



Heriot-Watt University
Research Gateway

Geometric inequalities via a symmetric differential operator defined by quantum calculus in the open unit disk

Citation for published version:

Ibrahim, RW, Elobaid, RM & Obaiys, SJ 2020, 'Geometric inequalities via a symmetric differential operator defined by quantum calculus in the open unit disk', *Journal of Function Spaces*, vol. 2020, 6932739. <https://doi.org/10.1155/2020/6932739>

Digital Object Identifier (DOI):

[10.1155/2020/6932739](https://doi.org/10.1155/2020/6932739)

Link:

[Link to publication record in Heriot-Watt Research Portal](#)

Document Version:

Publisher's PDF, also known as Version of record

Published In:

Journal of Function Spaces

Publisher Rights Statement:

Copyright © 2020 Rabha W. Ibrahim et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

General rights

Copyright for the publications made accessible via Heriot-Watt Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

Heriot-Watt University has made every reasonable effort to ensure that the content in Heriot-Watt Research Portal complies with UK legislation. If you believe that the public display of this file breaches copyright please contact open.access@hw.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

Research Article

Geometric Inequalities via a Symmetric Differential Operator Defined by Quantum Calculus in the Open Unit Disk

Rabha W. Ibrahim,^{1,2} Rafida M. Elobaid ,³ and Suzan J. Obaiys⁴

¹*Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam*

²*Faculty of Mathematics & Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam*

³*Deanship of Educational Services, Prince Sultan University, Saudi Arabia*

⁴*School of Mathematical and Computer Sciences, Heriot-Watt University Malaysia, Malaysia*

Correspondence should be addressed to Rafida M. Elobaid; robaid@psu.edu.sa

Received 11 February 2020; Revised 8 July 2020; Accepted 14 July 2020; Published 18 August 2020

Academic Editor: Adrian Petrusel

Copyright © 2020 Rabha W. Ibrahim et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The present investigation covenants with the concept of quantum calculus besides the convolution operation to impose a comprehensive symmetric q -differential operator defining new classes of analytic functions. We study the geometric representations with applications. The applications deliberated to indicate the certainty of resolutions of a category of symmetric differential equations type Briot-Bouquet.

1. Introduction

In this effort, we deal with the structure of q -calculus, which develops an interesting technique for calculations and organizes different classes of operators and specific transformations. The significance of q -calculus appeared in a huge number of applications including physical problems. The symmetric q -activation normally achieves q -difference equations (may involve derivative). A close connection between these operators and symmetries of q -symmetric operator is accordingly to be estimated (see [1–9]). In recent investigation, we deliver a process for deriving and interpreting from a symmetry possessions and infirm analogy with the traditional cases. By combining the q -calculus and the symmetric Salagean differential operator, we introduce a novel modified symmetric Salagean q -differential operator. Via employing this operator, we deliver new classes of analytic functions.

2. Preliminaries

This section gives out the mathematical processing to deliver the suggested SDOs and complex conformable operator for some classes of analytic functions in the open

unit disk $\cup = \{\xi \in \mathbb{C} : |\xi| < 1\}$. Let \wedge be the category of smooth function elicited as pursue

$$\gamma(\xi) = \xi + \sum_{n=2}^{\infty} \gamma_n \xi^n, \quad \xi \in \cup. \quad (1)$$

A function $\gamma \in \wedge$ is known as a starlike with respect to $\xi = 0$ if the straight line segment combining the origin to all else point of γ embedding completely in $\gamma(\xi : |\xi| < 1)$. The aim is that each point of $\gamma(\xi : |\xi| < 1)$ must be manifested via $(0,0)$. A univalent function ($\gamma \in \mathfrak{S}$;) is indicated to be convex in \cup if the linear slice combining two ends of $\gamma(\xi : |\xi| < 1)$ stays completely in $\gamma(\xi : |\xi| < 1)$. We denote these classes by \mathcal{S}^* and \mathcal{C} for starlike and convex, respectively. In addition, suppose that the category \mathcal{P} involves all functions γ analytic in \cup with a positive real part in \cup achieving $\gamma(0) = 1$. Mathematically, $\gamma \in \mathcal{S}^*$ if and only if $\xi \gamma'(\xi)/\gamma(\xi) \in \mathcal{P}$, and $\gamma \in \mathcal{C}$ if and only if $1 + \xi \gamma''(\xi)/\gamma'(\xi) \in \mathcal{P}$; equivalently,

$$\Re\left(\xi \gamma'(\xi)/\gamma(\xi)\right) > 0, \quad (2)$$

for starlikeness and

$$1 + \Re \left(\xi \gamma''(\xi) / \gamma'(\xi) \right) > 0, \quad (3)$$

for convexity.

For two functions, γ_1 and γ_2 belong to the category \wedge and are said to be subordinate, noting by $\gamma_1 < \gamma_2$, if we can find a Schwarz function τ with $\tau(0) = 0$ and $|\tau(\xi)| < 1$ achieving $\gamma_1(\xi) = \gamma_2(\tau(\xi))$, $\xi \in \cup$ (the detail can be located in [10]). Obviously, $\gamma_1(\xi) < \gamma_2(\xi)$ implies that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(\cup) \subset \gamma_2(\cup)$. We employ next facts, one can find it in [11].

