta \|\tilde{s}'\|_{L^3(0,\tau)}^3 \leq \delta \|\partial_t w_l\|_{L^2(\Omega_{l,\tau}^*)}^2 + \delta \|\partial_y F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 + C_\delta \|\tilde{s}'\|_{L^3(0,\tau)}^3 + C\tau. \end{aligned}$$

Notice that assuming the boundedness of  $\tilde{s}'_j$ , with  $j = 1, 2$ , we would have the  $L^2(0, T; L^2(\Omega_l^*))$ -norm instead of the  $L^6(0, T; L^2(\Omega_l^*))$ -norm of  $\partial_y F_l(w_l)$  in the last inequality. Using the results in (3.15) and (3.17), along with the boundedness of  $w_l$  and  $\tilde{w}_l$ , we estimate the next term in (3.13) as

$$\begin{aligned} \int_{\Omega_{l,\tau}^*} \left| \frac{R_l^2(\tilde{s})}{\tilde{w}_l^2} \partial_y F_l(w_l) \partial_t F_l(w_l) \partial_y \tilde{w}_l \right| dy dt &\leq \delta \left[ \|\partial_y F_l(w_l)\|_{L^4(\Omega_{l,\tau}^*)}^4 + \|\partial_t F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 \right] \\ &+ C_\delta \|\partial_y \tilde{w}_l\|_{L^4(\Omega_{l,\tau}^*)}^4 \leq \delta \left[ \|\partial_y F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 + \|\partial_t w_l\|_{L^2(\Omega_{l,\tau}^*)}^2 \right] + C_1 \|\tilde{s}'\|_{L^3(0,\tau)}^3 + C_\delta \|\partial_y \tilde{w}_l\|_{L^4(\Omega_{l,\tau}^*)}^4. \end{aligned}$$

For the fourth integral in (3.13) we have

$$\begin{aligned} \int_0^\tau \int_{\Omega_l^*} \left| \frac{1}{2} \frac{R_l^2(\tilde{s})}{\tilde{w}_l^2} |\partial_y F_l(w_l)|^2 \partial_t \tilde{w}_l - \frac{R_l(\tilde{s}) \partial_t R_l(\tilde{s})}{\tilde{w}_l} |\partial_y F_l(w_l)|^2 \right| dy dt \\ \leq C_\delta \|\tilde{s}'\|_{L^3(0,\tau)}^3 + \int_0^\tau \left[ C_1 \|\partial_y F_l(w_l)\|_{L^4(\Omega_l^*)}^2 \|\partial_t \tilde{w}_l\|_{L^2(\Omega_l^*)} + \delta \|\partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^3 \right] dt \\ \leq C_\delta \|\tilde{s}'\|_{L^3(0,\tau)}^3 + C_\delta \int_0^\tau \|\partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^2 \|\partial_t \tilde{w}_l\|_{L^2(\Omega_l^*)}^{\frac{4}{3}} dt + C_2 \tau^{\frac{1}{2}} \|\partial_t \tilde{w}_l\|_{L^2(\Omega_{l,\tau}^*)} \\ + \delta \left[ \|\partial_y F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 + \|\partial_t w_l\|_{L^2(\Omega_{l,\tau}^*)}^2 \right] + C_3 \tau, \end{aligned}$$

for  $\tau \in (0, T_1]$  and any fixed  $\delta > 0$ . Here we used the following estimate

$$\begin{aligned} \int_0^\tau \|\partial_y F_l(w_l)\|_{L^4(\Omega_l^*)}^2 \|\partial_t \tilde{w}_l\|_{L^2(\Omega_l^*)} dt &\leq C \int_0^\tau \left[ \|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_l^*)}^{\frac{1}{2}} \|\partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^{\frac{3}{2}} + 1 \right] \|\partial_t \tilde{w}_l\|_{L^2(\Omega_l^*)} dt \\ &\leq \delta \|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 + C_\delta \int_0^\tau \left[ \|\partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^2 \|\partial_t \tilde{w}_l\|_{L^2(\Omega_l^*)}^{\frac{4}{3}} + \|\partial_t \tilde{w}_l\|_{L^2(\Omega_l^*)} \right] dt, \end{aligned}$$

along with estimate (3.15). Using (3.17) and the boundedness of  $w_l$  we also obtain

$$\int_0^\tau \|\partial_y F_l(w_l)\|_{L^4(\Omega_l^*)}^2 \|\partial_t \tilde{w}_l\|_{L^2(\Omega_l^*)} dt \leq \delta \|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 + C_\delta \|\partial_t \tilde{w}_l\|_{L^2(\Omega_{l,\tau}^*)}^2 + C\tau.$$

The boundedness of  $\tilde{w}_A$ , along with the assumptions on  $G_A$ , implies

$$\int_{\Omega_{A,\tau}^*} \left| J_A(\tilde{s}_1) G_A(\tilde{w}_A) \frac{\partial_t F_A(w_A)}{\tilde{w}_A} \right| dy dt \leq C_\delta \tau + \delta \|\partial_t w_A\|_{L^2(\Omega_{A,\tau}^*)}^2,$$

for  $\tau \in (0, T_1]$  and any fixed  $\delta > 0$ .

Thus for  $\partial_t \tilde{w}_l \in L^2((0, T_1) \times \Omega_l^*)$  and  $\partial_y \tilde{w}_l \in L^6(0, T_1; L^4(\Omega_l^*))$ , combining the estimates from above, choosing  $\delta > 0$  sufficiently small, and applying the Gronwall inequality yields

$$\begin{aligned} \sum_{l=A,B} \left[ \|\partial_t w_l\|_{L^2(\Omega_{l,T_1}^*)}^2 + \|\partial_y F_l(w_l)\|_{L^\infty(0,T_1;L^2(\Omega_l^*))}^2 \right] &\leq C_1 (1 + \|\tilde{s}'_1\|_{L^3(0,T_1)}^3 + \|\tilde{s}'_2\|_{L^3(0,T_1)}^3) \\ (3.18) \quad + C_2 \sum_{l=A,B} \left[ T_1^{\frac{1}{3}} \|\partial_y \tilde{w}_l\|_{L^6(0,T_1;L^4(\Omega_l^*))}^4 + T_1^{\frac{1}{2}} \|\partial_t \tilde{w}_l\|_{L^2(\Omega_{l,T_1}^*)} + \exp(T_1^{\frac{1}{3}} \|\partial_t \tilde{w}_l\|_{L^2(\Omega_{l,T_1}^*)}^{\frac{4}{3}}) \right]. \end{aligned}$$

In a similar way we also obtain

$$\begin{aligned} \sum_{l=A,B} \left[ \|\partial_t w_l\|_{L^2(\Omega_{l,T_1}^*)}^2 + \|\partial_y F_l(w_l)\|_{L^\infty(0,T_1;L^2(\Omega_l^*))}^2 \right] &\leq C_1 (1 + \|\tilde{s}'_1\|_{L^3(0,T_1)}^3 + \|\tilde{s}'_2\|_{L^3(0,T_1)}^3) \\ (3.19) \quad + C_2 \sum_{l=A,B} \left[ \|\partial_y F_l(\tilde{w}_l)\|_{L^4(\Omega_{l,T_1}^*)}^4 + \|\partial_t \tilde{w}_l\|_{L^2(\Omega_{l,T_1}^*)}^2 \right]. \end{aligned}$$

Thus using (3.18) and (3.19), along with (3.15) and (3.17), and considering  $T_1$  sufficiently small, we find that the map  $K : \mathcal{W}_A \times \mathcal{W}_B \rightarrow \mathcal{W}_A \times \mathcal{W}_B$ , where  $(w_A, w_B) = K(\tilde{w}_A, \tilde{w}_B)$  is defined as a solution of problem (3.12) for a given  $(\tilde{s}_1, \tilde{s}_2) \in \mathcal{W}_s$ , is continuous.

Considering  $w_l$  in equation (3.13) instead of  $\tilde{w}_l$  and using the boundedness of  $w_l$  yield

$$\begin{aligned} \int_{\Omega_{i,\tau}^*} \left| Q_l(\tilde{s}') \frac{\partial_y F_l(w_l)}{w_l} \partial_t w_l \right| dy dt &\leq \int_0^\tau \delta \left[ \|\partial_y F_l(w_l)\|_{L^2(\Omega_i^*)}^6 + \|\partial_t w_l\|_{L^2(\Omega_i^*)}^2 \right] dt + C_\delta \|\tilde{s}'\|_{L^3(0,\tau)}^3 \\ &\leq \delta \left[ \|\partial_y F_l(w_l)\|_{L^2(\Omega_{i,\tau}^*)}^2 + \|\partial_t w_l\|_{L^2(\Omega_{i,\tau}^*)}^2 \right] + C_\delta \|\tilde{s}'\|_{L^3(0,\tau)}^3, \\ \int_{\Omega_{A,\tau}^*} \left| J_A(\tilde{s}_1) G_A(w_A) \frac{\partial_t F_A(w_A)}{w_A} \right| dy dt &\leq C_\delta \tau + \delta \|\partial_t w_A\|_{L^2(\Omega_{A,\tau}^*)}^2, \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega_{i,\tau}^*} \left| \frac{1}{2} \frac{R_l^2(\tilde{s})}{w_l^2} |\partial_y F_l(w_l)|^2 \partial_t w_l + \frac{R_l(\tilde{s}) \partial_t R_l(\tilde{s})}{w_l} |\partial_y F_l(w_l)|^2 \right| dy dt &\leq C_\delta \|\tilde{s}'\|_{L^3(0,\tau)}^3 \\ &+ \delta \left[ \|\partial_t w_l\|_{L^2(\Omega_{i,\tau}^*)}^2 + \|\partial_y F_l(w_l)\|_{L^2(\Omega_{i,\tau}^*)}^2 + 1 \right] + \int_0^\tau \|\partial_y F_l(w_l)\|_{L^4(\Omega_i^*)}^2 \|\partial_t w_l\|_{L^2(\Omega_i^*)} dt \\ &\leq C_\delta \|\tilde{s}'\|_{L^3(0,\tau)}^3 + \delta \left[ \|\partial_t w_l\|_{L^2(\Omega_{i,\tau}^*)}^2 + \|\partial_y F_l(w_l)\|_{L^2(\Omega_{i,\tau}^*)}^2 + 1 \right] + \tilde{C}_\delta \int_0^\tau \|\partial_y F_l(w_l)\|_{L^2(\Omega_i^*)}^6 dt, \end{aligned}$$

for  $\tau \in (0, T_1]$ . Choosing  $\delta > 0$  sufficiently small, applying the Gronwall inequality, and considering  $T_1$  such that

$$T_1 \leq \min_{l=A,B} \frac{\eta}{8C_\delta \rho_l^M (\|\partial_y F_l(1/\rho_l^0)\|_{L^2(\Omega_i^*)}^4 + 1)},$$

we obtain the following estimates for  $w_A$  and  $w_B$

$$(3.20) \quad \sum_{l=A,B} \left[ \|\partial_t w_l\|_{L^2(\Omega_{i,T_1}^*)}^2 + \|\partial_y F_l(w_l)\|_{L^\infty(0,T_1;L^2(\Omega_i^*))}^2 \right] \leq C + C_\delta \left[ \|\tilde{s}'_1\|_{L^3(0,T_1)}^3 + \|\tilde{s}'_2\|_{L^3(0,T_1)}^3 \right].$$

The estimate for  $\|\partial_t \partial_y w_l\|_{L^2(0,T_1;H^{-1}(\Omega_i^*))}$  in terms of  $\|\partial_y F_l(\tilde{w}_l)\|_{L^4(\Omega_{i,T_1}^*)}$  and  $\|\partial_t \tilde{w}_l\|_{L^2(\Omega_{i,T_1}^*)}$  follows directly from differentiating the equation for  $w_l$  with respect to  $y$  and using the boundedness of  $\|\partial_y^2 w_l\|_{L^2(\Omega_{i,T_1}^*)}$  and  $\|\partial_y F_l(w_l)\|_{L^\infty(0,T_1;L^2(\Omega_i^*))}$ , which is ensured by (3.18) and (3.15). Using (3.20) and (3.15) and differentiating the equation for  $w_l$  in (3.12) with respect to  $y$ , while considering  $w_l$  instead of  $\tilde{w}_l$ , gives

$$(3.21) \quad \sum_{l=A,B} \left[ \|\partial_t \partial_y w_l\|_{L^2(0,T_1;H^{-1}(\Omega_i^*))}^2 + \|\partial_y^2 w_l\|_{L^2(\Omega_{i,T_1}^*)}^2 \right] \leq C + C_\delta \left[ \|\tilde{s}'_1\|_{L^3(0,T_1)}^3 + \|\tilde{s}'_2\|_{L^3(0,T_1)}^3 \right].$$

Thus for a sufficiently small  $T_1$ , or small initial data,  $(w_A, w_B) = K(w_A, w_B)$  is uniformly bounded in  $\mathcal{W}_A \times \mathcal{W}_B$  and  $\partial_y w_l$  in  $\mathcal{V}_l$ , for  $l = A, B$ , where

$$\mathcal{V}_l = \{u \in L^2(0, T_1; H^1(\Omega_i^*)) \cap L^\infty(0, T_1; L^2(\Omega_i^*)), \partial_t u \in L^2(0, T_1; H^{-1}(\Omega_i^*))\}, \quad l = A, B.$$

The Aubin-Lions lemma, along with the Sobolev embedding theorem, ensures that  $\mathcal{V}_l$  is a compact subset of  $L^2(\Omega_{i,T_1}^*)$  and of  $L^2(0, T_1; C(\overline{\Omega_i^*}))$ , for  $l = A, B$ . Using inequality (3.17) we also obtain that the embedding  $\mathcal{V}_l \subset L^6(0, T_1; L^4(\Omega_i^*))$  is compact. Thus applying the Schauder fixed point theorem, see *e.g.* [45], gives that for a given pair  $(s_1, s_2) \in \mathcal{W}_s$  there exists a solution of problem (3.5) and (3.6) for  $t \in (0, T_1]$ , with an appropriate choice of  $T_1 > 0$ .

