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ABSTRACT

We investigated three-dimensional quantum systems with higher-order dispersion and nonlinear effects. The systems' soliton dynamics is studied based on the (3+1)-dimensional higher-order nonlinear Schrödinger equation (NLSE). Based on the self-similar approach and the bright soliton-type solution of the (1+1)-dimensional NLSE, we derived the analytical bright soliton solution for the (3+1)-dimensional NLSE with higher-order dispersion and nonlinear effects, with the typical soliton feature pictorially demonstrated. Our study illustrates that a higher-dimensional medium with higher-order dispersion and nonlinear effects supports soliton behavior. This demonstrates the applicability of the theoretical treatment presented in this work.

I. INTRODUCTION

The nonlinear phenomenon is one of the most intriguing topics in current physical science subjects and is investigated heavily in theory and experimental aspects. Solitons are the typical nonlinear phenomena, which are generated through the balance between dispersion and nonlinear interaction effects of the systems under study. For the theoretical investigation of nonlinearity, the nonlinear Schrödinger equation (NLSE)[1–7] has been proven to be one reliable model in the formulation of a significant number of nonlinear phenomena of quantum nature. This is simply because of the typical effects involved in many nonlinear problems. For instance, the NLSE plays a principal role in the theoretical description of wave propagation in fields such as condensed-matter physics,[8] plasma physics,[9] particle physics,[10] and nonlinear optics.[11]

As a one-dimensional setting is typical in numerous quantum systems for the specific integrability of the classical (1+1)-dimensional NLSE involved, numerous research work focuses on the main category of one-dimensional classical NLSE. When the system medium experiences drastic density changes from dynamical evolution, which involves three-body inter-particle interaction in dilute ultracold gas, for example, higher-order nonlinear interaction effects have to be taken into account. Additionally, incorporating higher-order nonlinear effects makes the system's spatial–temporal scale of the nonlinear structure particularly small, making it necessary to incorporate higher-order dispersion effects.[12,13] Even until now, the study of the fourth-order NLSE with second, third, and fourth-order dispersion effects, as well as higher-order interactions, including many-body interaction beyond regular two-body interaction effects, is relatively rare. As soliton-type solutions are very typical and have the ultimate goal of analytical solution study in the NLSE and NLSE related model, prior work utilizing the *F*-expansion method[14,15,22] analytically solves the (1+1)-dimensional higher-order NLSE. Additionally, it identifies bright soliton-type solutions at specific parametric settings and demonstrates certain nonlinear features supported by the higher-order NLSE under investigation.

For many actual quantum systems such as optical media, semiconductor electronic settings,[16] Bose–Einstein condensate (BEC),[17] and fluid mechanics,[18] the study for the impact of higher-order dispersion and nonlinear effects calls for the investigation of the three-dimensional setting of the system. Specifically, this is modeled by the corresponding (3+1)-dimensional NLSE. Notably, relatively rare analytical work is done on higher-dimensional systems with the corresponding NLSE. This is partly because under a normal experimental setting, weak and strong decay[19] occurrences prevent the formation of stable typical nonlinear features such as the solitons. The stabilization mechanism arising from the higher-order dispersion and nonlinear effects can induce the formation of the stable nonlinear feature,[20] in higher-dimensional quantum just as mentioned. Given the analytical soliton solution for the (1+1)-dimensional NLSE with higher-order dispersion and nonlinear interaction terms, the three-dimensional analytical solution for the corresponding (3+1)-dimensional NLSE with higher-order dispersion and nonlinearity can be derived from the analytical solution of its one-dimensional model. In this study, we derived the soliton-type solution for the corresponding (3+1)-dimensional NLSE model based on the self-similar approach,[10] with the dynamical evolution features of the derived bright soliton solution pictorially demonstrated.