For every nonnegative integer n , the q -integer number n symbolized by $[n, q]$ is structured by

$$[n, q] = \frac{1 - q^n}{1 - q}, \quad (4)$$

where $[0, q] = 0$, $[1, q] = 1$, and $\lim_{q \rightarrow 1^-} [n, q] = n$. Consequently, the analytic function γ is written by the formula

$$\Delta_q \gamma(\xi) = \frac{\gamma(q\xi) - \gamma(\xi)}{q\xi - \xi}, \quad \xi \in \cup. \quad (5)$$

Clearly, we have $\Delta_q \xi^n = [n, q] \xi^{n-1}$. Consequently, for $\gamma \in \wedge$, we attain

$$\Delta_q \gamma(\xi) = \sum_{n=1}^{\infty} \gamma_n [n, q] \xi^{n-1}, \quad \xi \in \cup, \gamma_1 = 1. \quad (6)$$

For $\gamma \in \wedge$, it realized that the Salàgean q -differential operator [12] has the formula

$$\begin{aligned} S_q^0 \gamma(\xi) &= \gamma(\xi), \\ S_q^1 \gamma(\xi) &= \xi \Delta_q \gamma(\xi), \\ &\dots, \\ S_q^k \gamma(\xi) &= \xi \Delta_q \left(S_q^{k-1} \gamma(\xi) \right), \end{aligned} \quad (7)$$

such that k represents as a positive integer. A computation based on the definition of Δ_q implies that $S_q^k \gamma(\xi) = \gamma(\xi) * \Pi_q^k(\xi)$, where $*$ is the convolution product.

$$\Pi_q^k(\xi) = \xi + \sum_{n=2}^{\infty} [n, q]^k \xi^n, \quad (8)$$

and

$$S_q^k \gamma(\xi) = \xi + \sum_{n=2}^{\infty} [n, q]^k \gamma_n \xi^n. \quad (9)$$

Clearly,

$$\lim_{q \rightarrow 1^-} S_q^k \gamma(\xi) = \xi + \sum_{n=2}^{\infty} n^k \gamma_n \xi^n, \quad (10)$$

the well-known Salagean differential operator [6].

Consider the role $\gamma(\xi)$ and a constant $0 \leq \lambda \leq 1$, we introduce the q -symmetric Salagean differential operator using the definition of Δ_q as follows:

$$\begin{aligned} \mathcal{S}_q^{\lambda,0} \gamma(\xi) &= \gamma(\xi), \\ \mathcal{S}_q^{\lambda,1} \gamma(\xi) &= \lambda \xi \Delta_q \gamma(\xi) - (1 - \lambda) \xi \Delta_q \gamma(-\xi) \\ &= \xi (\lambda \Delta_q \gamma(\xi) - (1 - \lambda) \Delta_q \gamma(-\xi)) \\ &= \lambda \left(\xi + \sum_{n=2}^{\infty} [n, q] \gamma_n \xi^n \right) - (1 - \lambda) \\ &\quad \cdot \left(-\xi + \sum_{n=2}^{\infty} [n, q] (-1)^n \gamma_n \xi^n \right) \\ &= \xi + \sum_{n=2}^{\infty} [n, q] [(\lambda - (1 - \lambda)(-1)^n)] \gamma_n \xi^n, \end{aligned} \quad (11)$$

$$\begin{aligned} \mathcal{S}_q^{\lambda,2} \gamma(\xi) &= \mathcal{S}_q^{\lambda,1} \left[\mathcal{S}_q^{\lambda,1} \gamma(\xi) \right] \\ &= \xi + \sum_{n=2}^{\infty} [n, q]^2 [(\lambda - (1 - \lambda)(-1)^n)]^2 \gamma_n \xi^n, \\ &\vdots \\ \mathcal{S}_q^{\lambda,k} \gamma(\xi) &= \mathcal{S}_q^{\lambda,1} \left[\mathcal{S}_q^{\lambda,k-1} \gamma(\xi) \right] \\ &= \xi + \sum_{n=2}^{\infty} [n, q]^k [(\lambda - (1 - \lambda)(-1)^n)]^k \gamma_n \xi^n. \end{aligned}$$

Obviously, we indicate that when $\lambda = 1 (\mathcal{S}_q^{1,k} = S_q^k)$, we get the Salagean q -differential operator. We can call (11) as the symmetric Salàgean q -differential operator in \cup . Also, we have the following two limits.

$$\lim_{q \rightarrow 1^-} \mathcal{S}_q^{1,k} \gamma(\xi) = \xi + \sum_{n=2}^{\infty} n^k \gamma_n \xi^n, \quad (12)$$

which are represented to the well-known Salagean differential operator [6] and

$$\lim_{q \rightarrow 1^-} \mathcal{S}_q^{\lambda,k} \gamma(\xi) = \xi + \sum_{n=2}^{\infty} n^k [(\lambda - (1 - \lambda)(-1)^n)]^k \gamma_n \xi^n, \quad (13)$$

the symmetric differential operator [8], respectively.

