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Multivariate Risk Measures Based on Conditional Expectation and Systemic Risk for Exponential Dispersion Models

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Abstract

Exponential dispersion models are well used and studied in quantitative risk management and actuarial science. One of the main interests is the risk measurement analysis of such models when facing extreme loss events. In this paper, we propose two multivariate risk measures based on conditional expectation and derive the explicit formulae for exponential dispersion models. In particular, our multivariate risk measures could facilitate a systemic risk measure with explicit expressions for exponential dispersion models subject to any pre-specified "systemic event." We provide two numerical examples based on practical data to show the advantages of our approach in the context of exponential dispersion models.

Keywords: Multivariate risk measures; Conditional expectation; Systemic risks; Capital allocation; Exponential dispersion models;

JEL: C46; D81

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1 Introduction

Investors and regulators in finance and insurance utilize risk measures to assess risks for various purposes. Most standard risk measures in the literature, such as Value-at-Risk (VaR), Tail Value-at-Risk (TVaR) and Expected Shortfall (ES), evaluate the risk by a single scale, which suggests an amount of capital added to the risk to make it acceptable. Studies on these "standalone" risk measures have been well-documented in literature. We refer to Denuit et al. (2005) and McNeil et al. (2005) for throughout reviews. In practice, decision-makers often work with systems involving multiple risks, in which the dependence among risks as well as external risk factors impose significant influences. Disregarding these factors could result in inappropriate regulations such as over-(under-)estimation of capital allocation and bias risk aggregation. Moreover, risks may come from heterogeneous sources and cannot be directly aggregate in practice; see, for example, Cousin and Di Bernardino (2013). In order to account for the impact of dependence and facilitate risk aggregation more realistically, multivariate risk measure, which assesses multiple risks by a vector of multiple scales, has been proposed in the literature.

Similar to univariate risk measures, there are many different types of multivariate risk measures in literature. Jouini et al. (2004) defined a type of vector-valued coherent risk measures. Mainik and Schaanning (2012) presented a vector-valued Conditional ES and used it to assess systemic risk. Cousin and Di Bernardino (2013, 2014) provided multivariate extensions to VaR and Conditional Tail Expectation based on their multivariate VaR. Cossette et al. (2016) proposed a bivariate TVaR and discussed its application in capital allocation. Merakli and Kucukyavuz (2018) defined a multivariate conditional VaR that differs from Cousin and Di Bernardino (2013). Landsman et al. (2016) presented explicit formulae of multivariate conditional tail expectation for Elliptical distribution. There are more relevant studies on multivariate risk measures in Torres et al. (2015), Di Bernardino et al. (2015), Cai et al. (2017), and references therein.

Despite the difference between these multivariate risk measures, a common feature of them is using conditional expectations. Intuitively, such a feature is consistent with practical application as these multivariate risk measures attempt to assess the expected loss under specific scenarios that are of interest and concern for stakeholders. For example, the regulators could consider the scenario where the whole economy is in a downturn and assess the risks for financial institutions exposed to such negative externalities and the systemic risk of the entire financial market. Likewise, the "contagion effect" can also be tested by imposing an extreme perilous event on one particular financial institution and assessing how other relevant business units' risks change in response to this scenario. Such multivariate risk measure based on conditional expectation is very suitable for these scenario analyses. It does not only provide a reasonable amount to compensate for the expected loss of underlying risks in the system but also takes the dependence among risks into account. Accordingly, decision-makers can take the conditioning scenarios as certain "systemic events" that are of their interest and concerns.

In the literature, Brownlees and Engle (2016) proposed a so-called SRISK measure, which is the expected capital shortfall of a financial entity conditioning on a prolonged market decline, and took the sum of SRISK across all firms as a measure of overall systemic risk for the entire financial system. The authors conducted an empirical analysis using their conditional multivariate risk measures based on U.S. financial firm's data. Similarly, Acharya et al. (2017) presented another empirical study using an analogue conditional multivariate risk measure, called systemic expected shortfall, to assess systemic risk. They emphasized the system's loss as a systemic event for conditioning. Both of the two empirical studies showed that such multivariate conditional expectation-based risk measure works well in assessing and predicting systemic risk. In theory, Feinstein et al. (2017) set up a general framework for measures of systemic risk. Notably, the conditional multivariate risk measures and its aggregation are in line with their acceptance criteria for system risk, i.e., it indeed provides an amount to make the systemic risk accepted and bail out the potential undercapitalization. Following the same clue, Biagini et al. (2019) proposed a unified approach to systemic risk measures using the acceptance sets and considered a scenario-dependent allocation and the contagion effect of risks under their framework. Dhaene et al. (2019) introduced a class of conditional distortion risk measures and discussed the connection between this class and the conventional stochastic orders.

By contrast with these well-documented studies, in this paper, we aim to derive explicit formulae for conditional multivariate risk measures instead of proposing a methodological framework. More specifically, we consider the multivariate Exponential Dispersion Model (EDM) in our work. The multivariate EDM is a class of multivariate distributions that contains many frequently-used probability distributions in finance and insurance, especially of modeling losses. Smyth and Jøgensen (2002) applied Tweedie's compound Poisson model, which is a particular case of EDM, to fit insurance claims data. It also plays a critical role in the Generalized Linear Model as the response variates are generally assumed to follow a specific distribution in the multivariate EDM (Blough et al. 1999). Many other applications of the multivariate EDM can be found in Landsman and Valdez (2005), De Jong and Heller (2008), Alai et al. (2015), Shi (2016), Shushi (2017), Bäuerle & Shushi (2019) and reference therein.

Our work could contribute to relevant studies in the literature in two folds.

First, we propose two multivariate risk measures (multivariate conditional expectation and multivariate conditional entropy) that include many other multivariate risk measures in the literature as special cases. Essentially, our multivariate risk measures are based on multivariate conditional expectations. As such, we can play with the conditioning region in our multivariate risk measure to assess the expected shortfall subject to various scenarios, i.e., our multivariate risk measures facilitate scenario analyses. Following this line, we further propose to use our multivariate risk measures to assess the systemic risk where the conditioning region is taken as "systemic events." More specifically, we take the straightforward aggregating of the multiple scales in our multivariate risk measure as a single-valued systemic risk measure. In particular, the weights derived from the normalized multivariate risk measure provide a top-down capital allocation rule, which is consistent with the so-called "scenario-dependent allocation" rule in Biagini et al. (2019). Both the systemic risk and allocation rule can serve for the investigation of how one financial institution's perilous situation affects other unities through the financial network, i.e., the contagion effect. In addition, our multivariate conditional entropy can be regarded as a natural multivariate extension to the entropy coherent measures in Laeven and Stadje (2013).

Second, we apply our multivariate risk measures to the EDM. We work out explicit formulae for these risk measures proposed in our paper. Note that given a specific model for regulators' consideration, then the financial network is identified, and the endogenous mechanism of the contagion effect is determined. However, one still needs to find a reasonable way to understand and explain the consequences of a possible event and a computational method to work out these risk measures for regulations. In this regard, our explicit formulae are not only useful in assessing the (systemic) risks of EDM but also provide an analytical method to study to what extent the individual risks contribute to the total systemic risk and how the contagion spreads through the financial system. Moreover, we can show that our explicit risk measure formulae characterize the EDM. Note that the multivariate EDM is fundamental in the generalized linear models and frequently applied in insurance models. Thus, our results are helpful for the sensitivity analysis and stress tests of relevant models. We present two numerical examples of EDM to illustrate the advantages of our results in this regard.

The rest of the paper is organized as follows. Section 2 proposes the two types of multivariate risk measures based on conditional expectation. In Section 3, we first briefly revise the definition of EDM, then we present the explicit formulae of our multivariate risk measures for EDM. As applications, we apply our results to two numerical examples based on practical data in Section 4. We also provide some discussions on the numerical examples. Section 5 concludes the paper.

2 Multivariate risk measures based on conditional expectation

For a random vector of mutually dependent risks $\mathbf{X} = (X_1, \ldots, X_n)^T$, $n \geq 2$, defined on a probability space $(\Omega, \mathcal{F}, P)$, a multivariate risk measure is a mapping from $\mathbf{X}$ to an n-dimensional scale vector in $\mathbb{R}^n$, i.e.,

$$\mathbf{\Psi} : \mathbf{X} \mapsto (x_1, \ldots, x_n)^T, \text{ where } (x_1, \ldots, x_n)^T \in \mathbb{R}^n.$$

We first propose using multivariate conditional expectation (MCE) as the multivariate risk measures. The definition is given as follows.