This work is organized as follows: In Sec. II, we present the formulation of the fourth-order NLSE model with higher-order dispersion and nonlinear interaction effects and also present relevant results on the typical analytical soliton solution of the corresponding (1+1)-dimensional higher-order NLSE. Section III offers the procedure details of reaching an analytical soliton solution of the corresponding (3+1)-dimensional higher-order NLSE, with the bright soliton-type solution features pictorially demonstrated. Section IV gives the concluding remarks.

II. THE (1+1)-DIMENSIONAL HIGHER-ORDER NLSE MODEL WITH BRIGHT SOLITON SOLUTION

A. (1+1)-dimensional NLSE model with higher-order dispersion and nonlinear interactions

The (1+1)-dimensional NLSE with higher-order dispersion[12,13,21] and nonlinear effects[22] takes the following form:

$$i\frac{\partial\psi(x,t)}{\partial t}+\gamma_1\frac{\partial^4\psi(x,t)}{\partial x^4}+i\gamma_2\frac{\partial^3\psi(x,t)}{\partial x^3}+\gamma_3\frac{\partial^2\psi(x,t)}{\partial x^2}+g_0\mid\psi(x,t)\mid^2\psi(x,t)+g_1\mid\psi(x,t)\mid^4\psi(x,t)+g_2\frac{\partial^2\mid\psi(x,t)\mid^2}{\partial x^2}\psi(x,t)=0, \quad (1)$$

where the space x and time t coordinates are the independent variables of ψ, and γ_1, γ_2, and γ_3 are coefficients of the dispersion term of different orders, $\gamma_1 \neq 0$. g_0 is the nonlinear two-body interaction coefficient, and g_1 is the nonlinear three-body interaction coefficient. The coefficient g_2 describes the higher-order scattering effects,[21] $g_2 = \frac{\alpha^2}{3} - \frac{\alpha r_e}{2}$; r_e is the effective interaction range; and α is the s-wave scattering length. We then summarized the established analytical solution of the one-dimensional equation (1) in Subsection II B and proceeded to derive the analytical soliton solution for the three-dimensional analog of Eq. (1) using a self-similar approach in Sec. III.

B. Typical bright soliton solution of higher-order (1+1)-dimensional NLSE

Equation (1) possesses the travel wave format solution as follows:

$$\psi(x,t)=\varphi(\zeta,)e^{i(At+Bx)}, \quad (2)$$

where $\zeta = px + qt$. Based on the F-expansion method, Qi[22] derived the exact analytical soliton solution for Eq. (1) as follows:

$$|\psi(x,t)|=\begin{cases}\left|\frac{a_2}{a_4}\right|^{\frac{1}{2}}\operatorname{sech}(\sqrt{a_2}(px+qt)),\ a_2>0, a_4<0 & (3a)\\ \left|\frac{a_2}{a_4}\right|^{\frac{1}{2}}\sec(\sqrt{|a_2|}(px+qt)),\ a_2<0, a_4>0, & (3b)\end{cases}$$

where the constants a_2, a_4, A, B, and q can be expressed by $\gamma_{1,2,3}$, $g_{0,1,2}$, and p, as shown in the work of Li. We can see that the solution Eq. (3) is of the bright soliton type for the higher-order 1D NlSE (1). We will see that the analytical soliton solution for the three-dimensional higher-order NLSE can also be derived based on the one-dimensional solution Eq. (3), as shown in Sec. III.

III. BRIGHT SOLITON BEHAVIOR FOR (3+1)-DIMENSIONAL HIGHER-ORDER SCHRÖDINGER EQUATION

Incorporating higher-order dispersion and nonlinear effects, the three-dimensional analog of Eq. (1) is

$$i\frac{\partial\psi(\vec{r},t)}{\partial t}+\alpha_1\sum_{i=1}^{3}\frac{\partial^4\psi(\vec{r},t)}{\partial x_i^4}+i\alpha_2\sum_{i=1}^{3}\frac{\partial^3\psi(\vec{r},t)}{\partial x_i^3}+\alpha_3\sum_{i=3}^{3}\frac{\partial^2\psi(\vec{r},t)}{\partial x_i^2}+g_0\mid\psi\mid^2\psi(\vec{r},t)+g_1\mid\psi(\vec{r},t)\mid^4\psi(\vec{r},t)+g_2\sum_{i=1}^{3}\frac{\partial^2\mid\psi(\vec{r},t)\mid^2}{\partial x_i^2}\psi(\vec{r},t)=0. \quad (4)$$

