Characterising actions on trees yielding non-trivial quasimorphisms

Citation for published version:
Iozzi, A, Pagliantini, C & Sisto, A 2020, 'Characterising actions on trees yielding non-trivial quasimorphisms', *Annales mathématiques du Québec*. https://doi.org/10.1007/s40316-020-00137-3

Digital Object Identifier (DOI):
10.1007/s40316-020-00137-3

Link:
Link to publication record in Heriot-Watt Research Portal

Document Version:
Peer reviewed version

Published In:
Annales mathématiques du Québec

CHARACTERISING ACTIONS ON TREES YIELDING NON-TRIVIAL QUASIMORPHISMS

ALESSANDRA IOZZI, CRISTINA PAGLIANTINI, AND ALESSANDRO SISTO

Abstract. Using a cocycle defined by Monod and Shalom in [MS04] we introduce the *median* quasimorphisms for groups acting on trees. Then we characterise actions on trees that give rise to non-trivial median quasimorphisms. Roughly speaking, either the action is highly transitive on geodesics, or it fixes a point in the boundary, or there exists an infinite family of non-trivial median quasimorphisms. In particular, in the last case the second bounded cohomology of the group is infinite dimensional as a vector space. As an application, we show that a cocompact lattice in the automorphism group of a product of trees has only trivial quasimorphisms if and only if the closures of the projections on each of the two factors are locally ∞-transitive.

Résumé. On utilise un cocycle introduit par Monod et Shalom in [MS04] pour définir le quasimorphisme *médian* d'un groupe agissant sur un arbre. Nous donnons une caractérisation d'actions sur un arbre pour lesquelles le quasimorphisme médian est non banal. Grosso modo soit l'action est "très" transitive sur les géodésiques, soit elle fixe un point du bord, soit il existe une famille infinie de quasimorphismes médians non banals. En particulier, dans le dernier cas la cohomologie bornée du groupe en degré deux a dimension infinie comme espace vectoriel. Nous appliquons les résultats ci-dessus pour montrer que un réseau cocompact dans le groupe d'automorphismes d'un produit d'arbres n'a que des quasimorphismes banals si et seulement si les fermetures des projections sur chacun des deux facteurs sont localement ∞-transitives.

Date: May 1, 2020.

2010 *Mathematics Subject Classification.* 55N, 20F65, 20E08.

All authors were supported by Swiss National Science Foundation project 144373.

1. Introduction

Since its inception in the 80s, bounded cohomology has proven itself to be a very efficient tool to approach rigidity questions (see for example [Gro82, Ghy87, Mat87, BI04, BIW10, BFS13]). While in degree 0 and 1 there are no enlightening differences between bounded and ordinary group cohomology, in degree 2 the study of the natural comparison map $\mathrm{H}^2_\mathrm{b} \to \mathrm{H}^2$ has also proven to be very fruitful. For example the surjectivity of the comparison map with all coefficients is a characterisation of Gromov hyperbolicity [Min01, Min02], while the injectivity is equivalent to the vanishing of the stable commutator length [Bav91]. It is hence natural that a lot of attention has been devoted to the study of the kernel of the comparison map, that is to say the space of quasimorphisms.

In this paper we study the space of quasimorphisms of a group Γ acting by automorphisms of a simplicial tree $\mathcal{T}$. On the one hand we give a trichotomy to show that, roughly speaking, either the action of Γ on $\mathcal{T}$ is highly transitive on geodesics, or it fixes a point in the boundary $\partial\mathcal{T}$ of $\mathcal{T}$, or there exists an infinite family of non-trivial quasimorphisms on Γ. As an application, we prove the converse of a result of Burger and Monod, [BM02, Corollary 26], thus giving a full characterization of locally ∞-transitive groups in terms of quasimorphisms. We recall that a group of automorphism of a locally finite tree is *locally ∞-transitive* if the stabiliser of each vertex is transitive on the n-spheres for all n.

More precisely, if $n \in \mathbb{N}$, we call n-geodesic a geodesic segment of length n. Let $\mathcal{E}^{(n)}$ be the set of connected *oriented* n-geodesics in $\mathcal{T}$ and let us consider the isometric action of Γ on $\ell^1(\mathcal{E}^{(n)})$, given by $\pi(g)f(s) := f(g^{-1}s)$. Let us fix a vertex $v \in \mathcal{T}$. We start by recalling the construction of a cocycle due to Monod and Shalom, [MS04], which is a discrete version of the Gromov–Sela cocycle. We define a map $\omega : \Gamma \to \ell^1(\mathcal{E}^{(n)})$ by

$$\omega(g) := \chi_{[\![v,gv]\!]} - \chi_{[\![gv,v]\!]},$$

where $[\![x, y]\!]$ is the set of connected geodesic segments of length n contained in the geodesic $[x, y]$ and oriented as $[x, y]$. It is easy to see that while ω is not bounded as a function of g, its coboundary turns out to be bounded

$$\sup_{g,h\in\Gamma} \|\delta^1\omega(g, h)\|_1 < \infty, \tag{1.1}$$

as $\delta^1\omega(g, h) := \pi(g)\omega(h) - \omega(gh) + \omega(g)$ is supported only on the geodesic paths that go through the median of the vertices v, gv and ghv.

Given an oriented n-geodesic s, we define the *median quasimorphism* $f_s : \Gamma \to \mathbb{Z}$ to be the evaluation of ω on the characteristic function $\chi_{\Gamma s} \in \ell^\infty(\mathcal{E}^{(n)})$ of the Γ-orbit Γs of s

$$f_s(g) := \langle \omega(g), \chi_{\Gamma s} \rangle = \sum_{\gamma \in \Gamma s} \chi_{[\![v,gv]\!]}(\gamma) - \chi_{[\![gv,v]\!]}(\gamma). \tag{1.2}$$

Remark 1.1. In other words, $f_s(g)$ counts the number of occurrences of a translate of s inside $[v, gv]$, minus the corresponding number for $[gv, v]$.

Notice that because of (1.1), the cocycle $\delta^1\omega : \Gamma \times \Gamma \to \ell^1(\mathcal{E}^{(n)})$ defines a bounded cohomology class $[\delta^1\omega] \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \ell^1(\mathcal{E}^{(n)}))$ and $\delta^1 f_s = \langle \delta^1\omega(g), \chi_{\Gamma s} \rangle$ defines a bounded cohomology class $[\delta^1 f_s] \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$.

Then, in Section 2 we prove the following result. Recall that a tree is semiregular if the full automorphism group is edge-transitive but not vertex-transitive, in which case we equivariantly color the vertices. Moreover an action is minimal if there is no invariant proper subtree.

Theorem 1. *Suppose that Γ acts minimally on the tree $\mathcal{T}$ and that every vertex of $\mathcal{T}$ has valence greater than 2. Then exactly one of the following holds.*

- *(i) Γ fixes a point a in the boundary $\partial\mathcal{T}$ of $\mathcal{T}$.*
- *(ii) $\mathcal{T}$ is semiregular and Γ acts transitively on the set of geodesics of any fixed length starting at any vertex of any fixed color.*
- *(iii) There exists an infinite family $\{[\delta^1 f_{s_n}]\}_{n\in\mathbb{N}}$ of linearly independent coclasses in $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$. In particular, there exists an injective $\mathbb{R}$-linear isometric map $\ell^1(\Gamma) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$.*

Remark 2. The conditions that Γ acts minimally on the tree $\mathcal{T}$ and that every vertex of $\mathcal{T}$ has valence greater than 2 are not restrictive because given any action on a tree $\mathcal{T}$ one can consider a minimal subtree $\mathcal{T}'$, which exists provided that Γ does not consist entirely of parabolic elements fixing the same point at infinity [Tit70, Corollary 3.5]. Furthermore there are no vertices of valence 1 in $\mathcal{T}'$ – that is, every geodesic can be extended – unless $\mathcal{T}'$ consists of a single edge only, and vertices of valence 2 can be removed by regarding each maximal path containing only vertices of valence 2 in their interior as a single edge. The latter procedure gives a well-defined tree provided that $\mathcal{T}'$ is not a subdivision of $\mathbb{R}$.

Note that in the case of proper CAT(0) spaces and actions with a rank one element, an analogous result relating the existence of a fixed point at infinity, the double transitivity of the action on the limit set and the finite dimensionality of the space of quasimorphisms was proven by Caprace and Fujiwara, [CF10, Theorem 1.8].

