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TIME EVOLUTION IN QUANTUM SYSTEMS AND STOCHASTICS

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Abstract

The time evolution problem for non-self adjoint second order differential operators is studied by means of the path integral formulation. Explicit computation of the path integral via the use of certain underlying stochastic differential equations, which naturally emerge when computing the path integral, leads to a universal expression for the associated measure regardless of the form of the differential operators. The discrete non-linear hierarchy (DNLS) is then considered and the corresponding hierarchy of solvable, in principle, SDEs is extracted. The first couple members of the hierarchy correspond to the discrete stochastic transport and heat equations. The discrete stochastic Burgers equation is also obtained through the analogue of the Cole-Hopf transformation. The continuum limit is also discussed.

1 Introduction

One of our main aims here is the solution of the time evolution problem associated to non self-adjoint operators using the path integral formulation. We consider the general second order differential operator \hat{L}_0 , and the associated time evolution problem:

$$-\partial_t f(\mathbf{x}, t) = \hat{L}f(\mathbf{x}, t) = \left(\hat{L}_0 + u(\mathbf{x})\right)f(\mathbf{x}, t), \quad (1.1)$$

$$\hat{L}_0 = \frac{1}{2} \sum_{i,j=1}^M g_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^M b_j(\mathbf{x}) \frac{\partial}{\partial x_j}, \quad g(\mathbf{x}) = \sigma(\mathbf{x})\sigma^T(\mathbf{x}) \quad (1.2)$$

where the diffusion matrix $g(\mathbf{x})$ and the matrix $\sigma(\mathbf{x})$ are in general dynamical (depending on the fields x_j) $M \times M$ matrices, while \mathbf{x} and the drift $b(\mathbf{x})$ are M vector fields with components x_j , b_j respectively, and T denotes usual transposition. The

¹This is based on a talk given by AD in ‘‘Quantum Theory and Symmetries XI’’, July 2019, Montreal, Canada.

operator \hat{L}_0 is not in general self-adjoint (Hermitian), therefore we also introduce the adjoint operator defined for any suitable function $f(\mathbf{x}, t)$ as

$$\hat{L}_0^\dagger f(\mathbf{x}, t) = \frac{1}{2} \sum_{i,j=1}^M \frac{\partial^2}{\partial x_i \partial x_j} \left(g_{ij}(\mathbf{x}) f(\mathbf{x}, t) \right) - \sum_{j=1}^M \frac{\partial}{\partial x_j} \left(b_j(\mathbf{x}) f(\mathbf{x}, t) \right). \quad (1.3)$$

Then, two distinct time evolution equations emerge.

1. The Fokker-Plank equation:

$$\partial_{t_1} f(\mathbf{x}, t_1) = \hat{L}_0^\dagger f(\mathbf{x}, t_1) \quad (1.4)$$

$t_1 \geq t_2$, with known initial condition $f(\mathbf{x}, t_2) = f_0(\mathbf{x})$.

2. The Kolmogorov backward equation:

$$-\partial_{t_2} f(\mathbf{x}, t_2) = \hat{L}_0 f(\mathbf{x}, t_2) \quad (1.5)$$

$t_2 \leq t_1$, with known final condition $f(\mathbf{x}, t_1) = f_f(\mathbf{x})$.

To simplify the complicated situation of non-constant diffusion coefficients we employ a local change of frame, which reduces the \hat{L} operator to the simpler form with constant diffusion coefficients, and an effective drift. We then compute the generic path integral by requiring that the fields involved satisfy discrete time analogues of stochastic differential equations (SDEs). These equations naturally emerge when computing the path integral via the time discretization scheme. This leads to the computation of the measure, which turns out to be an infinite product of Gaussians.

Our second objective is to explore links between SDEs, and quantum integrable systems. To illustrate these associations we discuss a typical exactly solvable discrete quantum system, the discrete non-linear Schrödinger hierarchy. We express the quantum integrals of motion as second order differential operators after a suitable rescaling of the fields and we then extract a hierarchy of associated SDEs, which can be in principle solved by means of suitable integrator factors. The first two non-trivial members of the hierarchy correspond to the discrete stochastic transport and heat equations. The discrete stochastic Burgers equation is also obtained from the discrete stochastic heat equation through the analogue of the Cole-Hopf transformation (see also relevant [1]). More details on the derivation of the reported results can be found in [2].