To complete the proof we shall show that  $\mathcal{M} : L^3(0, T_1)^2 \rightarrow L^3(0, T_1)^2$  given by

$$\mathcal{M}(r_1, r_2) = \left( -\frac{\partial_y F_A(w_A(t, s_1^*))}{w_A}, -\frac{\partial_y F_B(w_B(t, s_2^*))}{w_B} \right),$$

where  $s_j(t) = s_j^* + \int_0^t r_j(\tau) d\tau$  for  $j = 1, 2$ , maps  $\mathcal{W}'_s = \{(r_1, r_2) \in L^3(0, T_1)^2 : \|r_j - s_j^{*,1}\|_{L^3(0,T_1)} \leq 1\}$  into itself and is precompact. Considering  $(r_1, r_2) \in \mathcal{W}'_s$  we have

$$\begin{aligned} \int_0^{T_1} |\mathcal{M}(r_1, r_2) - (s_1^{*,1}, s_2^{*,1})|^3 dt &\leq \sum_{l=A,B} \delta \int_0^{T_1} \|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_i^*)}^2 dt \\ &+ \sum_{l=A,B} T_1 \left[ C_\delta \sup_{(0,T)} \|\partial_y F_l(w_l)\|_{L^2(\Omega_i^*)}^6 + C \right] \leq 1, \end{aligned}$$

for an appropriate choice of  $\delta > 0$  and  $T_1 > 0$ . To show that  $\mathcal{M} : L^3(0, T_1)^2 \rightarrow L^3(0, T_1)^2$  is precompact we consider two sequences  $\{r_1^n\}$  and  $\{r_2^n\}$  bounded in  $L^3(0, T_1)$  and obtain

$$(3.22) \quad \begin{aligned} & \|\partial_y F_l(w_l^n)\|_{L^\infty(0, T_1; L^2(\Omega_l^*))} + \|\partial_y^2 F_l(w_l^n)\|_{L^2((0, T_1) \times \Omega_l^*)} \\ & + \|\partial_t w_l^n\|_{L^2(\Omega_l^*, T_1)} + \|\partial_t \partial_y F_l(w_l^n)\|_{L^2(0, T_1; H^{-1}(\Omega_l^*))} \leq C, \end{aligned}$$

for  $l = A, B$ , with a constant  $C$  independent of  $n$ . Using the fact that the embedding  $H^1(\Omega_l^*) \subset C(\overline{\Omega_l^*})$  is compact and applying the Aubin-Lions lemma we obtain the strong convergence  $w_l^n \rightarrow w_l$  in  $L^p(0, T_1; C(\overline{\Omega_l^*}))$ , for any  $1 < p < \infty$ , and  $\partial_y F_l(w_l^n) \rightarrow \partial_y F_l(w_l)$  in  $L^2(0, T_1; C(\overline{\Omega_l^*}))$  as  $n \rightarrow \infty$ . This combined with the estimate (3.22) ensures that

$$(3.23) \quad \begin{aligned} & \int_0^{T_1} \left\| \frac{\partial_y F_l(w_l^n)}{w_l^n} - \frac{\partial_y F_l(w_l)}{w_l} \right\|_{L^\infty(\Omega_l^*)}^3 dt \leq C_1 \int_0^{T_1} \|\partial_y F_l(w_l^n) - \partial_y F_l(w_l)\|_{L^\infty(\Omega_l^*)}^3 dt \\ & + C_2 \int_0^{T_1} \|\partial_y F_l(w_l)\|_{L^\infty(\Omega_l^*)}^3 \|w_l^n - w_l\|_{L^\infty(\Omega_l^*)}^3 dt \\ & \leq C_3 \left[ \int_0^{T_1} \|\partial_y F_l(w_l^n) - \partial_y F_l(w_l)\|_{L^\infty(\Omega_l^*)}^2 dt \right]^{\frac{1}{2}} + C_4 \left[ \int_0^{T_1} \|w_l^n - w_l\|_{L^\infty(\Omega_l^*)}^{12} dt \right]^{\frac{1}{4}} \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ , where we used the fact that

$$\begin{aligned} \int_0^{T_1} \|\partial_y F_l(w_l^n) - \partial_y F_l(w_l)\|_{L^\infty(\Omega_l^*)}^4 dt & \leq \int_0^{T_1} \left[ \|\partial_y^2 F_l(w_l^n)\|_{L^2(\Omega_l^*)}^2 \|\partial_y F_l(w_l^n)\|_{L^2(\Omega_l^*)}^2 \right. \\ & \left. + \|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_l^*)}^2 \|\partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^2 \right] dt \leq C. \end{aligned}$$

The convergence in (3.23) implies the strong convergence  $\mathcal{M}(r_1^n, r_2^n) \rightarrow \mathcal{M}(r_1, r_2)$  in  $L^3(0, T_1)^2$  as  $n \rightarrow \infty$ . Hence, we have proved the existence of a solution of problem (3.5)-(3.7) in  $(0, \hat{T}) \times \Omega_l^*$  with  $\hat{T} = \min\{T_1, T_2\}$ , where

$$\begin{aligned} T_1 &= \min \left\{ \left[ ((s_1^* - s_0)/8)^{\frac{3}{2}} (1 + \|s_1^{*,1}\|_{L^2(0, T_1)})^{-\frac{3}{2}} \right], \left[ ((s_2^* - s_1^*)/8)^{\frac{3}{2}} (1 + \|s_2^{*,1}\|_{L^2(0, T_1)})^{-\frac{3}{2}} \right] \right\}, \\ T_2 &= \min_{l=A, B} \frac{\eta_l}{8C_\delta \rho_l^M (\|\partial_x F_l(1/\rho_l^0)\|_{L^2(\Omega_l^*)}^4 + 1)}. \end{aligned}$$

Now we show that  $s_1'(t)$  and  $s_2'(t)$  are uniformly bounded, which will allow us to iterate over successive time intervals and obtain that  $\hat{T} \leq T_2$ . The uniform boundedness of  $\rho_B$ , the assumptions on  $F_B$ , and equations (3.2) ensure that  $s_2'(t)$  is uniformly bounded. To show the boundedness of  $s_1'(t)$ , we consider the original problem (2.18), (2.19), (2.22), and (2.23) and apply the comparison principle to the following problem for  $v = \eta_A^{-1} F_A(1/\rho_A)$ :

$$(3.24) \quad \begin{aligned} & \partial_t v = \tilde{D}_A(v) \partial_x^2 v + \tilde{D}_A(v) \tilde{G}_A(v) \quad \text{in } (s_0, s_1(t)), \quad t > 0, \\ & \partial_x v(t, x) = 0 \quad \text{for } x = s_0, \quad t > 0, \\ & v(t, x) = \frac{1}{\eta_B} F_B\left(\frac{1}{\rho_B}\right) \quad \text{for } x = s_1(t), \quad t > 0, \\ & v(0, x) = \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A^0}\right) \quad \text{in } (s_0, s_1^*), \end{aligned}$$

where  $\tilde{G}_A(v) = G_A([F_A^{-1}(\eta_A v)]^{-1})$  and  $\tilde{D}_A(v) = D_A([F_A^{-1}(\eta_A v)]^{-1})$ .

Consider the interval  $(s_1(t) - \delta, s_1(t))$ , with  $t > 0$ , and the function

$$\omega(t, x) = \frac{1}{\eta_B} F_B\left(\frac{1}{\rho_B^M}\right) + \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A^M}\right) \left[ \frac{2}{\delta} (s_1(t) - x) - \frac{1}{\delta^2} (s_1(t) - x)^2 \right],$$

for some  $\delta > 0$ . A similar idea was used in [21]. Since  $F_A$  and  $F_B$  are monotonically decreasing functions, we obtain

$$\begin{aligned} \omega(t, s_1(t)) &= \frac{1}{\eta_B} F_B\left(\frac{1}{\rho_B^M}\right) \geq \frac{1}{\eta_B} F_B\left(\frac{1}{\rho_B(t, s_1(t))}\right) = v(t, s_1(t)), \\ \omega(t, s_1(t) - \delta) &= \frac{1}{\eta_B} F_B\left(\frac{1}{\rho_B^M}\right) + \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A^M}\right) \geq \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A(t, s_1(t) - \delta)}\right) = v(t, s_1(t) - \delta). \end{aligned}$$

For the derivatives of  $\omega$ , since  $s_1'(t) \geq 0$ , we have

$$\begin{aligned}\partial_t \omega(t, x) &= \frac{2}{\delta} \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A^M}\right) s_1'(t) \left[1 - \frac{s_1(t) - x}{\delta}\right] \geq 0 \quad \text{for } x \in [s_1(t) - \delta, s_1(t)], \quad t \geq 0, \\ \partial_x \omega(t, x) &= \frac{2}{\delta} \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A^M}\right) \left[\frac{s_1(t) - x}{\delta} - 1\right], \quad \partial_x^2 \omega(t, x) = -\frac{2}{\delta^2} \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A^M}\right).\end{aligned}$$

Using the assumptions on  $G_A$ , for  $\delta > 0$  sufficiently small we obtain

$$\partial_t(\omega - v) - \tilde{D}_A(v) \partial_x^2(\omega - v) \geq \tilde{D}_A(v) \left[ \frac{2}{\delta^2} \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A^M}\right) - \tilde{G}_A(v) \right] \geq 0.$$

Since  $F_A$  is continuous and  $\eta_A^{-1} F_A(1/\rho_A^0) = \eta_B^{-1} F_B(1/\rho_B^0)$  at  $x = s_1^*$ , there exists a sufficiently small  $\delta > 0$  such that

$$\omega(0, x) \geq v(0, x) \quad \text{for } x \in [s_1^* - \delta, s_1^*].$$

Then applying the comparison principle for parabolic equations gives

$$\omega(t, x) \geq v(t, x) \quad \text{for } t \in (0, T) \text{ and } x \in [s_1(t) - \delta, s_1(t)].$$

Hence we have

$$-\frac{2}{\delta} \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A^M}\right) = \partial_x \omega \leq \partial_x v = \frac{1}{\eta_A} \partial_x F_A\left(\frac{1}{\rho_A}\right) \leq 0 \quad \text{at } x = s_1(t)$$

and for some sufficiently small fixed  $\delta > 0$

$$0 \leq \frac{ds_1(t)}{dt} \leq \frac{2}{\delta \rho_A^{\text{eq}}} \frac{1}{\eta_A} F_A\left(\frac{1}{\rho_A^M}\right) \quad \text{in } (0, T).$$

Therefore, provided that  $\partial_x F_l(1/\rho_l)$  is uniformly bounded in  $L^\infty(0, T; L^2(\Omega_l(t)))$ , for  $l = A, B$ , the uniform boundedness of  $s_1'$  and  $s_2'$  allows us to iterate over successive time intervals and prove the existence of a global solution of the free-boundary problem (2.18), (2.19), (2.22) and (2.23).