Depending on the definition of (11), we impose the recognizing classes:

Definition 1. A member $\gamma \in \wedge$ is called in the category S_q^* (λ, k, \hbar) when

$$S_q^*(\lambda, k, \hbar) = \left\{ \gamma \in \wedge \frac{\xi \left(\mathcal{S}_q^{\lambda, k} \gamma(\xi) \right)''}{\mathcal{S}_q^{\lambda, k} \gamma(\xi)} < \hbar(\xi), \hbar \in \mathcal{C} \right\}. \quad (14)$$

We obtain the following special cases:

- (i) $\lim_{q \rightarrow 1^-} S_q^*(\lambda, k, \hbar) = S_1^*(\lambda, k, \hbar)$ ([8])
- (ii) $S_q^*(\lambda, 0, (1 + A\xi/1 + B\xi))$ ([13–15])
- (iii) $S_q^*(\lambda, 0, (2/1 + e^{-\xi}))$ ([16])
- (iv) $S_q^*(\lambda, 0, (1 + \varepsilon^2 \xi^2/1 - \varepsilon^2 - \varepsilon^2 \xi^2))$, $\varepsilon = 1 - \sqrt{5}/2$ ([17, 18])
- (v) $S_q^*(\lambda, 0, 1 + (\beta - \alpha/\pi)i \log(1 - \exp(2\pi i((1 - \alpha)/(\beta - \alpha)))\xi/1 - \xi))$ ([19])
- (vi) $S_q^*(\lambda, 0, 1 + (2/\pi(1 - \alpha))i \log(1 - \exp(\pi i(1 - \alpha)^2 \xi/1 - \xi))$ ([20])
- (vii) $S_q^*(\lambda, 0, \sqrt{1 + \xi})$ ([21])
- (viii) $S_q^*(\lambda, 0, \sin(\xi) + 1)$ ([22])
- (ix) $S_q^*(\lambda, 0, \cos(\xi) + 1)$ ([23])
- (x) $S_q^*(\lambda, 0, (1 + \xi/1 + ((1 - \nu)/\nu)\xi)^{1/\nu})$, $\nu \geq 0.5, \nu \geq 1$ ([24])

Definition 2. If $\gamma \in \wedge$, then $\gamma \in \mathbb{J}_q^{\lambda, b}(A, B, k)$ if and only if

$$1 + \frac{1}{b} \left(\frac{2\mathcal{S}_q^{\lambda, k+1} \gamma(\xi)}{\mathcal{S}_q^{\lambda, k} \mathbb{E}(\xi) - \mathcal{S}_q^{\lambda, k} \gamma(-\xi)} \right) < \frac{1 + A\xi}{1 + B\xi} \quad (\xi \in \mathbb{U}, -1 \leq B < A \leq 1, k \in \mathbb{Z}_+, b \in \mathbb{C} \setminus \{0\}, \lambda \in [0, 1]). \quad (15)$$

- (i) $\lambda = 1, q \rightarrow 1^- \Rightarrow$ [25]
- (ii) $\lambda = 1, B = 0, q \rightarrow 1^- \Rightarrow$ [26]
- (iii) $\lambda = 1, A = 1, B = -1, \# = 2, q \rightarrow 1^- \Rightarrow$ [27]
- (iv) $q \rightarrow 1^- \Rightarrow$ [8]

In our investigation, we focus on the geometric presentation of the special classes $S_q^*(\lambda, k, \hbar)$ and $\mathbb{J}_q^{\lambda, b}(A, B, k)$ via utilizing the basis information given in [11].

Lemma 3. Let $a \in \mathbb{C}$, let n be a positive integer and let

$$\mathfrak{H}[a, n] = \left\{ \gamma : \gamma(\xi) = a + \xi + \gamma_n \xi^n + \gamma_{n+1} \xi^{n+1} + \dots \right\}. \quad (16)$$

(i) Suppose that $\wp \in \mathbb{R}$ such that $\Re(\gamma(\xi) + \wp \xi \gamma'(\xi)) > 0 \Rightarrow \Re(\gamma(\xi)) > 0$. In addition, let $\wp > 0$ and $\gamma \in \mathfrak{H}[1, n]$; thereafter, there are two fixed positive numbers $\ell > 0$ and $b > 0$ with $b = b(\wp, \ell, n)$ so that

$$\gamma(\xi) + \wp \xi \gamma'(\xi) < \left[\frac{1 + \xi}{1 - \xi} \right]^b \Rightarrow \gamma(\xi) < \left[\frac{1 + \xi}{1 - \xi} \right]^\ell. \quad (17)$$

(ii) Assume that $\nu \in [0, 1)$ and $\nu \in \mathfrak{H}[1, n]$, subsequently, there occurs a fixed number $\ell > 0$ with ℓ such that

$$\Re(\gamma^2(\xi) + 2\gamma(\xi) \cdot \xi \gamma'(\xi)) > \nu \Rightarrow \Re(\gamma(\xi)) > \ell. \quad (18)$$

(iii) Consider $\gamma \in \mathfrak{H}[a, n]$ with $\Re(a) > 0$ posterity

$$\Re(\gamma(\xi) + \xi \gamma'(\xi) + \xi^2 \gamma''(\xi)) > 0, \quad (19)$$

or for $\theta : \mathbb{U} \rightarrow \mathbb{R}$ such that

$$\Re\left(\gamma(\xi) + \theta(\xi) \frac{\xi \gamma'(\xi)}{\gamma(\xi)}\right) > 0, \quad (20)$$

thus $\Re(\gamma(\xi)) > 0$.

