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Bose-Einstein condensate with quintic-order nonlinearity

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We study the dynamics of sonic black hole horizon formation of quasi-one-dimensional (1D) Bose-Einstein condensate incorporating higher-order quintic nonlinear interaction. Based on the one dimensional Gross-Pitaevskii equation with nonlinearity up to the quintic order, we derived a novel analytical formula for the key dynamical variables of sonic horizon formation using the modified variational method and exact F-expansion method. We obtained good agreement between the key dynamical variables from the two different methods. The stabilization effects of higher-order nonlinear interaction along with the more precise location of the sonic horizon boundary were illustrated. The theoretical results obtained in this work can be used to guide relevant experimental observations of sonic black hole-related dynamics incorporating the effects of higher-order nonlinear interaction.

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Black hole-related physics is a fascinating subject that is extensively investigated both in astrophysical observation and theory. Black hole-related problems have also drawn the attention of physicists in the field of ultracold physics. As typical entity in astrophysics, black hole was originally predicted by General Relativity as the region with compact mass possessing strong gravitational effects so that even light cannot escape from the region. As pointed out by Unruh [1], if we identify supersonic excitation as fields and system fluid flow as curved space-time in an ultracold atomic system setting, then a sonic black hole exists with the formation of the sonic horizon. The sonic horizon is the boundary inside which fluid flow is supersonic, and outside which the flow is subsonic. Unlike and unrepeatable, uncontrollable and high cost occurrence in astrophysical setting for black hole related physics observation, the sonic black hole with similar characteristic like Hawking radiation [2, 3], can be studied with a flexible and controllable ultracold atomic system setting.

In the study of an ultracold atomic system like the Bose-Einstein condensate (BEC), a flexible way to modulate the atomic system is to tune the interparticle nonlinear interaction using the Feshbach resonance technique. It has been found that owing to the wide system density variations, the three-body scattering effect must be taken into account; and higher-order nonlinear interaction [4–8], such as the quintic interaction term must be incorporated into the theoretical model for studying the system dynamics, like sonic black hole horizon formation and evolution. In fact, recent studies showed that in practical implementations, higher-order nonlinear interaction plays an important role in stabilizing the evolution mode in optically-controlled ultracold systems, such as the stable remote transport of vortex solitons [9].

In this work, based on the one-dimensional Gross-Pitaevskii equation (GPE) [10–17] incorporating a higher-order quintic nonlinear interaction term, we study the sonic black horizon formation problem in a quasi-one-dimensional BEC trapped in elongated harmonic potential. For validation purposes, calculations were carried out using two methods: the modified variational method [18] and exact F-expansion method [19, 20]. The key sonic black hole horizon dynamical variables were derived from the two methods. The analytical formula of the system fluid flow velocities derived from the two methods showed good agreement. The evolution of the BEC distribution width parametric function and sonic horizon
interaction is illustrated by plotting the "potential" curve that affects the parameters of system evolution. We also compared our theoretically-derived sonic horizon location with higher-order nonlinear interaction and the scenario without higher-order nonlinear interaction. The sonic horizon location showed better agreement with the sonic horizon location from the numerical simulation when higher-order nonlinear effects were incorporated. The theoretical results obtained can be used to guide relevant experimental observations of sonic black hole dynamics incorporating higher-order nonlinear effects.

The remainder of this paper is organized as follows. The next section presents the theoretical one-dimensional GPE model incorporating nonlinear interaction up to quintic order and the calculation details of the system wave function and sonic horizon variables based on the variational method. Section III presents the same quantities calculated in Section II using the exact F-expansion method. Section IV discusses and compares the theoretically-derived results; and the final section presents some concluding remarks.

II. OSCILLATION MODE FOR BOSE-EINSTEIN CONDENSATE WITH HIGHER-ORDER NONLINEAR INTERACTION BASED ON THE VARIATIONAL METHOD

As the promising model for the study of Bose-Einstein condensate, the form of one-dimensional GPE equation incorporating nonlinear interaction up to quintic order is as follows

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \Omega^2 x^2 + g_1 |\Psi|^2 + g_2 |\Psi|^4$$ (1)

We assume the following ansatz for the wave function, $\Psi(x,t) = \Psi(\frac{x}{\rho(t)}, t)$, $\rho(t = 0) = \rho_0$.