Local ∞-transitivity implies already very strong properties on the locally finite tree $\mathcal{T}$, that will be either regular or semiregular. Moreover, in some sense locally ∞-transitive groups are the most rigid. For example if $H < \mathrm{Aut}(\mathcal{T})$ is a closed subgroup that is locally ∞-transitive, then for all $N \trianglelefteq H$, $N \neq \{e\}$, the quotient H/N is compact, [BM00, Proposition 3.1.2]. Or else, if $\Gamma < H_1 \times H_2$ is a cocompact lattice with $H_i < \mathrm{Aut}(\mathcal{T}_i)$ and H_i is locally ∞-transitive, then Γ satisfies the Normal Subgroup Theorem, that is every normal subgroup is finite or of finite index, [BS06].

We prove the following result in Section 4:

Corollary 3. *Let $\mathcal{T}_i$, $i = 1, 2, \ldots, k$ be regular trees of finite valence greater than 2, let $\Gamma < Aut(\mathcal{T}_1) \times \cdots \times Aut(\mathcal{T}_k)$ be a cocompact lattice and let $\mathrm{pr}_i : Aut(\mathcal{T}_1) \times \cdots \times Aut(\mathcal{T}_k) \to Aut(\mathcal{T}_i)$ be the projection maps. Then the following are equivalent:*

(i) *the $H_i := \overline{\mathrm{pr}_i(\Gamma)}$ are locally ∞-transitive, $i = 1, 2, \ldots, k$;*
(ii) *any quasimorphism $\Gamma \to \mathbb{R}$ is bounded;*
(iii) *any median quasimorphism $\Gamma \to \mathbb{R}$ is bounded.*

As a consequence of this corollary we can construct infinite families of examples of groups for which Theorem 1(iii) holds, as appropriate extensions of irreducible cocompact lattices in the product of $\mathrm{PSL}(n, \mathbb{Q}_p) \times \mathrm{PSL}(n, \mathbb{Q}_\ell)$, with $n \geq 3$, and p, ℓ prime (see Section 5).

Notice that an analogous result for products of geodesically complete cocompact CAT(0) cube complexes was obtained by Caprace and Sageev [CS11, Theorem H]. In that case they also deduce that the vanishing of the space of quasimorphisms implies that the CAT(0) cube complexes are indeed semiregular trees.

Theorem 1 applies in particular to amalgamated products and HNN extensions. Fujiwara [Fuj00] showed that, under mild hypotheses, these have infinite dimensional second bounded cohomology. We show that Fujiwara's result can be deduced from ours, which gives in fact a strictly larger class of amalgamated products with infinite dimensional second bounded cohomology. For instance the group $S_3 *_{\mathbb{Z}_2} \mathbb{Z}_4$, where S_3 is the symmetric group on 3 elements, satisfies condition (iii) of Theorem 1 but is not covered by Fujiwara's result, [Fuj00, Theorem 1.1]. See Section 5 for details.

Another very large class of groups known to have infinite dimensional second bounded cohomology is the class of *acylindrically hyperbolic groups* [HO13] (see also [FPS13, BBF13]), and under rather general conditions a group acting on a tree is acylindrically hyperbolic [MO13, Theorem 2.1]. However, we remark that groups satisfying condition (iii) of Theorem 1 include examples that are not covered by [MO13] (once again, see Section 5 for details).

Acknowledgments: We thank the referee for comments and suggestions that improved the exposition of the paper.

2. Proof of Theorem 1

2.1. **Notation.** We fix the notation of the theorem, until the end of Section 3.

We fix a vertex v of $\mathcal{T}$ as basepoint. All geodesics segments have vertices of $\mathcal{T}$ as endpoints.

Definition 2.1. The *orbit length* (or *o-length*) of a geodesic γ is the number of vertices in the orbit Γv crossed by γ minus one. We denote the o-length by $|\cdot|_o$.

An $o(n)$*-geodesic* is a geodesic segment of orbit length n with vertices in Γv as endpoints. If we do not need to emphasize the length we will use the notation o-geodesic. We call *o-edge* an $o(1)$-geodesic and *o-vertex* a vertex in the orbit Γv. The set of oriented $o(n)$-geodesics is denoted by $\mathcal{E}_v^{(n)}$.

Definition 2.2. Let γ_1 and γ_2 be two o-geodesics. We say that γ_1 is *chainable* with γ_2 if $\xi\gamma_1$ concatenates with $g\gamma_2$ in an o-vertex for some $g, \xi \in \Gamma$ and they only intersect in such o-vertex.

2.2. **Mutual exclusiveness.** First of all, we observe that cases (i), (ii) and (iii) are mutually exclusive. It is clear that (i) and (ii) cannot simultaneously occur.

Assuming now that (ii) holds, we show that, for any oriented geodesic s, f_s is identically 0. Recall from Remark 1.1 that f_s counts occurrences of translates of s. In view of our transitivity assumption, the number of translates of s contained in some o-geodesic γ only depends on the o-length of γ. In fact, γ contains a unique subgeodesic γ' with vertices in the orbit Γv, and the length of γ' is the o-length of γ. Moreover, any translate of s contained in γ is in fact contained in γ'. If α is now a different geodesic with the same o-length of γ, we see that the corresponding subgeodesic α' as above has the same length as γ', and hence by our assumption α' is a translate of γ'. Therefore, α' and γ' contain the same number of translates of s, and hence the same is true for α and γ. Now, since for any γ we have that γ and γ^{-1} have the same o-length, we have that γ and γ^{-1} contain the same number of translates of s. By Remark 1.1, this implies that f_s is identically 0.

To see that if (i) holds then (iii) cannot occur, observe first that any oriented geodesic is obtained concatenating two (possibly trivial) subgeodesics γ_1, γ_2 so that γ_1 and γ_2^{-1} are each contained in some ray towards the fixed point $a \in \partial\mathcal{T}$. Furthermore, γ_1 and γ_2^{-1} are translates of each other, and this decomposition is preserved by the action of Γ. If the decomposition for the oriented geodesic s is trivial, meaning that one of the two subgeodesics is trivial, then, ignoring orientations for now, any translate of s contained in a geodesic $[v, gv]$ is contained in one of the subgeodesics γ_1, γ_2 in the subdivision of $[v, gv]$ described above. Moreover, if the translate s' is contained in γ_1 as an oriented subgeodesic, then gs' is contained in γ_2^{-1} (not γ_2!) as an oriented subgeodesic. Therefore, using similar observations switching the roles of γ_1 and γ_2 and the possible orientations, we see that for any translate s' of s contained (with the correct orientation) in $[v, gv]$ we have a corresponding translate contained (with the correct orientation) in $[gv, v]$. Hence, all contributions cancel out and we see that f_s is identically 0. If the decomposition is non-trivial, any oriented geodesic contains at most one translate of s, whence f_s is bounded.

2.3. **The strategy.** We now have to show that when (i), (ii) do not hold, then (iii) must hold.

Here is an informal description of the strategy that we will follow. We will define an equivariant labelling on the oriented o-geodesics of $\mathcal{T}$, the label of an oriented o-geodesic being a word in the alphabet $\{a, b, c\}$. We will show that certain "sufficiently complicated" words can be realised as labels of oriented o-geodesics (when (i) and (ii) do not hold). The main property of such words is that they only have short common subwords compared to their length, which translates into o-geodesics realizing such words having short overlaps compared to their o-length. This property will allow us to control the counting procedure that defines f_s and in particular to show, for suitable choices of s, that the homogenization of f_s is not a homomorphism.

2.4. **Labellings.** A *labelling* for us will consist of the following data. First, a positive integer k. Second, the choice for each orbit of $o(k)$-geodesics of a letter from $\{a, b, c\}$, its label. With a slight abuse, we sometimes refer to the label of an $o(k)$-geodesic to mean the label of its orbit. Finally, the final piece of data is the assignment of a word to each o-geodesic according to the following procedure. Let γ be an o-geodesic. If γ_1 is the initial oriented subpath of γ of orbit length

k, we define the first letter of the label of γ to be the label of γ_1. In general, let γ_i be the oriented subpath of γ of orbit length k that starts from the i-th o-vertex of γ. Then the i-th letter of the label of γ is the label of γ_i. Once again, we call the word associated to an o-geodesic the label of the o-geodesic. Notice that if an o-geodesic has orbit length $L \geq k$ then the associated word has length $L - k + 1$. (The label of o-geodesics of orbit length less than k is empty.)