Remark 4. Concerning Lemma 3 (i), more information about ℓ and b can be found in [11] (Theorem 3.1c, p. 73). About (ii), more information about ν and ℓ can be found in [11] (Theorem 3.1e or Corollary 3.1e.2, p. 77 and p. 79, respectively). And for (iii), one can see Theorem 4.1g in [11] (p. 198) and, for $\theta(\xi) = 1$, the inequality contains a Briot-Bouquet formula.

3. Main Results

Here, we focus in the geometric representations of the classes $S_q^*(\lambda, k, \hbar)$ and $\mathbb{J}_q^{\lambda, b}(A, B, k)$ and the outcomes of these classes.

Theorem 5. For $\gamma \in \wedge$ if one of the recognizing determinations are indicated by

- (i) $\mathcal{S}_q^{\lambda, k} \gamma(\xi)$ is of bounded turning

(ii) γ satisfies the subordination formula

$$\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' < \left(\frac{1+\xi}{1-\xi}\right)^b, \quad b > 0, \xi \in \mathcal{U}; \quad (21)$$

(iii) γ achieves the formula

$$\Re\left(\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' \frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > \frac{\varsigma}{2}, \quad \varsigma \in [0, 1), \xi \in \mathcal{U}, \quad (22)$$

(iv) ψ satisfies the relation

$$\Re\left(\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'' - \left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' + 2\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > 0, \quad (23)$$

(v) γ admits the inequality

$$\Re\left(\frac{z\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)} + 2\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > 1, \quad (24)$$

then $(\mathcal{S}_q^{\lambda,k}\gamma(\xi)/\xi) \in \mathcal{P}(\sigma)$ where $0 \leq \sigma < 1$.

Proof. Sort out a function ρ in the following construction:

$$\rho(\xi) = \frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi} \Rightarrow \xi\gamma'(\xi) + \rho(\xi) = \left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'. \quad (25)$$

Via the main information, $\mathcal{S}_q^{\lambda,k}\gamma(\xi)$ is of constrained limit turning; it infers that $\Re(\xi\rho'(\xi) + \rho(\xi)) > 0$. Accordingly, via Lemma 3 (i), we acquire $\Re(\rho(\xi)) > 0$ which incomes the first requested statement of the theorem. In opinion of the additional information, we obligate the subsequent subordination relative

$$\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' = \xi\rho'(\xi) + \rho(\xi) < \left[\frac{1+\xi}{1-\xi}\right]^b. \quad (26)$$

Now, based on Lemma 3 (i), there exists a fixed positive number $\ell > 0$ satisfying $b = b(\ell)$ and the subordination inequality

$$\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi} < \left(\frac{1+\xi}{1-\xi}\right)^\ell. \quad (27)$$

This leads to the conclusion

$$\Re\left(\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > \sigma, \quad \sigma \in [0, 1). \quad (28)$$

Lastly, we assume the third fact, a direct reckoning reaches to

$$\Re\left(\rho^2(\xi) + 2\rho(\xi) \cdot \xi\rho'(\xi)\right) = 2\Re\left(\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' \frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) \geq \varsigma. \quad (29)$$

According to the virtue of Lemma 3 (ii), there occurs a fixed positive number, say $\ell > 0$ achieving $\Re(\rho(\xi)) > \ell$. Consequently, we obtain

$$\rho(\xi) = \frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi} \in \mathcal{P}(\sigma), \quad \sigma \in [0, 1). \quad (30)$$

Hence, via Equation (29), it indicates that

$$\Re\left(\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'\right) > 0. \quad (31)$$

So by Noshiro-Warschawski and Kaplan Propositions, $\mathcal{S}_q^{\lambda,k}\gamma(\xi) \in \mathfrak{S}$ and of bounded turning in \mathcal{U} .

Via the derivative (25) and operating the real, one can attain the real relation

$$\Re\left(\rho(\xi) + \xi\rho'(\xi) + \xi^2\rho''(\xi)\right) = \Re\left(\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)'' - \left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)' + 2\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi}\right) > 0. \quad (32)$$

Hence, according to Lemma 3 (ii), one get $\Re(\mathcal{S}_q^{\lambda,k}\gamma(\xi)/\xi) > 0$.

Via considering the logarithmic derivative on (25) and acting as areal part, we obtain the consequence conversation:

$$\begin{aligned} & \Re\left(\rho(\xi) + \frac{\xi\rho'(\xi)}{\rho(\xi)} + \xi^2\rho''(\xi)\right) \\ &= \Re\left(\frac{\xi\left(\mathcal{S}_q^{\lambda,k}\gamma(\xi)\right)''}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)} + 2\frac{\mathcal{S}_q^{\lambda,k}\gamma(\xi)}{\xi} - 1\right) > 0. \end{aligned} \quad (33)$$

Thus, according to Lemma 3 (iii), where $\theta(\xi) = 1$, we attain that $\Re(\mathcal{S}_q^{\lambda,k}\gamma(\xi)/\xi) > 0$.