Thus, as next we prove the uniform boundedness of  $\|\partial_x F_l(1/\rho_l)\|_{L^\infty(0, T; L^2(\Omega_l(t)))}$ , or equivalently of  $\|\partial_y F_l(w_l)\|_{L^\infty(0, T; L^2(\Omega_l^*))}$ , for  $l = A, B$ . First we show higher regularity of the solutions of problem (3.5)-(3.7) by differentiating the equations in (3.5) with respect to the time variable and considering  $\phi = \partial_t^2 F_A(w_A)/w_A$  and  $\psi = \partial_t^2 F_B(w_B)/w_B$  as test functions, respectively,

$$\begin{aligned}& \sum_{l=A, B} \left\{ \int_{\Omega_{l, \tau}^*} \left( \frac{J_l D_l}{w_l} |\partial_t^2 w_l|^2 + \left[ J_l D_l \frac{|\partial_t w_l|^2}{w_l} + \partial_t J_l D_l \frac{\partial_t w_l}{w_l} \right] \partial_t^2 w_l + \partial_t J_l D_l \frac{(\partial_t w_l)^3}{w_l} \right) dy dt \right. \\ & \quad + \frac{1}{2} \int_0^\tau \frac{d}{dt} \int_{\Omega_l^*} \frac{R_l^2(s)}{w_l} |\partial_y \partial_t F_l(w_l)|^2 dy dt - \int_{\Omega_{l, \tau}^*} \partial_y (R_l \partial_t R_l \partial_y F_l(w_l)) \frac{\partial_t^2 F_l(w_l)}{w_l} dy dt \\ & \quad + \int_{\Omega_{l, \tau}^*} \left[ |\partial_y \partial_t F_l(w_l)|^2 \frac{1}{2} \left( \frac{R_l^2}{w_l^2} \partial_t w_l - \frac{\partial_t R_l^2(s)}{w_l} \right) - \frac{R_l^2(s)}{w_l^2} \partial_y \partial_t F_l(w_l) \partial_t^2 F_l(w_l) \partial_y w_l \right] dy dt \\ & \quad \left. + \int_{\Omega_{l, \tau}^*} \partial_t (Q_l(s')) \partial_y F_l(w_l) \frac{\partial_t^2 F_l(w_l)}{w_l} dy dt \right\} - \int_{\Omega_{A, \tau}^*} \partial_t G_A(w_A) \frac{\partial_t^2 F_A(w_A)}{w_A} dy dt \\ & \quad = \int_0^\tau \frac{\partial_t \partial_y F_A(w_A)}{w_A} \partial_t^2 F_A(w_A) \Big|_{y=s_1^*} dt + \int_0^\tau \frac{\partial_t \partial_y F_B(w_B)}{w_B} \partial_t^2 F_B(w_B) \Big|_{s_1^*}^{s_2^*} dt.\end{aligned}$$

The second term in the equation above can be estimated as

$$\begin{aligned}\int_{\Omega_{l, \tau}^*} |\partial_t w_l|^2 |\partial_t^2 w_l| dy dt &\leq \delta \|\partial_t^2 w_l\|_{L^2(\Omega_{l, \tau}^*)}^2 + C_\delta \|\partial_t w_l\|_{L^4(\Omega_{l, \tau}^*)}^4 \\ &\leq \delta \|\partial_t^2 w_l\|_{L^2(\Omega_{l, \tau}^*)}^2 + C_\delta \int_0^\tau \left[ \|\partial_t \partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^2 \|\partial_t w_l\|_{L^1(\Omega_l^*)}^2 + \|\partial_t w_l\|_{L^1(\Omega_l^*)}^4 \right] dt,\end{aligned}$$

for  $\tau \in (0, T]$ . For the third and fourth terms we have

$$\begin{aligned}\int_{\Omega_{l, \tau}^*} [|\partial_t w_l| |s'| |\partial_t^2 w_l| + |\partial_t w_l|^3 |s'|] dy dt &\leq \delta \left[ \|\partial_t \partial_y w_l\|_{L^2(\Omega_{l, \tau}^*)}^2 + \|\partial_t^2 w_l\|_{L^2(\Omega_{l, \tau}^*)}^2 \right] \\ &\quad + C_\delta \left[ \|s'\|_{L^\infty(0, \tau)}^2 \|\partial_t w_l\|_{L^2(\Omega_{l, \tau}^*)}^2 + \|\partial_t w_l\|_{L^2(\Omega_{l, \tau}^*)}^{\frac{5}{2}} \|s'\|_{L^3(0, \tau)}^3 \right].\end{aligned}$$

Moreover,

$$\begin{aligned} & \int_{\Omega_{l,\tau}^*} \left( |\partial_y \partial_t F_l(w_l)|^2 \left| \frac{R_l^2(s)}{w_l^2} \partial_t w_l - \frac{\partial_t R_l^2(s)}{w_l} \right| + \frac{R_l^2(s)}{w_l^2} |\partial_y \partial_t F_l(w_l) \partial_y w_l| |\partial_t^2 F_l(w_l)| \right) dy dt \\ & \leq C_\delta \int_0^\tau \|\partial_y \partial_t F_l(w_l)\|_{L^2(\Omega_l^*)}^2 \left[ \|\partial_t w_l\|_{L^2(\Omega_l^*)}^{\frac{4}{3}} + \|\partial_y w_l\|_{L^2(\Omega_l^*)}^2 \|\partial_t w_l\|_{L^2(\Omega_l^*)}^{\frac{2}{3}} + \|\partial_y w_l\|_{L^2(\Omega_l^*)}^{\frac{8}{3}} \right. \\ & \quad \left. + |s'| + 1 \right] dt + \delta \|\partial_t^2 F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2. \end{aligned}$$

We estimate the next terms as

$$\begin{aligned} & \int_{\Omega_{l,\tau}^*} \left[ |\partial_y (R_l \partial_t R_l \partial_y F_l(w_l))| + |\partial_t (Q_l(s') \partial_y F_l(w_l))| \right] \frac{|\partial_t^2 F_l(w_l)|}{w_l} dy dt \leq \delta \|\partial_t^2 F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 \\ & + C_\delta \int_0^\tau [ |s'|^2 + |s''|^2 ] \|\partial_y F_l(w_l)\|_{L^2(\Omega_l^*)}^2 + |s'|^2 (\|\partial_y^2 F_l(w_l)\|_{L^2(\Omega_l^*)}^2 + \|\partial_y \partial_t F_l(w_l)\|_{L^2(\Omega_l^*)}^2) dt. \end{aligned}$$

The reaction term is estimated by

$$\delta \|\partial_t^2 F_A(w_A)\|_{L^2(\Omega_{1,\tau}^*)}^2 + C_\delta \int_0^\tau [ |s_1'|^2 + (1 + |s_1|^2) \|\partial_t w_A\|_{L^2(\Omega_A^*)}^2 ] dt.$$

For the non-zero contributions from the boundary terms we have

$$\begin{aligned} & \int_0^\tau \partial_y F_l(w_l) \frac{\partial_t w_l}{w_l^2} \partial_t^2 F_l(w_l) \Big|_{y=s_1^*} dt = \frac{\partial_y w_l(t, s_1^*)}{2w_l^2(t, s_1^*)} |\partial_t F_l(w_l(t, s_1^*))|^2 \Big|_0^\tau \\ & - \frac{1}{2} \int_0^\tau \left[ \frac{\partial_y \partial_t w_l}{w_l^2} - \frac{\partial_y w_l \partial_t w_l}{w_l^3} \right] |\partial_t F_l(w_l)|^2 \Big|_{y=s_1^*} dt = J_1 + J_2 \end{aligned}$$

for  $l = A, B$ , where

$$\begin{aligned} |J_1| & \leq \delta [\|\partial_t \partial_y F_l(w_l(\tau))\|_{L^2(\Omega_l^*)}^2 + \|\partial_t^2 w_l\|_{L^2(\Omega_{l,\tau}^*)}^2] + C_\delta [\|\partial_y w_l\|_{L^\infty(0,\tau;L^2(\Omega_l^*))}^4 \|\partial_t w_l\|_{L^2(\Omega_{l,\tau}^*)}^6 + 1], \\ |J_2| & \leq \delta \|\partial_t^2 F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 + C_\delta \|\partial_y w_l\|_{L^\infty(0,\tau;L^2(\Omega_l^*))}^2 [\|\partial_t w_l\|_{L^2(\Omega_{l,\tau}^*)}^2 + \|\partial_y w_l\|_{L^2(\Omega_{l,\tau}^*)}^2 + 1] \\ & + C \int_0^\tau \|\partial_y \partial_t F_l(w_l)\|_{L^2(\Omega_l^*)}^2 [\|\partial_t F_l(w_l)\|_{L^2(\Omega_l^*)}^{\frac{4}{3}} + \delta \|\partial_t F_l(w_l)\|_{L^2(\Omega_l^*)}^2] dt, \end{aligned}$$

and

$$\int_0^\tau \frac{\partial}{\partial t} \frac{\partial_y F_B(w_B)}{w_B} \partial_t^2 F_B(w_B) \Big|_{y=s_2^*} dt = -|\partial_t F_B(w_B(\tau, s_2^*))|^2 + |\partial_t F_B(w_B(0, s_2^*))|^2.$$

Hence, applying the Gronwall inequality and using the estimates (3.20) and (3.21) yields

$$\begin{aligned} (3.25) \quad & \sum_{l=A,B} \|\partial_t^2 w_l\|_{L^2(\Omega_{l,\hat{T}}^*)} + \|\partial_y \partial_t F_l(w_l)\|_{L^\infty(0,\hat{T};L^2(\Omega_l^*))} \leq C_1 \\ & + C_2 \sum_{l=A,B} [\|s_1''\|_{L^2(0,\hat{T})} + \|s_2''\|_{L^2(0,\hat{T})}] \exp(C_3 [1 + \|\partial_y w_l\|_{L^\infty(0,\hat{T};L^2(\Omega_l^*))}^{\frac{8}{3}} + \|\partial_t w_l\|_{L^2(\Omega_{l,\hat{T}}^*)}^2]), \end{aligned}$$

with constants  $C_j$ , for  $j = 1, 2, 3$ , depending on  $\|s_1'\|_{L^\infty(0,\hat{T})}$  and  $\|s_2'\|_{L^\infty(0,\hat{T})}$ .

Using the Gagliardo-Nirenberg inequality and boundedness of  $\partial_y F_A(w_A(t, s_1^*))$  we obtain

$$\begin{aligned} \|s_1''\|_{L^2(0,\tau)}^2 & \leq C_1 \left[ \|\partial_t \partial_y F_A(w_A)\|_{L^2(0,\tau;L^\infty(\Omega_A^*))}^2 + \|\partial_t w_A\|_{L^2(0,\tau;L^\infty(\Omega_A^*))}^2 \|\partial_y F_A(w_A(\cdot, s_1^*))\|_{L^2(0,\tau)}^2 \right] \\ & \leq C_2 \left[ \|\partial_t \partial_y F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)} \|\partial_t \partial_y^2 F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)} + \|\partial_t \partial_y F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 + \|\partial_t w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 \right] \\ & \leq \delta \|\partial_t \partial_y^2 F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 + C_\delta \|\partial_t \partial_y F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 + C_3 \|\partial_t w_A\|_{L^2(\Omega_{A,\tau}^*)}^2. \end{aligned}$$

Differentiating (3.5) with respect to the time variable and using definition of  $J_A$ ,  $R_A$ , and  $Q_A$  yields

$$\begin{aligned} \|\partial_t \partial_y^2 F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 & \leq C \left[ \|\partial_t^2 w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 + \|(|s_1| + s_1') \partial_t \partial_y F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 + \|s_1' w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 \right. \\ & \quad \left. + \|(|s_1| + s_1') \partial_t w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 + \|s_1' (|\partial_y^2 F_A(w_A)| + |\partial_y F_A(w_A)|)\|_{L^2(\Omega_{A,\tau}^*)}^2 + \|s_1'' \partial_y w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 \right]. \end{aligned}$$



The last term is estimated by

$$\begin{aligned} \|s_1'' \partial_y w_A\|_{L^2(\Omega_{A,\tau}^*)} &\leq \|\partial_y w_A\|_{L^\infty(0,\tau;L^2(\Omega_A^*))}^2 \|s_1''\|_{L^2(0,\tau)}^2 \leq \delta \|\partial_t \partial_y^2 F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 \\ &+ C_\delta \left[ \|\partial_y w_A\|_{L^\infty(0,\tau;L^2(\Omega_A^*))}^4 + \|\partial_y w_A\|_{L^\infty(0,\tau;L^2(\Omega_A^*))}^2 \right] \left[ \|\partial_t \partial_y F_A(w_A)\|_{L^2(\Omega_{A,\tau}^*)}^2 + \|\partial_t w_A\|_{L^2(\Omega_{A,\tau}^*)}^2 \right]. \end{aligned}$$

Similar estimates, using the boundedness of  $\partial_y F_B(w_B(t, s_2^*))$ , hold for  $\partial_t \partial_y^2 F_B(w_B)$  and  $s_2''$ . Thus choosing  $\delta > 0$  sufficiently small and using the boundedness of  $s_1'$  and  $s_2'$  we obtain

$$\begin{aligned} \|\partial_t \partial_y^2 F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 &\leq C_1 \left[ \|\partial_t^2 w_l\|_{L^2(\Omega_{l,\tau}^*)}^2 + \|\partial_y w_l\|_{L^\infty(0,\tau;L^2(\Omega_l^*))}^2 \right] + C_2 \tau \\ &+ C_3 (1 + \|\partial_y w_l\|_{L^\infty(0,\tau;L^2(\Omega_l^*))}^4) \left[ \|\partial_t \partial_y F_l(w_l)\|_{L^2(\Omega_{l,\tau}^*)}^2 + \|\partial_t w_l\|_{L^2(\Omega_{l,\tau}^*)}^2 \right] \end{aligned}$$

and, considering the same calculations as in the derivation of (3.25), conclude

$$(3.26) \quad \sum_{l=A,B} \left[ \|\partial_t^2 w_l\|_{L^2(\Omega_{l,\hat{T}}^*)} + \|\partial_y \partial_t F_l(w_l)\|_{L^\infty(0,\hat{T};L^2(\Omega_l^*))} \right] \leq C,$$

where the constant  $C$  depends on  $\|\partial_y w_l\|_{L^\infty(0,\hat{T};L^2(\Omega_l^*))}$ ,  $\|\partial_t w_l\|_{L^2(\Omega_{l,\hat{T}}^*)}$  for  $l = A, B$ , and  $\|s_j'\|_{L^\infty(0,\hat{T})}$  for  $j = 1, 2$ .

