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Polygon Offsetting with Squares Erected on Its Sides

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This column is a place for those bits of contagious mathematics that travel from person to person in the community, because they are so elegant, surprising, or appealing that one has an urge to pass them on.

Contributions are most welcome.

Elementary geometry is a fascinating research topic. In spite of the existence of countless beautiful and unexpected results, there is still room for inventing or discovering something new. In particular, the surprising result of Napoleon's theorem [2, 8] inspired fruitful studies on harmonic analysis of polygons and iterative constructions on polygons, as seen, for example, in [1, 4, 6].

Napoleon's theorem states that if one erects equilateral triangles outward and inward on the sides of a given triangle, the centers of the outward equilateral triangles form an equilateral triangle, and the centers of the inward equilateral triangles also form an equilateral triangle. These triangles formed by the inward and outward centers are often called the inner and outer Napoleon triangles.

Sometimes the formulation of Napoleon's theorem includes the following relationships between the area of the original triangle and its outer and inner Napoleon triangles: the signed area of the original triangle is equal to the sum of the signed areas of the outer and inner Napoleon triangles [2, 5].

A main contribution of our paper consists in extending the above area result to an arbitrary planar polygon when squares are erected on the polygon's sides. Working with squares is easier than with equilateral triangles: given a polygon drawn on graph paper such that its vertices are situated at grid points, it is easy to construct outer and inner squares on the polygon's sides and formulate and test various hypotheses. Erecting squares establishes a link with van Aubel's theorem, another elementary geometry gem, which describes a relationship between the centers of the squares erected on the sides of a quadrilateral [7].

Given a planar polygon $\mathcal{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n]$, consider the outer and inner squares erected on the polygon's sides and denote the squares' centers by $\boldsymbol{b}_k$ and $\boldsymbol{c}_k$, respectively, for $k = 1, \ldots, n$. Figure 1 (left) illustrates the construction, and Figure 1 (right) gives an example of a hexagon and two of its corresponding hexagons formed by the outer and inner squares' centers. Denote by $F(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ its signed area and by $S(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ the sum of its squared sides $\sum_k |\boldsymbol{a}_{k+1} - \boldsymbol{a}_k|^2$.

Playing with simple polygons drawn on graph paper leads to interesting observations. Let us begin with two very simple examples. For the triangle $[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$ shown in the left-hand image of Figure 2, we have

$$2F(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = 4 = 6 - 2 = F(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3) + F(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$$

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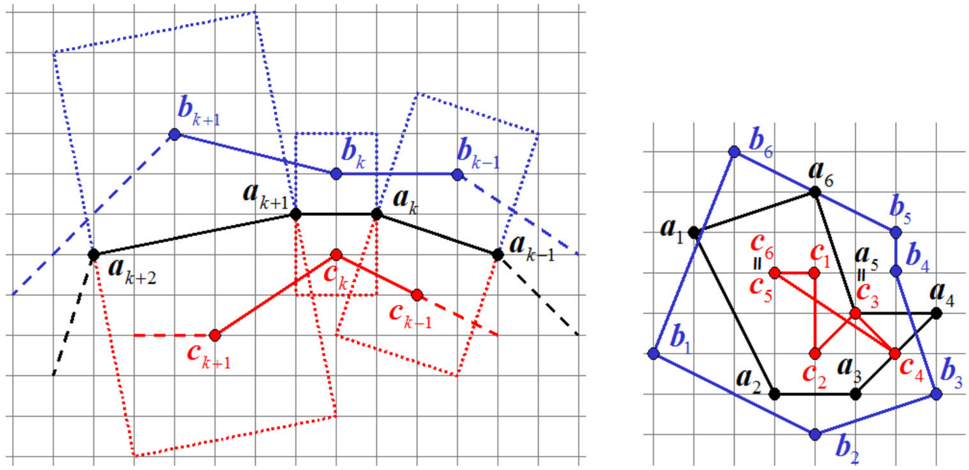

Figure 1. Left: the centers of the the outer and inner squares erected on the sides of a polygon form two new polygons. Right: a hexagon and two of its corresponding hexagons formed by the centers of the outer and inner squares.