Theorem 6. Suppose that $\gamma \in \mathcal{S}_q^*(\lambda, k, \hbar)$, with $\hbar(\xi) \in \mathcal{C}$. Then

$$\mathcal{S}_q^{\lambda,k} < \xi e^{\int_0^\xi \hbar(\partial(u)) - 1/u du}, \quad (34)$$

where $\delta \in \mathcal{U}$ (analytic) satisfies that $\delta(0) = 0$ and $|\delta(\xi)| < 1$. In addition, for $|\xi| = \chi$, $\mathcal{S}_q^{\lambda,k}$ reckoning fulfills the formula statement

$$\exp \left(\int_0^1 \frac{\hbar(\delta(-\chi)) - 1}{\chi} d\chi \right) \leq \left| \frac{\mathcal{S}_q^{\lambda,k}}{\xi} \right| \leq \exp \left(\int_0^1 \frac{\hbar(\delta(\chi)) - 1}{\chi} d\chi \right). \tag{35}$$

Proof. We note that $\gamma \in \mathcal{S}_q^*(\lambda, k, \hbar)$, then one can gain

$$\left(\frac{z \left(\mathcal{S}_q^{\lambda,k} \gamma(\xi) \right)'}{\mathcal{S}_q^{\lambda,k} \gamma(\xi)} \right) < \hbar(\xi), \xi \in \mathcal{U}, \tag{36}$$

this confirms that there occurs analytic function type Schwarz achieving the relations $\delta(0) = 0, |\delta(\xi)| < 1$ and

$$\left(\frac{\xi \left(\mathcal{S}_q^{\lambda,k} \gamma(\xi) \right)'}{\mathcal{S}_q^{\lambda,k} \gamma(\xi)} \right) = \hbar(\delta(\xi)), \xi \in \mathcal{U}. \tag{37}$$

A calculation gives us

$$\left(\frac{\left(\mathcal{S}_q^{\lambda,k} \gamma(\xi) \right)'}{\mathcal{S}_q^{\lambda,k} \gamma(\xi)} \right) - \frac{1}{\xi} = \frac{\hbar(\delta(\xi)) - 1}{\xi}. \tag{38}$$

Via integrating left and right parts, one can achieve

$$\log \mathcal{S}_q^{\lambda,k} \gamma(\xi) - \log \xi = \int_0^\xi \frac{\hbar(\delta(u)) - 1}{u} du. \tag{39}$$

Thus, we have

$$\log \frac{\mathcal{S}_q^{\lambda,k} \gamma(\xi)}{\xi} = \int_0^\xi \frac{\hbar(\delta(u)) - 1}{u} du. \tag{40}$$

Via the definition and the properties of subordination, one can have

$$\mathcal{S}_q^{\lambda,k} \gamma(\xi) < \xi \exp \left(\int_0^\xi \frac{\hbar(\delta(u)) - 1}{u} du \right). \tag{41}$$

Furthermore, we deliver that $\hbar(z)$ maps the disk $0 < |z| < \chi < 1$ onto a convex symmetric domain corresponding to the real axis, that is

$$\hbar(-\chi | \xi |) \leq \Re(\hbar(\delta(\chi \xi))) \leq \hbar(\chi | \xi |), \chi \in (0, 1), \tag{42}$$

which implies that

$$\begin{aligned} \hbar(-\chi) &\leq \hbar(-\chi | \xi |), \quad \hbar(\chi | \xi |) \leq \hbar(\chi), \\ \int_0^1 \frac{\hbar(\delta(-\chi | \xi |)) - 1}{\chi} d\chi &\leq \Re \left(\int_0^1 \frac{\hbar(\delta(\chi)) - 1}{\chi} d\chi \right) \\ &\leq \int_0^1 \frac{\hbar(\delta(\chi | \xi |)) - 1}{\chi} d\chi. \end{aligned} \tag{43}$$

Via applying Equation (40), one can indicate that

$$\begin{aligned} \int_0^1 \frac{h(\delta(-\chi | \xi |)) - 1}{\chi} d\chi &\leq \log \left| \frac{\mathcal{S}_q^{\lambda,k} \gamma(\xi)}{\xi} \right| \\ &\leq \int_0^1 \frac{\hbar(\delta(\chi | \xi |)) - 1}{\chi} d\chi, \end{aligned} \tag{44}$$

which leads to

$$\begin{aligned} \exp \left(\int_0^1 \frac{h(\delta(-\chi | \xi |)) - 1}{\chi} d\chi \right) &\leq \left| \frac{\mathcal{S}_q^{\lambda,k} \gamma(\xi)}{\xi} \right| \\ &\leq \exp \left(\int_0^1 \frac{\hbar(\delta(\chi | \xi |)) - 1}{\chi} d\chi \right). \end{aligned} \tag{45}$$

Hence, we have

$$\begin{aligned} \exp \left(\int_0^1 \frac{h(\delta(-\chi)) - 1}{\eta} d\chi \right) d\chi &\leq \left| \frac{\mathcal{S}_q^{\lambda,k} \gamma(\xi)}{\xi} \right| \\ &\leq \exp \left(\int_0^1 \frac{\hbar(\delta(\chi)) - 1}{\chi} d\chi \right). \end{aligned} \tag{46}$$

Corollary 7 [8]. If $q \rightarrow 1$ in Theorem 6, then

$$\mathcal{S}_1^{\lambda,k} \gamma(\xi) < \xi \exp \left(\int_0^\xi \frac{\hbar(\delta(u)) - 1}{u} du \right). \tag{47}$$