2 Time evolution and the Feynman-Kac formula

Before we compute the solution of the time evolution problem via the path integral formulation we shall implement the quantum canonical transform, that turns the dynamical diffusion matrix in (1.2) into identity at the level of the PDEs. This result will be then used for the explicit computation of the general path integral, and the derivation of the Feynman-Kac formula [2].

2.1 The quantum canonical transformation

We will show in what follows that the general \hat{L} operator can be brought into the less involved form:

$$\hat{L} = \frac{1}{2} \sum_{j=1}^M \frac{\partial^2}{\partial y_j^2} + \sum_{j=1}^M \tilde{b}_j(y) \frac{\partial}{\partial y_j} + u(y) \quad (2.1)$$

with an induced drift $\tilde{b}(y)$. This can be achieved via a simple change of the parameters x_j , which geometrically is nothing but a change of frame. Indeed, let us introduce a new set of parameters y_j such that [2]:

$$dy_i = \sum_j \sigma_{ij}^{-1}(x) dx_j, \quad \det \sigma \neq 0, \quad (2.2)$$

then \hat{L} can be expressed in the form (2.1), and the induced drift components are given as

$$\tilde{b}_k(y) = \sum_j \sigma_{kj}^{-1}(y) b_j(y) + \frac{1}{2} \sum_{j,l} \sigma_{jl}(y) \partial_{y_l} \sigma_{kj}^{-1}(y). \quad (2.3)$$

Bearing also in mind that $\sum_j \sigma_{jl} \sigma_{kj}^{-1} = \delta_{kl}$, we can write in the compact vector/matrix notation:

$$\tilde{b}(y) = \sigma^{-1}(y) \left(b(y) - \frac{1}{2} (\nabla_y \sigma^T(y))^T \right), \quad \nabla_y = \left(\partial_{y_1}, \dots, \partial_{y_M} \right) \quad (2.4)$$

where one first solves for $x = x(y)$ via (2.2). The transformation discussed above corresponds to a generalization of the so called Lamperti transform at the level of SDEs (we refer the interested reader to [2] and references therein).

2.2 The path integral: Feynmann-Kac formula

We are now in the position to solve the time evolution problem for the considerably simpler operator (2.1). Our starting point is the time evolution equation (1.3), (1.4), (2.1):

$$\partial_t f(y, t) = \hat{L}^\dagger f(y, t),$$

we then explicitly compute the propagator $K(y_f, y_i | t, t')$:

$$f(y, t) = \int \prod_{j=1}^M dy'_j K(y, y' | t, t') f(y', t') \quad (2.5)$$

$$= \int \prod_{n=1}^N \prod_{j=1}^M dy_{jn} \prod_{n=1}^N K(y_{n+1}, y_n | t_{n+1}, t_n) f(y_1, t_1). \quad (2.6)$$

We employ the standard time discretization scheme as shown above, (see also for instance [3]), we insert the unit N times, $(\frac{1}{2\pi} \int dy_{jn} dp_{jn} e^{ip_{jn}(y_{jn}-a)} = 1)$, for each

component y_j , and we perform the Gaussian integrals with respect to each p_{jn} parameter. We then conclude that the path integral can be expressed as

$$K(y_f, y_i | t, t') = \int d\mathbf{q} \exp \left[- \sum_j \sum_n \frac{(\Delta y_{jn} - \delta \tilde{b}_{jn}(y))^2}{2\delta} + \delta \sum_n u_n(y) \right] \quad (2.7)$$

$$d\mathbf{q} = \frac{1}{(2\pi\delta)^{\frac{NM}{2}}} \prod_{n=2}^N \prod_{j=1}^M dy_{jn} \quad (2.8)$$

where $f_n = f_n(y_n)$ and $\Delta y_{jn} = y_{jn+1} - y_{jn}$. where $\delta = t_{n+1} - t_n$ and with boundary conditions: $y_f = y_{N+1}$, $y_i = y_1$, $t_i = t' = 0$ (t' will be dropped henceforth for brevity), $t_f = t$.