2.5. **Choice of words.** If w is a word in the alphabet $\{a, b\}$ we denote by $\overline{w}$ the word obtained reading w from right to left.

We wish to realise certain words as labels of o-geodesics. The words satisfy the following requirements.

LEMMA 2.3. *There exists a family of words* $\{w_{ni}\}_{n\geq 1, i=1,2,3}$ *in the alphabet* $\{a, b\}$ *such that*

(i) each w_{ni} *is the concatenation of words of the form* ab^N *for* $N \geq 2$*;*
(ii) for each integer k *there exists* $n_0(k)$ *so that if* $n, m \geq n_0(k)$ *and if* w_{ni} *and* w_{mj} *share a subword of length at least* $\min\{|w_{ni}| + k - 1, |w_{mj}| + k - 1\}/10 - k$ *then* $n = m$ *and* $i = j$*;*
(iii) Once again for $n, m \geq n_0(k)$, w_{ni} *does not share a common subword of length at least* $\min\{|w_{ni}| + k - 1, |\overline{w_{mj}}| + k - 1\}/10 - k$ *with* $\overline{w_{mj}}$*;*

Proof. Let us pick $v = 100(3n + i)$ with $n \geq 1$ and $i \in \{1, 2, 3\}$. We choose $w_{ni} = ab^v ab^{v+1} \cdots ab^{v+100}$. Since all the exponents of the b's are different the family of words $\{w_{ni}\}$ easily satisfies the lemma. $\square$

We say that a word in $\{a, b\}$ is *good* if it satisfies the property in Lemma 2.3(i). In the next subsection we will exhibit, in all cases where (i) and (ii) of Theorem 1 do not apply, a labelling so that any good word can be realised as the label of some o-geodesic. The reader might wish to skip this on first reading.

REMARK 2.4. We will also ensure that the labellings satisfy one of the following two conditions:

(1) Either we assign the label a (resp. b) to a unique orbit of o-geodesics, or
(2) the inverse of any element labelled a is not in an orbit labelled b.

2.6. **Definition of the labelling.** The labellings have to be defined in different ways depending on the properties of the action of Γ. We also show case by case why any good word can be realised as an oriented o-geodesic.

Case (1). Suppose that Γ does not act transitively on the set $\mathcal{E}_v^{(1)}$ of o-edges, and furthermore that there exist representatives e_1, e_2, e_3 of three distinct orbits of o-edges that only meet at their common starting point v.

The first required piece of data of the labelling is a positive integer: we pick $k = 1$.

Let us point out the following fact that will be used in this subsection.

Remark 2.5. Every incoming o-edge e in v is chainable with at least two o-edges among e_1, e_2 and e_3. Indeed, the o-edge e (the thickened one in Figure 1) could share its final edge with at most one among e_1, e_2 and e_3.

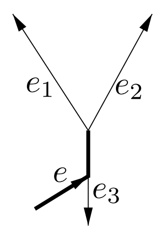

Figure 1. e is chainable with e_1 and e_2

Similarly, if we suppose that two o-edges only meet at their common starting point v, then every incoming o-edge e in v is chainable with at least one of those (see Figure 2). In particular, if both the emanating

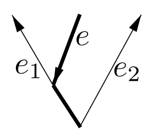

Figure 2. e is chainable with e_2

o-edges are in the orbit of e_1, for instance, then every incoming o-edge in v is chainable with e_1.

Let us now form a directed graph Λ (possibly with loops) with vertices marked $1, 2, 3$ and edges as follows. If, for some i, j, e_i is chainable with e_j, then connect i and j with a directed edge $i \to j$ in the graph Λ. By Remark 2.5 it follows that at least two edges emanate from any vertex of Λ.

Lemma 2.6. *Up to permuting the indices, either Λ*

(a) contains the subgraph described in Figure 3-(a), or

(b) it is the graph of Figure 3-(b), or

(c) it is the graph of Figure 3-(c).

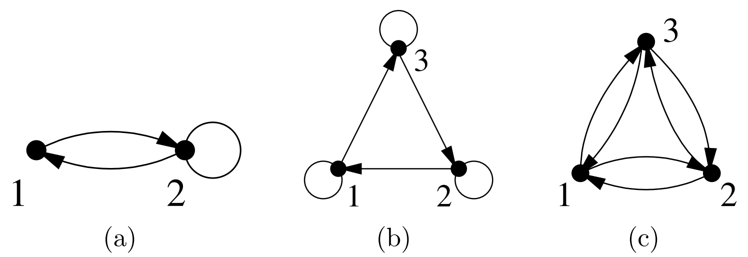

Figure 3. Directed graph Λ

Proof. Let us suppose that Λ does not contain a subgraph of type bigon-loop (Figure 3-(a)). Let us focus on the vertex $j \in \{1, 2, 3\}$. Either one edge emanating from j is a loop or no edge is a loop. If a loop is based at j then the same happens at all vertices of Λ otherwise we can realise a bigon-loop. Hence, we have the configuration described in Figure 3-(b). On the other hand, if no edge emanating from j is a loop then there are no loops in Λ, and the only possible configuration is the one described in Figure 3-(c). □

Case (1a). Suppose that the graph Λ contains a bigon-loop as in Figure 3-(a).

Assign label a to the orbit of the o-edge e_1 and label b to the orbit of the o-edge e_2. Finally, assign label c to all other orbits.

Figure 3-(a) implies that e_2 is chainable with e_1 and this realises the syllable ba. Similarly, e_1 is chainable with e_2 so that we have the syllable ab. Finally, e_2 is chainable with e_2, which corresponds to bb. Therefore all good words can be realised as labels of o-geodesics.

Case (1b). The graph Λ is of type described in Figure 3-(b).

Assign label b to the orbit either of the o-edge e_2 or of the o-edge e_3. Also, assign label a to the orbit of the o-edge e_1 and label c to all other orbits.

Similarly to Case (1a), Figure 3-(b) implies that all good words can be realised as labels of o-geodesics.

Recall that we need to verify that our labelling satisfies one of the two conditions listed in Remark 2.4. Since condition (1) does not hold, let us verify that the inverse of any element labelled a is not in an orbit labelled b (condition (2)). First of all notice that the only possible configuration of the o-edges associated to Figure 3-(b) is described by Figure 4. Therefore it can be easily seen that the inverse of e_1 is neither

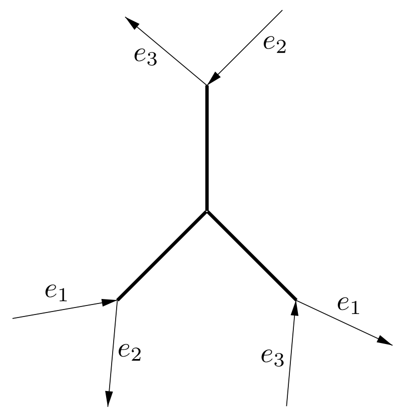

Figure 4. configuration of o-edges associated to Figure 3-(b)

in the orbit of e_2 nor in the orbit of e_3, for otherwise in both cases e_2 would be chainable with e_3.

Case (1c). The graph Λ is of type described in Figure 3-(c).

Assign label a to the orbit of the o-edge e_1. Also, assign label b to the orbits of the o-edges e_2 and e_3, and label c to all other orbits. Then similarly to Case (1a) we can realise the syllables ab, ba, bb concatenating elements in the orbits of e_1, e_2, e_3.

Moreover, as in Case (1b), let us verify that condition (2) of Remark 2.4 holds. The inverse of e_1 is neither in the orbit of e_2 nor in the orbit of e_3, for otherwise e_2 (resp. e_3) would be chainable with e_2 (resp. e_3). Therefore the orbit of e_1^{-1} is labelled with an a or with a c.

This completes the definition of the labelling in Case (1).

Case (2). Suppose that Γ *does not act transitively on the set* $\mathcal{E}_v^{(1)}$ *of o-edges, and that there do not exist o-edges* e_1, e_2, e_3 *as in Case (1).*

As in Case (1) we choose as positive integer $k = 1$.

Since the valence of v is at least 3, there exist o-edges e_1, $e_1' = g(e_1)$ with $g \in \Gamma$ and $e_2 \notin \Gamma e_1$ emanating from v and pairwise only intersecting in v, see Figure 5.