Theorem 8. Suppose that $\gamma \in \mathbb{J}_q^{\lambda,b}(A, B, k)$, then the odd construction formula

$$\mathfrak{M}(\xi) = \frac{1}{2} [\gamma(\xi) - \gamma(-\xi)], \quad \xi \in \mathcal{U}, \tag{48}$$

fulfills the consequently subordination

$$1 + \frac{1}{b} \left(\frac{\mathcal{S}_q^{\lambda,k+1} \mathfrak{M}(\xi)}{\mathcal{S}_q^{\lambda,k} \mathfrak{M}(\xi)} - 1 \right) < \frac{1 + A\xi}{1 + B\xi},$$

$$\Re \left(\frac{\xi \mathfrak{M}(\xi)'}{\mathfrak{M}(\xi)} \right) \geq \frac{1 - \varrho^2}{1 + \varrho^2}, \quad |\xi| = \varrho < 1,$$

$$(-1 \leq B < A \leq 1, \xi \in \mathcal{U}, k = 1, 2, \dots, \lambda \in [0, 1], b \in \mathbb{C} \setminus \{0\}). \tag{49}$$

Proof. Let $\gamma \in \mathbb{J}_q^{\lambda,b}(A, B, k)$. Subsequently, we get that there occurs a function $\mathfrak{Z} \in \mathbb{J}(A, B)$ with the formula

$$\begin{aligned} (\mathfrak{Z}(\xi) - 1).b &= \left(\frac{2\mathcal{S}_q^{\lambda,k+1}\gamma(\xi)}{\mathcal{S}_q^{\lambda,k}\gamma(\xi) - \mathcal{S}_q^{\lambda,k}\gamma(-\xi)} \right), \\ (\mathfrak{Z}(-\xi) - 1).b &= \left(\frac{-2\mathcal{S}_q^{\lambda,k+1}\gamma(-\xi)}{\mathcal{S}_q^{\lambda,k}\gamma(\xi) - \mathcal{S}_q^{\lambda,k}\gamma(-\xi)} \right). \end{aligned} \tag{50}$$

This yields

$$1 + \frac{1}{b} \left(\frac{\mathcal{S}_q^{\lambda,k+1}\mathfrak{M}(\xi)}{\mathcal{S}_q^{\lambda,k}\mathfrak{M}(\xi)} - 1 \right) = \frac{\mathfrak{Z}(\xi) + \mathfrak{Z}(-\xi)}{2}. \tag{51}$$

In addition, since \mathfrak{Z} achieves the inequality

$$\mathfrak{Z}(\xi) < \frac{1 + A\xi}{1 + B\xi}, \tag{52}$$

taking account that the fractional functional express $(1 + A\xi)/(1 + B\xi)$ is univalent and hence, consequently, we attain the relation

$$1 + \frac{1}{b} \left(\frac{\mathcal{S}_q^{\lambda,k+1}\mathfrak{M}(\xi)}{\mathcal{S}_q^{\lambda,k}\mathfrak{M}(\xi)} - 1 \right) < \frac{1 + A\xi}{1 + B\xi}. \tag{53}$$

Furthermore, the expression $\mathfrak{M}(\xi) \in \mathcal{S}^*$, which leads to the inequality

$$\frac{\xi\mathfrak{M}(\xi)'}{\mathfrak{M}(\xi)} < \frac{1 - \xi^2}{1 + \xi^2}, \tag{54}$$

that is, there occurs a Schwarz function $\kappa \in \mathcal{U}$, $|\kappa(\xi)| \leq |\xi| < 1$, $\kappa(0) = 0$ with the property

$$\Psi(\xi) := \frac{z\mathfrak{M}(\xi)'}{\mathfrak{M}(\xi)} < \frac{1 - \kappa(\xi)^2}{1 + \kappa(\xi)^2}, \tag{55}$$

which implies that there is $\zeta, |\zeta| = r < 1$ such that

$$\kappa^2(\zeta) = \frac{1 - \Psi(\zeta)}{1 + \Psi(\zeta)}, \quad \zeta \in \mathcal{U}. \tag{56}$$

An operation, one can indicate that

$$\left| \frac{1 - \Psi(\zeta)}{1 + \Psi(\zeta)} \right| = |\kappa(\zeta)|^2 \leq |\zeta|^2 < 1. \tag{57}$$

Thus, one can inform the recognizing inequality

$$|\Psi(z) - \frac{|\zeta|^4 + 1}{1 - |\zeta|^4}| \leq \frac{2|\zeta|^2}{(1 - |\zeta|^4)}. \tag{58}$$

This implies the consequence result

$$\Re(\Psi(\xi)) \geq \frac{1 - \mathcal{Q}^2}{1 + \mathcal{Q}^2}. \tag{59}$$

Next consequence outcomes of the above result can be located in [8, 25, 26] accordingly.