We recall expression (2.7) and we make the fundamental assumption [2]:

$$\Delta y_n - \delta \tilde{b}_n(y) = \Delta w_n \quad (2.9)$$

assuming also that w_{nj} are Brownian paths (see for instance [4] on Wiener processes), i.e. (2.9) is the discrete time analogue of an SDE. After a change of the volume element in (2.7), subject to (2.9), we conclude (see [2] for the detailed computation):

$$K(y_f, y_i | t) = \int d\mathbf{M} e^{\int_0^t u(y_s) ds}, \quad (2.10)$$

$$d\mathbf{M} = \lim_{\delta \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{(2\pi\delta)^{\frac{NM}{2}}} \exp \left[- \frac{1}{2\delta} \sum_{n=1}^N \Delta w_n^T \Delta w_n \right] \prod_{n=2}^N \prod_{j=1}^M dw_{jn}. \quad (2.11)$$

We may now evaluate the measure: in the continuum time limit (2.11), we consider the Fourier representation on $[0, t]$ for w_s , i.e. Wiener's representation of the Brownian path [4]:

$$w_s = \frac{f_0}{\sqrt{t}} s + \sqrt{\frac{2}{t}} \sum_{k>0} \frac{f_k}{\omega_k} \sin(\omega_k s), \quad \omega_k = \frac{2\pi k}{t}. \quad (2.12)$$

$f_0 = \frac{w_t}{\sqrt{t}}$ and f_k , $k \in \{0, 1, \dots\}$ are M vectors with components f_{kj} , $j \in \{1, 2, \dots, M\}$ being standard normal variables. We are interested in the computation of the measure in the continuum limit $N \rightarrow \infty$, $\delta \rightarrow 0$, and we also recall the following boundary conditions: $w(s=0) = 0$, $w(s=t) = w_t$, then

$$\begin{aligned} d\mathbf{M} &= \frac{e^{-\frac{1}{2t} w_t^T w_t}}{(2\pi t)^{\frac{M}{2}}} d\mathbf{M}_0 \\ d\mathbf{M}_0 &= \prod_{k \geq 1} \prod_{j=1}^M \frac{df_{kj}}{\sqrt{2\pi}} \exp \left[-\frac{1}{2} \sum_{k \geq 1} \sum_j f_{kj}^2 \right]. \end{aligned} \quad (2.13)$$

The measure naturally is expressed as an infinite product of Gaussians regardless of the specific forms of the diffusion coefficients and the drift.

Having computed the propagator explicitly (2.10) we conclude that equation (2.6) can be then expressed as

$$f(x_f, t_f) = \int d\mathbf{M} e^{\int_0^t u(x_s) ds} f_0(x_0), \quad f_0(x_0) = f(x_0, t_0)$$

which is precisely the Feynman-Kac formula, and describes the time evolution of a given initial profile $f_0(\mathbf{x}_0)$ to $f(\mathbf{x}_f, t_f)$ a solution of the Fokker-Planck equation. One could have started from the Kolmogorov backward equation and computed the path integral backwards in time:

$$f(\mathbf{x}_0, t_0) = \int d\mathbf{M} e^{\int_0^t u(\mathbf{x}_s) ds} f_f(\mathbf{x}_f), \quad f_f(\mathbf{x}_f) = f(\mathbf{x}_f, t_f).$$

In this case the Feynman-Kac formula describes the reversed time evolution of a given final state $f_f(\mathbf{x}_f)$, to a previous state $f(\mathbf{x}_0, t_0)$ a solution of the Kolmogorov backward equation.

One of the main aims is the computation of expectation values:

$$\langle \mathcal{O}(\mathbf{x}_s) \rangle = \frac{\mathbf{E}_t \left(\mathcal{O}(\mathbf{x}_s) e^{\int_0^t u(\mathbf{x}_s) ds} \right)}{\mathbf{E}_t \left(e^{\int_0^t u(\mathbf{x}_s) ds} \right)}, \quad 0 \leq s \leq t \quad (2.14)$$

where we define via (2.10), (2.13)

$$\mathbf{E}_t \left(\mathcal{O}(\mathbf{x}_s) \right) = \int d\mathbf{w}_t d\mathbf{M} \mathcal{O}(\mathbf{x}_s) \quad 0 \leq s \leq t. \quad (2.15)$$

(2.14) can be used provided that solutions of the associated SDEs are available, so that the fields \mathbf{x}_{tj} are expressed in terms of the variables \mathbf{w}_{tj} .

3 The quantum (D)NLS and a hierarchy of S(P)DEs

We start our analysis with the DNLS model, with the corresponding quantum Lax operator given by [5], [6],

$$L_j(\lambda) = \begin{pmatrix} \lambda + \Theta_j + z_j Z_j & z_j \\ Z_j & 1 \end{pmatrix}$$

z_j, Z_j are canonical $[z_i, Z_j] = -\delta_{ij}$, and we consider the map:

$$z_j \mapsto \mathbf{x}_j, \quad Z_j \mapsto \partial_{\mathbf{x}_j}. \quad (3.1)$$