Definition 2.1 (MCE) *Let $\Omega_{\mathbf{X}} = \Omega_{X_1,\ldots,X_n} \subset \Omega$ be a measurable sub-set of $\mathbf{X}$ in $\mathcal{F}$. The multivariate conditional expectation of $\mathbf{X}$, defined as*

$$\mathbf{\Psi}_E(\mathbf{X}; \mathbf{\Omega_X}) := E[\mathbf{X}|\mathbf{X} \in \Omega_{\mathbf{X}}] \in \mathbb{R}^n, \tag{1}$$

is a multivariate risk measure for $\mathbf{X}$.

Note that, when $n = 1$, $\Psi_E(X; \Omega_X)$ collapses to the conventional univariate conditional expectation of X. Moreover, in our MCE, one is free to choose her preferred $\Omega_{\mathbf{X}}$ for the risk measures. We list some examples of $\Omega_{\mathbf{X}}$ that include other proposed multivariate risk measures as special cases.

Example 2.2

1. Cossette et al. (2016) proposed multivariate lower-orthant VaR (notation $\underline{VaR}_q(\mathbf{X})$)

$$\underline{VaR}_q(\mathbf{X}) = E\left(\mathbf{X} | F_{\mathbf{X}}(\mathbf{x}) = q\right),$$

   and multivariate upper-orthant VaR (notation $\overline{VaR}_q(\mathbf{X})$)

$$\overline{VaR}_q(\mathbf{X}) = E\left(\mathbf{X} | \overline{F}_{\mathbf{X}}(\mathbf{x}) = 1 - q\right),$$

   where $q \in (0, 1)$ is the quantile level and $F_{\mathbf{X}}(\mathbf{x})$ is the cumulative distribution function (cdf) of random vector $\mathbf{X}$. $\Omega_{\mathbf{X}}$ is taken as $F_{\mathbf{X}}(\mathbf{x}) = q$ and $\overline{F}_{\mathbf{X}}(\mathbf{x}) = 1 - q$ respectively.

2. Similarly with the multivariate lower-(upper-)orther VaR, Cousin and Di Bernardino (2014) defined the multivariate lower-orthant tail conditional expectation (TCE)

$$\underline{TCE}_q(\mathbf{X}) = E\left(\mathbf{X} | F_{\mathbf{X}}(\mathbf{X}) \geq q\right) \tag{2}$$

   and multivariate upper-orthant TCE

$$\overline{TCE}_q(\mathbf{X}) = E\left(\mathbf{X} | \overline{F}_{\mathbf{X}}(\mathbf{X}) \leq 1 - q\right). \tag{3}$$

3. Landsman et al. (2016) proposed another extension of multivariate TCE (MTCE).

$$MTCE_q(\mathbf{X}) = E\left(\mathbf{X} | X_1 > VaR_q(X_1), X_2 > VaR_q(X_2), \ldots, X_n > VaR_q(X_n)\right), \tag{4}$$

   where $\Omega_{\mathbf{X}}$ is taken as

$$\Omega_{\mathbf{X}} = \{X_1 \geq VaR_q(X_1)\} \cap \ldots \cap \{X_n \geq VaR_q(X_n)\}. \tag{5}$$

In the axiomatic studies in literature, a risk measure is often expected to satisfy some desirable properties (Artzner et al., 1999). For example, the MTCE in (4) satisfies positive homogeneity, translation invariance, semi-subadditivity, and monotonicity; the multivariate TCE and MVaR in (2) and (3) also satisfy the positive homogeneity property, translation invariance and monotonicity, and sub-additivity in the case of mutually independent risks. The general setting of MCE may incorporate these multivariate risk measures as specials but also impede some desirable properties. Nevertheless, it is straightforward to show that translation-invariance and positive homogeneity are reserved.

Proposition 2.3 *We say that a multivariate risk measure is translation-invariant and positive homogeneous if it satisfies the following properties:*

- *Translation invariance: For any* $\mathbf{X}$ *and any vector of constants* $\boldsymbol{\alpha} \in \mathbb{R}_+^n$

$$\Psi(\mathbf{X} + \boldsymbol{\alpha}) = \Psi(\mathbf{X}) + \boldsymbol{\alpha}.$$

- *Positive homogeneity: For any* $n \times 1$ *random vector of risks* $\mathbf{X}$ *and a positive constant* λ*, we have*

$$\Psi(\lambda \mathbf{X}) = \lambda \Psi(\mathbf{X}).$$

Moreover, the MCE (1) in Definition 2.1 satisfies the two properties.

In addition to MCE, one may also define other analogue risk measures using multivariate conditional expectations. In particular, when the moment generating function (mgf) of $\mathbf{X}$ exists, we propose the multivariate conditional entropy (MCEn) as another multivariate risk measure in the following definition.

Definition 2.4 (MCEn) *Let* $\Omega_{\mathbf{X}} = \Omega_{X_1,\ldots,X_n} \subset \Omega$ *be a measurable sub-set of* $\mathbf{X}$ *in* $\mathcal{F}$*. The multivariate conditional entropy of* $\mathbf{X}$*, defined as*

$$\boldsymbol{\Psi}_{En}(\mathbf{X}; \boldsymbol{\Omega}_{\mathbf{X}}) := \left(\gamma_1^{-1} \ln E\left(e^{-\gamma_1 X_1} | \mathbf{X} \in \Omega_{\mathbf{X}}\right), \ldots, \gamma_n^{-1} \ln E\left(e^{-\gamma_n X_n} | \mathbf{X} \in \Omega_{\mathbf{X}}\right)\right)^T, \tag{6}$$

where $\gamma_i > 0, i = 1, 2, \ldots, n$*, is a multivariate risk measure for* $\mathbf{X}$*.*

MCEn is a natural extension to the single-scale conditional entropy, which is defined as $\gamma^{-1} \ln E\left(e^{-\gamma X} | X \in \Omega_X\right)$ for a $\Omega_X \subset \Omega$ and $\gamma > 0$. In Detlefsen and Scandolo (2005), the parameter $\gamma > 0$ is explained as the degree of risk aversion for investors with exponential utility functions. It also can be regarded as a multivariate extension to the entropy coherent measures in Laeven and Stadje (2013).

Both $\boldsymbol{\Psi}_E(\mathbf{X}; \boldsymbol{\Omega}_{\mathbf{X}})$ and $\boldsymbol{\Psi}_{En}(\mathbf{X}; \boldsymbol{\Omega}_{\mathbf{X}})$ give different scale vectors to measure $\mathbf{X}$ by considering various conditioning regions. In this regard, regulators could choose *conditioning regions* to assess the risk of $\mathbf{X}$ according to realistic situations, which fits the appetite of practical applications. On one hand, some conditioning regions, such as the *tail regions* (see, e.g., Cai et al., 2017), are of particular interests as it reflects a downturn economic background.

In this case, the components of $\mathbf{\Psi}_E(\mathbf{X};\mathbf{\Omega}_\mathbf{X})$ and $\mathbf{\Psi}_{En}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})$ are not only assessments to the expected losses of each risk but also provide implications on how individual risks interact with each other in this distress condition. By contrast with a single scale for each X_i, MCE and MCEn take the dependence structure among the risks into account. In particular, it suggests a top-down capital allocation rule for an available amount (cf. Cossette et al. 2016 and (8)). On the other hand, by taking different conditioning regions into account, it is straightforward to implement scenarios analysis on the entire system of $\mathbf{X}$. Hence, $\mathbf{\Psi}_E(\mathbf{X};\mathbf{\Omega}_\mathbf{X})$ and $\mathbf{\Psi}_{En}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})$ also account for the systemic risk of $\mathbf{X}$ where the conditioning regions are the so-called "*systemic events*" in Acharya et al. (2016). More specifically, it is easy to assess how the systemic risk varies along with different "systemic events" using $\mathbf{\Psi}_E(\mathbf{X};\mathbf{\Omega}_\mathbf{X})$ and $\mathbf{\Psi}_{En}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})$.