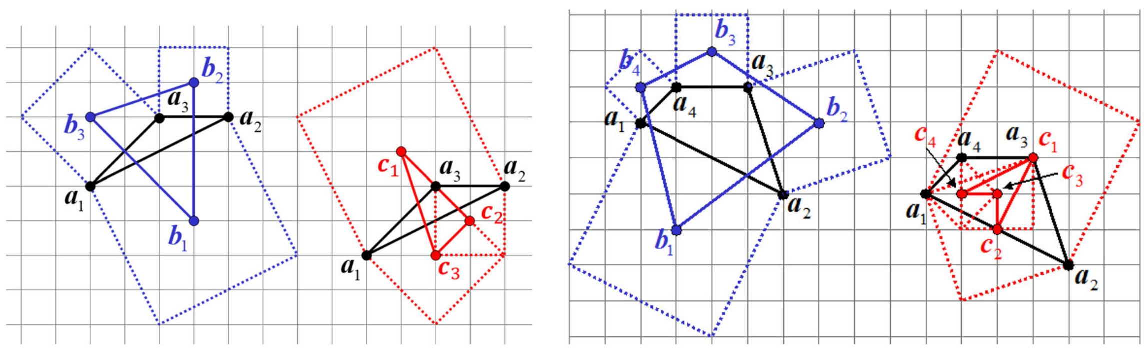

Figure 2. A triangle and quadrilateral drawn on graph paper are used to demonstrate our results for area, sum of squared sides, and polar moment of inertia.

and

$$\begin{aligned}2S(\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3) &= 2(20+4+8)\\ &= (16+10+18)+(2+8+10)\\ &= S(\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3)+S(\boldsymbol{c}_1,\boldsymbol{c}_2,\boldsymbol{c}_3)\,.\end{aligned}$$

Similarly, for the quadrilateral $[\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3,\boldsymbol{a}_4]$ shown in the right-hand image of Figure 2, we have

$$\begin{aligned}2F(\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3,\boldsymbol{a}_4) = 12 = 13-1 = F(\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4)\\ +F(\boldsymbol{c}_1,\boldsymbol{c}_2,\boldsymbol{c}_3,\boldsymbol{c}_4)\end{aligned}$$

and

$$\begin{aligned}2S(\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3,\boldsymbol{a}_4) &= 2(20+10+4+2)\\ &= (25+13+5+17)+(5+1+1+5)\\ &= S(\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3,\boldsymbol{b}_4)+S(\boldsymbol{c}_1,\boldsymbol{c}_2,\boldsymbol{c}_3,\boldsymbol{c}_4).\end{aligned}$$

Similar results can be obtained if instead of the signed area or sum of its squared sides, we consider the polar moment of inertia of the vertices of a polygon $I(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n)=\sum|\boldsymbol{o}-\boldsymbol{a}_k|^2$, where $\boldsymbol{o}$ is a fixed point.

It turns out that more general results hold.

Theorem 1. *For a general planar polygon $\mathcal{A}=[\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n]$, the following formulas are satisfied:*

$$\begin{aligned}2F(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n) &= F(\boldsymbol{b}_1,\boldsymbol{b}_2,\dots,\boldsymbol{b}_n)+F(\boldsymbol{c}_1,\boldsymbol{c}_2,\dots,\boldsymbol{c}_n)\,,\\ 2S(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n) &= S(\boldsymbol{b}_1,\boldsymbol{b}_2,\dots,\boldsymbol{b}_n)+S(\boldsymbol{c}_1,\boldsymbol{c}_2,\dots,\boldsymbol{c}_n)\,,\\ 2I(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n) &= I(\boldsymbol{b}_1,\boldsymbol{b}_2,\dots,\boldsymbol{b}_n)+I(\boldsymbol{c}_1,\boldsymbol{c}_2,\dots,\boldsymbol{c}_n)\,.\end{aligned}$$

While the second identity can be easily verified using the law of cosines, the first and third seem less trivial.