Let us now define the generating function of the integrals of motion of the system:

$$\mathbf{t}(\lambda) = \text{tr} \left(L_M(\lambda) \dots L_2(\lambda) L_1(\lambda) \right). \quad (3.2)$$

Indeed, the expansion of $\ln(\mathbf{t}(\lambda)) = \sum_{k=0}^M \frac{I_k}{\lambda^k}$ provides the local integrals of motion (see e.g. [8]). We keep here terms up to third order in the expansion of $\ln(\mathbf{t})$ and by suitably scaling the involved fields, we obtain the first three local integrals of motion

of the quantum DNLS hierarchy (keeping the suitably scaled terms) [5]:

$$\begin{aligned}
H_1 &= \sum_j^M x_j \partial_{x_j} \\
H_2 &= \frac{1}{2} \sum_{j=1}^M x_j^2 \partial_{x_j}^2 - \sum_{j=1}^M \Delta^{(1)}(x_j) \partial_{x_j} \\
H_3 &= \frac{1}{2} \sum_{j=1}^M x_j^2 \partial_{x_j}^2 - \nu \sum_{j=1}^M \Delta^{(2)}(x_j) \partial_{x_j} + (\text{higher order terms})\dots \quad (3.3)
\end{aligned}$$

where we have chosen $\Theta_j = 1$, and $H_1 = I_1$, $H_2 = -I_2 + \frac{1}{2}I_1$, $H_3 = -\frac{1}{3}(I_3 + I_2 - \frac{1}{2}I_1)$, $\nu = \frac{1}{3}$. We also define: $\Delta^{(1)}z_j = z_{j+1} - z_j$, $\Delta^{(2)}z_j = z_{j+2} - 2z_{j+1} + z_j$.

The next order in the expansion provides H_4 , which is the Hamiltonian of the quantum version of complex mKdV system and so on. The equations of motion (classical and quantum) associated e.g. to $H^{(2)}$ can be derived via the zero curvature condition or Heisenberg's equation (recall also (3.1)):

$$\frac{dz_j}{dt} = -\Delta^{(1)}z_j + z_j^2 Z_j. \quad (3.4)$$

Similar equations can be obtained for H_3 , but are omitted here for brevity. The Hamiltonians $H_{2,3}$ are of the form (1.2), and the corresponding set of SDEs are [2]

$$dx_{tj} = -\nu_k \Delta^{(k-1)}x_t dt + x_{tj} dw_{tj}. \quad (3.5)$$

where $k \in \{2, 3\}$ and $\nu_2 = 1$, $\nu_3 = \frac{1}{3}$. ν_k can be set equal to one henceforth, after suitably rescaling time. By comparing (3.4) and (3.5) ($k = 2$) we observe that the non-linearity appearing in (3.4) is replaced by the multiplicative noise in (3.5).

Let us now derive the solution of the set of SDEs (3.5) introducing suitable integrator factors (see e.g. [10]). Let us consider the general set of SDES

$$dx_{tj} = b_j(x_t)dt + x_{tj}dw_{tj}.$$

We introduce the following set of integrator factors:

$$\mathcal{F}_j(t) = \exp\left(-\int_0^t dw_{sj} + \frac{1}{2}\int_0^t ds\right) \quad (3.6)$$

and define the new fields: $y_{tj} = \mathcal{F}_j(t)x_{tj}$, then one obtains a differential equation for the vector field y :

$$\frac{dy_t}{dt} = \mathcal{A}(t)y_t \Rightarrow y_t = \mathcal{P} \exp\left(\int_0^t \mathcal{A}(s)ds\right)y_0. \quad (3.7)$$

For instance in the case of (3.5), for $k = 2$, the $M \times M$ matrix \mathcal{A} is given as

$$\mathcal{A}(t) = \sum_{j=1}^M \left(e_{jj} - \mathcal{B}_j(t)e_{j,j+1} \right), \quad \mathcal{B}_j(t) = \exp\left(\Delta^{(1)}(w_{tj})\right),$$

where e_{ij} are $M \times M$ matrices with entries $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. For $k = 3$, the \mathcal{A} matrix involves also terms e_{jj+2} , and so on. The solution (3.7) can be expressed as a formal series expansion

$$\begin{aligned} \mathcal{P} \exp \left(\int_0^t \mathcal{A}(s) ds \right) = \\ \sum_{n=0}^{\infty} \int_0^t \int_0^{t_n} \dots \int_0^{t_2} dt_n dt_{n-1} \dots dt_1 \mathcal{A}(t_n) \mathcal{A}(t_{n-1}) \dots \mathcal{A}(t_1), \\ t \geq t_n \geq t_{n-1} \dots \geq t_2. \end{aligned}$$

Remark 1.