Case (2a). Suppose that e_1 is chainable with e_2.

As emphasized in Remark 2.5, any o-edge is chainable with e_1. In this case assign label a to the orbit of the o-edge e_2 and label b to the orbit of the o-edge e_1. Moreover, assign label c to all other orbits. Then the above chainable pairs allow us to realise the syllables ab, ba, bb.

Case (2b). Suppose that e_1 is not chainable with e_2, but that e_1 is chainable with e_1^{-1}.

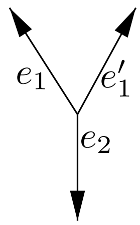

Figure 5. Case (2)

This case cannot occur. In fact, by hypothesis, there are two o-edges in the orbit of e_1 only intersecting in their common final point v. This easily implies that e_1 is chainable with any o-edge.

Case (2c). Suppose that e_1 is not chainable with e_2, and e_1 is not chainable with e_1^{-1}. We show that condition (i) of Theorem 1 applies.

We prove that one can put an orientation on $\mathcal{T}$ with the following properties:

- the (oriented) edge e is oriented positively (resp. negatively) if and only if it is contained in an o-edge in the orbit of e_1 (resp. e_1^{-1}).
- Every o-vertex has exactly one incoming edge.

Let us start with the remark that e_1 is not in the orbit of e_1^{-1}, for otherwise e_1 would be chainable with e_2.

Also, we claim that any o-edge e emanating from v and sharing the first edge with some o-edge in the same orbit as e_1 has to be in the same orbit as e_1 (Figure 6). In fact, if this was not the case we would

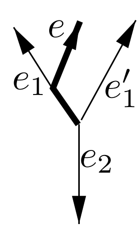

Figure 6

have o-edges in the orbits of e_1, e_2, e sharing only their common starting point. Hence, this would be a triple as in Case (1) because the orbits of e_1, e_2, e would be pairwise distinct: e_1 is not in the same orbit as e_2 by hypothesis, and if e was in the same orbit as e_2 there would be two o-edges emanating from v in the orbit of e_2 that only intersect at v, and this would imply that e_1 is chainable with e_2 (see Remark 2.5).

Finally, all o-edges in the orbit of e_1^{-1} emanating from v must share the first edge, for otherwise e_1 would be chainable with e_1^{-1} (see Remark 2.5).

The three observations above imply that all o-edges in the orbit of e_1^{-1} emanating from v share the first edge with e_2, for otherwise e_1 would be chainable with e_2.

Also, any other edge emanating from v is contained only in o-edges in the orbit of e_1, for otherwise we would have a triple as in Case (1). Notice that (once we show that the orientation is well-defined) this gives us the second property above.

Let us now show that any edge e is contained in some o-edge in the orbit of either e_1 or e_1^{-1}.

By minimality of the action, e is contained in some o-edge e'. If e' is in the orbit of either e_1 or e_1^{-1} we are done, so suppose not. By what we have shown so far, e' shares the initial edge with an o-edge in the orbit of e_1^{-1}. It is readily seen that e is either contained in an o-edge in the orbit of e_1^{-1} or in an o-edge e'' (the thickened o-edge in Figure 7) that lies in the orbit of e_1 since it shares the initial edge with an o-edge in the orbit of e_1.

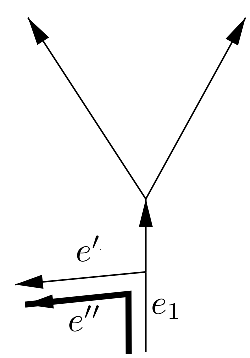

Figure 7

Finally, we are left to show that the orientation is well-defined, namely that any given edge cannot be contained both in an o-edge in the orbit of e_1 and an o-edge in the orbit of e_1^{-1}. If not, we could construct an o-edge e' (the thickened o-edge in Figure 8) sharing the first edge with e_1 and the final edge with an o-edge in the orbit of e_1^{-1}. The inverse of such o-edge would be in the orbit of e_1^{-1} but would share the first edge with some o-edge in the orbit of e_1, a contradiction.

The orientation we just constructed is clearly Γ-invariant, and there is only one point at infinity a corresponding to geodesic rays obtained

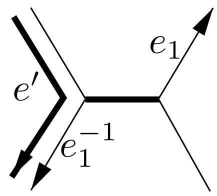

Figure 8

concatenating negatively oriented edges. Such point is a fixed point of Γ.

Case (3a). There exists a minimal integer k so that Γ acts transitively on $\mathcal{E}_v^{(1)}, \ldots, \mathcal{E}_v^{(k-1)}$ but not on $\mathcal{E}_v^{(k)}$, and also $k > 1$.

As positive integer we pick k.

By hypothesis there exist at least two orbits of oriented geodesics of orbit length k in $\mathcal{T}$. We pick one orbit of oriented geodesics, and we label as a that orbit, and we label as b some other orbit of oriented geodesics. Finally, we assign label c to all other orbits.

Let us now realise every word w in $\{a, b, c\}$ as (the label of) an oriented o-geodesic in $\mathcal{T}$. Consider any $o(k)$-geodesic ξ starting from the fixed vertex v with label the first letter of w. (The geodesic segment ξ exists because Γ is transitive on Γv.) Let ξ_{k-1} be the final subpath of ξ of orbit length $k-1$, and let θ be an oriented $o(k)$-geodesic containing ξ_{k-1} and labelled as the second letter of w (here we use the transitivity of Γ on $o(k-1)$-geodesics). We can now concatenate ξ with θ_1, where θ_1 is the final subpath of θ of orbit length 1 (see Figure 9). Iterating

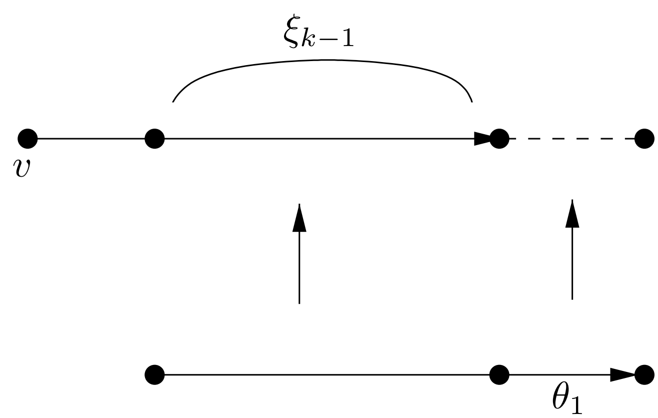

Figure 9. Realising a word as an oriented o-geodesic

this construction, we can construct an oriented o-geodesic with any required label.

Case (3b). Γ *acts transitively on* $\mathcal{E}_v^{(k)}$ *for each* k.

Notice that all o-edges have the same length, since they are in the same orbit. Also, such length cannot be larger than 2 for otherwise there would exist an o-edge of length 2, the thickened one in Figure 10. Therefore, the transitivity of the Γ-action on $\mathcal{E}^{(k)}$ for every $k > 0$

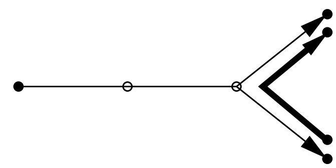

Figure 10. o-edges of length 2 and 3 in the same orbit

implies that we are in the situation described in point (ii) of Theorem 1.

2.7. **Preparatory lemmas.** From now on we assume that conditions (i) and (ii) of Theorem 1 do not hold, so that there exists a labelling with the property that all good words can be realised as the label of some o-geodesic. Recall that part of the data of a labelling is an integer k, which we fix from now on.

To justify what comes next, we mention now that we want to use the following simple criterion to prove linear independence.

Lemma 2.7. *Let* $\{c_n\}$ *be non-trivial quasimorphisms on a group* Γ. *Suppose that there exist elements* $\{\eta_n, h_n\}$ *of* Γ *so that for each positive integers* z, m, n *we have*

$$c_n(\eta_m^z) = c_n(h_m^z) = 0$$

and

$$\begin{cases} c_n((\eta_m h_m)^z) = 0 & \text{if } m \neq n \\ c_n((\eta_m h_m)^z) \geq z - 1 & \text{if } m = n \end{cases}$$

Then $\{[\delta^1 c_n]\}$ *are linearly independent in* $\mathrm{H}_\mathrm{b}^2(\Gamma, \mathbb{R})$.