However, such multivariate risk measures using conditioning are subject to all sub-sets $\Omega_\mathbf{X}$ in $\mathcal{F}$; hence they are generally under partial order, which brings difficulties in comparison for the systemic risk with respect to different systemic events $\Omega_\mathbf{X}$. A straightforward way to tackle this problem is to consider the aggregation of all components in multivariate risk measures, i.e.

$$S_{\mathbf{\Psi}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})} := \sum_{j=1}^{n} \mathbf{\Psi}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})_j. \tag{7}$$

As aforementioned, $\mathbf{\Psi}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})$ could translate heterogeneous risks that may not be able to aggregate directly into a scalar vector. By taking the aggregation, (7) reduces the information in this vector to a single scale, which can be interpreted as the expected financial resources that are needed to cover the loss of system under event $\Omega_\mathbf{X}$. In other words, (7) could represent the cost of a bail-out for the systemic risk. The aggregation in (7) also provides a top-down capital allocation rule. Specifically, for an available capital K, we may assign the amount $w_{\mathbf{\Psi}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})}[i] * K$ to i-th risk X_i where

$$w_{\mathbf{\Psi}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})}[i] = \frac{\mathbf{\Psi}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})_i}{S_{\mathbf{\Psi}(\mathbf{X};\mathbf{\Omega}_\mathbf{X})}}, i = 1, 2, \ldots, n. \tag{8}$$

3 Multivariate risk measures for multivariate EDM

In this section, we derive explicit formulae to these multivariate risk measures for EDM.

3.1 Multivariate EDM

We first provide a very concise introdution to the family of multivariate exponential dispersion models (EDM). The multivariate EDM is a class of multivariate distributions that contains many frequently-used distributions in modeling financial risks. We say that the distribution of $\mathbf{X}$ has an exponential dispersion (ED) distribution if its probability measure takes the form (Jørgensen 1987)

$$dF_{\mathbf{X}}(\mathbf{x}) = e^{\lambda\left(\mathbf{x}^T\boldsymbol{\theta}-k(\boldsymbol{\theta})\right)}dQ\left(\lambda,\mathbf{x}\right), \mathbf{x}\in\mathbf{R}^n, \boldsymbol{\theta}\in\Theta, \lambda>0,$$

where $Q\left(\lambda,\mathbf{x}\right)$ is a distribution on $\mathbb{R}^n$, and

$$k\left(\boldsymbol{\theta}\right) = \log\int\exp\left(\mathbf{x}^T\boldsymbol{\theta}\right)dQ\left(\mathbf{x}\right)$$

is called the cumulant generating function of Q and the set Θ of $\boldsymbol{\theta}$ satisfies $\Theta = \{\boldsymbol{\theta}\in\mathbb{R}^n : k\left(\boldsymbol{\theta}\right)<\infty\}$. The probability density function (pdf) of $\mathbf{X}$ is then given by

$$f_{\mathbf{X}}\left(\mathbf{x};\lambda,\boldsymbol{\theta}\right) = a\left(\lambda,\mathbf{x}\right)e^{\lambda\left(\mathbf{x}^T\boldsymbol{\theta}-k(\boldsymbol{\theta})\right)}, \mathbf{x}\in\mathbb{R}^n, \quad (9)$$

where a and k are given functions, and we denote it as $\mathbf{X}\backsim ED_n\left(\boldsymbol{\theta},\lambda\right)$.[1]

Moreover, one could define the location and dispersion of $\mathbf{X}$ as

$$\begin{aligned}\boldsymbol{\mu} &= \tau\left(\boldsymbol{\theta}\right) = \nabla_{\boldsymbol{\theta}}k\left(\boldsymbol{\theta}\right), \quad (10)\\ \sigma^2 &= 1/\lambda,\end{aligned}$$

[1] Let Λ be a subset of $\mathbb{R}_+ = (0,\infty)$

$$\Lambda = \left\{\lambda>0 \mid \lambda k\left(\boldsymbol{\theta}\right) = \log\int_{\mathbb{R}}e^{\mathbf{x}^T\boldsymbol{\theta}}dQ^*\left(\lambda,\mathbf{x}\right)\right\},$$

for some $Q^*\left(\lambda,\mathbf{x}\right)$. Then, Λ is the set of λ such that $\lambda k(\cdot)$ is the cumulant generating function of some distribution Q_λ, where τ is also called the mean value mapping in the literature[2]. Then, we may obtain the covariance matrix and the moment generating function of $\mathbf{X}$ as

$$\begin{aligned} Cov\left(\mathbf{X}\right) &= \sigma^2\Sigma = \sigma^2\nabla_{\boldsymbol{\theta}}\nabla_{\boldsymbol{\theta}}^T k|_{\boldsymbol{\theta}=\tau^{-1}(\boldsymbol{\mu})}, \\ M_{\mathbf{X}}\left(\mathbf{t}\right) &= \exp\left\{\lambda\left(k\left(\boldsymbol{\theta}+\mathbf{t}/\lambda\right)-k\left(\boldsymbol{\theta}\right)\right)\right\}, \mathbf{t}\in\mathbb{R}^n, \end{aligned} \tag{11}$$

Note that many celebrated and well-used distributions belong to the ED family of distributions, e.g., the multivariate normal, multinomial, Poisson, and Gamma distributions. Thus, many types of risk models based on the ED family in practice fall in the class of EDM. For more details on the EDM, we refer to Jørgensen, B. (1987).

3.2 MCE formula for EDM

We present the formula of MCE for EDM in the following theorem.

Theorem 3.1 *Let* $\mathbf{X} \backsim ED_n\left(\boldsymbol{\theta},\lambda\right)$ *be an exponential dispersion random vector. Then, the MCE of* $\mathbf{X}$ *is given by,*

$$\Psi_E\left(\mathbf{X};\Omega_{\mathbf{X}}\right) = \boldsymbol{\mu} + \sigma^2\nabla_{\boldsymbol{\theta}}\log F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}|\boldsymbol{\theta},\lambda\right), \tag{12}$$

where $\nabla_{\boldsymbol{\theta}} = \left(d/d\theta_1,\dots,d/d\theta_n\right)^T$ *is the vector of derivatives with respect to* $\boldsymbol{\theta}$*,* $\boldsymbol{\mu}$ *is the location vector defined in (10), and* $F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}|\boldsymbol{\theta},\lambda\right) := \Pr\left(\mathbf{X}\in\Omega_{\mathbf{X}}\right)$ *is the probability of* $\mathbf{X}$ *to be in the region* $\Omega_{\mathbf{X}}$*.*

Proof. Similar to the univariate case shown in Theorem 1 in Landsman and Valdez (2005), we observe that

$$\begin{aligned} \nabla_{\boldsymbol{\theta}}\log F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}|\boldsymbol{\theta},\lambda\right) &= \frac{1}{F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}|\boldsymbol{\theta},\lambda\right)}\nabla_{\boldsymbol{\theta}}\int_{\{\mathbf{x}:\mathbf{x}\in\Omega_{\mathbf{X}}\}} e^{\lambda\left[\boldsymbol{\theta}^T\mathbf{x}-k(\boldsymbol{\theta})\right]}dQ_\lambda\left(\mathbf{x}\right) \\ &= \frac{1}{F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}|\boldsymbol{\theta},\lambda\right)}\int_{\{\mathbf{x}:\mathbf{x}\in\Omega_{\mathbf{X}}\}} \lambda\left[\mathbf{x}-\nabla_{\boldsymbol{\theta}}k\left(\boldsymbol{\theta}\right)\right]e^{\lambda\left[\boldsymbol{\theta}^T\mathbf{x}-k(\boldsymbol{\theta})\right]}dQ_\lambda\left(\mathbf{x}\right) \\ &= \frac{\lambda}{F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}|\boldsymbol{\theta},\lambda\right)}[\int_{\{\mathbf{x}:\mathbf{x}\in\Omega_{\mathbf{X}}\}} \mathbf{x}e^{\lambda\left[\boldsymbol{\theta}^T\mathbf{x}-k(\boldsymbol{\theta})\right]}dQ_\lambda\left(\mathbf{x}\right) \\ &- \nabla_{\boldsymbol{\theta}}k\left(\boldsymbol{\theta}\right)\int_{\{\mathbf{x}:\mathbf{x}\in\Omega_{\mathbf{X}}\}} e^{\lambda\left[\boldsymbol{\theta}^T\mathbf{x}-k(\boldsymbol{\theta})\right]}dQ_\lambda\left(\mathbf{x}\right)] \\ &= \lambda\left(\Psi_E\left(\mathbf{X};\Omega_{\mathbf{X}}\right)-\nabla_{\boldsymbol{\theta}}k\left(\boldsymbol{\theta}\right)\right) \\ &= \lambda\left(\Psi_E\left(\mathbf{X};\Omega_{\mathbf{X}}\right)-\boldsymbol{\mu}\right), \end{aligned}$$