Hence estimates (3.21) and (3.26) imply that  $\partial_y F_l(w_l) \in C(\overline{\Omega_{l,\hat{T}}^*})$ . Since  $\partial_y F_A(w_A(t, s_1^*))$  and  $\partial_y F_B(w_B(t, s_j^*))$ , for  $j = 1, 2$ , are bounded, then  $\partial_y F_A(w_A)$  is bounded in  $[s_1^* - \delta, s_1^*]$  and  $\partial_y F_B(w_B)$  is bounded in  $[s_1^*, s_1^* + \delta]$  and  $[s_2^* - \delta, s_2^*]$ , for  $t \in [0, \hat{T}]$  and for a sufficiently small  $\delta > 0$ . Taking  $F_l(w_l)$  as test function in (3.5) and using boundedness of  $\partial_y F_A(w_A(t, s_1^*))$  and  $\partial_y F_B(w_B(t, s_j^*))$ , for  $j = 1, 2$ , yield

$$\sum_{l=A,B} \left[ \|w_l\|_{L^\infty(0,\hat{T};L^2(\Omega_l^*))} + \|\partial_y F_l(w_l)\|_{L^2(\Omega_{l,\hat{T}}^*)} \right] \leq C.$$

Considering a cut-off function  $\zeta_l \in C^2(\overline{\Omega_l^*})$ , such that  $\zeta_A(y) = 1$  for  $y \in [s_0, s_1^* - \delta/2]$  and  $\zeta_A(y) = 0$  for  $y \in [s_1^* - \delta/4, s_1^*]$  and  $\zeta_B(y) = 1$  for  $y \in [s_1^* + \delta/2, s_2^* - \delta/2]$  and  $\zeta_B(y) = 0$  for  $y \in [s_1^*, s_1^* + \delta/4]$  and  $y \in [s_2^* - \delta/4, s_2^*]$  and taking  $\partial_t F_l(w_l) \zeta_l^2$  as test function in (3.5) gives

$$\sum_{l=A,B} \left[ \|\partial_t w_l \zeta_l\|_{L^2(\Omega_{l,\hat{T}}^*)} + \|\partial_y F_l(w_l) \zeta_l\|_{L^\infty(0,\hat{T};L^2(\Omega_l^*))} \right] \leq C_1 \left( 1 + \sum_{l=A,B} \|F_l(w_l)\|_{L^2(0,\hat{T};H^1(\Omega_l^*))} \right) \leq C_2.$$

Therefore,  $\|\partial_y F_A(w_A)\|_{L^\infty(0,\hat{T};L^2(\Omega_A^*))}$  and  $\|\partial_y F_B(w_B)\|_{L^\infty(0,\hat{T};L^2(\Omega_B^*))}$  are uniformly bounded and  $\hat{T}$  can be chosen independently of the initial data. Then iterating over  $\hat{T}$ , we can conclude that there exists a global solution of problem (2.18), (2.19), (2.22), (2.23).  $\square$

#### 4. TRAVELLING-WAVE SOLUTIONS OF THE FREE-BOUNDARY PROBLEM

In this section, we carry out a travelling-wave analysis for the free-boundary problem (2.18), (2.19) and (2.22).

We begin by noting that under assumptions (2.25), (2.27) and (2.28), the free-boundary problem (2.18), (2.19) and (2.22) can be written in terms of the cell pressures  $P_A$  and  $P_B$  defined according to the following barotropic relation

$$(4.1) \quad P_l(\rho_l) = 0 \quad \text{for } \rho_l \leq \rho_l^{\text{eq}} \quad \text{and} \quad \frac{dP_l(\rho_l)}{d\rho_l} = \frac{D_l(\rho_l)}{\rho_l} \quad \text{for } \rho_l > \rho_l^{\text{eq}}, \quad l = A, B.$$

Under assumptions (2.27), the barotropic relation (4.1) is such that

$$(4.2) \quad \frac{dP_l(\rho_l)}{d\rho_l} > 0 \quad \text{for } \rho_l > \rho_l^{\text{eq}}, \quad l = A, B.$$

The monotonicity conditions (4.2) allow one to write both the force terms  $F_l(1/\rho_l)$  and the growth rate  $G(\rho_A)$  as functions of the cell pressure  $\tilde{F}_A(P_A)$ ,  $\tilde{F}_B(P_B)$  and  $\tilde{G}(P_A)$ . Moreover, with the notation

$$(4.3) \quad P_l^{\text{eq}} = P_l(\rho_l^{\text{eq}}) \geq 0 \quad \text{and} \quad P^{\text{M}} = P_A(\rho^{\text{M}}) > P_A^{\text{eq}},$$

the monotonicity conditions (4.2) make it possible to reformulate assumptions (2.25) and (2.28) on the functions  $F_l$  and  $G$  as

$$(4.4) \quad \tilde{F}_l(P_l) = 0 \quad \text{for } P_l(\rho_l) \leq P_l^{\text{eq}}, \quad \frac{d\tilde{F}_l(P_l)}{dP_l} > 0 \quad \text{for } P_l > P_l^{\text{eq}}, \quad l = A, B$$

and

$$(4.5) \quad \tilde{G}(\cdot) > 0 \quad \text{in } (0, P^M), \quad \tilde{G}(\cdot) = 0 \quad \text{in } [P^M, \infty), \quad \frac{d\tilde{G}(\cdot)}{dP_A} < 0 \quad \text{in } (0, P^M].$$

Hence, under assumptions (2.25), (2.27) and (2.28), using the barotropic relation (4.1), we can rewrite the free-boundary problem (2.18), (2.19) and (2.22) in the following alternative form:

$$(4.6) \quad \begin{aligned} \partial_t \rho_A - \partial_x(\rho_A \partial_x P_A) &= \tilde{G}(P_A) \rho_A & \text{for } x \in (s_0, s_1(t)), \quad t > 0, \\ \partial_t \rho_B - \partial_x(\rho_B \partial_x P_B) &= 0 & \text{for } x \in (s_1(t), s_2(t)), \quad t > 0, \\ \partial_x P_A &= 0 & \text{at } x = s_0, \\ \frac{1}{\eta_A} \tilde{F}_A(P_A) &= \frac{1}{\eta_B} \tilde{F}_B(P_B) & \text{at } x = s_1(t), \\ \frac{ds_1}{dt} &= -\partial_x P_A & \text{at } x = s_1(t), \\ \frac{ds_1}{dt}(\rho_A - \rho_B) &= -(\rho_A \partial_x P_A - \rho_B \partial_x P_B) & \text{at } x = s_1(t), \\ \frac{ds_2}{dt} &= \frac{1}{\eta_B} \tilde{F}_B(P_B) - \frac{1}{2} \partial_x P_B & \text{at } x = s_2(t), \\ \frac{ds_2}{dt} &= -\partial_x P_B & \text{at } x = s_2(t). \end{aligned}$$

Having rewritten the problem in this form allows us to construct travelling-wave solutions using an approach that builds on the method of proof recently presented in [12, 31]. For the sake of brevity, in this section we drop the tildes from all the quantities in problem (4.6).

We construct travelling-wave solutions of the free-boundary problem (4.6) such that both the position of the inner free boundary,  $s_1(t)$ , and the position of the outer free boundary,  $s_2(t)$ , move at a given constant speed  $c > 0$ . Without loss of generality, we let  $s_0$  go to  $-\infty$  and consider the case where

$$(4.7) \quad s_1(t) = (c + o(1))t \quad \text{and} \quad s_2(t) = \ell + (c + o(1))t,$$

for some  $\ell > 0$ , so that

$$s_1(0) = s_1^* = 0, \quad s_2(0) = s_2^* = \ell \quad \text{and} \quad \frac{ds_1}{dt} = \frac{ds_2}{dt} = c.$$

We make the following travelling-wave ansatz for the cell densities  $\rho_A$  and  $\rho_B$

$$(4.8) \quad \rho_A(t, x) = \rho_A(z) \quad \text{and} \quad \rho_B(t, x) = \rho_B(z) \quad \text{with } z = x - ct,$$

which are related to the cell pressures  $P_A(z)$  and  $P_B(z)$  through the barotropic relation (4.1). In this framework, substituting the travelling-wave ansatz (4.8) into problem (4.6) we find

$$(4.9) \quad \begin{aligned} -c \rho'_A &= (\rho_A P'_A)' + G(P_A) \rho_A & \text{in } -\infty < z < 0, \\ -c \rho'_B &= (\rho_B P'_B)' & \text{in } 0 < z < \ell, \\ \frac{1}{\eta_A} F_A(P_A) &= \frac{1}{\eta_B} F_B(P_B) & \text{at } z = 0, \\ P'_A &= -c & \text{at } z = 0, \\ c(\rho_A - \rho_B) &= -\rho_A P'_A + \rho_B P'_B & \text{at } z = 0, \\ \frac{1}{\eta_B} F_B(P_B) &= c + \frac{1}{2} P'_B & \text{at } z = \ell, \\ P'_B &= -c & \text{at } z = \ell, \end{aligned}$$

where  $\rho'_l$  and  $P'_l$  denote the derivatives of  $\rho_l$  and  $P_l$  with respect to the variable  $z$ , with  $l = A, B$ . We consider the case where the following condition holds

$$(4.10) \quad \rho_A(z) \xrightarrow{z \rightarrow -\infty} \rho^M,$$

which implies  $P_A(z) \xrightarrow{z \rightarrow -\infty} P^M$ . Moreover, we note that the principle of mass conservation ensures that

$$(4.11) \quad \int_0^\ell \rho_B(z) dz = M,$$

for some  $M > 0$ . The results of our travelling-wave analysis are summarised in the following theorem.

**Theorem 4.1.** *Under Assumptions 3.1(i)-(iii), for any  $M > 0$  given there exist  $c > 0$  and  $\ell > 0$  such that the travelling-wave problem defined by system (4.9), complemented with the asymptotic condition (4.10), admits a solution whereby:*

(i)  $\rho_A(z)$  is decreasing in  $(-\infty, 0)$  and satisfies the condition

$$(4.12) \quad \rho_A^{\text{eq}} < \rho_A(0) < \rho_A(z) < \rho^M \quad \text{for all } z \in (-\infty, 0);$$

(ii)  $\rho_B(z)$  is decreasing in  $(0, \ell)$  and satisfies the condition

$$(4.13) \quad \rho_B^{\text{eq}} < \rho_B(\ell) < \rho_B(z) < \rho_B(0) \quad \text{for all } z \in (0, \ell)$$

along with the condition (4.11).

Moreover, in the case where  $F_A(\cdot) = F_B(\cdot)$ , the following jump condition holds:

$$(4.14) \quad \text{sgn}(\rho_A(0) - \rho_B(0)) = \text{sgn}(\eta_A - \eta_B).$$

*Proof.* We prove Theorem 4.1 in five steps.