The discrete version of the stochastic Burgers equation can be obtained from the discrete stochastic heat equation through the analogue of the Cole-Hopf transformation. Indeed, by setting $x_j = e^{y_j}$, in (3.5) ($k = 3$):

$$dy_j = - \left(e^{\Delta y_j} (e^{\Delta y_{j+1}} - 1) - (e^{\Delta y_j} + 1) \right) dt + dw_j, \quad (3.8)$$

where for simplicity we have set $\Delta^{(1)} = \Delta$. By also setting $u_j = \Delta y_j$, we obtain a discrete version of the stochastic Burgers equation

$$du_j = - \left(e^{u_{j+1}} (e^{u_{j+2}} - e^{u_j}) - 2(e^{u_{j+1}} - e^{u_j}) \right) dt + \Delta dw_j. \quad (3.9)$$

Assuming the scaling $\Delta y_j \sim \delta$, we expand the exponentials and keep up to second order terms in (3.8), (3.9):

$$dy_j = - \left(\Delta^{(2)} y_j + (\Delta y_j)^2 + \mathcal{O}(\delta^3) \right) + dw_j \quad (3.10)$$

$$du_j = - \left(\Delta^{(2)} u_j + \Delta u_j^2 + \mathcal{O}(\delta^3) \right) + \Delta dw_j. \quad (3.11)$$

The second of the equations above provides a good approximation for the discrete viscous Burgers equation, as will be also clear in the next subsection.

3.1 The continuum models and SPDEs

It will be instructive to consider the continuum limits of the Hamiltonians H_2 , H_3 (3.3) and the respective SDEs. After considering the thermodynamic limit $M \rightarrow \infty$, $\delta \rightarrow 0$ ($\delta \sim \frac{1}{M}$) we obtain

$$\begin{aligned} x_{tj} &\rightarrow \varphi(x, t), & \frac{x_{tj+1} - x_{tj}}{\delta} &\rightarrow \partial_x \varphi(x, t), \\ \delta \sum_j f_j &\rightarrow \int dx f(x), & w_{tj} &\rightarrow W(x, t), \end{aligned} \quad (3.12)$$

where the Wiener field or Brownian sheet $W(x, t)$ is periodic and square integrable in $[-L, L]$, and is represented as [4]

$$W(x, t) = \frac{\sqrt{L}}{\pi} \sum_{n \geq 1} \frac{1}{n} \left(X_t^{(n)} \cos \left(\frac{n\pi x}{L} \right) + Y_t^{(n)} \sin \left(\frac{n\pi x}{L} \right) \right), \quad (3.13)$$

$X_t^{(n)}, Y_t^{(n)}$ are independent Brownian motions. In the continuum limit the Hamiltonians (3.3) become the Hamiltonians of quantum NLS hierarchy:

$$H_c^{(k)} = \int dx \left(\frac{1}{2} \varphi^2(x) \hat{\varphi}^2(x) - \partial_x^{(k-1)} \varphi(x) \hat{\varphi}(x) \right), \quad k = 2, 3 \quad (3.14)$$

where $[\varphi(x), \hat{\varphi}(y)] = \delta(x-y)$, $(\hat{\varphi}(x) \sim \frac{\partial}{\partial \varphi(x)})$ and the SDEs (3.5) become the stochastic transport ($k=2$) and heat equation ($k=3$) with multiplicative noise:

$$\partial_t \varphi(x, t) = -\partial_x^{k-1} \varphi(x, t) + \varphi(x, t) \dot{W}(x, t).$$

The stochastic heat equation can be mapped to the stochastic Hamilton-Jacobi and viscous Burgers equations [1]. Indeed, we set: $\varphi = e^h$, $u = \partial_x h$ then (3.15):

$$\begin{aligned} \partial_t h(x, t) &= -\partial_x^2 h(x, t) - (\partial_x h(x, t))^2 + \dot{W}(x, t) \\ \partial_t u(x, t) &= -\partial_x^2 u(x, t) - 2u(x, t) \partial_x u(x, t) + \partial_x \dot{W}(x, t). \end{aligned} \quad (3.15)$$

Connections between the SDEs and the quantum Darboux transforms [11], [12] can be also studied. The classical Darboux-Bäcklund transformation [13], [14] provides an efficient way to find solutions of integrable PDEs. The key question is how this transformation can facilitate the solution of SDEs [1], [15], [16].

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