[2]An ED random vector $\mathbf{X}$ is also often denoted as $\mathbf{X} \backsim ED_n\left(\boldsymbol{\mu},\lambda\right)$ in literature.

which leads to the conclusion that

$$\begin{aligned}\Psi_E(\mathbf{X};\Omega_{\mathbf{X}}) &= \boldsymbol{\mu} + \lambda^{-1}\nabla_{\boldsymbol{\theta}} \log F_{\mathbf{X}}(\Omega_{\mathbf{X}}|\boldsymbol{\theta},\lambda) \\ &= \boldsymbol{\mu} + \sigma^2\nabla_{\boldsymbol{\theta}} \log F_{\mathbf{X}}(\Omega_{\mathbf{X}}|\boldsymbol{\theta},\lambda).\end{aligned}$$

■

From Theorem 3.1, we can derive those specific multivariate risk measures mentioned earlier in the context of EDM.

Example 3.2

1. The multivariate lower and upper orthant TCE measures $\underline{TCE}_q(\mathbf{X})$ and $\overline{TCE}_q\mathbf{X}$:

$$\begin{aligned}\underline{TCE}_q(\mathbf{X}) &= \boldsymbol{\mu} + \sigma^2\nabla_{\boldsymbol{\theta}} \log \Pr(F_{\mathbf{X}}(\mathbf{X}) \geq q|\boldsymbol{\theta},\lambda), \\ \overline{TCE}_q(\mathbf{X}) &= \boldsymbol{\mu} + \sigma^2\nabla_{\boldsymbol{\theta}} \log \Pr\left(\overline{F}_{\mathbf{X}}(\mathbf{X}) \leq 1-q|\boldsymbol{\theta},\lambda\right).\end{aligned}$$

2. The MTCE measure:

$$MTCE_q(\mathbf{X}) = \boldsymbol{\mu} + \sigma^2\nabla_{\boldsymbol{\theta}} \log \overline{F}_{\mathbf{X}}(VaR_q(\mathbf{X})|\boldsymbol{\theta},\lambda). \tag{13}$$

In accord with Example 2.2, the two examples are of particular interest in literature as it corresponds to the tail regions where individual risks are beyond certain levels of loss under idiosyncratic risk measures. Note that in (13), we take an identical univariate risk measure VaR_q to define the conditioning region. More generally, we could have broader results than this setting. Let

$$\rho_i : X_i \in \mathbb{R} \Rightarrow \rho_i(X_i) \in \mathbb{R}, i = 1, 2, \ldots, n$$

be a single-scale risk measure (e.g., $VaR_q(X_i)$, $TCE_q(X_i)$ or $E(X_i)$) of the risk X_i. We can define a vector of these single-scale risk measures as $\rho(\mathbf{X}) = (\rho_1(X_1), \ldots, \rho_n(X_n))^T$ and a more general conditioning region, namely $\bigcap_{i=1}^n \{X_i \geq \rho_i(\mathbf{X_i})\}$. For instance, one may take $\rho_1(X_1) = VaR_{99.9\%}[X_1], \rho_2(X_2) = VaR_{99.5\%}[X_2]$ while $\rho_3(X_3) = CTE_{95\%}[X_3]$. In our EDM framework, the explicit formula of MCE with this general conditioning region is also available. A similar idea of conditional systemic events can be found in Dhaene et al. (2019).

Proposition 3.3 *For a random vector* $\mathbf{X} \backsim ED_n(\boldsymbol{\theta}, \lambda)$, *we have*

$$\Psi_E(\mathbf{X}; \bigcap_{i=1}^{n} \{X_i \geq \rho_i(\mathbf{X}_i)\}) = \boldsymbol{\rho}(\mathbf{X}) + \mathbf{z}_{\boldsymbol{\rho}(\mathbf{X})}, \tag{14}$$

where $\mathbf{z}_{\boldsymbol{\rho}(\mathbf{X})} = \left(z_{\boldsymbol{\rho}(\mathbf{X}),1}, \ldots, z_{\boldsymbol{\rho}(\mathbf{X}),n}\right)^T$,

$$z_{\boldsymbol{\rho}(\mathbf{X}),j} = \frac{1}{\overline{F}_{\mathbf{X}}(\rho(\mathbf{x})|\boldsymbol{\theta}, \lambda)} \int_{\rho(\mathbf{x})_j}^{\infty} \overline{F}_{\mathbf{X}}(\rho(\mathbf{x})_1, \ldots, u_j, \ldots, \rho(\mathbf{x})_n |\boldsymbol{\theta}, \lambda)\, du_j,$$

is a vector of tail functions of $\mathbf{X}$.

Proof.

$$\Psi_E\left(\mathbf{X}; \bigcap_{i=1}^{n} \{X_i \geq \rho_i(\mathbf{X})\}\right) = E(\mathbf{X}|\mathbf{X} \geq \boldsymbol{\rho}(\mathbf{X})) = \frac{1}{\overline{F}_{\mathbf{X}}(\boldsymbol{\rho}(\mathbf{x})|\boldsymbol{\theta}, \lambda)} \int_{\{\mathbf{x}:\mathbf{x} \geq \boldsymbol{\rho}(\mathbf{x})\}} \mathbf{x}\, dF_{\mathbf{X}}(\mathbf{x}).$$

Now, for every component of (14), we have

$$\Psi_E\left(\mathbf{X}; \bigcap_{i=1}^{n} \{X_i \geq \rho_i(\mathbf{X}_i)\}\right)_j$$
$$= \rho(\mathbf{x})_j + \frac{1}{\overline{F}_{\mathbf{X}}(\boldsymbol{\rho}(\mathbf{x})|\boldsymbol{\theta}, \lambda)} \int_{\rho(\mathbf{x})_j}^{\infty} \overline{F}_{\mathbf{X}}(\rho(\mathbf{x})_1, \ldots, u_j, \ldots, \rho(\mathbf{x})_n |\boldsymbol{\theta}, \lambda)\, du_j,$$
$$j = 1, 2, \ldots, n,$$

which immediately leads to the following representation

$$\Psi_E\left(\mathbf{X}; \bigcap_{i=1}^{n} \{X_i \geq \rho_i(\mathbf{X_i})\}\right) = \boldsymbol{\rho}(\mathbf{x}) + \frac{1}{\overline{F}_{\mathbf{X}}(\rho(\mathbf{x})|\boldsymbol{\theta}, \lambda)}$$
$$\cdot \left(\int_{\rho(\mathbf{x})_1}^{\infty} \overline{F}_{\mathbf{X}}(u_1, \rho(\mathbf{x})_2, \ldots, \rho(\mathbf{x})_n |\boldsymbol{\theta}, \lambda)\, du_1, \right.$$
$$\left. \ldots, \int_{\rho(\mathbf{x})_n}^{\infty} \overline{F}_{\mathbf{X}}\left(\rho(\mathbf{x})_1, \ldots, \rho(\mathbf{x})_{n-1}, u_n|\boldsymbol{\theta}, \lambda\right) du_n\right)^T$$
$$= \boldsymbol{\rho}(\mathbf{x}) + \mathbf{z}_{\boldsymbol{\rho}(\mathbf{x})}.$$

■

We can see the MCE formula is of peculiar form for EDM. In fact, the formula (14) characterizes EDM as Theorem 2 in Landsman and Valdez (2005), where the authors only consider the univariate case. We extend their result to the multivariate EDM.