Proof. In our proof, we use the approach developed in [3]. It is convenient to think about planar points as complex numbers and represent polygons with n vertices as vectors in $\mathbb{C}^n$. Given a polygon $\mathcal{A}=[\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n]$, denote by $x\mathcal{A}$ the polygon obtained from $\mathcal{A}$ by a one-place cyclic shift of the components of $\mathcal{A}$, namely, $x\mathcal{A}=[\boldsymbol{a}_2,\boldsymbol{a}_3,\dots,\boldsymbol{a}_n,\boldsymbol{a}_1]$. Similarly, $\frac{1}{x}\mathcal{A}=[\boldsymbol{a}_n,\boldsymbol{a}_1\boldsymbol{a}_2,\dots,\boldsymbol{a}_{n-2},\boldsymbol{a}_{n-1}]$. Let $P:\mathbb{C}^n\times\mathbb{C}^n$ be a Hermitian form. Following [3], let us call $P(\cdot,\cdot)$ a polygonal form if $P(x\mathcal{A},x\mathcal{B})=P(\mathcal{A},\mathcal{B})$ for arbitrary n-gons $\mathcal{A}$ and $\mathcal{B}$.

The three important polygonal forms

$$F(\mathcal{A},\mathcal{B})=\frac{1}{4i}\sum\left(\boldsymbol{a}_{k+1}\boldsymbol{b}_k^*-\boldsymbol{a}_k\boldsymbol{b}_{k+1}^*\right),$$

where $\mathbf{i}=\sqrt{-1}$ and the asterisk denotes the complex conjugate,

$$S(\mathcal{A},\mathcal{B}) = \sum (\boldsymbol{a}_{k+1} - \boldsymbol{a}_k)\left(\boldsymbol{b}^*_{k+1} - \boldsymbol{b}^*_k\right),$$

and

$$I(\mathcal{A},\mathcal{B}) = \sum \boldsymbol{a}_k \boldsymbol{b}^*_k$$

correspond to the signed area of a polygon $F(\mathcal{A}) = F(\mathcal{A},\mathcal{A})$, the sum of the squared sides of a polygon $S(\mathcal{A}) = S(\mathcal{A},\mathcal{A})$, and the polar moment of inertia of the vertices of a polygon $I(\mathcal{A}) = I(\mathcal{A},\mathcal{A})$, respectively.

Consider now a polygon $\mathcal{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n]$ and two polygons $\mathcal{B}$ and $\mathcal{C}$ formed by the centers of the outer and inner squares constructed on the sides of $\mathcal{A}$, respectively. It is sufficient to show that

$$2P(\mathcal{A},\mathcal{A}) = P(\mathcal{B},\mathcal{B}) + P(\mathcal{C},\mathcal{C}). \tag{1}$$

Obviously, we have

$$\mathbf{i}(\boldsymbol{a}_{k+1} - \boldsymbol{b}_k) = \boldsymbol{a}_k - \boldsymbol{b}_k.$$

In other words, we can write

$$\mathbf{i}(x\mathcal{A} - \mathcal{B}) = \mathcal{A} - \mathcal{B},$$

or equivalently,

$$(x + \mathbf{i})\mathcal{A} = (1 + \mathbf{i})\mathcal{B}. \tag{2}$$

Similarly,

$$(x - \mathbf{i})\mathcal{A} = (1 - \mathbf{i})\mathcal{C}. \tag{3}$$

Subtracting (3) from (2) leads to

$$\mathcal{A} = \frac{1+\mathbf{i}}{2\mathbf{i}}\mathcal{B} + \frac{1-\mathbf{i}}{2\mathbf{i}}\mathcal{C}. \tag{4}$$

Our task is to show that

$$P(\mathcal{A},\mathcal{A}) = \frac{|1+\mathbf{i}|^2}{|2\mathbf{i}|^2}P(\mathcal{B},\mathcal{B}) + \frac{|1-\mathbf{i}|^2}{|2\mathbf{i}|^2}P(\mathcal{C},\mathcal{C}),$$