Proof. The conclusion easily follows by considering the homogenization of c_n. □

For w a word in a, b, we denote w^{-1} the word obtained reading w from right to left and replacing each label of an $o(k)$-geodesic with the label of the $o(k)$-geodesic with opposite orientation.

Fix from now on a choice of words $\{w_{ni}\}$ as in Lemma 2.3, as well as $n_0 = n_0(k)$.

LEMMA 2.8. *Suppose that for some integers $n, m \geq n_0$, some $i, j \in \{1,2,3\}$ and some $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ we have o-geodesics γ_1, γ_2 in $\mathcal{T}$ labelled, respectively, by $w_{ni}^{\epsilon_1}$ and $w_{mj}^{\epsilon_2}$ that share a common o-subgeodesic of o-length at least* $\min\{|\gamma_1|_o, |\gamma_2|_o\}/10 - 1$. *Then $\epsilon_1 = \epsilon_2, i = j, m = n$.*

Notice that $|\gamma_1|_o = |w_{ni}^{\epsilon_1}| + k - 1$.

Proof. Notice that if two o-geodesics share a common subgeodesic of o-length, say L, then their labels have a common subword of length $L - k + 1$.

If $\epsilon_1 = \epsilon_2$, then it is easy to get $i = j, m = n$ because w_{ni} and w_{mj} can share a long subword only if they coincide.

We will now argue that we cannot have, say $\epsilon_1 = +1$ and $\epsilon_2 = -1$. Suppose by contradiction that this is the case. Notice that by Lemma 2.3-(iii) there must be an $o(k)$-geodesic labelled a or b (contained in γ_1) that coincides with the inverse of an $o(k)$-geodesic labelled, respectively, b or a (contained in γ_2). In the second case of Remark 2.4, this cannot happen, hence we can suppose that the labels a, b correspond to one orbit of $o(k)$-geodesics each. In this case we get that the label w'_{mj} of γ_2 is obtained reading w_{mj} right-to-left and replacing each a with b and vice versa. It is easily seen that w'_{mj} and w_{nj} cannot share a long subword in view of Lemma 2.3(i). □

DEFINITION 2.9. Let us call $w_{ni}^{\pm 1}$*-subgeodesic* an o-subgeodesic of an o-geodesic labelled $w_{ni}^{\pm 1}$. A *long $w_{ni}^{\pm 1}$-subgeodesic* is a $w_{ni}^{\pm 1}$-subgeodesic of o-length at least $(|w_{ni}| + k - 1)/2$.

DEFINITION 2.10. The *almost concatenation* of the o-geodesics γ_1 and γ_2 (if it exists) is the o-geodesic obtained concatenating either γ_1 and γ_2 or γ_1, an o-edge and γ_2. Almost concatenation of more than two o-geodesics can be defined similarly.

LEMMA 2.11. *There exist elements $g_{ni} \in \Gamma$ so that the following hold.*

(1) *There exists an o-geodesic s_n obtained almost concatenating a long w_{n3}-subgeodesic and a long w_{n1}-subgeodesic with the following property. For each positive integer N, the o-geodesic from v to $(g_{n1}g_{n3})^N v$ is the almost concatenation of a long w_{n1}-subgeodesic, $N-1$ translates of s_n and a long w_{n3}-subgeodesic.*
(2) *For each positive integer N, the o-geodesic from v to $(g_{n1}g_{n2})^N v$ (resp. $(g_{n2}^{-1}g_{n3})^N v$) is obtained alternately almost concatenating long w_{n1}-subgeodesics and long w_{n2}-subgeodesics (resp. long w_{n2}^{-1}-subgeodesic and long w_{n3}-subgeodesics).*

Proof. Chose g_{ni} so that the geodesic from v to $g_{ni}v$ is labelled w_{ni}. Let us study the o-geodesic $[v, (g_{n1}g_{n3})^2 v]$, the other cases being similar.

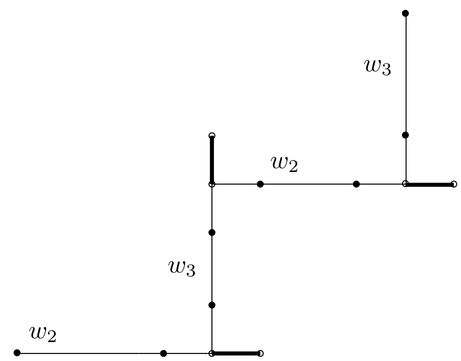

Figure 11. the o-geodesic $[v, (g_{n1}g_{n3})^2 v]$

We can form a path from v to $(g_{n1}g_{n3})^2 v$ by concatenating 4 geodesics, alternately labelled w_{n1} and w_{n3}. Since by Lemma 2.8 we can bound the overlap of geodesics labelled w_{n1} (resp. w_{n3}) and geodesics labelled w_{n3}^{-1} (resp. w_{n1}^{-1}), the conclusion easily follows, see Figure 11. □

2.8. **Conclusion of proof of Theorem 1.** Let s_n be as in Lemma 2.11. We want to apply Lemma 2.7 to $c_n = f_{s_n}$, $\eta_n = g_{n1}g_{n2}$, $h_n = g_{n2}^{-1}g_{n3}$ for n sufficiently large. Let us for example show $c_n((g_{m2}^{-1}g_{m3})^z) = 0$, the other cases being similar (it is also worthwhile to point out that the geodesic $[v, (g_{n1}g_{n3})^z v]$ contains at least $z-1$ translates of s_n by Lemma 2.11-(1)). If there were a translate of $s_n^{\pm 1}$ contained in $[v, (g_{m2}^{-1}g_{m3})^z v]$, then a suitable translate of s_n would contain a $w_{n1}^{\pm 1}$-subgeodesic γ of length $(|w_{n1}^{\pm 1}| + k - 1)/4 - 1$ with one of the following properties:

- γ is contained either in a long w_{m2}^{-1}-subgeodesic or in a long w_{m3}-subgeodesic, or
- γ contains either a long w_{m2}^{-1}-subgeodesic or a long w_{m3}-subgeodesic.

All configurations are excluded by Lemma 2.8. By definition of f_{s_n}, this implies $c_n((g_{m2}^{-1}g_{m3})^z) = 0$. □

3. Remarks on actions fixing a point at infinity

In this section we make an observation about actions as in Theorem 1 (i).

For $a \in \partial\mathcal{T}$ we denote by β_a any Busemann function based at a. Recall that, given a vertex w of $\mathcal{T}$ this is defined as

$$\beta_a(w) := \lim_{t\to\infty} \left(d(\gamma(t), w) - t\right),$$

where γ is a geodesic ray defining $a \in \partial\mathcal{T}$. Notice that $\beta_a(w)$ is an integer for every vertex w. Also, for $a \in \partial\mathcal{T}$ we denote by $\mathcal{E}_\beta^{(n)}$ the set of *monotone geodesics* of length n, meaning geodesics of length n along which β_a is increasing.

The following proposition says that Γ-orbits of monotone geodesics are determined by two parameters, namely the length n and the Busemann function modulo l at the starting point, where l is the translation length of any fixed hyperbolic element.

PROPOSITION 3.1. *Suppose that Γ acts minimally on the tree $\mathcal{T}$ and that every vertex of $\mathcal{T}$ has valence greater than 2. Suppose that Γ fixes a point a in the boundary $\partial\mathcal{T}$ of $\mathcal{T}$. Then Γ acts transitively on the set of monotone geodesics of any fixed length starting at any vertex of any fixed color.*

Moreover, l is the translation length of any fixed element of Γ that acts hyperbolically on $\mathcal{T}$.

Proof. First of all, let us show that there exists some $h \in \Gamma$ that acts hyperbolically on $\mathcal{T}$. Pick any $v \in \mathcal{T}$. It is easily seen that any element $h \in \Gamma$ so that $\beta_a(hv) \neq \beta_a(v)$ acts hyperbolically, because, up to exchanging v and hv and replacing h with h^{-1}, a subray of $[v, a)$ gets mapped by h into a proper subray. Since there are no leaves in $\mathcal{T}$, such h must exist.