Corollary 9. *If $\lambda = 1$ in Theorem 8, then*

$$1 + \frac{1}{b} \left(\frac{\mathcal{S}_q^{1,k+1}\mathfrak{M}(\xi)}{\mathcal{S}_q^{1,k}\mathfrak{M}(\xi)} - 1 \right) < \frac{1 + A\xi}{1 + B\xi}. \tag{60}$$

Corollary 10. *If $\lambda = 1, k = 1$, and $q \rightarrow 1$ in Theorem 8, then*

$$1 + \frac{1}{b} \left(\frac{\mathcal{S}_q^{1,2}\mathfrak{M}(\xi)}{\mathcal{S}_q^{1,1}\mathfrak{M}(\xi)} - 1 \right) < \frac{1 + A\xi}{1 + B\xi}. \tag{61}$$

Corollary 11. *If $q \rightarrow 1$ in Theorem 8, then*

$$1 + \frac{1}{b} \left(\frac{\mathcal{S}_q^{\lambda,k+1}\mathfrak{M}(\xi)}{\mathcal{S}_q^{\lambda,k}\mathfrak{M}(\xi)} - 1 \right) < \frac{1 + A\xi}{1 + B\xi}. \tag{62}$$

4. Applications

We introduce an application of our outcomes based on finding the outcome of Briot-Bouquet equation (BBE) (see [11] for more information). This category of ODE is an association of ODE whose outcomes are formulas in the complex plane. Existence and uniqueness theorems include the utility of majors and minors (or subordination and superordination concepts) (see [28–31]). Investigation of rational first ODEs in the complex region implies the finding of new transcendental special functions which are now known as symmetric BBE

$$\beta\gamma(\xi) + (1 - \beta) \frac{\xi(\gamma(\xi))'}{\gamma(\xi)} = \hbar(\xi), \quad \hbar(0) = \gamma(0), \beta \in [0, 1]. \tag{63}$$

By employing the q -differential operator (11), we have the q -formula of BBE

$$\beta\gamma(\xi) + (1 - \beta) \left(\frac{\xi(\mathcal{S}_q^{\lambda,k}\gamma(\xi))'}{\mathcal{S}_q^{\lambda,k}\gamma(\xi)} \right) = \hbar(\xi), \quad \hbar(0) = \gamma(0), \xi \in \mathcal{U}. \tag{64}$$

A simple result of (64) can be recognized at $\beta = 1$. Thus, we investigate the status, $\gamma \in \wedge$ and $\beta = 0$. The initial condition will be $\gamma(0) = \hbar(0) = 0$.

Theorem 12. Suppose that Equation (64) with $\beta = 0$ and $\gamma \in \Lambda$ with nonnegative coefficients. If $\tilde{h}(\xi)$, $\xi \in \cup$ is convex in \cup , then there occurs a major solution achieving the inequality

$$\mathcal{S}_q^{\lambda,k} \gamma(\xi) < \xi \exp \left(\int_0^\xi \frac{h(\tilde{\delta}(u)) - 1}{u} du \right), \quad (65)$$

such that $\tilde{\delta}(\xi)$ indicates analytic function in \cup , satisfying the relations $|\tilde{\delta}(\xi)| < 1$ and $\tilde{\delta}(0) = 0$.

Proof. In virtue of every conditions indicated by (64), and $\gamma(\xi) \in \Lambda$, we realize the information

$$\begin{aligned} & \Re \left(\frac{\xi \left(\mathcal{S}_q^{\lambda,k} \gamma(\xi) \right)'}{\mathcal{S}_q^{\lambda,k} \gamma(\xi)} \right) > 0 \\ \Leftrightarrow & \Re \left(\frac{\xi + \sum_{n=2}^{\infty} n[n, q]^k [(\lambda - (1 - \lambda)(-1)^n)]^k \gamma_n \xi^n}{\xi + \sum_{n=2}^{\infty} [n, q]^k [(\lambda - (1 - \lambda)(-1)^n)]^k \gamma_n \xi^n} \right) > 0 \\ \Leftrightarrow & \Re \left(\frac{1 + \sum_{n=2}^{\infty} n[n, q]^k [(\lambda - (1 - \lambda)(-1)^n)]^k \gamma_n \xi^{n-1}}{1 + \sum_{n=2}^{\infty} [n, q]^k [(\lambda - (1 - \lambda)(-1)^n)]^k \gamma_n \xi^{n-1}} \right) > 0 \\ \Leftrightarrow & \left(\frac{1 + \sum_{n=2}^{\infty} n[n, q]^k [(\lambda - (1 - \lambda)(-1)^n)]^k \gamma_n}{1 + \sum_{n=2}^{\infty} [n, q]^k [(\lambda - (1 - \lambda)(-1)^n)]^k \gamma_n} \right) > 0, \end{aligned} \quad (66)$$

where $\Re(\xi) \neq 0$. Moreover, by the definition of $\mathcal{S}_q^{\lambda,k} \gamma(\xi)$, we indicate that $(\mathcal{S}_q^{\lambda,k} \gamma)(0) = 0$. Consequently, yields that

$$\frac{\xi \left(\mathcal{S}_q^{\lambda,k} \gamma(\xi) \right)'}{\mathcal{S}_q^{\lambda,k} \gamma(\xi)} \in \mathcal{D} \Rightarrow \gamma(\xi) \in S_q^*(\lambda, k, \tilde{h}). \quad (67)$$

Thus, via Theorem 6, we reach the desired outcome in (65).