Here, the various order differential terms correspond to operators of $\hat{p} = i\hbar\vec{\nabla}$ (in natural unit $\hbar = 1$), that is, $\hat{p}, \hat{p}^2, \hat{p}^3, \hat{p}^4$. To solve (4), we can assume that the analytical solution of Eq. (4) takes the following form:

$$\psi(x,y,z,t)=\psi_{3D}(\vec{r},t). \quad (5)$$

To proceed further, we assume that the self-similar solution (5) of Eq. (4) takes the following form:

$$\psi_{3D}(\vec{r},t)=\psi_{1D}(\xi(x,y,z),\tau(t))e^{iA(x,y,z)}. \quad (6)$$

Substituting ansatz (6) into Eq. (4), we obtain

$$
\begin{aligned}
&i\frac{\tau(t)}{\partial t}\frac{\partial\psi_{1D}(\vec{r},t)}{\partial\tau}+\alpha_1\sum_{i=1}^{3}\left(\frac{\partial\xi}{\partial x_i}\right)^4\frac{\partial^4\psi_{1D}}{\partial\xi^4}\\
&\quad+i\sum_{i=1}^{3}\left[4\frac{A(x,y,z)}{\partial x_i}\alpha_1+\alpha_2\right]\left(\frac{\partial\xi}{\partial x_i}\right)^3\frac{\partial^3\psi_{1D}}{\partial\xi^3}\\
&\quad+\sum_{i=3}^{3}\left\{\left[-6\left(\frac{A}{\partial x_i}\right)^2+6i\frac{\partial^2A}{\partial x_i^2}\right]-3\frac{\partial A}{\partial x_i}\alpha_2+\alpha_3\right\}\left(\frac{\partial\xi}{\partial x_i}\right)^2\\
&\quad\times\frac{\partial^2\psi_{1D}}{\partial\xi^2}+i\sum_{i=1}^{3}\left\{\left[-4\frac{\partial^3A}{\partial x_i^3}-8\frac{\partial^2A}{\partial x_i^2}\frac{\partial A}{\partial x_i}-4\left(\frac{\partial A}{\partial x_i}\right)^3\right]\alpha_1\right.\\
&\quad\left.-3\left[\frac{\partial^2A}{\partial x_i^2}+\left(\frac{\partial A}{\partial x_i}\right)^2\right]\alpha_2+2\alpha_3\frac{\partial A}{\partial x_i}\right\}\left(\frac{\partial\xi}{\partial x_i}\right)\frac{\partial\psi_{1D}}{\partial\xi}\\
&\quad+g_0\mid\psi\mid^2\psi+g_1\mid\psi\mid^2\psi+g_2\left[\sum_{i=1}^{3}\left(\frac{\partial\xi}{\partial x_i}\right)^2\right]\frac{\partial^2\mid\psi_{1D}\mid^2\psi_{1D}}{\partial\xi^2}=0,
\end{aligned}
\tag{7}
$$

where $(x_1, x_2, x_3) = (x, y, z)$. We expect that the resultant equation (7) will take the same format as Eq. (1), which corresponds to the (1+1)-dimensional higher-order Schrödinger equation with ξ, τ coordinate notations. Therefore, it is necessary that the coefficient formula of each term in Eq. (7) does not contain the spatial x, y, z coordinates, which requires functions $\xi(x,y,z)$ and $A(x,y,z)$ to be linear functions of spatial x, y, z coordinates and the primary partial derivative coefficient β_4 (for the term of the first-order partial derivative of ψ with ξ) to be zero. So,

$$\xi(x,y,z) = k_1x + k_2y + k_3z, \tag{8}$$

$$A(x,y,z) = a_1x + a_2y + a_3z. \tag{9}$$

Here, k_1, k_2, k_3 satisfies normalization conditions $k_1^2 + k_2^2 + k_3^2 = 1$.