In order to prove the proposition, we can fix a basepoint v on the axis of h and show that for any integer k, Γ acts transitively on the set of monotone $o(k)$-geodesics. This is clear when we fix the axis of a conjugate of h and we restrict to monotone $o(k)$-geodesics contained in that axis. Notice now that for any $g \in \Gamma$, the axis of h^g passes through gv and shares a common subray with the axis of h. Hence, for any $g \in \Gamma$, the monotone $o(k)$-geodesic starting at gv is in the same orbit as a monotone $o(k)$-geodesic contained in the axis of h, and hence it is in the same orbit as the monotone $o(k)$-geodesic starting at v. □

4. Local ∞-transitivity and quasimorphisms

In this section we prove Corollary 3. For future reference we record the following obvious observation:

LEMMA 4.1. *Let $\mathcal{T}$ be a simplicial tree and let $\Gamma < Aut(\mathcal{T})$. If Γ acts transitively on all geodesic segments of a given length starting on a set $\mathcal{S}$ of vertices, then for every vertex $v \in \mathcal{S}$ the stabiliser $\mathrm{Stab}_\Gamma(v)$ acts transitively on all spheres centered at v.*

Let $\mathcal{T}$ be a locally finite tree with $\mathcal{V}$ the set of vertices and $\mathcal{E}$ the set of edges. If $e \in \mathcal{E}$, we denote by $\mathcal{T}_e^+$ the connected component of $\mathcal{T} \smallsetminus \{e\}$ containing the terminus $t(e)$ of e.

Lemma 4.2. *Let $\mathcal{T}$ be a locally finite tree such that each vertex has valence at least 2. Let $\Lambda < Aut(\mathcal{T})$ be a subgroup such that $\Lambda\backslash\mathcal{T}$ is finite and there is no fixed point in the boundary $\partial\mathcal{T}$. Then Λ acts minimally on $\mathcal{T}$.*

Proof. We will show first that the Λ-action on the boundary $\partial\mathcal{T}$ is minimal, and conclude that the action on the tree is minimal as well.

Let $F \subseteq \partial\mathcal{T}$ be a closed non-empty Λ-invariant subset. We suppose that $F \neq \partial\mathcal{T}$. Let $\mathcal{T}'$ be the convex hull of F, that is the subtree consisting of all bi-infinite geodesics connecting two elements in F. Let $\xi \in \partial\mathcal{T} \smallsetminus F$ and let $e \in \mathcal{E}$ be an edge in $\mathcal{T}$ such that $\xi \in \partial\mathcal{T}_e^+ \subset \partial\mathcal{T} \smallsetminus F$. Let $v_n \in \mathcal{T}_e^+$ be a vertex with $d(v_n, t(e)) = n$. Since $\mathcal{T}_e^+$ and $\mathcal{T}'$ are disjoint, $d(\mathcal{T}', v_n) \geq n$ and, by invariance, $d(\mathcal{T}', \lambda v_n) \geq n$ for all $\lambda \in \Lambda$. Let $D \subset \mathcal{V}$ be a fundamental domain for the Λ-action in $\mathcal{V}$. Since D is finite, there exists $n_0 \in \mathbb{N}$ such that $d(v, \mathcal{T}') \leq n_0$ for all $v \in D$. By choosing n with $n > n_0$ we see that D cannot intersect the Λ-orbit of v_n, which is a contradiction. Thus Λ acts minimally on $\partial\mathcal{T}$.

Let now $\mathcal{T}' \subset \mathcal{T}$ be a Λ-invariant non-empty subtree. Then $\partial\mathcal{T}' \subseteq \partial\mathcal{T}$ is a closed Λ-invariant subset that is non-empty since $\Lambda\backslash\mathcal{T} < \infty$. We will show that $\partial\mathcal{T}' \neq \partial\mathcal{T}$. Since $\mathcal{T}' \neq \mathcal{T}$, there exists a vertex $v \in \mathcal{T}'$ such that the valence of v in $\mathcal{T}'$ is strictly smaller than the valence of v in $\mathcal{T}$. Let $v' \in \mathcal{V}(\mathcal{T})$ be an adjacent vertex with $v' \notin \mathcal{V}(\mathcal{T}')$ and let $e := (v, v')$ be the corresponding oriented edge in $\mathcal{T}$. Since the valence of each vertex is at least 2, this edge can be extended to an infinite ray, so that $\partial\mathcal{T}_e^+ \neq \varnothing$. But $\mathcal{T}_e^+ \cap \mathcal{T}' = \varnothing$, so that $\partial\mathcal{T}_e^+ \cap \partial\mathcal{T}' = \varnothing$ and hence $\partial\mathcal{T}' \neq \partial\mathcal{T}$. □

In order to prove Corollary 3 , we recall the following lemma from [BM00]:

Lemma 4.3 ([BM00, Lemma 3.1.1]). *Let $\mathcal{T}$ be a regular tree with vertex set $\mathcal{V}$ and $H < Aut(\mathcal{T})$ a closed subgroup. Then the following assertions are equivalent:*

(1) H is locally ∞-transitive;
(2) $\mathrm{Stab}_H(v)$ is transitive on $\partial\mathcal{T}$ for every vertex $v \in \mathcal{V}$;
(3) H is non-compact and acts transitively on $\partial\mathcal{T}$;
(4) H is doubly transitive on $\partial\mathcal{T}$.

Proof of Corollary 3. The implication (1)⇒(2) is in [BM02, Corollary 26], while (2)⇒(3) is obvious.

To show that (3)⇒(1) observe that since the $\mathcal{T}_i$, $i = 1, \ldots, k$ are regular and $\Gamma < \mathrm{Aut}(\mathcal{T}_1) \times \cdots \times \mathrm{Aut}(\mathcal{T}_k)$ is cocompact, then Γ-action on $\mathcal{T}_i$ is cofinite. If $H_i := \overline{\mathrm{pr}_i(\Gamma)}$, the projection

$$\mathrm{pr}_i : \mathrm{Aut}(\mathcal{T}_1) \times \cdots \times \mathrm{Aut}(\mathcal{T}_k) \to \mathrm{Aut}(\mathcal{T}_i)$$

induces an equivariant map

$$(\mathrm{Aut}(\mathcal{T}_1) \times \cdots \times \mathrm{Aut}(\mathcal{T}_k))/\Gamma \to \mathrm{Aut}(\mathcal{T}_i)/H_i\,,$$

so that on $\mathrm{Aut}(\mathcal{T}_i)/H_i$ there is a finite invariant measure. If H_i were to fix a point $\xi \in \partial\mathcal{T}_i$, it would be amenable (see for example [FTN91, Chapter 1, Section 8]) and because of the existence of a finite invariant measure on $\mathrm{Aut}(\mathcal{T}_i)/H_i$, $\mathrm{Aut}(\mathcal{T}_i)$ would be amenable as well. Hence H_i does not fix any point $\xi \in \partial\mathcal{T}_i$, so that by Lemma 4.2 the action on $\mathcal{T}_i$ is minimal. Thus the Γ-action on $\mathcal{T}_i$ satisfies the hypothesis of Theorem 1 and we must be in case (ii). By Lemma 4.1 there exists a vertex $x \in \mathcal{T}_i$ such that $\mathrm{Stab}_\Gamma(x)$ acts transitively on all spheres centered at x. By continuity $\mathrm{Stab}_{H_i}(x)$ acts transitively on $\partial\mathcal{T}_i$ and hence so does H_i. Since H_i is non-compact, by the implication (3)⇒(1) of Lemma 4.3, H_i is locally ∞-transitive. □

5. Examples

We show in this section that our Theorem 1 is not covered by the results of Fujiwara [Fuj00] and of Minasyan and Osin [MO13].

5.1. **Amalgamated free product of groups.** Theorem 1.1 [Fuj00] states that if $G = A *_C B$ is such that $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$ then there exists an injective $\mathbb{R}$-linear map $\ell^1(\Gamma) \to \mathrm{H}^2_{\mathrm{b}}(G,\mathbb{R})$. We now show how to deduce Fujiwara's result from Theorem 1.

According to Bass-Serre theory, G acts on a tree $\mathcal{T}$ whose vertices are the left G-cosets of A and B and whose edges are the left G-cosets of C.