Theorem 3.4 (Characterization of EDM via MCE) *Let* $\mathbf{X} = (X_1, \ldots, X_n)^T \backsim ED_n(\boldsymbol{\theta}, \lambda)$ *be an ED random vector with* $X_1, \ldots, X_n \in (a, b)$ *where the interval* (a, b) *can be finite or infinite. Then, the multivariate risk measure MCE takes the form*

$$\Psi_E\left(\mathbf{X}; \bigcap_{i=1}^{n}\{X_i \geq \rho_i(\mathbf{X}_i)\}\right) = \boldsymbol{\rho}(\mathbf{X}) + \boldsymbol{\gamma}, \boldsymbol{\gamma} \in \mathbf{R}^n \cap \{\mathbf{0}\},$$

if and only if $X_1, \ldots, X_n$ *are mutually independent random variables and every* $X_i, i = 1, 2, \ldots, n$ *is a shifted exponential distribution with* $\gamma_i = E(X_i)$ *.*

Proof. Define $U_j(t) = \overline{F}_{\mathbf{X}}(\rho(\mathbf{x})_1, \ldots, t, \ldots, \rho(\mathbf{x})_n | \boldsymbol{\theta}, \lambda)$, where $t \in \mathbb{R}$ is the $j-$th component of $\overline{F}_{\mathbf{X}}$. From Proposition 3.3, we observe that for the $j-$th component

$$z_{\rho(\mathbf{x}),j} = \frac{1}{U_j(t)}\int_t^{\infty} U_j(u_j)du_j, t = \rho_j(\mathbf{X}) \in (a, \infty).$$

Then, for a constant $z_{\rho(\mathbf{x}),j}$ the following equation holds

$$\frac{1}{U_j(t)}\int_t^{\infty} U_j(u_j)du_j = \gamma_j,$$

which leads to the differential equation $-U_j(t) = \gamma_j U_j'(t)$, that has the following simple solution

$$U_j(t) = k_j e^{-\eta_j t}, \tag{15}$$

where $\eta_j = 1/\gamma_j$. Recalling that $U_j(t)$ is a tail function, from its properties and from the form of $U_j(t)$ (15), we observe that $U_j(t)$ describes a tail function of a univariate shifted exponential distribution, where $k_j = e^{\eta_j a}$. ■

Comparing with Proposition 3.3, Theorem 3.4 requires additional condition (mutual independence) but it provides sufficient and necessary condition to characterize the EDM.

3.3 MCEn for Multivariate EDM

The following lemma is useful to derive the MCEn formulae of EDM.

Lemma 3.5 *Let* $\mathbf{X}$ *be some random vector with a finite mgf and the support of* $\mathbf{X}, \chi \subseteq \mathbb{R}^n$. *Then, the mgf of* $\mathbf{X}|\left(\mathbf{X} \in \Omega_{\mathbf{X}}\right), \Omega_{\mathbf{X}} \subset \chi$, *takes the following form*

$$M_{\mathbf{X}}^{\Omega_{\mathbf{X}}}\left(\mathbf{t}\right) = M_{\mathbf{X}}\left(\mathbf{t}\right) \frac{F_{\mathbf{X}_{\mathbf{t}}}\left(\Omega_{\mathbf{X}}\right)}{F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}\right)}, \mathbf{t} \in \mathbb{R}^n.$$

Here $F_{\mathbf{X}_{\mathbf{t}}}\left(\Omega_{\mathbf{X}}\right) := \Pr\left(\mathbf{X}_{\mathbf{t}} \in \Omega_{\mathbf{X}}\right)$, *where* $\mathbf{X}_{\mathbf{t}} \in \mathbb{R}^n$ *is an Esscher transform random vector with the probability measure* $dF_{\mathbf{X}_{\mathbf{t}}}\left(\mathbf{u}\right) = e^{\mathbf{t}^T\mathbf{u}} dF_{\mathbf{X}}\left(\mathbf{u}\right) M_{\mathbf{X}}\left(\mathbf{t}\right)^{-1}, \mathbf{u} \in \chi$.

Proof. From the definition of the conditional mgf of $\mathbf{X}$, we have

$$M_{\mathbf{X}}^{\Omega_{\mathbf{X}}}\left(\mathbf{t}\right) = \frac{\int_{\Omega_{\mathbf{X}}} e^{\mathbf{t}^T\mathbf{x}} dF_{\mathbf{X}}\left(\mathbf{x}\right)}{F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}\right)}.$$

Using the fact that

$$M_{\mathbf{X}}\left(\mathbf{t}\right)^{-1} \int_{\chi} e^{\mathbf{t}^T\mathbf{x}} dF_{\mathbf{X}}\left(\mathbf{x}\right) = 1,$$

we define a random vector $\mathbf{X}_{\mathbf{t}} \in \mathbb{R}^n$ with the following probability measure

$$dF_{\mathbf{X}_{\mathbf{t}}}\left(\mathbf{u}\right) = \exp\left(\mathbf{t}^T\mathbf{u}\right) dF_{\mathbf{X}}\left(\mathbf{u}\right) M_{\mathbf{X}}\left(\mathbf{t}\right)^{-1}, \mathbf{u} \in \chi. \tag{16}$$

Taking (16) into account, we finally reformulate $M_{\mathbf{X}}^{\Omega_{\mathbf{X}}}\left(\mathbf{t}\right)$, as follows:

$$M_{\mathbf{X}}^{\Omega_{\mathbf{X}}}\left(\mathbf{t}\right) = M_{\mathbf{X}}\left(\mathbf{t}\right) \frac{F_{\mathbf{X}_{\mathbf{t}}}\left(\Omega_{\mathbf{X}}\right)}{F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}\right)}.$$

■

The proof of Lemma 3.5 takes use of a so-called "exponential tilting" techniques. Note that Lemma 3.5 holds for any continuous distribution with moment generating function; thus, this result covers several previous results in the literature (Butler and Wood, 2004, Valdez et al., 2009, and Landsman et al., 2016). In particular, we have the mgf of EDM given a conditioning region as follows.

Proposition 3.6 *Let* $\mathbf{X} \backsim ED_n\left(\boldsymbol{\theta}, \lambda\right)$ *be an* $n \times 1$ *ED random vector. Then, the mgf of* $\mathbf{X}|\left(\mathbf{X} \in \Omega_{\mathbf{X}}\right)$ *is given by*

$$M_{\mathbf{X}}^{\Omega_{\mathbf{X}}}\left(\mathbf{t}\right) = M_{\mathbf{X}}\left(\mathbf{t}\right) \frac{F_{\mathbf{Z}}\left(\Omega_{\mathbf{X}}; \mathbf{t}\right)}{F_{\mathbf{X}}\left(\Omega_{\mathbf{X}}\right)}, \mathbf{t} \in \mathbb{R}^n, \tag{17}$$

where $\mathbf{Z} \in \mathbb{R}^n$ *is an ED random vector with the probability measure pdf*

$$dF_{\mathbf{Z}}\left(\mathbf{z}\right) = e^{\lambda\left(\mathbf{u}^T\left(\boldsymbol{\theta}+\mathbf{t}\right) - k\left(\boldsymbol{\theta}+\mathbf{t}/\lambda\right)\right)} dQ\left(\lambda, \mathbf{u}\right).$$

Proof. Following Proposition 3.3, this proposition immediately follows by observing that $f_{\mathbf{X_t}}$ (16) is an exponential dispersion probability measure,

$$dF_{\mathbf{X_t}}(\mathbf{u}) = \exp\left(\mathbf{t}^T\mathbf{u}\right) dF_{\mathbf{X}}(\mathbf{u})\, M_{\mathbf{X}}(\mathbf{t})^{-1} = e^{\mathbf{t}^T\mathbf{u}+\lambda\left(\mathbf{u}^T\boldsymbol{\theta}-k(\boldsymbol{\theta})\right)-\lambda(k(\boldsymbol{\theta}+\mathbf{t}/\lambda)-k(\boldsymbol{\theta}))} dQ(\lambda, \mathbf{u})$$
$$= \exp\left\{\lambda\left(\mathbf{u}^T(\boldsymbol{\theta}+\mathbf{t}) - k(\boldsymbol{\theta}+\mathbf{t}/\lambda)\right)\right\} dQ(\lambda, \mathbf{u}).$$