*Step 1: existence of a solution of problem (4.9).* For  $c > 0$  given, we have the following problem for  $P_B(z)$

$$\begin{aligned} -c \rho'_B &= (\rho_B P'_B)' && \text{in } 0 < z < \ell, \\ P'_B &= -c && \text{at } z = 0, \\ \frac{1}{\eta_B} F_B(P_B) &= \frac{c}{2} && \text{at } z = \ell. \end{aligned}$$

Integrating the equation for  $P_B$  over  $(0, z)$ , with  $z < \ell$ , and using the condition at  $z = 0$  we obtain an ordinary differential equation with final condition at  $z = \ell$ , that is,

$$\begin{aligned} \rho_B P'_B &= -c \rho_B && \text{in } 0 < z < \ell, \\ \frac{1}{\eta_B} F_B(P_B) &= \frac{c}{2} && \text{at } z = \ell, \end{aligned}$$

which can be solved explicitly giving  $P_B(z) = c(\ell - z) + F_B^{-1}\left(\frac{\eta_B}{2}c\right)$ . Notice that since  $c > 0$  we have  $P_B(\ell) > P_B^{\text{eq}}$  and  $F_B$  is invertible. Knowing  $P_B(z)$ , we have the following problem for  $P_A(z)$

$$\begin{aligned} -c \rho'_A &= (\rho_A P'_A)' + G(P_A) \rho_A && \text{in } -\infty < z < 0, \\ \frac{1}{\eta_A} F_A(P_A) &= \frac{1}{\eta_B} F_B(P_B) && \text{at } z = 0, \\ P'_A &= -c && \text{at } z = 0. \end{aligned}$$

Integrating the equation for  $P_A$  over  $(z, 0)$ , with  $z < 0$ , and using the second condition at  $z = 0$  we obtain an ordinary differential equation with final condition at  $z = 0$ , that is,

$$\begin{aligned} \rho_A P'_A &= -c \rho_A + \int_z^0 G(P_A) \rho_A d\xi && \text{in } -\infty < z < 0, \\ \frac{1}{\eta_A} F_A(P_A) &= \frac{1}{\eta_B} F_B(P_B) && \text{at } z = 0. \end{aligned}$$

Under Assumptions 3.1(i), (iii) on the functions  $F_A$ ,  $F_B$  and  $G$ , the above problem admits a solution. Hence, for a given  $c > 0$  there exists a solution of problem (4.9).

*Step 2: monotonicity of  $\rho_B$  in  $(0, \ell)$  and proof of the condition (4.13).* Integrating (4.9)<sub>2</sub> between a generic point  $z \in [0, \ell)$  and  $\ell$ , and using the boundary condition (4.9)<sub>7</sub>, we find

$$(4.15) \quad P'_B(z) = -c < 0 \quad \text{for all } z \in [0, \ell].$$

Moreover, integrating (4.15) between a generic point  $z \in [0, \ell)$  and  $\ell$  gives

$$(4.16) \quad P_B(z) = c(\ell - z) + P_B(\ell) \quad \text{for } z \in [0, \ell].$$

Therefore,

$$(4.17) \quad P_B(0) - P_B(\ell) = c\ell, \quad P_B(\ell) < P_B(z) < P_B(0) \quad \text{for all } z \in (0, \ell).$$

Furthermore, note that (4.15) allows us to rewrite the boundary condition (4.9)<sub>6</sub> as

$$(4.18) \quad F_B(P_B(\ell)) = \eta_B \frac{c}{2} > 0.$$

Since under assumptions (4.4) we have that  $F_B(P_B(\ell)) > 0$  if and only if  $P_B(\ell) > P_B^{\text{eq}}$ , we conclude that

$$(4.19) \quad P_B(\ell) > P_B^{\text{eq}}.$$

Finally, since the function  $F_B(P_B)$  is monotone for  $P_B > P_B^{\text{eq}}$ , cf. (4.4), the value of  $P_B(\ell)$  is uniquely determined by (4.18).

Using the results (4.15), (4.17) and (4.19) along with the fact that, under assumptions (4.1) and (4.2),  $P_B > 0$  if and only if  $\rho_B > \rho_B^{\text{eq}}$  and  $P_B$  is a monotonically increasing and continuous function of  $\rho_B$  for  $\rho_B > \rho_B^{\text{eq}}$ , we conclude that the function  $\rho_B$  is continuous in  $(0, \ell)$  and satisfies the following conditions

$$(4.20) \quad \rho'_B(z) < 0 \quad \text{for all } z \in (0, \ell)$$

and

$$(4.21) \quad \rho_B^{\text{eq}} < \rho_B(\ell) < \rho_B(z) < \rho_B(0) \quad \text{for all } z \in (0, \ell).$$

*Step 3: identification of  $\ell$ .* For  $M > 0$  given, since the value of  $\rho_B(z)$  is uniquely determined for all  $z \in [0, \ell]$ , the value of  $\ell$  is uniquely defined by the integral identity (4.11).

*Step 4: monotonicity of  $\rho_A$  in  $(-\infty, 0]$  and proof of the condition (4.12).* Recalling that  $\frac{dP_A(\rho_A)}{d\rho_A} > 0$

for  $\rho_A > \rho_A^{\text{eq}}$ , we multiply both sides of (4.9)<sub>1</sub> by  $\frac{dP_A}{d\rho_A}$  and use assumption (4.10) to obtain the following boundary-value problem for  $P_A$

$$(4.22) \quad -P'_A(c + P'_A) - P''_A \rho_A \frac{dP_A}{d\rho_A} = G(P_A) \rho_A \frac{dP_A}{d\rho_A} \quad \text{in } (-\infty, 0]$$

$$(4.23) \quad P_A(z) \xrightarrow{z \rightarrow -\infty} P^M \quad \text{and} \quad P_A(0) = P_A^0.$$

Let  $z^* \in (-\infty, 0)$  be a critical point of  $P_A$ . Using (4.22) we conclude that

$$P''_A(z^*) = -G(P_A(z^*)).$$

Therefore, under the conditions (4.5) and (4.23), the strong maximum principle ensures that  $P_A < P^M$  in  $(-\infty, 0]$  and that  $P_A$  cannot have a local minimum in  $(-\infty, 0)$ , *i.e.*

$$(4.24) \quad P'_A(z) < 0 \quad \text{for all } z \in (-\infty, 0).$$

Hence the solution  $P_A$  of (4.22), (4.23) is a continuous and nonincreasing function that satisfies

$$(4.25) \quad P_A^{\text{eq}} < P_A(0) < P_A(z) < P^M \quad \text{for all } z \in (-\infty, 0).$$

Using the results (4.24) and (4.25) along with the fact that, under assumptions (4.1) and (4.2),  $P_A > 0$  if and only if  $\rho_A > \rho_A^{\text{eq}}$  and  $P_A$  is a monotonically increasing and continuous function of  $\rho_A$  for  $\rho_A > \rho_A^{\text{eq}}$ , we conclude that the function  $\rho_A$  is continuous in  $(-\infty, 0)$  and satisfies the following conditions

$$(4.26) \quad \rho'_A(z) < 0 \quad \text{for all } z \in (-\infty, 0)$$

and

$$(4.27) \quad \rho_A^{\text{eq}} < \rho_A(0) < \rho_A(z) < \rho^M \quad \text{for all } z \in (-\infty, 0),$$

with  $\rho^M$  being related to  $P^M$  by (4.3).

*Step 5: proof of the jump condition* (4.14). The transmission conditions (4.9)<sub>4</sub> and (4.9)<sub>5</sub> give

$$(4.28) \quad P'_A(0) = -c = P'_B(0).$$

Furthermore, due to the uniqueness of the value of  $P_B(0) > P_B^{\text{eq}}$ , under the monotonicity assumptions (4.4), the transmission condition (4.9)<sub>3</sub> allows one to uniquely determine the value of  $P_A(0) > P_A^{\text{eq}}$ . In particular, in the case where  $F_A(\cdot) = F_B(\cdot) \equiv F(\cdot)$ , the transmission condition (4.9)<sub>3</sub> gives

$$\frac{F(P_A(0))}{F(P_B(0))} = \frac{\eta_A}{\eta_B} \implies \text{sgn}(F(P_A(0)) - F(P_B(0))) = \text{sgn}(\eta_A - \eta_B),$$

from which, using the monotonicity assumptions (4.1) and (4.4), one finds the jump condition (4.14).  $\square$

## 5. NUMERICAL SOLUTIONS OF THE FREE-BOUNDARY PROBLEM AND COMPUTATIONAL SIMULATIONS FOR THE INDIVIDUAL-BASED MODEL

In this section, we illustrate the results established in Theorem 4.1 by presenting a sample of numerical solutions of the free-boundary problem (2.18), (2.19) and (2.22). Moreover, we compare numerical solutions of the continuum model with computational simulations for the individual-based model (2.3)-(2.5). We focus on the case where the force terms  $F_A(d_{ij})$  and  $F_B(d_{ij})$  are both given by the following cubic approximation of the JKR force law [27]

$$(5.1) \quad F(d_{ij}) = a_1(d_{ij} - d^{\text{eq}}) + a_2(d_{ij} - d^{\text{eq}})^2 + a_3(d_{ij} - d^{\text{eq}})^3 \quad \text{for } d_{ij} < d^{\text{eq}},$$

where  $d_{ij} = |r_i - r_j|$  and  $d^{\text{eq}}$  stands for the equilibrium intercellular distance, which is the distance between cell centres above which cells do not exert any force upon one another (*i.e.*  $F(d_{ij}) = 0$  for all  $d_{ij} \geq d^{\text{eq}}$ ). The equilibrium distance  $d^{\text{eq}}$  and the coefficients  $a_1$ ,  $a_2$  and  $a_3$  depend on the biophysical characteristics of the cells and are defined as

$$(5.2) \quad \begin{aligned} d^{\text{eq}} &= 2R - \frac{1}{2} \frac{(\pi\gamma)^{2/3} (3R)^{1/3}}{\tilde{E}^{2/3}}, & a_1 &= -\frac{3}{5} (3R\tilde{E})^{2/3} (\pi\gamma)^{1/3} d^{\text{eq}}, \\ a_2 &= \frac{33}{125} \frac{\tilde{E}^{4/3} (3R)^{1/3}}{(\pi\gamma)^{1/3}} (d^{\text{eq}})^2, & a_3 &= \frac{209}{3125} \frac{\tilde{E}^2}{\pi\gamma} (d^{\text{eq}})^3. \end{aligned}$$

In the formulas (5.2),  $R$  is the cell radius, the parameter  $\gamma$  measures the strength of cell-cell adhesion and  $\tilde{E}$  is an effective Young's modulus defined as

$$(5.3) \quad \tilde{E} = \frac{E}{2(1 - \nu^2)},$$

with  $E$  and  $\nu$  being, respectively, the Young's modulus and the Poisson's ratio of the cells. We refer the interested reader to Appendix A for a detailed derivation of the approximate representation of the JKR force law given by (5.1)-(5.3).

Using the formal relations between the intercellular distance and the cell density (2.13) and (2.14), we compute the cell densities and the equilibrium cell density as

$$(5.4) \quad \begin{aligned} \rho_A(t, r_i) &= \frac{1}{r_{i+1} - r_i} & \text{for } i = 1, \dots, m, \\ \rho_B(t, r_i) &= \frac{1}{r_{i+1} - r_i} & \text{for } i = m + 1, \dots, n, \end{aligned}$$

and  $\rho_A^{\text{eq}} = \rho_B^{\text{eq}} = \rho^{\text{eq}} = 1/d^{\text{eq}}$ . The approximation of the JKR force law (5.1) can be rewritten in terms of the cell densities  $\rho_l$  and  $\rho^{\text{eq}}$  as follows

$$(5.5) \quad F(1/\rho_l) = a_1 \left( \frac{1}{\rho_l} - \frac{1}{\rho^{\text{eq}}} \right) + a_2 \left( \frac{1}{\rho_l} - \frac{1}{\rho^{\text{eq}}} \right)^2 + a_3 \left( \frac{1}{\rho_l} - \frac{1}{\rho^{\text{eq}}} \right)^3.$$

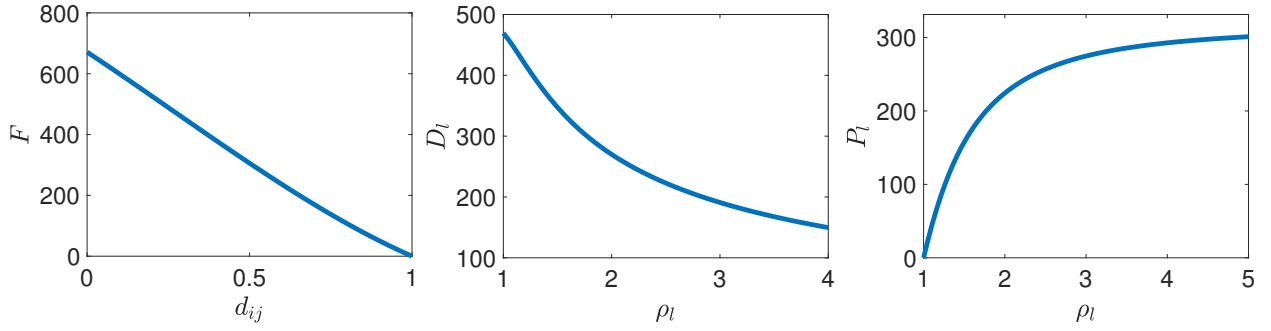


FIGURE 1. *Representations of the JKR model. Left panel.* The force between neighbouring cells  $F$ , defined by (5.5), is plotted against the intercellular distance  $d_{ij}$ . *Central panel.* The nonlinear diffusion coefficient  $D_l$ , defined by (5.6), is plotted against the cell density  $\rho_l$ . *Right panel.* The cell pressure  $P_l$ , defined by (5.7), is plotted against the cell density  $\rho_l$ . The parameter values are as in (5.9) and  $\eta_l = 0.5 \times 10^{-2} \text{ kg s}^{-1}$ . The values of  $d_{ij}$  and  $\rho_l$  are nondimensionalised by  $d^{\text{eq}}$  and  $\rho^{\text{eq}}$ , respectively.