If $|B/C| = 2$, as explained in Remark 2, we replace $\mathcal{T}$ with the tree $\mathcal{T}'$ obtained merging edges that share a vertex of degree 2, if not we set $\mathcal{T} = \mathcal{T}'$. Then $\mathcal{T}'$ satisfies the conditions of Theorem 1. We now claim that if $1, a_1, a_2 \in A$ lie in pairwise distinct double-cosets, then the oriented geodesics $[B, a_1B], [B, a_2B]$ in $\mathcal{T}$ are not in the same G-orbit. This easily implies that $\mathcal{T}'$ has at least two orbits of geodesics of length 2 and hence Theorem 1(ii) does not hold for $\mathcal{T}'$. To prove the claim, just observe that an element $g \in G$ mapping $[B, a_1B]$ to $[B, a_2B]$ stabilises $[B, A]$, and hence belongs to C, and maps the edge labelled a_1C to the one labelled a_2C, whence it also belongs to $a_2Ca_1^{-1}$, which contradicts the fact that a_1, a_2 are not in the same double coset (see Figure 12). Theorem 1(i) is well-known not to hold (for $\mathcal{T}'$), for example because the stabiliser in $\mathcal{T}$ of each A-vertex acts transitively on the edges emanating from it. Therefore, condition (iii) of Theorem 1 holds and we can deduce that $\mathrm{H}^2_{\mathrm{b}}(G,\mathbb{R})$ contains a copy of ℓ^1.

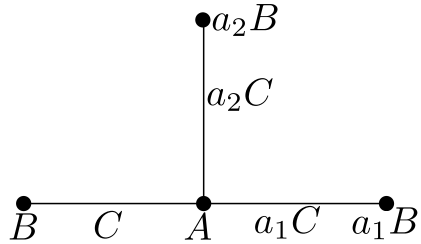

Figure 12

Our result is more general even when restricted to amalgamated products. Indeed, let us consider the group $S_3 *_{\mathbb{Z}_2} \mathbb{Z}_4$ where S_3 is the symmetric group on 3 elements and its associated tree $\mathcal{T}$. Since the stabilizer of each vertex of $\mathcal{T}$ is a finite group, it easily follows that conditions (i) and (ii) of Theorem 1 cannot be satisfied. Hence, $\mathrm{H}^2_{\mathrm{b}}(S_3 *_{\mathbb{Z}_2} \mathbb{Z}_4, \mathbb{R})$ is infinite dimensional, a fact that can also be deduced from the fact that it virtually free. However, it cannot be deduced from Fujiwara's result because $|\mathbb{Z}_2 \backslash S_3 / \mathbb{Z}_2| \geq 3$ does not hold.

5.1.1. *A family of examples.* Based on [BM00, Example 1.2.1] we can construct an infinite family of examples of (cocompact) irreducible lattices $\Gamma < \mathrm{PSL}(n, \mathbb{Q}_{p_1}) \times \mathrm{PSL}(n, \mathbb{Q}_{p_2})$, where p_1, p_2 are primes and $n \geq 3$. We illustrate the example for $n = 3$.

For $\ell \in \{p_1, p_2\}$ we consider the Bruhat–Tits building Δ_ℓ associated to $\mathrm{PSL}(3, \mathbb{Q}_\ell)$, that we proceed to recall (see [Bro89, Chapter V, 8B]). Let V be a 3-dimensional vector space over $\mathbb{Q}_\ell$. Let $\mathcal{L}'$ be the space of lattices $L \subset V$, that is of sub-$\mathbb{Z}_\ell$-modules $L = \mathbb{Z}_\ell v_1 + \mathbb{Z}_\ell v_2 + \mathbb{Z}_\ell v_3$, where $\{v_1, v_2, v_3\}$ is a $\mathbb{Q}_\ell$-basis of V. We consider on $\mathcal{L}'$ an equivalence relation where $L_1 \sim L_2$ if there exists $\lambda \in \mathbb{Q}_\ell^*$ such that $L_1 = \lambda L_2$, and let $\mathcal{L} := \mathcal{L}'/\sim$. On $\mathcal{L}$ we define an incidence relation, where we say that $[L_1], [L_2] \in \mathcal{L}$ are *incident* if $[L_1] \neq [L_2]$ and there exists representatives L_1, L_2 such that $\ell L_1 \subset L_2 \subset L_1$. Consider now the *flag complex*, that is the simplicial complex whose 0-cells are equivalence classes of lattices $[L] \in \mathcal{L}$, whose 1-cells are pairs of incident 0-cells and whose 2-cells are triples of pairwise incident 0-cells. We denote Δ_ℓ the flag complex with the natural $\mathrm{PSL}(3, \mathbb{Q}_\ell)$-action.

To an 0-cell $[L] \in \mathcal{L}$ we can associate a *type* as follows. Let L be a representative of $[L]$ with $L = \mathbb{Z}_\ell v_1 + \mathbb{Z}_\ell v_2 + \mathbb{Z}_\ell v_3$; if $\det(v_1, v_2, v_3) = \ell^n k$, where $k \in \mathbb{Z}_\ell^\times$, $n \in \mathbb{Z}$, then the type of $[L]$ is $n \mod 3$. It is easy to see that the type of an 0-cell is well defined. As an example, the type of $\mathbb{Z}_\ell e_1 + \mathbb{Z}_\ell e_2 + \mathbb{Z}_\ell e_3$ is 0, the type of $\mathbb{Z}_\ell e_1 + \mathbb{Z}_\ell e_2 + \mathbb{Z}_\ell \ell e_3$ is 1, while the type of $\mathbb{Z}_\ell e_1 + \mathbb{Z}_\ell \ell e_2 + \mathbb{Z}_\ell \ell e_3$ is 2, where e_1, e_2, e_3 are the standard basis elements. In addition to 3 types of 0-cells, there are also 3-types of 1-cells. Let $\mathcal{G}_\ell$ be the subgraph of Δ_ℓ consisting of edges of a given type, for example (02). Fix a vertex $[L_0]$ in $\mathcal{G}_\ell$, for example of type 0, $L_0 = \mathbb{Z}_\ell e_1 + \mathbb{Z}_\ell e_2 + \mathbb{Z}_\ell e_3$. The vertices $[L]$ incident to $[L_0]$ are all of type 2 and, by definition, all satisfy the condition that $\ell L_0 \subset L \subset L_0$. In other words they are in one-to-one correspondence with subspaces (in this case lines) in $L_0/\ell L_0 \simeq \mathbb{F}_\ell^3$. The stabiliser of $[L_0]$ in $\mathrm{PSL}(3,\mathbb{Q}_\ell)$ is $\mathrm{PSL}(3,\mathbb{Z}_\ell)$ and the effective action of $\mathrm{Stab}_{\mathrm{PSL}(3,\mathbb{Q}_\ell)}([L_0]) = \mathrm{PSL}(3,\mathbb{Z}_\ell)$ on the $\ell^2+\ell+1$ vertices incident to $[L_0]$ is the action of $\mathrm{PSL}(3,\mathbb{F}_\ell)$ on the $\ell^2+\ell+1$ lines in $\mathbb{F}_\ell^3$. Since $\mathrm{PSL}(3,\mathbb{F}_\ell)$ is doubly transitive on the set of pairs of lines, it follows that $\mathrm{Stab}_{\mathrm{PSL}(3,\mathbb{Q}_\ell)}([L_0])$ is doubly transitive on the set of incident vertices. It is easily shown that, given an action on a tree, if all vertex stabilisers are doubly transitive on the sphere of radius 1 around the corresponding point, then all stabilisers act transitively on the sphere of radius 2.

We have hence given a sketch of the proof the following:

Lemma 5.1. *The* $\mathrm{PSL}(3,\mathbb{Q}_\ell)$*-action on the graph* $\mathcal{G}_\ell$ *is locally* 2*-transitive.*

Let $\mathcal{T}_{\ell^2+\ell+1}$ be the regular tree that is the universal covering of $\mathcal{G}_\ell$. Let

$$0 \longrightarrow \pi_1(\mathcal{G}_\ell) \hookrightarrow H_\ell \twoheadrightarrow \mathrm{PSL}(3,\mathbb{Q}_\ell) \longrightarrow 0$$

be the exact sequence associated to the universal covering projection $\pi : \mathcal{T}_{\ell^2+\ell+1} \to \mathcal{G}_\ell$, that is

$$H_\ell := \{g \in \mathrm{Aut}(\mathcal{T}_{\ell^2+\ell+1}) : \pi \circ g = h \circ \pi \text{ for some } h \in \mathrm{PSL}(3,\mathbb{Q}_\ell)\},$$

with

$$\pi_1(\mathcal{G}_\ell) = \{g \in \mathrm{Aut}(\mathcal{T}_{\ell^2+\ell+1}) : \pi \circ g = \pi\} \triangleleft H_\ell . \tag{5.1}$$

The following lemma shows that H_ℓ cannot be locally ∞-transitive.