We can have

$$\tau(t) = t.$$

Substituting Eqs. (8) and (9) into Eq. (7), we obtain

$$
\begin{aligned}
&i\frac{\partial\psi_{1D}(\vec{r},t)}{\partial t}+\beta_1\frac{\partial^4\psi_{1D}(\vec{r},t)}{\partial\xi^4}+i\beta_2\frac{\partial^3\psi_{1D}(\vec{r},t)}{\partial\xi^3}\\
&\quad+\beta_3\frac{\partial^2\psi_{1D}(\vec{r},t)}{\partial\xi}+i\beta_4\frac{\partial\psi_{1D}(\vec{r},t)}{\partial\xi}\\
&\quad+g_0\mid\psi_{1D}\mid^2\psi_{1D}+g_1\mid\psi_{1D}\mid^4\psi_{1D}\\
&\quad+g_2'\frac{\partial^2\mid\psi_{1D}\mid^2\psi_{1D}}{\partial\xi^2}=0,
\end{aligned}
\tag{10}
$$

where

$$\beta_1 = \alpha_1\left(k_1^4 + k_2^4 + k_3^4\right), \tag{11a}$$

$$\beta_2 = 4\alpha_1\left(a_1k_1^3 + a_2k_2^3 + a_3k_3^3\right) + \alpha_2\left(k_1^3 + k_2^3 + k_3^3\right), \tag{11b}$$

$$
\begin{aligned}
\beta_3 = &-6\alpha_1\left(a_1^2k_1^2 + a_2^2k_2^2 + a_3^2k_3^2\right)\\
&- 3\alpha_2\left(a_1k_1^2 + a_2k_2^2 + a_3k_3^2\right) + \alpha_3\left(k_1^2 + k_2^2 + k_3^2\right),
\end{aligned}
\tag{11c}
$$

$$g_2' = g_2\left(k_1^2 + k_2^2 + k_3^2\right), \tag{11d}$$

$$
\begin{aligned}
\beta_4 = &-4\alpha_1\left(a_1^3k_1 + a_2^3k_2 + a_3^3k_3\right) - 3\alpha_2\left(a_1^2k_1 + a_2^2k_2 + a_3^2k_3\right)\\
&+ 2\alpha_3\left(a_1k_1 + a_2k_2 + a_3k_3\right) = 0.
\end{aligned}
\tag{11e}
$$

Because the degree of freedom for the selection of the reference axis direction is two, we choose the axis direction so that $a_2 = a_3 = 0$ in (9). At the same time, using the normalization condition of k_i, Eqs. (11a)–(11e) became

$$\beta_1 = \alpha_1\left(k_1^4 + k_2^4 + k_3^4\right), \tag{12a}$$

$$\beta_2 = 4\alpha_1a_1k_1^3 + \alpha_2\left(k_1^3 + k_2^3 + k_3^3\right), \tag{12b}$$

$$\beta_3 = -6\alpha_1a_1^2k_1^2 - 3\alpha_2a_1k_1^2 + \alpha_3, \tag{12c}$$

$$g_2' = g_2, \tag{12d}$$

$$\beta_4 = -\left(4a_1^2 + 3a_1 - 2\right)a_1k_1 = 0. \tag{12e}$$

From the constraint formula (12e), we obtained

$$a_1 = \frac{-3+\sqrt{41}}{8}.$$

The analytical solution ψ_{1D} of Eq. (4) is the typical bright soliton solution presented in Eq. (3),

$$
\begin{aligned}
\psi_{1D} &= \mathrm{sech}\left[p\xi(x,y,z) + qt\right]e^{i[c\xi(x,y,z)+Bt]}\\
&= \mathrm{sech}\left[p(k_1x + k_2y + k_3z) + qt\right]e^{i[c(k_1x+k_2y+k_3z)+Bt]}.
\end{aligned}
\tag{13}
$$