Inserting the latter expression for  $F(1/\rho_l)$  into the definition (2.20) of the nonlinear diffusion coefficient  $D_l(\rho_l)$  yields

$$(5.6) \quad D_l(\rho_l) = \begin{cases} -\frac{3a_3(\rho^{\text{eq}} - \rho_l)^2 + 2a_2(\rho^{\text{eq}} - \rho_l)\rho_l\rho^{\text{eq}} + a_1(\rho_l\rho^{\text{eq}})^2}{\eta_l(\rho^{\text{eq}})^2\rho_l^4} & \text{if } \rho_l > \rho^{\text{eq}}, \\ 0 & \text{if } \rho_l \leq \rho^{\text{eq}}, \end{cases}$$

from which, using (4.1), we obtain the following barotropic relation for the cell pressure  $P_l$

$$(5.7) \quad P_l(\rho_l) = \begin{cases} \frac{1}{\eta_l(\rho^{\text{eq}})^2\rho_l^4} \left( \frac{\alpha_1}{2}\rho_l^2 + \frac{2\alpha_2}{3}\rho_l + \frac{3\alpha_3}{4} \right) + P_0^l & \text{if } \rho_l \geq \rho^{\text{eq}}, \\ 0 & \text{if } \rho_l < \rho^{\text{eq}}. \end{cases}$$

In (5.7), the term  $P_0^l$  is an integration constant such that  $P_l(\rho^{\text{eq}}) = 0$  and

$$(5.8) \quad \alpha_1 = a_1(\rho^{\text{eq}})^2 - 2a_2\rho^{\text{eq}} + 3a_3, \quad \alpha_2 = a_2(\rho^{\text{eq}})^2 - 3a_3\rho^{\text{eq}}, \quad \alpha_3 = a_3(\rho^{\text{eq}})^2.$$

Numerical simulations were performed using parameter values chosen in agreement with those used in [20], that is,

$$(5.9) \quad E = 300 \text{ Pa}, \quad \nu = 0.4, \quad \gamma = \zeta k_B T, \quad R = 7.5 \times 10^{-6} \text{ m},$$

where  $\zeta = 10^{15} \text{ m}^2$  is the density of cell-cell adhesion molecules in the cell membrane,  $k_B$  is the Boltzmann constant and  $T = 298 \text{ K}$  is an absolute temperature. Figure 1 displays the plots of the force  $F$  between neighbouring cells, the nonlinear diffusion coefficient  $D_l$ , and the cell pressure  $P_l$ , obtained using the parameter values given by (5.9).

We let the cell damping coefficients of population  $A$  be  $\eta_A = 0.5 \times 10^{-2} \text{ kg s}^{-1}$ , and considered the cases where  $\eta_A = \eta_B$  or  $\eta_B = 2\eta_A$  or  $\eta_B = 0.5\eta_A$ . Moreover, for the cell proliferation term we assumed

$$g(1/\rho_A) = \tilde{H}(1/\rho_A - 1/\rho^{\text{M}}) \quad \text{and} \quad G(\rho_A) = \alpha \tilde{H}(\rho^{\text{M}} - \rho_A), \quad \text{with} \quad \rho^{\text{M}} = \frac{4}{3}\rho^{\text{eq}} \quad \text{and} \quad \alpha = \frac{1}{2},$$

where  $\tilde{H}$  is a smooth approximation to the Heaviside function.

To construct numerical solutions, the free-boundary problem (2.18), (2.19) and (2.22) was transformed to a Lagrangian reference frame and the method of lines was employed to solve the resultant equations. The resulting system of ordinary differential equations, as well as the ordinary differential equations (2.4) and (2.5) of the individual-based model, were numerically solved using the MATLAB routine ODE15S.

The plots in Figures 2-4 show sample dynamics of the cell density  $\rho$  defined as

$$(5.10) \quad \rho(t, x) = \begin{cases} \rho_A(t, x), & \text{if } x \leq s_1(t), \\ \rho_B(t, x), & \text{if } x > s_1(t), \end{cases}$$

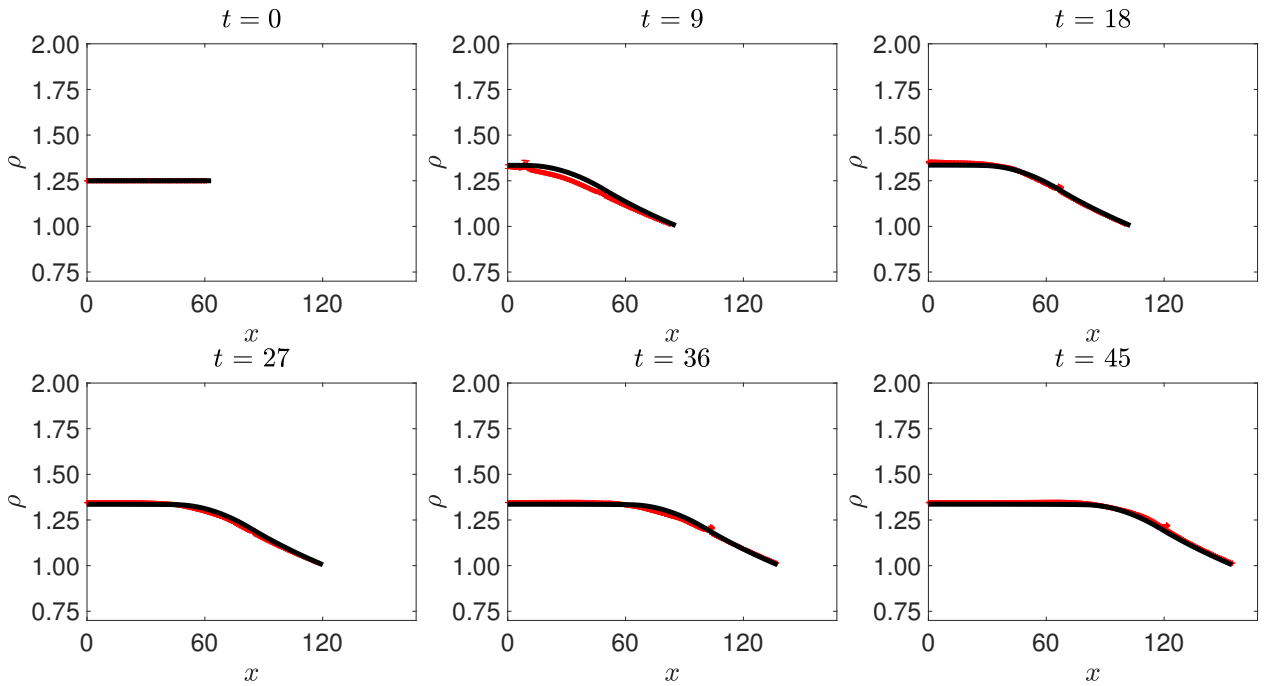


FIGURE 2. Comparison between the free-boundary problem and the individual-based model for  $\eta_A = \eta_B$ . The cell density,  $\rho(t, x)$ , given by (5.10) is plotted against  $x$  for increasing values of  $t$ . The cell densities  $\rho_A$  and  $\rho_B$  are either numerical solutions of the free-boundary problem (black lines) or approximate cell densities computed from simulation results for the individual-based model using (5.4) (red markers). The values of  $x$  are nondimensionalised by  $d^{\text{eq}}$ , while the values of  $\rho$  are nondimensionalised by  $\rho^{\text{eq}}$ .

with  $\rho_A$  and  $\rho_B$  being either numerical solutions of the free-boundary problem (2.18), (2.19), (2.22) (black lines) or approximate cell densities computed from numerical solutions of the individual-based model (2.3)-(2.5) using (5.4) (red markers). In agreement with the results established in Theorem 4.1, we observe the emergence of travelling-wave solutions, whereby the positions of the inner free boundary  $s_1(t)$  and the outer free boundary  $s_2(t)$  move at the same constant speed, and the cell densities  $\rho_A$  and  $\rho_B$  are monotonically decreasing.

Moreover,  $\rho$  is continuous at  $s_1$  if  $\eta_A = \eta_B$ , cf. the plots in Figure 2, whereas it has a jump discontinuity at  $s_1$  both for  $\eta_A < \eta_B$  and for  $\eta_A > \eta_B$ . The sign of the jump  $\rho(s_1^+) - \rho(s_1^-)$  satisfies condition (4.14), cf. the plots in Figures 3 and 4, and, once that the travelling-wave is formed, the size of the jump is constant and such that the transmission condition (4.9)<sub>3</sub> is met, see Supplementary Figure B.1(a). As shown by these plots, there is an excellent match between the numerical solutions of the free-boundary problem and the computational simulation results for the individual-based model.

## 6. DISCUSSION

We presented an off-lattice individual-based model that describes the dynamics of two contiguous cell populations with different proliferative and mechanical characteristics.

We formally showed that this discrete model can be represented in the continuum limit as a free-boundary problem for the cell densities. We proved an existence result for the free-boundary problem and constructed travelling-wave solutions. We performed numerical simulations in the case where the cellular interaction forces are described by the celebrated JKR model of elastic contact, and we found excellent agreement between the computational simulation results for the individual-based model, the numerical solutions of the corresponding free-boundary problem and the travelling-wave analysis. Taken together, the results of numerical simulations demonstrate that the solutions of the free-boundary problem faithfully capture the qualitative and quantitative properties of the outcomes of the off-lattice individual-based model.

In this paper, we focussed on a one-dimensional scenario where the two cell populations do not mix. It would be interesting to extend the individual-based mechanical model presented here, and the

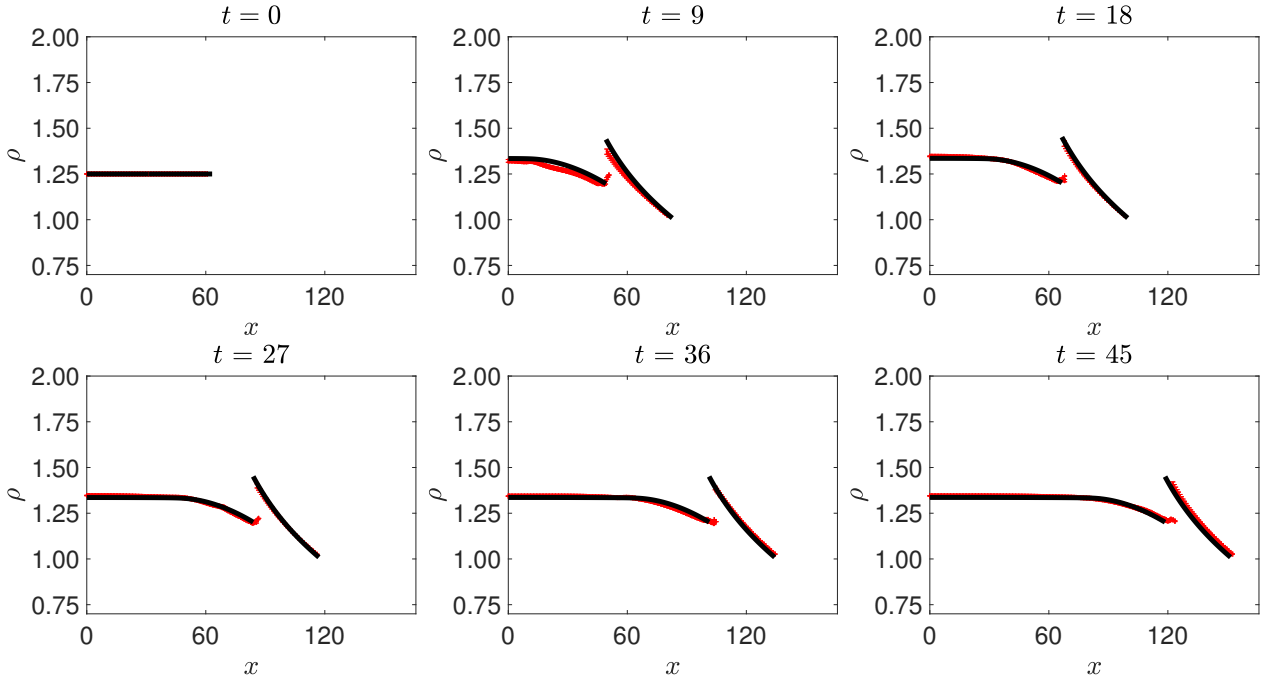


FIGURE 3. Comparison between the free-boundary problem and the individual-based model for  $\eta_A < \eta_B$ . The cell density,  $\rho(t, x)$ , given by (5.10), is plotted against  $x$  for increasing values of  $t$ . The cell densities  $\rho_A$  and  $\rho_B$  are either numerical solutions of the free-boundary problem (black lines) or approximate cell densities computed from simulation results for the individual-based model using (5.4) (red markers). The values of  $x$  are nondimensionalised by  $d^{\text{eq}}$ , while the values of  $\rho$  are nondimensionalised by  $\rho^{\text{eq}}$ .