Eq. (1) is derived from the following Lagrangian density,

$$L = \Psi^* \left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{1}{2} m \Omega^2 x^2 - \frac{1}{2} g_1(t) |\Psi|^2 - \frac{1}{3} g_2(t) |\Psi|^4 \right] \Psi$$ (2)

We aim to investigate the ground state perturbed mode of the system, more explicit form of $\Psi$ is as follows,

$$\Psi(\frac{x}{\rho(t)}, t) = \frac{C_0}{\rho^{1/2}(t)} \varphi(\frac{x}{\rho(t)}) \exp(-i\beta(t)x^2)$$ (3)
\[ \beta(t) = \frac{m \rho'(t)}{2\hbar \rho(t)} \]  

(4)

Firstly integrating over the spatial coordinates, the Euler-Langer equation of \( \rho(t) \) can be derived from Eq. (2) as follows,

\[ m\rho''(t) = -\frac{\partial V(\rho)}{\partial \rho} \]  

(5)

where \( V_0(\rho) = m\frac{\Omega^2}{2} \sigma^2 + \frac{\hbar^2}{2m} \frac{C_1}{C_3} \sigma^2 \), \( V_1(\rho) = g_1(t)C_3/C_1(1/\sigma^2) \), \( V_2(\rho) = \frac{g_2(t)C_2}{C_1} \frac{1}{\sigma^4} \), \( C_1 = \int_0^\infty x^2 \sigma \varphi^2(x) dx \), \( C_2 = \int_0^\infty \frac{d(\varphi^2)}{dx} dx \), \( C_3 = \int_0^\infty \varphi^2 dx \), \( C_4 = \int_0^\infty \varphi^4 dx \). When the nonlinear interaction strength \( g_1, g_2 \) are very small, we can make the approximation \( V = V_0 \), So Eq. (5) is reduced to

\[ m\rho''(t) = -\frac{\varphi V_0(\rho)}{\rho} \]  

(6)

Eq. (6) can be analytically solved with the solution \( \rho^2(t) = a_0 \sin(\omega t) + b_0 \) where \( a_0 = \sqrt{(\frac{\omega^2}{2} + \frac{\hbar^2}{2m^2\varphi_0^2})^2 - \frac{C_2\hbar^2}{2C_1m^2\varphi_0^2}} \), \( b_0 = \frac{\omega^2}{2} + \frac{C_2\hbar^2}{2C_1m^2\varphi_0^2} \), \( \omega = 2\Omega \). Widening \( \rho(t) \) oscillates at frequency \( \omega \) between maximum \( \sqrt{b_0 + a_0} \) and minimum \( \sqrt{b_0 + a_0} \). When considering the influence of \( g_1(t) \) and \( g_2(t) \) on the nonlinear interaction, the evolution mode of \( \rho(t) \) is affected by \( V(\rho) \). When \( g_1(t) << \frac{\hbar^2}{2m^2\varphi_0^2}, |g_2| < |g_1| \), for three cases of (I) \( g_2 = 0 \), (II) \( g_2 < 0 \), (III) \( g_2 > 0 \), we draw diagrams of \( V(\rho) \) for typical nonlinear interaction strength setting.

The sonic horizon formation analysis is carried out based on the calculation of the system flow velocity and sound velocity. First we calculate the flow velocity from Eq. (3) and Eq. (4),

\[ v_{f1} = \frac{\hbar}{2mi}(\nabla^* \nabla \Psi - \Psi \nabla \nabla^*) = \frac{\hbar}{m} \frac{\partial}{\partial x} (-\beta(t)x^2) = \frac{\rho'(t)}{\rho(t)} x \]  

(7)

This result comes from the ansatz (4), we will verify Eq. (7) from exact solution of Eq. (1) based on a different method-F expansion method. Now we perform stability analysis regarding system distribution width \( \rho(t) \), whose analytical form depends on the 'potential' \( V(\rho) \) in Eq. (5). We plot \( V(\rho) \) vs. \( \rho \) regarding the contribution of nonlinear strength constant \( g_1 \) and \( g_2 \), as shown in Fig. (1). We can see from the plot that while the system distribution width \( \rho(t) \) oscillates around the minimum of \( V(\rho) \), The nonzero quintic order nonlinear interaction \( g_2 \) tends to narrow the oscillation range with higher value of \( g_2 \). So the (positive) higher-order nonlinear interaction has the stabilization effect.
FIG. 1: $V(\rho)$ vs. system distribution width $\rho(t)$ for the cases of $g_1 = 0.1$ and (I) $g_2 = 0$ (black solid line), (II) $g_2 = 0.01$ (dotted red line), (III) $g_2 = 0.05$ (dashed blue line). ($g_2$ is in units of $\frac{4\pi\hbar^2a_0}{m}$, and $a_0$ is the initial s-wave scattering length)