■

Theorem 3.7 *Let* $\mathbf{X} \backsim ED_n(\boldsymbol{\theta}, \lambda)$ *be an* n*-variate ED random vector. Then, under the condition that the mgf of* $\mathbf{X}$ *is finite, the* i*-th component of the MCEn measure takes the form*

$$\Psi_{En}(\mathbf{X}; \Omega_{\mathbf{X}})_i = \frac{\lambda}{\gamma_i}\left(k(\theta_i - \gamma_i/\lambda) - k(\theta_i)\right) + \ln\left(\frac{F_{\mathbf{Z}}(\Omega_{\mathbf{X}}; \boldsymbol{\gamma}_{0.i})}{F_{\mathbf{X}}(\Omega_{\mathbf{X}})}\right)^{1/\gamma_i},$$

where $\boldsymbol{\gamma}_{0.i}$ *is* $n \times 1$ *vector of zeros except the* i*-th component that is* $-\gamma_i$*.*

Proof. From Theorem 2, we observe that the i-th component of the MCEn measure (6) is given by

$$\Psi_{En}(\mathbf{X}; \Omega_{\mathbf{X}})_i = \frac{1}{\gamma_i}\left(\ln M_{\mathbf{X}}(\boldsymbol{\gamma}_{0.i}) + \ln\left(\frac{F_{\mathbf{Z}}(\Omega_{\mathbf{X}}; \boldsymbol{\gamma}_{0.i})}{F_{\mathbf{X}}(\Omega_{\mathbf{X}})}\right)\right). \tag{18}$$

Now, from the marginal property of the mgf, we note that $M_{\mathbf{X}}(\boldsymbol{\gamma}_{0.i}) = M_{X_i}(-\gamma_i)$ takes the form of a univariate mgf of X_i at the point $-\gamma_i$. Finally, from the mgf of ED random variables (17), and after algebraic calculations we conclude that

$$\Psi_{En}(\mathbf{X}; \Omega_{\mathbf{X}})_i = \gamma_i^{-1}\ln\left(M_{X_i}(-\gamma_i)\frac{F_{\mathbf{Z}}(\Omega_{\mathbf{X}}; \boldsymbol{\gamma}_{0.i})}{F_{\mathbf{X}}(\Omega_{\mathbf{X}})}\right)$$
$$= \gamma_i^{-1}\left(\ln\exp\left\{\lambda\left(k(\theta_i - \gamma_i/\lambda) - k(\theta_i)\right)\right\} + \ln\left(\frac{F_{\mathbf{Z}}(\Omega_{\mathbf{X}}; \boldsymbol{\gamma}_{0.i})}{F_{\mathbf{X}}(\Omega_{\mathbf{X}})}\right)\right)$$
$$= \frac{\lambda}{\gamma_i}\left(k(\theta_i - \gamma_i/\lambda) - k(\theta_i)\right) + \ln\left(\frac{F_{\mathbf{Z}}(\Omega_{\mathbf{X}}; \boldsymbol{\gamma}_{0.i})}{F_{\mathbf{X}}(\Omega_{\mathbf{X}})}\right)^{1/\gamma_i}.$$

_ ■

3.4 Systemic risk for EDM

In this subsection, we provide explicit formulae of systemic risk measures based on aggregation using MCE, namely, $S_{\mathbf{\Psi}_E(\mathbf{X};\mathbf{\Omega_X})}$ and $w_{\mathbf{\Psi}_E(\mathbf{X};\mathbf{\Omega_X})}[i]$ defined in (7) and (8). Note that, we now replace the $\mathbf{\Psi}(\mathbf{X};\mathbf{\Omega_X})$ by $\mathbf{\Psi}_E(\mathbf{X};\mathbf{\Omega_X})$ accordingly in notations as we focus on the case where MCE applies.

Proposition 3.8 *Suppose that* $\mathbf{X} \backsim ED_n(\boldsymbol{\theta},\lambda)$, *and let* $\mu_S = \sum_{j=1}^n \mu_j$. *Then, the measure for the total amount of risk,* $S_{\mathbf{\Psi}_E(\mathbf{X})}$, *is given by*

$$S_{\mathbf{\Psi}_E(\mathbf{X};\Omega_\mathbf{X})} = \mu_S + \sigma^2 \sum_{j=1}^n \frac{d}{d\theta_j} \ln F_\mathbf{X}\left(\Omega_\mathbf{X}|\boldsymbol{\theta},\lambda\right), \tag{19}$$

and the weight $w_{\mathbf{\Psi}_E(\mathbf{X};\Omega_\mathbf{X}),i}$ *for the* i*-th risk* $X_i, i = 1,2,\ldots,n$, *to the financial system with* $\mathbf{X} \backsim ED_n(\boldsymbol{\mu},\lambda)$ *and a multivariate risk measure* $\mathbf{\Psi}_E(\mathbf{X};\Omega_\mathbf{X})$, *is given by*

$$w_{\mathbf{\Psi}_E(\mathbf{X};\Omega_\mathbf{X})} = \mu_i\beta_1 + \sigma^2\beta_2 \tag{20}$$

where β_1 *and* β_2 *are, respectively,*

$$\beta_1 = \frac{1}{\mu_S + \sigma^2 \sum_{j=1}^n \frac{d}{d\theta_j} \ln F_\mathbf{X}\left(\Omega_\mathbf{X}|\boldsymbol{\theta},\lambda\right)},$$

and

$$\beta_2 = \frac{\frac{d}{d\theta_i} \ln F_\mathbf{X}\left(\Omega_\mathbf{X}|\boldsymbol{\theta},\lambda\right)}{\mu_S + \sigma^2 \sum_{j=1}^n \frac{d}{d\theta_j} \ln F_\mathbf{X}\left(\Omega_\mathbf{X}|\boldsymbol{\theta},\lambda\right)}.$$

Although Proposition 3.8 is a natural result following directly from Theorem 3.1, we present it here to stress how our explicit formulae for EDM can be useful for practical scenario analysis, especially for systemic risks. As aforementioned, once we set up an EDM for the financial network, the systemic risk, and how it spreads across the system would be determined. However, the riskiness and contagion effect do not reveal themselves automatically; one still needs to employ a reasonable methodology to calculate the risk measures and the interactions among risks under various scenarios. Therefore, (19) and (20) could serve for this purpose in our EDM framework. We state the usefulness of Proposition 3.8 as follows.

First, from a regulator's point of view, MCE is more reasonable than the vector composed by the marginal expected shortfalls ($\mathrm{ES}_p[X] := E[X|X >$

$VaR_p[X]]$) as the dependence among X_i is taken into account. Therefore, $S_{\mathbf{\Psi}_E}$ provides a more realistic amount to safeguard or bail out the system when it is subject to a possible distress situation $\Omega_{\mathbf{X}}$. Because our explicit formulae are applicable for any chosen $\Omega_{\mathbf{X}}$, the regulator can plug in any interested "systemic event" to calculate the corresponding quantity of riskiness. By doing so, he or she can find out to what extent the systemic risk varies along with different events. *Second*, for a well-selected $\Omega_{\mathbf{X}}$, the translation-invariant and positive homogeneous properties of MCE guarantee a type of risk measure defined via acceptance set (see, e.g., Biagini et al. 2019 and Feinstein 2017 for more details). To put it simply, under the mapping of conditional expectation, $S_{\mathbf{\Psi}_E}$ would be the minimal amount to add to the system such that the payoff of systemic risk (defined by the direct aggregation) is acceptable. *Third,* for an amount of available capital, $w_{\mathbf{\Psi}_E}$ suggests a top-down allocation rule subject to the scenario $\Omega_{\mathbf{X}}$. Hence, $w_{\mathbf{\Psi}_E}$ is assorted with the so-called scenario-dependent allocation in Biagini et al. (2019). When $S_{\mathbf{\Psi}_E}$ is interpreted as the expected capital shortfall of the system under a market decline, $w_{\mathbf{\Psi}_E}$ reflects how the marginal unities contribute the total risk and how the risk spread across the system from one to others. *Forth,* from an investor's point of view, taking excessive risk is often motivated by higher profit opportunities. By considering different $\Omega_{\mathbf{X}}$, the explicit formula in Proposition 3.8 allows a portfolio manager to investigate the performance of risky assets under both flourish and decline market, and thus to compare and balance the risk-reward of his or her decision-making. In a word, in the context of EDM, it clear from Proposition 3.8 that only $\Omega_{\mathbf{X}}$ can adapt $S_{\mathbf{\Psi}_E}$ and $w_{\mathbf{\Psi}_E}$ once a EDM has been specified. Thus, the advantage of Proposition 3.8 and former explicit formulae lies in the free choice of conditioning regions. We can to play with the formulae to conduct scenario analyses for the models, such as stress tests, sensitivity analysis, and robustness tests.