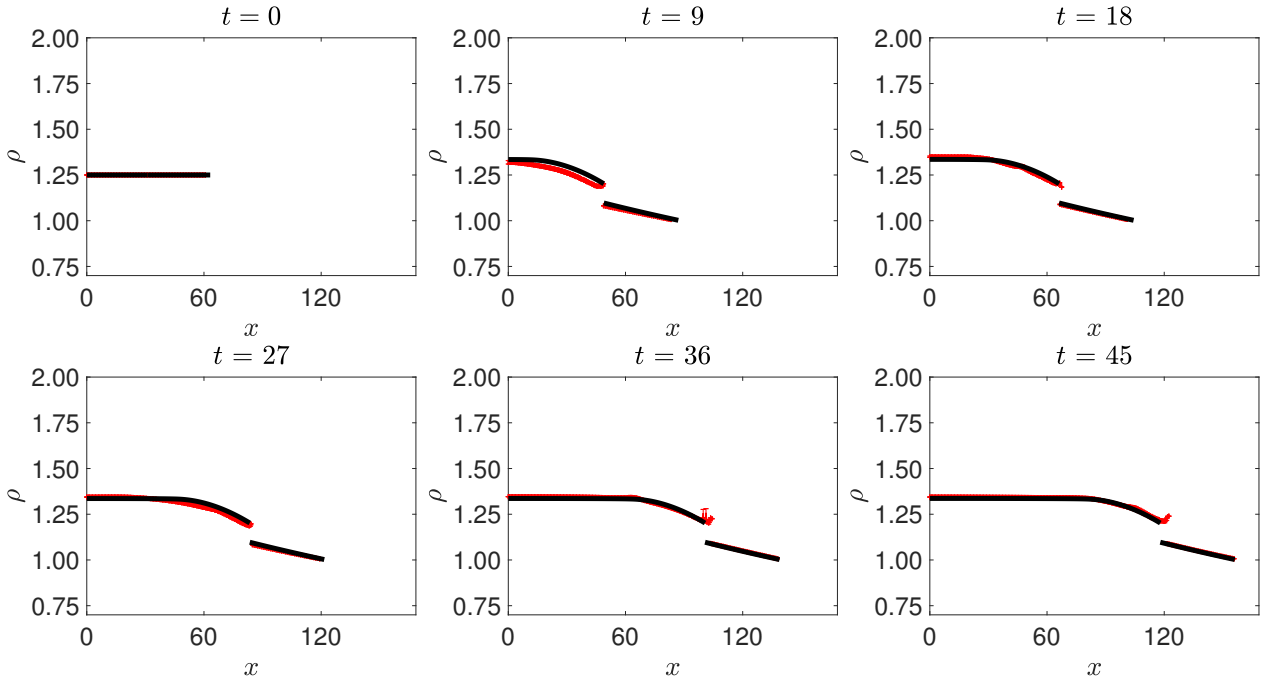


FIGURE 4. Comparison between the free-boundary problem and the individual-based model for  $\eta_A > \eta_B$ . The cell density  $\rho(t, x)$ , given by (5.10), is plotted against  $x$  for increasing values of  $t$ . The cell densities  $\rho_A$  and  $\rho_B$  are either numerical solutions of the free-boundary problem (black lines) or approximate cell densities computed from simulation results for the individual-based model using (5.4) (red markers). The values of  $x$  are nondimensionalised by  $d^{\text{eq}}$ , while the values of  $\rho$  are nondimensionalised by  $\rho^{\text{eq}}$ .



related formal method of derivation of the corresponding continuum model as well, to more realistic two-dimensional cases whereby spatial mixing between the two cell populations can occur. In this regard, an additional development of our study would be to formulate probabilistic discrete mechanical models of interacting cell populations and, using asymptotic methods analogous to those employed, for instance, in [11, 24, 38, 39], to perform a rigorous derivation of their continuum counterparts. These are all lines of research that we will be pursuing in the near future.

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#### APPENDIX A. APPROXIMATE REPRESENTATION OF THE JKR FORCE LAW

The nonlinear function  $F^{JKR}(d_{ij})$  that gives the JKR force law between the  $i^{th}$  cell and the  $j^{th}$  cell, with centres at distance  $d_{ij}$ , is implicitly defined by the following formulas [20]

$$(A.1) \quad \begin{aligned} \delta_{ij} &= \frac{a_{ij}^2}{R_{ij}} - \sqrt{\frac{2\pi\gamma a_{ij}}{E_{ij}}}, \\ a_{ij}^3 &= a_{ij}^3(F_{ij}^{JKR}) = \frac{3R_{ij}}{4E_{ij}} \left[ F_{ij}^{JKR} + 3\pi\gamma R_{ij} + \sqrt{6\pi\gamma R_{ij} F_{ij}^{JKR} + (3\pi\gamma R_{ij})^2} \right], \end{aligned}$$

where

$$R_{ij}^{-1} = R_i^{-1} + R_j^{-1}, \quad d_{ij} = R_i + R_j - \delta_{ij} \quad \text{and} \quad E_{ij}^{-1} = (1 - \nu_i^2)E_i^{-1} + (1 - \nu_j^2)E_j^{-1}.$$

In the formulas (A.1), the parameter  $\gamma$  models the strength of cell-cell adhesion,  $R_i$  stands for the radius of the  $i^{th}$  cell,  $E_i$  is the Young’s modulus of the  $i^{th}$  cell, and  $\nu_i$  denotes the Poisson’s ratio of

the  $i^{\text{th}}$  cell. Analogous considerations hold for the parameters of the  $j^{\text{th}}$  cell. Moreover,  $\delta_{ij}$  is the sum of the deformations undergone by the  $i^{\text{th}}$  cell and the  $j^{\text{th}}$  cell.

As the computational cost of numerical simulations carried out by solving implicitly for  $F^{JKR}(d_{ij})$  is prohibitive, we derive an approximate representation of this function based on a third degree polynomial of the form

$$(A.2) \quad F^{JKR}(d_{ij}) \approx F(d_{ij}) = a_3^{ij} (d_{ij} - d_{ij}^{\text{eq}})^3 + a_2^{ij} (d_{ij} - d_{ij}^{\text{eq}})^2 + a_1^{ij} (d_{ij} - d_{ij}^{\text{eq}}),$$

with

$$(A.3) \quad a_1^{ij} = F'(d_{ij}^{\text{eq}}) d_{ij}^{\text{eq}}, \quad a_2^{ij} = \frac{1}{2} F''(d_{ij}^{\text{eq}}) (d_{ij}^{\text{eq}})^2, \quad a_3^{ij} = \frac{1}{6} F'''(d_{ij}^{\text{eq}}) (d_{ij}^{\text{eq}})^3.$$

In the above equations,  $d_{ij}^{\text{eq}}$  denotes the equilibrium distance between the centres of cell  $i$  and cell  $j$  (*i.e.* the distance  $d_{ij}$  such that  $F(d_{ij}) = 0$  for all  $d_{ij} \geq d_{ij}^{\text{eq}}$ ).

With this goal in mind, we look for explicit expressions of  $d_{ij}^{\text{eq}}$ ,  $F'(d_{ij}^{\text{eq}})$ ,  $F''(d_{ij}^{\text{eq}})$  and  $F'''(d_{ij}^{\text{eq}})$  in terms of the cell radii and the mechanical parameters of the cells. In the rest of this appendix we will use the abridged notation  $F_{ij}$  for  $F(d_{ij})$ .

Expression for  $d_{ij}^{\text{eq}}$ . The equilibrium distance  $d_{ij}^{\text{eq}}$  can be directly computed from the formulas (A.1). In fact, choosing  $d_{ij} = d_{ij}^{\text{eq}}$  and using the fact that  $F(d_{ij}^{\text{eq}}) = 0$ , from the second formula in (A.1) we obtain

$$a_{ij}^3(0) = \frac{9\pi\gamma R_{ij}^2}{2E_{ij}}.$$

Substituting this expression into the first formula in (A.1) yields

$$\delta_{ij}^{\text{eq}} = \frac{R_{ij}^{1/3} (9\pi\gamma)^{2/3}}{2^{2/3} E_{ij}^{2/3}} - \frac{3^{1/3} 2^{1/3} (\pi\gamma)^{2/3} R_{ij}^{1/3}}{E_{ij}^{2/3}} = \frac{1}{2} \frac{(\pi\gamma)^{2/3} (6R_{ij})^{1/3}}{E_{ij}^{2/3}}$$

and noting that  $d_{ij}^{\text{eq}} = R_i + R_j - \delta_{ij}^{\text{eq}}$  we find the equilibrium distance  $d_{ij}^{\text{eq}}$ .

Expression for  $F'(d_{ij}^{\text{eq}})$ . We substitute the second formula in (A.1) into the first formula to obtain

$$(A.4) \quad R_i + R_j - d_{ij} = \frac{1}{R_{ij}} \left( \frac{3R_{ij}}{4E_{ij}} \right)^{2/3} f(F_{ij})^{2/3} - \left( \frac{2\pi\gamma}{E_{ij}} \right)^{1/2} \left( \frac{3R_{ij}}{4E_{ij}} \right)^{1/6} f(F_{ij})^{1/6},$$

where

$$f(F_{ij}) = F_{ij} + \alpha + \sqrt{2\alpha} \sqrt{F_{ij} + \alpha/2} \quad \text{with} \quad \alpha = 3\pi\gamma R.$$

Differentiating  $f$  with respect to  $F_{ij}$  yields

$$f'(F_{ij}) = 1 + \frac{\sqrt{2\alpha}}{2} \frac{1}{\sqrt{F_{ij} + \alpha/2}}, \quad f''(F_{ij}) = -\frac{\sqrt{2\alpha}}{4} \frac{1}{(F_{ij} + \alpha/2)^{3/2}}$$

and

$$f'''(F_{ij}) = \frac{3\sqrt{2\alpha}}{8} \frac{1}{(F_{ij} + \alpha/2)^{5/2}}.$$

Hence, for  $d_{ij} = d_{ij}^{\text{eq}}$  we have

$$(A.5) \quad f(0) = 2\alpha = 6\pi\gamma R, \quad f'(0) = 2, \quad f''(0) = -\frac{1}{\alpha} = -\frac{1}{3\pi\gamma R_{ij}}, \quad f'''(0) = \frac{3}{\alpha^2} = \frac{3}{(3\pi\gamma R_{ij})^2}.$$

Differentiating both sides of (A.4) with respect to  $d_{ij}$  we find

$$-1 = \left[ A \frac{2}{3} (f(F_{ij}))^{-1/3} - \frac{B}{6} (f(F_{ij}))^{-5/6} \right] f'(F_{ij}) F'_{ij},$$

where

$$A = \frac{1}{R} \left( \frac{3R_{ij}}{4E_{ij}} \right)^{2/3} = \frac{3}{2} \frac{1}{(6R_{ij})^{1/3} E_{ij}^{2/3}}, \quad B = \left( \frac{2\pi\gamma}{E_{ij}} \right)^{1/2} \left( \frac{3R_{ij}}{4E_{ij}} \right)^{1/6} = \frac{(6R_{ij})^{1/6} (\pi\gamma)^{1/2}}{E_{ij}^{2/3}}.$$

Rearranging terms in the latter equation yields

$$F'_{ij} = - (f'(F_{ij}))^{-1} \left[ A \frac{2}{3} (f(F_{ij}))^{-1/3} - \frac{B}{6} (f(F_{ij}))^{-5/6} \right]^{-1}$$

and, therefore, for  $d_{ij} = d_{ij}^{\text{eq}}$  we have

$$F'(d_{ij}^{\text{eq}}) = - (f'(0))^{-1} \left[ A \frac{2}{3} (f(0))^{-1/3} - \frac{B}{6} (f(0))^{-5/6} \right]^{-1}.$$