III. EXACT OSCILLATION MODE STUDY FOR BOSE-EINSTEIN CONDENSATE WITH HIGHER-ORDER NONLINEAR INTERACTION

In order to have a double-check for the quantitative results obtained in previous section from variational method, we try to strictly solve Eq. (1) based on the $F$-expansion method. We assume that the exact analytical solution of Eq. (1) has the following form

$$
\Psi(x,t) = \sigma^{1/2}(t) \exp[i\frac{\sigma(t)}{\sigma(t)} x^2] V(x,t) e^{i\theta(x,t)}
$$

where

$$
V(x,t) = h(t) G(\xi)
$$

$$
\xi = p(t)x + q(t)
$$

$$
\theta(x,t) = \Phi_1(t)x^2 + \Phi_2(t)x + \Phi_0(t)
$$

$G(\xi)(\xi = p(t)x + q(t))$ is the base function in the $F$-expansion approach that satisfies

$$
\left(\frac{dG(\xi)}{d\xi}\right)^2 = T(G) = b_8G^8 + b_6G^6 + b_4G^4 + b_2G^2 + b_0
$$
Eqs. (8), (9) and (10) into Eq. (1), A polynomial consisting of \( x^iG^j \) or \( x^i(G^1)^j \) terms (\( i, j \) are integers) can be obtained by using Eq. (11). The coefficients of various terms of the polynomial are ordinary differential equations of timing variable \( t \), they are as follows,

\[
\begin{align*}
  x^2G^i(\xi) : \Phi_2'(t) + 4\sigma^2(t)\Phi_2'(t) + \alpha(t) &= 0, \\
  xG^i(\xi) : \Phi_1'(t) + 4\sigma^2(t)\Phi_2(t)\Phi_1(t) &= 0, \\
  G^8(\xi) : 8\sigma^2(t)p^2(t)a_8 - g_3\sigma^3(t)h^8(t) &= 0, \\
  G^6(\xi) : 6\sigma^2(t)p^2(t)a_6 - g_2\sigma^2(t)h^4(t) &= 0, \\
  G^4(\xi) : 4\sigma^2(t)p^2(t)a_4 - g_1\sigma(t)h^2(t) &= 0, \\
  G^2(\xi) : 2\sigma^2(t)p^2(t)a_2 - \Phi'_0(t) &= 0, \\
  xG'(\xi) : p'(t) + 4\sigma^2(t)\Phi_2(t)p(t) &= 0, \\
  G'(\xi) : q'(t) + 2\sigma^2(t)p(t)\Phi_1(t) &= 0, \\
  G(\xi) : h'(t) + 2\sigma^2(t)\Phi_2(t)h(t) &= 0
\end{align*}
\]  

(12a)-(12i)

we can see from Eq. (12) that

\[
\begin{align*}
  \sigma(t)p(t) &= D_0, \quad \sigma(t)\Phi_1(t) = D_1 \\
  \sigma(t)h^2(t) &= D_2 \\
  b_6 &= \frac{g_2D_2^2}{6\sigma D_1^2} \\
  b_4 &= \frac{g_1D_2}{4\sigma D_1^2}
\end{align*}
\]  

(13a)-(13d)

where \( D_0, D_1, D_2 \) are constants to be determined by the initial experimental conditions. And \( b_0, b_1 \) can be adjusted freely, the exact value depends on the experimental conditions. Considering full \textit{Cubic – Quintic} nonlinear interaction \((g_2 \neq 0)\)

\[
T(G) = b_0G^6 + b_4G^4 + b_2G^2 = k_0G^6[(G^{−2} − c)^2 − d]
\]  

(14)

where \(c = -\frac{b_4}{2b_6}, d = \frac{b_2}{4b_6} - \frac{b_4}{b_2}, k_0 = b_2\).

In combination with Eq. (11) and Eq. (14), when \(d > 0\), \(G(x, t)\) can be obtained from definition formula (11), (14) of \(G\),

\[
G(x, t) = \sqrt{\frac{1}{d^{1/2}\cosh(b_2^{1/2}\xi) + c}}
\]  

(15)
The solution of Eq. (1) has the following form:

\[ |\psi(x, t)| = C^{1/2} \sqrt{\frac{1}{d^{1/2} \cosh(b_2^{1/2} \xi) + c}} \]  

(16)

This is a solution of bright soliton type. Eq. (16) can be regarded as an exact analytical description of the pseudo-form \( \varphi \) of Eq. (3) wave function.