Lemma 5.2. *Let $\mathcal{T}$ be a locally finite tree and let $H < Aut(\mathcal{T})$ be a closed locally ∞-transitive subgroup. Any normal discrete subgroup $N \triangleleft H$ consisting of hyperbolic elements must be trivial.*

Proof. Since it is discrete, the group N is countable and, in particular, the set of axes $\{a_n : n \in N, n \neq e\}$ is countable. If $h \in H$, we have that $h(a_n) = a_{hnh^{-1}}$. But Lemma 4.3 (1)⇒(4) implies that H acts doubly transitive on $\partial\mathcal{T}$, so that $\{h(a_n) : h \in H\}$ would be uncountable. □

Let us consider now an irreducible (cocompact) lattice

$$\Gamma < \mathrm{PSL}(3,\mathbb{Q}_{p_1}) \times \mathrm{PSL}(3,\mathbb{Q}_{p_2})$$

and let us consider the inverse image $\widetilde{\Gamma} < H_{p_1} \times H_{p_2}$ of Γ with respect to the exact sequence

$$\pi_1(\mathcal{G}_{p_1}) \times \pi_1(\mathcal{G}_{p_2}) \hookrightarrow H_{p_1} \times H_{p_2} \twoheadrightarrow \mathrm{PSL}(3, \mathbb{Q}_{p_1}) \times \mathrm{PSL}(3, \mathbb{Q}_{p_2})$$

Since Γ is irreducible, it has dense projections and hence, by Lemma 5.1 the action of $\mathrm{pr}_i(\Gamma)$ on $\mathcal{G}_{p_i}$ is locally 2-transitive, for $i = 1, 2$. It follows that also the action of $\mathrm{pr}_i(\widetilde{\Gamma})$ on $\mathcal{T}_{p_i^2+p_i+1}$ is locally 2-transitive; however by Lemma 5.2, the closures of these projections are not locally ∞-transitive.

By Corollary 3 we deduce that $\widetilde{\Gamma}$ has an infinite dimensional set of linearly independent median quasimorphisms. Notice that Γ, being an irreducible lattice in a high rank Lie group, has no non-trivial quasimorphisms, [BM02].

The same construction can of course being performed for an irreducible lattice $\Gamma < \mathrm{PSL}(n, \mathbb{Q}_{p_1}) \times \mathrm{PSL}(n, \mathbb{Q}_{p_2})$, for any $n \geq 3$, in which case we will have an $(n-1)$-dimensional flag complex.

Notice that Fujiwara's condition on the number of double cosets is in particular incompatible with the local 2-transitivity of $\widetilde{\Gamma}$. In fact, if we realise $\widetilde{\Gamma}$ as the product of $\mathrm{Stab}_{\widetilde{\Gamma}}(v_1)$ and $\mathrm{Stab}_{\widetilde{\Gamma}}(v_2)$, amalgamated over $\mathrm{Stab}_{\widetilde{\Gamma}}(v_1, v_2)$, the local 2-transitivity is equivalent to $\left|\mathrm{Stab}_{\widetilde{\Gamma}}(v_1, v_2) \backslash \mathrm{Stab}_{\widetilde{\Gamma}}(v_i) / \mathrm{Stab}_{\widetilde{\Gamma}}(v_1, v_2)\right| = 2$, for $i = 1, 2$.

5.2. **Simple groups with many median quasimorphisms.** Examples of groups satisfying Theorem 1 (iii) are the Burger–Mozes universal groups $U(F)$, where F is a permutation group that is not 2-transitive, [BM00]. Burger–Mozes point out that if F is transitive and generated by its point stabilizers, then $U(F)$ has a simple subgroup of index 2 ([BM00, Proposition 3.2.1] and [Tit70, Theorem 4.5]). Thus in particular these $U(F)$ are not acylindrically hyperbolic and hence the techniques of [HO13] to detect infinite dimensionality of $\mathrm{H}^2_{\mathrm{b}}(G, \mathbb{R})$ do not apply.

References

[Bav91] Ch. Bavard. Longueur stable des commutateurs. *Enseign. Math. (2)*, 37(1-2):109–150, 1991.

[BBF13] M. Bestvina, K. Bromberg, and K. Fujiwara. Bounded cohomology via quasi-trees. `arXiv:1306.1542`, 2013.

[BFS13] U. Bader, A. Furman, and R. Sauer. Integrable measure equivalence and rigidity of hyperbolic lattices. *Invent. Math.*, 194(2):313–379, 2013.

[BI04] M. Burger and A. Iozzi. Bounded Kähler class rigidity of actions on Hermitian symmetric spaces. *Ann. Sci. École Norm. Sup. (4)*, 37(1):77–103, 2004.

[BIW10] M. Burger, A. Iozzi, and A. Wienhard. Surface group representations with maximal Toledo invariant. *Ann. of Math. (2)*, 172(1):517–566, 2010.

[BM00] M. Burger and S. Mozes. Groups acting on trees: from local to global structure. *Inst. Hautes Études Sci. Publ. Math.*, (92):113–150 (2001), 2000.

[BM02] M. Burger and N. Monod. Continuous bounded cohomology and applications to rigidity theory. *Geom. Funct. Anal.*, 12(2):219–280, 2002.

[Bro89] K. S. Brown. *Buildings*. Springer-Verlag, New York, 1989.

[BS06] U. Bader and Y. Shalom. Factor and normal subgroup theorems for lattices in products of groups. *Invent. Math.*, 163(2):415–454, 2006.

[CF10] P.-E. Caprace and K. Fujiwara. Rank-one isometries of buildings and quasi-morphisms of Kac-Moody groups. *Geom. Funct. Anal.*, 19(5):1296–1319, 2010.

[CS11] P.-E. Caprace and M. Sageev. Rank rigidity for CAT(0) cube complexes. *Geom. Funct. Anal.*, 21(4):851–891, 2011.

[FPS13] R. Frigerio, M. B. Pozzetti, and A. Sisto. Extending higher dimensional quasi-cocycles. `arXiv:1311.7633`, 2013.

[FTN91] A. Figà-Talamanca and C. Nebbia. *Harmonic analysis and representation theory for groups acting on homogeneous trees*, volume 162 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1991.

[Fuj00] K. Fujiwara. The second bounded cohomology of an amalgamated free product of groups. *Trans. Amer. Math. Soc.*, 352(3):1113–1129, 2000.

[Ghy87] É. Ghys. Groupes d'homéomorphismes du cercle et cohomologie bornée. In *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 81–106. Amer. Math. Soc., Providence, RI, 1987.

[Gro82] M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982.

[HO13] M. Hull and D. Osin. Induced quasicocycles on groups with hyperbolically embedded subgroups. *Algebr. Geom. Topol.*, 13(5):2635–2665, 2013.

[Mat87] S. Matsumoto. Some remarks on foliated S^1 bundles. *Invent. Math.*, 90(2):343–358, 1987.

[Min01] I. Mineyev. Straightening and bounded cohomology of hyperbolic groups. *Geom. Funct. Anal.*, 11(4):807–839, 2001.

[Min02] I. Mineyev. Bounded cohomology characterizes hyperbolic groups. *Q. J. Math.*, 53(1):59–73, 2002.

[MO13] A. Minasyan and D. Osin. Acylindrical hyperbolicity of groups acting on trees. `arXiv:1310.6289`, 2013.

[MS04] N. Monod and Y. Shalom. Cocycle superrigidity and bounded cohomology for negatively curved spaces. *J. Differential Geom.*, 67(3):395–455, 2004.

[Tit70] J. Tits. Sur le groupe des automorphismes d'un arbre. In *Essays on topology and related topics (Mémoires dédiés à Georges de Rham)*, pages 188–211. Springer, New York, 1970.

Department Mathematik, ETHZ, 8092 Zürich,
Email address: alessandra.iozzi@math.ethz.ch

Department Mathematik, ETHZ, 8092 Zürich,
Email address: cristina.pagliantini@math.ethz.ch

Department Mathematik, ETHZ 8092 Zürich,
Email address: alessandro.sisto@math.ethz.ch