4 Numerical Examples

In this section, we present numerical studies to further illustrate the advantage of our explicit method in scenario analysis. We consider a continuous model based on financial data from a stock market and a discrete model based on count data of car insurance claims from an insurance company, respectively.

4.1 Continuous case: market-based financial system risk

We collect monthly stock return data in London stock exchange from April 2013 to November 2019 in the finance sector of the market[3]. In order to account for the systemic risk and the contagion effect, we take three industry segments in finance (Banks X_1, Insurance X_2, Financial and Credit Service X_3). Similar empirical studies on the market-based systemic risk can be found in Kleinow et al. (2017) and Brownlees & Engle (2017). We use the multivariate normal distribution to fit the data. In the context of EDM, the density function of a multivariate normal distribution can be rewritten in the form of (9) as follow.

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{\sqrt{2\pi^n|\boldsymbol{\Sigma}|}}\exp(\frac{1}{2}\mathbf{x}\boldsymbol{\Sigma}^{-1}\mathbf{x}) * \exp(\mathbf{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) \\ &= \frac{1}{\sqrt{2\pi^n|\boldsymbol{\Sigma}|}}\exp(\frac{1}{2}\mathbf{x}\boldsymbol{\Sigma}^{-1}\mathbf{x}) * \exp(x^T\boldsymbol{\theta} - \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta}), \\ &= a\left(\lambda, \mathbf{x}\right) e^{\lambda\left(\mathbf{x}^T\boldsymbol{\theta} - k(\boldsymbol{\theta})\right)}, \end{aligned}$$

where $\boldsymbol{\theta} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $k(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} = \frac{1}{2}\boldsymbol{\mu}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and $\lambda = 1$. The parameters are estimated via maximum likelihood estimation.

$$\boldsymbol{\mu} = \begin{pmatrix} -0.1140677 \\ 0.5896240 \\ 0.2107343 \end{pmatrix}\%, \boldsymbol{\Sigma} = \begin{pmatrix} 19.088935 & 12.503116 & -3.720492 \\ 12.503116 & 20.268816 & -3.162601 \\ -3.720492 & -3.162601 & 8.851913 \end{pmatrix}\%\%.$$

We rely on our explicit formula to calculate the MCE Ψ_E for the three industry indices. For comparisons, we also calculate the marginal expected shortfall ES_p for each segment. The numerical results are presented in Table 1.

There are two types of scenarios in our consideration in Table 1; both reflect a depression market to different extents. In the upper part(scenario $\Omega^1_{\mathbf{X}}$), the level of the depression on the three financial segments goes severe simultaneously along with the same descending p. By contrast, further extreme scenarios are imposed to the bank segment (X_1) in the lower part

[3] The data is available on https://finance.yahoo.com/ and https://www.londonstockexchange.com.

	Ω_X^1					
	$p = 0.20$		$p = 0.1$		$p = 0.05$	
	$\Psi_E\ (w_{\mathbf{\Psi}_E})$	ES_p	$\Psi_E\ (w_{\mathbf{\Psi}_E})$	ES_p	$\Psi_E\ (w_{\mathbf{\Psi}_E})$	ES_p
X_1	-15.33(70.28%)	-15.26	-19.14(69.87%)	-19.06	-22.38(69.65%)	-22.35
X_2	-6.28(28.79%)	-6.04	-7.99(29.17%)	-7.73	-9.44(29.38%)	-9.20
X_3	-0.202(0.93%)	-0.22	-0.262(0.96%)	-0.28	-0.313(0.97%)	-0.331
S_{Ψ_E}	-21.81	–	-27.39	–	-32.14	–
	$\Omega_{\mathbf{X}}^2$					
	$p = 0.15$		$p = 0.05$		$p = 0.01$	
	$\Psi_E\ (w_{\mathbf{\Psi}_E})$	ES_p	$\Psi_E\ (w_{\mathbf{\Psi}_E})$	ES_p	$\Psi_E\ (w_{\mathbf{\Psi}_E})$	ES_p
X_1	-16.84(71.76%)	-16.91	-22.03(75.32%)	-22.35	-28.40(77.70%)	-28.80
X_2	-6.43(27.39%)	-6.04	-7.02(24.00%)	-6.04	-7.95(21.77%)	-6.04
X_3	-0.20(0.85%)	-0.22	-0.19(0.68%)	-0.22	-0.19(0.53%)	-0.22
S_{Ψ_E}	-23.47	–	-29.24	–	-36.54	–

Table 1: MCE and systemic risk based on the multivariate normal model. $\Omega_{\mathbf{X}}^1 = \{X_1 < VaR_p(X_1)\} \cap \{X_2 < VaR_p(X_2)\} \cap \{X_3 < VaR_p(X_3)\}$. $\Omega_{\mathbf{X}}^2 = \{X_1 < VaR_p(X_1)\} \cap \{X_2 < VaR_{0.2}(X_2)\} \cap \{X_3 < VaR_{0.2}(X_3)\}$

(scenario $\Omega_{\mathbf{X}}^2$), while $p = 0.2$ is reserved for the other two segments (Insurance X_2 and Financial and Credit Service X_3). We are particularly interested in the extreme scenarios of the bank sector due to its largest capitalization among the three.

Note that the results in Table 1 have been rescaled by the market capitalization of the three segments[4], i.e., the multiple negative values of MCE indicate the magnitude of the expected drop-off when the market is in a "big recession". Through the direct aggregation, systemic risk measure S_{Ψ_E} suggests the expected amount (taking absolute value) to bail out the expected shortfall of financial segments of the market due to the recession, i.e., to add this amount to the system to make it acceptable for regulations. Accordingly, both the components of Ψ_E and the systemic risk S_{Ψ_E} are of greater magnitude along with p, whereas the marginal ES_p leads to over-(under-) estimates as it ignores the dependence among the three segments. In particular, ES_p overestimates the risk of X_1 when it is in extreme upper tail () because the other two segments can absorb some risk of X_1. By comparing MCE under $\Omega_{\mathbf{X}}^1$ and $\Omega_{\mathbf{X}}^2$, we can understand to what extent the risk

[4] 4The capitalizations are 244.95, 105.74 and 5.59 bi£.

spreads in the system. For instance, some extreme perilous status of the bank segment could lead to even higher systemic risk ($\Omega^1_{\mathbf{X}}$ at $p = 0.05$ v.s. $\Omega^2_{\mathbf{X}}$ at $p = 0.01$). Moreover, w_{Ψ_E} provides a top-down allocation rule for an amount of available capital. It is worth noticing that the contribution of the insurance segment (X_2) to the total systemic risk increases when the extreme scenario goes more severe. From a portfolio manager's point of view, this also motivates further investigation on a blooming market, e.g., $\Omega_{\mathbf{X}} = \{X_1 > VaR_p(X_1)\} \cap \{X_2 > VaR_p(X_2)\} \cap \{X_3 > VaR_p(X_3)\}$ with $p = 0.95$ to test to what extent, he or she should allocate the investment to the three industry segments.

4.2 Discrete example: count data for insurance company

The second case we consider is based on the count data of car insurance claims from an insurance company in the third quarter of 1973. The data is available in R package MASS. The claims were collected from four districts; each district contains 16 samples. We treat the number of claims in the four districts as dependent individual insurance business lines exposed to risk. We then employ negative multinomial distribution to fit the data using maximum likelihood estimation. In the context of EDM, the density of negative multinomial distribution can be written as (see, Jørgensen, 1987).

$$f_{\mathbf{X}}(\mathbf{x}; \lambda, \boldsymbol{\theta}) = \frac{\Gamma(\lambda + x_1 + \ldots + x_n)}{\Gamma(\lambda + x_1)\Gamma(\lambda + x_n)\Gamma(\lambda)} \exp(\sum_{i=1}^{n} x_i\theta_i + \lambda \ln(1 - \sum_{i=1}^{n} e^{\theta_i})),$$

and the parameters for the data is $e^{\boldsymbol{\theta}} = (0.436001, 0.281301, 0.174590, 0.102923)$ and $\lambda = 1.026603$.