From Eqs. (12), with proper setting of free parametric function \( \Phi_0(t) \), we can get

\[ \Phi_2(t) = 0 \]  

(17)

and

\[ \Phi_1(t) = -\frac{m \dot{\sigma}(t)}{4\hbar \sigma(t)} \]  

(18)

Also from Eqs. (12), we can get the equation satisfied by \( \sigma(t) \) as follows,

\[ \frac{d}{dt}(\frac{\dot{\sigma}}{\sigma^3}) + 4\frac{\ddot{\sigma}^2}{\sigma^4} + k \frac{\ddot{\sigma}}{\sigma^2} - \frac{\dot{\sigma}}{2\sigma} + \frac{d}{dt}(\frac{\dot{\sigma}}{4\sigma}) = 0 \]  

(19)

Eq. (19) has the following analytical solution,

\[ \sigma(t) = A_0 \sin 2\Omega t + B_0 \]  

(20)

From the phase function (10), we can reach the analytical formula for the system flow velocity as follows

\[ v_{f2} = \frac{\hbar}{m} \frac{\partial}{\partial x}(-\Phi_1(t)x^2) = \frac{\dot{\sigma}(t)}{2\sigma(t)} x \]  

(21)

Using Eqs. (18) and (20) and comparing Eq. (21) with Eq. (7), we can see that the analytical expressions for system flow velocity \( v_{f1} \) and \( v_{f2} \) are identical. The reason is that not only the derivation for \( v_{f2} \) is exact, the derivation for \( v_{f1} \) is based on the imaginary part of Eq. (1) which is also exact (although the modulus of \( \Psi \) is not strictly exact in variational method). This demonstrates the validity of our theoretical treatments based on the two different analytical methods for the sonic black hole horizon dynamics study. While the variational method clearly demonstrates the oscillation mode for sonic horizon formation and evolution, the exact \( F \)-expansion method gives a more precise description of the system wave function.
The main task in the study of sonic black hole horizon formation is to discover the timing and spatial coordinates for the sonic horizon appearance. And it is the flow velocity and sound velocity that determine the location of the sonic horizon of the system. The position of the sonic horizon is the intersection point(s) of the sound velocity and the flow velocity of the system within certain time range of its existence. The change of the general sound velocity ($\propto |\varphi|$) near the intersection is relatively slow, and the slight change of the form $\varphi$ has little effect on the position of the sonic horizon. The analytical expression of the system sound velocity is

$$c_s(x, t) \simeq \sqrt{\frac{g_0}{m} |\psi(x, t)|} = \sqrt{\frac{g_0}{m} \rho^{3/2}(t_0)} \sqrt{\frac{1}{d^{1/2} \cosh\left(\frac{b_1}{2} \frac{1}{\rho(t)} x \right) + c}}$$

(22)

Here we choose the exact expression (16) for $\psi$ with the fact $\sigma(t) = \dot{\rho}(t)$ just identified. The location of the sonic horizon $x = x_s$ is determined by the solution of the following boundary equation,

$$[v_f(x, t) - c_s(x, t)]|_{x=x_s} = 0$$

(23)

where $v_f = v_{f1} = v_{f2}$ is the system flow velocity. The timing range of the sonic horizon existence is the timing set for which Eq. (23) has real solution $x_s$. It is not hard to see that for Eq. (23) to have solution, we generally require $\dot{\rho}(t) > 0$ in one oscillation period of system width $\rho(t)$. So the sonic horizon appears and disappears (at least for the timing range when $\dot{\rho}(t) < 0$) periodically. Fig (2) gives the plot showing the location of sonic horizon-at the coordinate of intersection point between the flow velocity $c_f$ curve and sound velocity curve $c_s$. We can see that, the match between theoretical calculated curves and numerically simulated curves is good, and the theoretical curve incorporating higher-order nonlinear effect matches with numerical curve better comparing with case without higher-order nonlinear interaction, showing the importance of higher-order nonlinear interaction contribution for the sonic horizon dynamics.
FIG. 2: System flow velocity $v_f(x)$ and sound velocity $c_s(x)$ vs. $x$ at the middle point of the timing range when the sonic horizon exists, based on Eqs. (7), (21) and (22) (in units of $(\hbar\omega/m)^{1/2}$): Curve I and Curve II are sound velocity from analytical result (22) and simulation, the dashed green line is flow velocity $v_f$ from numerical simulation, the blue straight line shows the analytical derived $v_f$ incorporating nonzero quintic-order nonlinear interaction $g_2$, the dotted red line shows the analytical $v_f$ without considering the quintic-order nonlinearity $g_2 = 0$.

V. CONCLUSION

In this work, we studied the effects of higher-order nonlinear interaction on the sonic analog of an astrophysical black hole in quasi-one-dimensional BEC. Using the modified variational method and exact F-expansion method, we derived the key dynamical variables for sonic black hole horizon dynamics based on the one-dimensional GPE incorporating higher quintic nonlinearity. We obtained good agreement between the kinetic quantities derived from the two methods, demonstrating the validity of the theoretical treatment proposed in this work. The stabilization effects together with more precise determination of the sonic horizon location from higher-order nonlinear interaction effects were illustrated. The theoretical results obtained can be used to guide experimental observations of sonic black hole-related phenomena incorporating higher-order nonlinear effects.
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