Finally, noting that

$$-\frac{1}{2} \left[ A \frac{2}{3} f(0)^{-1/3} - \frac{B}{6} f(0)^{-5/6} \right]^{-1} = \frac{6}{5} (6R_{ij}E_{ij})^{2/3} (\pi\gamma)^{1/3}$$

we find

$$(A.6) \quad F'(d_{ij}^{\text{eq}}) = -\frac{3}{5} (6R_{ij}E_{ij})^{2/3}.$$

Expression for  $F''(d_{ij}^{\text{eq}})$ . Differentiating twice both sides of (A.4) with respect to  $d_{ij}$  yields

$$\begin{aligned} 0 &= \left[ A \frac{2}{3} (f(F_{ij}))^{-1/3} - \frac{B}{6} (f(F_{ij}))^{-5/6} \right] f''(F_{ij}) \left( (F'_{ij})^2 + f'(F_{ij}) F''_{ij} \right) \\ &\quad + \left[ -A \frac{2}{9} (f(F_{ij}))^{-4/3} + \frac{5B}{36} (f(F_{ij}))^{-11/6} \right] (f'(F_{ij}) F'_{ij})^2. \end{aligned}$$

Rearranging terms in the latter equation gives

$$F''_{ij} = - (f'(F_{ij}))^{-1} f''(F_{ij}) (F'_{ij})^2 - (f'(F_{ij}))^2 (F'_{ij})^3 \left[ A \frac{2}{9} f(F_{ij})^{-4/3} - B \frac{5}{36} f(F_{ij})^{-11/6} \right].$$

Hence, choosing  $d_{ij} = d_{ij}^{\text{eq}}$  and using the fact that  $F(d_{ij}^{\text{eq}}) = 0$  along with the expressions (A.5) for  $f'(0)$  and  $f''(0)$ , we find

$$(A.7) \quad F''(d_{ij}^{\text{eq}}) = \frac{1}{2\alpha} \left( F'(d_{ij}^{\text{eq}}) \right)^2 - 4 \left( F'(d_{ij}^{\text{eq}}) \right)^3 \left[ A \frac{2}{9} f(0)^{-4/3} - B \frac{5}{36} f(0)^{-11/6} \right].$$

Noting that [cf. the expression of  $f(0)$  in (A.5)]

$$A \frac{2}{9} f(0)^{-4/3} - B \frac{5}{36} f(0)^{-11/6} = \frac{7}{36} \frac{1}{(6R_{ij})^{5/3} E_{ij}^{2/3} (\pi\gamma)^{4/3}}$$

and inserting the expression (A.6) of  $F'(d_{ij}^{\text{eq}})$  into (A.7) yields

$$F''(d_{ij}^{\text{eq}}) = \left( \frac{9}{25} + 4 \frac{27}{125} \frac{7}{36} \right) (6R_{ij}E_{ij})^{4/3} (\pi\gamma)^{2/3} \frac{1}{6R_{ij}\pi\gamma} = \frac{66}{125} \frac{E_{ij}^{4/3} (6R_{ij})^{1/3}}{(\pi\gamma)^{1/3}}.$$

Expression for  $F'''(d_{ij}^{\text{eq}})$ . Differentiating thrice both sides of (A.4) with respect to  $d_{ij}$  yields

$$\begin{aligned} 0 &= \left[ A \frac{2}{3} f(F_{ij})^{-1/3} - \frac{B}{6} f(F_{ij})^{-5/6} \right] \left[ f'''(F_{ij}) (F'_{ij})^3 + 3f''(F_{ij}) F'_{ij} F''_{ij} + f'(F_{ij}) F'''_{ij} \right] \\ &\quad + \left[ -A \frac{2}{9} f(F_{ij})^{-4/3} + \frac{5B}{36} f(F_{ij})^{-11/6} \right] f'(F_{ij}) F'_{ij} \left[ f''(F_{ij}) (F'_{ij})^2 + f'(F_{ij}) F''_{ij} \right] \\ &\quad + \left[ A \frac{8}{27} f(F_{ij})^{-7/3} - B \frac{55}{216} f(F_{ij})^{-17/6} \right] (f'(F_{ij}) F'_{ij})^3. \end{aligned}$$

Rearranging terms in the latter equation we obtain

$$\begin{aligned} F'''_{ij} &= \left\{ 3 \left[ \frac{2A}{9} f(F_{ij})^{-4/3} - \frac{5B}{36} f(F_{ij})^{-11/6} \right] F'_{ij} \left[ f''(F_{ij}) (F'_{ij})^2 + f'(F_{ij}) F''_{ij} \right] \right. \\ &\quad \left. - \left[ \frac{8A}{27} f(F_{ij})^{-7/3} - \frac{55B}{216} f(F_{ij})^{-17/6} \right] (f'(F_{ij}))^2 (F'_{ij})^3 \right\} \left[ A \frac{2}{3} f(F_{ij})^{-1/3} - \frac{B}{6} f(F_{ij})^{-5/6} \right]^{-1} \\ &\quad - \left( f'''(F_{ij}) (F'_{ij})^3 + 3f''(F_{ij}) F'_{ij} F''_{ij} \right) (f'(F_{ij}))^{-1}. \end{aligned}$$

Hence, choosing  $d_{ij} = d_{ij}^{\text{eq}}$  and using the fact that  $F(d_{ij}^{\text{eq}}) = 0$ , along with the expression (A.7) for  $F''(d_{ij}^{\text{eq}})$  and the expressions (A.5) for  $f'(0)$ ,  $f''(0)$  and  $f'''(0)$ , we find

$$(A.8) \quad F'''(d_{ij}^{\text{eq}}) = 48C_1^2 \left( F'(d_{ij}^{\text{eq}}) \right)^5 + \left( 8C_2 - \frac{6}{\alpha} C_1 \right) \left( F'(d_{ij}^{\text{eq}}) \right)^4 - \frac{3}{4\alpha^2} \left( F'(d_{ij}^{\text{eq}}) \right)^3,$$

where

$$\begin{aligned}
C_1 &= \left[ A \frac{2}{9} f(0)^{-4/3} - \frac{5B}{36} f(0)^{-11/6} \right] = \frac{7}{36} \frac{1}{E_{ij}^{2/3} (6R_{ij})^{5/3} (\pi\gamma)^{4/3}}, \\
C_2 &= A \frac{8}{27} f(0)^{-7/3} - B \frac{55}{216} f(0)^{-17/6} \\
&= \frac{1}{27} (6R_{ij}\pi\gamma)^{-7/3} \left[ 8 \frac{1}{R_{ij}^{1/3} E_{ij}^{2/3}} \left( \frac{3}{4} \right)^{2/3} - \frac{55}{8} \frac{(6R_{ij})^{1/6} (\pi\gamma)^{1/2}}{E_{ij}^{2/3} (6R_{ij}\pi\gamma)^{1/2}} \right] \\
&= \frac{41}{216} \frac{1}{E_{ij}^{2/3} (6R_{ij})^{8/3} (\pi\gamma)^{7/3}}.
\end{aligned}$$

Finally, inserting the expression (A.6) for  $F'(d_{ij}^{\text{eq}})$  into (A.8) we obtain

$$F'''(d_{ij}^{\text{eq}}) = \frac{1254}{3125} \frac{E_{ij}^2}{\pi\gamma}.$$

Expressions for  $d_{ij}^{\text{eq}}$ ,  $a_1^{ij}$ ,  $a_2^{ij}$  and  $a_3^{ij}$ . Taken together the results from above give

$$\begin{aligned}
d_{ij}^{\text{eq}} &= R_i + R_j - \frac{1}{2} \frac{(\pi\gamma)^{2/3} (6R_{ij})^{1/3}}{E_{ij}^{2/3}}, \\
a_1^{ij} &= d_{ij}^{\text{eq}} F'(d_{ij}^{\text{eq}}) = -\frac{3}{5} (6R_{ij} E_{ij})^{2/3} (\pi\gamma)^{1/3} d_{ij}^{\text{eq}}, \\
a_2^{ij} &= \frac{1}{2} F''(d_{ij}^{\text{eq}}) (d_{ij}^{\text{eq}})^2 = \frac{33}{125} \frac{E_{ij}^{4/3} (6R_{ij})^{1/3}}{(\pi\gamma)^{1/3}} (d_{ij}^{\text{eq}})^2, \\
a_3^{ij} &= \frac{1}{6} F'''(d_{ij}^{\text{eq}}) (d_{ij}^{\text{eq}})^3 = \frac{209}{3125} \frac{E_{ij}^2}{\pi\gamma} (d_{ij}^{\text{eq}})^3.
\end{aligned} \tag{A.9}$$

Approximate representation of  $F^{JKR}(d_{ij})$  used to perform numerical simulations. To perform numerical simulations, we assumed that all cells have the same radius  $R$ , Young's modulus  $E$  and Poisson's ratio  $\nu$ , *i.e.*

$$R_i = R, \quad E_i = E \quad \text{and} \quad \nu_i = \nu \quad \text{for all } i = 1, \dots, n.$$

Under these assumptions, we have that

$$R_{ij} = \frac{R}{2} \quad \text{and} \quad E_{ij} = \tilde{E} = \frac{E}{2(1-\nu^2)} \quad \text{for all } i, j = 1, \dots, n,$$

and the approximate representation of the JKR force law given by (A.2) and (A.9) reads as

$$F^{JKR}(d_{ij}) \approx a_3 (d_{ij} - d^{\text{eq}})^3 + a_2 (d_{ij} - d^{\text{eq}})^2 + a_1 (d_{ij} - d^{\text{eq}}) \quad \text{for all } i, j = 1, \dots, n, \tag{A.10}$$

with

$$\begin{aligned}
d^{\text{eq}} &= 2R - \frac{1}{2} \frac{(\pi\gamma)^{2/3} (3R)^{1/3}}{\tilde{E}^{2/3}}, & a_1 &= -\frac{3}{5} (3R\tilde{E})^{2/3} (\pi\gamma)^{1/3} d^{\text{eq}}, \\
a_2 &= \frac{33}{125} \frac{\tilde{E}^{4/3} (3R)^{1/3}}{(\pi\gamma)^{1/3}} (d^{\text{eq}})^2, & a_3 &= \frac{209}{3125} \frac{\tilde{E}^2}{\pi\gamma} (d^{\text{eq}})^3.
\end{aligned} \tag{A.11}$$

## APPENDIX B. SUPPLEMENTARY FIGURE

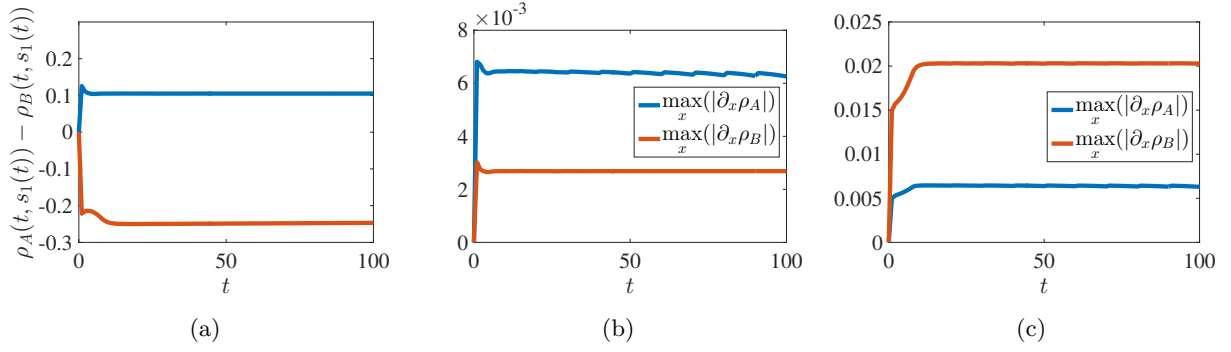


FIGURE B.1. (a) Plot of  $\rho_A(t, s_1(t)) - \rho_B(t, s_1(t))$  against  $t$  in the case where  $\eta_A < \eta_B$  (orange line), which corresponds to Figure 3, and in the case where  $\eta_A > \eta_B$  (blue line), which corresponds to Figure 4. (b) Plot of  $\max_x(|\partial_x \rho_A|)$  (blue) and  $\max_x(|\partial_x \rho_B|)$  (orange) against  $t$  in the case where  $\eta_A < \eta_B$ , which corresponds to Figure 3. (c) Plot of  $\max_x(|\partial_x \rho_A|)$  (blue) and  $\max_x(|\partial_x \rho_B|)$  (orange) against  $t$  in the case where  $\eta_A > \eta_B$ , which corresponds to Figure 4. See Figures 3 and 4 for further details.