We consider four types of scenarios in this case. In the scenarios of $\Omega^1_{\mathbf{X}}$ and $\Omega^2_{\mathbf{X}}$, individual risks are in tail regions simultaneously whereas there are at least one of the individual risks in tail regions for scenarios of $\Omega^3_{\mathbf{X}}$ and $\Omega^4_{\mathbf{X}}$. We also choose X_1, which has the highest contribution to the total systemic risk, for a stress test to show the contagion effect of the model ($\Omega^2_{\mathbf{X}}$ and $\Omega^4_{\mathbf{X}}$). Note that, the scenarios $\Omega^1_{\mathbf{X}}$ and $\Omega^2_{\mathbf{X}}$ are the same for $p = 0.95$; so are $\Omega^3_{\mathbf{X}}$ and $\Omega^4_{\mathbf{X}}$. In order to keep consistent with the discrete model, the results in Table 2 have been rounded to integers.

Similarly to the continuous case, when the conditioning tail region moves to extreme upper zones, i.e., $p \to 1$, the systemic risk increases accordingly.

Ω_X^1 & Ω_X^2								
	$p = 0.95$		$p = 0.99$			$p = 0.995$		
	$\Psi_E^{\Omega_X^1}\ (w_{\Psi_E}^{\Omega_X^1})$	ES_p	$\Psi_E^{\Omega_X^1}\ (w_{\Psi_E}^{\Omega_X^1})$	$\Psi_E^{\Omega_X^2}\ (w_{\Psi_E}^{\Omega_X^2})$	ES_p	$\Psi_E^{\Omega_X^1}\ (w_{\Psi_E}^{\Omega_X^1})$	$\Psi_E^{\Omega_X^2}\ (w_{\Psi_E}^{\Omega_X^2})$	ES_p
X_1	510(43.66%)	480	571(43.69%)	543(43.97%)	539	712(43.71%)	675(44.03%)	676
X_2	330(28.25%)	311	369(28.23%)	348(28.18%)	349	460(28.24%)	432(28.18%)	438
X_3	206(17.64%)	194	230(17.60%)	216(17.49%)	218	287(17.62%)	268(17.48%)	273
X_4	122(10.45%)	116	137(10.48%)	128(10.36%)	130	170(10.44%)	158(10.31%)	163
S_{Ψ_E}	1168	–	1307	1235	–	1629	1533	–

Ω_X^3 & Ω_X^4								
	$p = 0.95$		$p = 0.99$			$p = 0.995$		
	$\Psi_E^{\Omega_X^3}\ (w_{\Psi_E}^{\Omega_X^3})$	ES_p	$\Psi_E^{\Omega_X^3}\ (w_{\Psi_E}^{\Omega_X^3})$	$\Psi_E^{\Omega_X^4}\ (w_{\Psi_E}^{\Omega_X^4})$	ES_p	$\Psi_E^{\Omega_X^3}\ (w_{\Psi_E}^{\Omega_X^3})$	$\Psi_E^{\Omega_X^4}\ (w_{\Psi_E}^{\Omega_X^4})$	ES_p
X_1	508(43.72%)	480	568(43.69%)	542(43.96%)	539	709(43.68%)	674(44.02%)	676
X_2	328(28.23%)	311	367(28.23%)	347(28.14%)	349	458(28.22%)	431(28.15%)	438
X_3	204(17.56%)	194	229(17.62%)	216(17.52%)	218	286(17.62%)	268(17.50%)	273
X_4	122(10.50%)	116	136(10.46%)	128(10.38%)	130	170(10.47%)	158(10.32%)	163
S_{Ψ_E}	1162	–	1300	1233	–	1623	1531	–

Table 2: MCE and systemic risk based on negative multinomial model.
$\Omega_{\mathbf{X}}^1 = \{X_1 > VaR_p(X_1)\} \cap \{X_2 > VaR_p(X_2)\} \cap \{X_3 > VaR_p(X_3)\} \cap \{X_4 > VaR_p(X_4)\}$.
$\Omega_{\mathbf{X}}^2 = \{X_1 > VaR_p(X_1)\} \cap \{X_2 > VaR_{.95}(X_2)\} \cap \{X_3 > VaR_{.95}(X_3)\} \cap \{X_4 > VaR_{.95}(X_4)\}$.
$\Omega_{\mathbf{X}}^3 = \{X_1 > VaR_p(X_1)\} \cup \{X_2 > VaR_p(X_2)\} \cup \{X_3 > VaR_p(X_3)\} \cup \{X_4 > VaR_p(X4)\}$.
$\Omega_{\mathbf{X}}^4 = \{X_1 > VaR_p(X_1)\} \cup \{X_2 > VaR_{.95}(X_2)\} \cup \{X_3 > VaR_{.95}(X_3)\} \cup \{X_4 > VaR_{.95}(X4)\}$.

By looking into $\Psi_E^{\Omega_{\mathbf{X}}}$ and $w_{\mathbf{\Psi}_E}^{\Omega_{\mathbf{X}}}$, we are able to find out to which extent, each marginal risk contributes to the total risk and how they interact with each other. Likewise, when p only increases for X_1 (scenarios $\Omega_{\mathbf{X}}^2$ and $\Omega_{\mathbf{X}}^4$), the other risks also increase, while the marginal ES_p generally underestimates the risks since the dependence has not been taken into account. More interestingly, we can observe that there are only slight changes in $w_{\mathbf{\Psi}_E}$ across different scenarios. Furthermore, although the upper part and lower part of Table 2 consider different scenarios, there are only little difference between $\Psi_E^{\Omega_{\mathbf{X}}^1}$ and $\Psi_E^{\Omega_{\mathbf{X}}^3}$ for different p level; the same holds for $\Psi_E^{\Omega_{\mathbf{X}}^2}$ and $\Psi_E^{\Omega_{\mathbf{X}}^4}$. Such results suggest a strong positive tail dependence among individual risks. From a practical viewpoint, we may explain such results as a consequence due to exogenous factors that lead to catastrophe loss to all business lines and the entire system. When one business line suffers from extreme loss, so do the others.

4.3 Discussions

Based on the two numerical examples above, we have shown the advantageous strength of our explicit formulae in facilitating scenarios analysis subject to any specific conditioning zones (systemic events). We may apply similar analyses based on the multivariate conditional entropy Ψ_{En} defined in Theorem 3.7. Concerning the continuous example, we choose a downside region for conditioning. Our results could be of interest not only for regulators who need to assess the expected shortfall of the system when the market is in a downturn but also for the investors who study the downside risk of a portfolio (see, Klenbar et al. 2017). As for the discrete example, we consider the count data of car insurance claims. Note that, in the classic collective model (cf. Dickson, 2016), there are additional random variables to account for the claim size. We did not include this in our study because we would like to keep consistent with a discrete model. However, since the claim size and the claim number are often assumed to independent, our results could also be useful for the approximation method to collective models in Zhou et al. (2019). Moreover, we can consider other multivariate distribution in the EDM family for scenario analysis, such as the multivariate Inverse Gaussian distribution for the continuous case and the Multinomial distribution for the discrete case. In other words, our explicit approaches also apply to robustness and misspecification tests of models in the context of EDM.

5 Conclusion

Motivated by the popularity of the exponential dispersion models in quantitative risk management and actuarial science, we propose two multivariate risk measures based on conditional expectation and derive the explicit formulae subject to a free choice of conditioning region. As an application, we further utilize these results to study the systemic risk and conduct scenarios analysis with respect to different "systemic events." Our explicit formulae of the multivariate risk measures allow quantifying the amount of each risk from a system consisting of dependent individuals. We further present two numerical examples based on data from a financial market and an insurance company. Our paper provides both theoretical and practical methods to measure systemic risks and to conduct the scenario analysis for exponential dispersion models. Therefore, we believe that further investigation of the tail systemic risk measures of exponential dispersion models is of interest, from both theoretical and practical perspectives.

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