Substituting (13) into (6), we obtained the analytical solution of the three-dimensional higher-order NLSE (4) as

$$
\begin{aligned}
\psi_{3D}(\vec{r},t) = &\,\mathrm{sech}\left[\left(k_1'x + k_2'y + k_3'z\right) + qt\right]\\
&\times e^{i[(ck_1+a_1)x+ck_2y+ck_3z+Bt]}.
\end{aligned}
\tag{14}
$$

Here, q, c, B are constants to be determined by the initial setting of the system under study, and

$$k_i' = pk_i (i = 1, 2, 3). \tag{15}$$

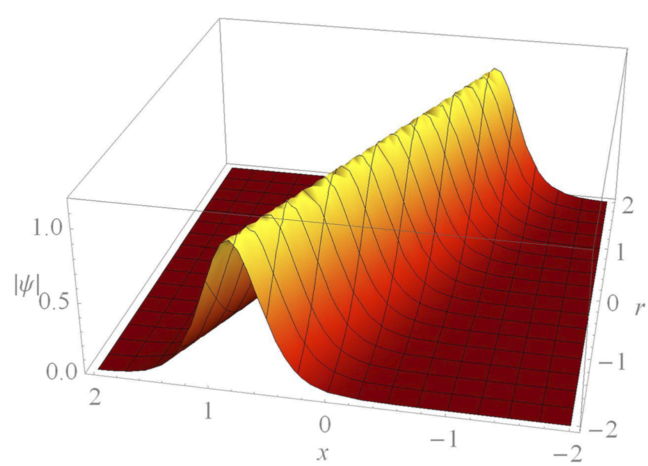

FIG. 1. Bright soliton $|\psi|$ for the three-dimensional case($r = y + z, k_1 = \frac{1}{\sqrt{2}}, k_2 = k_3 = 1/2, p = 1.0, q = 3.0, t = 0.0$).

Figure 1 shows the typical waveform of the three-dimensional bright soliton solution (14) just derived. We can see that the system flow velocity is

$$\vec{v} = \frac{\hbar}{2mi}\left(\psi^{*}\nabla\psi - \psi\nabla\psi^{*}\right) = \frac{\hbar}{m}\left[(ck_1 + a_1)\mathbf{e}_x + ck_2\mathbf{e}_y + ck_3\mathbf{e}_z\right], \tag{16}$$

which is constant. We can see that $\vec{v}$ and (k_1', k_2', k_3') are generally not in the same direction; the solution (14) is a truly three-dimensional solution for Eq. (4). We can also see for Eq. (16) that the flow velocity of the system supporting bright soliton behavior is constant. This means that there is no mass accumulation and dissipation (for incompressible fluids) in space for such constant velocity distribution ($\nabla \cdot \vec{v} = 0$), which means that the bright soliton waveform will not change with time within the valid range of the theoretical three-dimensional nonlinear Schrödinger equation model with higher-order nonlinear interaction.

IV. CONCLUSION

In this study, we investigated the soliton dynamics for three-dimensional quantum systems with higher-order dispersion and nonlinear effects, which are modeled by the (3+1) dimensional higher-order nonlinear Schrödinger equation. We derived the bright soliton solution for the (3+1)-dimensional higher-order NLSE based on the self-similar approach. The dynamical features of the derived bright soliton solution are pictorially demonstrated. Our work illustrates that the three-dimensional system with higher-order dispersion and nonlinear effects supports bright soliton behavior, and the theoretical results derived here can be utilized to guide experimental observation of soliton dynamics in a higher-order dispersion and nonlinear medium.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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