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] R. D. Carmichael, "The general theory of linear q -difference equations," *American Journal of Mathematics*, vol. 34, no. 2, pp. 147–168, 1912.
- [2] D. O. Jackson, T. Fukuda, O. Dunn, and English Majors and Core Skill-based, "On q -definite integrals," *Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.
- [3] T. E. Mason, "On properties of the Solutions of linear q -difference equations with entire function coefficients," *American Journal of Mathematics*, vol. 37, no. 4, pp. 439–444, 1915.
- [4] W. J. Trjitzinsky, "Analytic theory of linear q -difference equations," *Acta Mathematica*, vol. 61, pp. 1–38, 1933.
- [5] M. E. H. Ismail, E. Merkes, and D. Styer, "A generalization of starlike functions," *Complex Variables, Theory and Application: An International Journal*, vol. 14, pp. 77–84, 2007.
- [6] G. S. Salagean, *Subclasses of univalent functions, Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981)*, vol. 1013 of Lecture Notes in Mathematics, Springer, Berlin, 1983.
- [7] F. M. Al-Oboudi, "On univalent functions defined by a generalized Sălăgean operator," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 27, Article ID 172525, pp. 1429–1436, 2004.
- [8] R. W. Ibrahim and M. Darus, "New symmetric differential and integral operators defined in the complex domain," *Symmetry*, vol. 11, no. 7, p. 906, 2019.
- [9] M. Naeem, S. Hussain, T. Mahmood, S. Khan, and M. Darus, "A new subclass of analytic functions defined by using Salagean q -differential operator," *Mathematics*, vol. 7, no. 5, p. 458, 2019.
- [10] P. Duren, *Univalent Functions*, vol. 259 of Grundlehren der mathematischen Wissenschaften, Springer-Verlag New York Inc, 1983.
- [11] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, CRC Press, 2000.
- [12] M. Govindaraj and S. Sivasubramanian, "On a class of analytic functions related to conic domains involving q -calculus," *Analysis Mathematica*, vol. 43, no. 3, pp. 475–487, 2017.
- [13] Z. J. Jakubowski and J. Kaminski, "On some properties of Mocanu-Janowski functions," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 23, pp. 1523–1532, 1978.
- [14] M. Obradović and S. Owa, "On certain properties for some classes of starlike functions," *Journal of Mathematical Analysis and Applications*, vol. 145, no. 2, pp. 357–364, 1990.
- [15] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, pp. 19–23, Tianjin, China, 1992.
- [16] P. Goel and S. S. Kumar, "Certain class of starlike functions associated with modified sigmoid function," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 1, pp. 957–991, 2020.
- [17] J. Dziok, R. K. Raina, and J. Sokół, "Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers," *Computers & Mathematics with Applications*, vol. 61, no. 9, pp. 2605–2613, 2011.
- [18] J. Dziok, R. K. Raina, and J. Sokół, "On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers," *Mathematical and Computer Modelling*, vol. 57, no. 5–6, pp. 1203–1211, 2013.
- [19] R. Kargar, A. Ebadian, and J. Sokol, "On subordination of some analytic functions," *Siberian Mathematical Journal*, vol. 57, no. 4, pp. 599–605, 2016.
- [20] R. Kargar, A. Ebadian, and J. Sokół, "On booth lemniscate and starlike functions," *Analysis and Mathematical Physics*, vol. 9, no. 1, pp. 143–154, 2019.
- [21] J. Sokół, "On some subclass of strongly starlike functions," *Demonstratio Mathematica*, vol. 31, no. 1, pp. 81–86, 1998.
- [22] N. E. Cho, V. Kumar, S. S. Kumar, and V. Ravichandran, "Radius problems for starlike functions associated with the sine function," *Bulletin of the Iranian Mathematical Society*, vol. 45, no. 1, pp. 213–232, 2019.
- [23] H. Tang, H. M. Srivastava, S.-H. Li, and G.-T. Deng, "Majorization results for subclasses of starlike functions based on the

- sine and cosine functions,” *Bulletin of the Iranian Mathematical Society*, vol. 46, no. 2, pp. 381–388, 2020.
- [24] S. Sivasubramanian, M. Govindaraj, G. Murugusundaramoorthy, and N. E. Cho, “Differential subordination for analytic functions associated with left-like domains,” *Journal of Computational Analysis & Applications*, vol. 26, no. 1, 2019.
- [25] M. Arif, K. Ahmad, J. L. Liu, and J. Sokół, “A new class of analytic functions associated with Sălăgean operator,” *Journal of Function Spaces*, vol. 2019, Article ID 6157394, 8 pages, 2019.
- [26] K. Sakaguchi, “On a certain univalent mapping,” *Journal of the Mathematical Society of Japan*, vol. 11, no. 1, pp. 72–75, 1959.
- [27] R. N. Das and P. Singh, “On subclasses of schlicht mapping,” *Indian Journal of Pure and Applied Mathematics*, vol. 8, pp. 864–872, 1977.
- [28] R. W. Ibrahim and M. Darus, “Subordination and superordination for univalent solutions for fractional differential equations,” *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 871–879, 2008.
- [29] R. W. Ibrahim, “On holomorphic solutions for nonlinear singular fractional differential equations,” *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1084–1090, 2011.
- [30] R. W. Ibrahim, “Existence and uniqueness of holomorphic solutions for fractional Cauchy problem,” *Journal of Mathematical Analysis and Applications*, vol. 380, no. 1, pp. 232–240, 2011.
- [31] R. W. Ibrahim, “Fractional complex transforms for fractional differential equations,” *Advances in Difference Equations*, vol. 2012, no. 1, 2012.