which is equivalent to (1). We have

$$\begin{aligned}
P(\mathcal{A}) &= P\left(\frac{1+\mathbf{i}}{2\mathbf{i}}\mathcal{B} + \frac{1-\mathbf{i}}{2\mathbf{i}}\mathcal{C}\right)\\
&= P\left(\frac{1+\mathbf{i}}{2\mathbf{i}}\mathcal{B} + \frac{1-\mathbf{i}}{2\mathbf{i}}\mathcal{C}, \frac{1+\mathbf{i}}{2\mathbf{i}}\mathcal{B} + \frac{1-\mathbf{i}}{2\mathbf{i}}\mathcal{C}\right)\\
&= P\left(\frac{1+\mathbf{i}}{2\mathbf{i}}\mathcal{B}\right) + P\left(\frac{1-\mathbf{i}}{2\mathbf{i}}\mathcal{C}\right) + P\left(\frac{1+\mathbf{i}}{2\mathbf{i}}\mathcal{B}, \frac{1-\mathbf{i}}{2\mathbf{i}}\mathcal{C}\right)\\
&\quad + P\left(\frac{1-\mathbf{i}}{2\mathbf{i}}\mathcal{C}, \frac{1+\mathbf{i}}{2\mathbf{i}}\mathcal{B}\right).
\end{aligned}$$

Thus we need to show that

$$P\left(\frac{1+\mathbf{i}}{2\mathbf{i}}\mathcal{B}, \frac{1-\mathbf{i}}{2\mathbf{i}}\mathcal{C}\right) + P\left(\frac{1-\mathbf{i}}{2\mathbf{i}}\mathcal{C}, \frac{1+\mathbf{i}}{2\mathbf{i}}\mathcal{B}\right) = 0,$$

or equivalently, using (2) and (3), that

$$P((x+\mathbf{i})\mathcal{A}, (x-\mathbf{i})\mathcal{A}) + P((x-\mathbf{i})\mathcal{A}, (x+\mathbf{i})\mathcal{A}) = 0.$$

We have

$$\begin{aligned}
&P((x+\mathbf{i})\mathcal{A}, (x-\mathbf{i})\mathcal{A}) + P((x-\mathbf{i})\mathcal{A}, (x+\mathbf{i})\mathcal{A})\\
&\quad = P((x+\mathbf{i})(1/x+\mathbf{i})\mathcal{A}, \mathcal{A}) + P((x-\mathbf{i})(1/x-\mathbf{i})\mathcal{A}, \mathcal{A})\\
&\quad = \mathbf{i}P((x+1/x)\mathcal{A}, \mathcal{A}) - \mathbf{i}P((x+1/x)\mathcal{A}, \mathcal{A}) = 0.
\end{aligned}$$

This completes the proof. □

Possible Extensions

A simple way to extend our results is to consider polygons formed by combinations of the centers of the outer and inner squares. For example, for a quadrilateral $[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4]$, it is easy to show that

$$F(\boldsymbol{b}_1, \boldsymbol{c}_2, \boldsymbol{b}_3, \boldsymbol{c}_4) + F(\boldsymbol{c}_1, \boldsymbol{b}_2, \boldsymbol{c}_3, \boldsymbol{b}_4) = 0.$$

For a general polygon $\mathcal{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n]$, it is not difficult to derive the following result. Let $\mathcal{A}' = [\dots, \boldsymbol{a}_{k-1}, \boldsymbol{a}_{k+1}, \dots]$ be the polygon constructed from the vertices of $\mathcal{A}$ by joining $\boldsymbol{a}_{k-1}$ and $\boldsymbol{a}_{k+1}$ and skipping the vertex $\boldsymbol{a}_k$. Note that if n is even, this actually corresponds to two disconnected polygons bounded by $[\boldsymbol{a}_1, \boldsymbol{a}_3, \dots, \boldsymbol{a}_{n-1}]$ and by $[\boldsymbol{a}_2, \boldsymbol{a}_4, \dots, \boldsymbol{a}_n]$. We have

$$\begin{aligned}
&F(\boldsymbol{c}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \dots, \boldsymbol{b}_n) + F(\boldsymbol{b}_1, \boldsymbol{c}_2, \boldsymbol{b}_3, \dots, \boldsymbol{b}_n) + \cdots + F(\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{c}_n)\\
&\quad = (n-2)F(\mathcal{B}) + F(\mathcal{A}')
\end{aligned}$$

and

$$\begin{aligned}
&F(\boldsymbol{b}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \dots, \boldsymbol{c}_n) + F(\boldsymbol{c}_1, \boldsymbol{b}_2, \boldsymbol{c}_3, \dots, \boldsymbol{c}_n) + \cdots + F(\boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{b}_n)\\
&\quad = (n-2)F(\mathcal{C}) + F(\mathcal{A}').
\end{aligned}$$

Similar formulas also hold for the sums of squared sides.

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