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STABILITY OF SOLUTIONS OF A 1-DIMENSIONAL, p-LAPLACIAN PROBLEM AND THE SHAPE OF THE BIFURCATION CURVE

BRYAN P. RYNNE

ABSTRACT. We consider the p-Laplacian boundary-value problem

$$-\varphi_p(u')' = \lambda f(u), \quad \text{on } (-1,1), \tag{1}$$
$$u(\pm 1) = 0, \tag{2}$$

where $p > 1$ $(p \neq 2)$, $\varphi_p(z) := |z|^{p-1} \operatorname{sgn} z$, $z \in \mathbb{R}$, $\lambda \geqslant 0$, $f : \mathbb{R} \to \mathbb{R}$ is C^2 and $f > 0$ on $\mathbb{R}$. Under these conditions the set of solutions (λ, u) of (1)-(2) consists of the trivial solution $(\lambda, u) = (0,0)$ together with a single (connected) C^2 curve $\mathcal{S} \subset \mathbb{R}_+ \times C^1_0[-1,1]$ $(\mathbb{R}_+ = (0,\infty))$. Under additional conditions on f the 'shape' of $\mathcal{S}$ can be determined.

Solutions of (1)-(2) are equilibrium solutions of a related time-dependent, parabolic problem, and in this time-dependent setting the stability of these equilibria is of interest. It will be shown that the stability of solutions on $\mathcal{S}$ is determined by the shape of $\mathcal{S}$. This will first be discussed in a general setting, and the results will then be applied to the specific case where $\mathcal{S}$ is 'S-shaped'. Finally, similar results will be obtained, for 'generic' λ, without any additional conditions on f.

1. Introduction

We consider the p-Laplacian boundary-value problem

$$-\varphi_p(u')' = \lambda f(u), \quad \text{on } (-1,1), \tag{1.1}$$
$$u(\pm 1) = 0, \tag{1.2}$$

where $p > 1$ $(p \neq 2)$, $\varphi_p(z) := |z|^{p-1} \operatorname{sgn} z$, $z \in \mathbb{R}$, $\lambda \geqslant 0$, and the function $f : \mathbb{R} \to \mathbb{R}$ is C^2 and

$$f(\xi) > 0, \quad \xi \in \mathbb{R}. \tag{1.3}$$

It is well-known that under these conditions on f the set of solutions (λ, u) of (1.1)-(1.2) consists of the trivial solution $(\lambda, u) = (0,0)$ together with a single (connected) C^2 curve $\mathcal{S} \subset \mathbb{R}_+ \times C^1_0[-1,1]$ (where $\mathbb{R}_+ = (0,\infty)$). Under additional conditions on f the 'shape' of the curve $\mathcal{S}$ can be determined. For example, under the conditions in [6] the curve $\mathcal{S}$ has a single 'bend', to the left, while in [11] it has two bends, so is said to be 'S-shaped' (this is illustrated in Fig. 1 below, and described more precisely in Theorem 4.1 and the remarks following this theorem).

Solutions of (1.1)-(1.2) can be regarded as equilibrium solutions of a related time-dependent, parabolic problem and, in this time-dependent setting, the stability of these equilibria is of interest, and this will be determined here. It will be shown that the stability of the solutions lying on $\mathcal{S}$ is determined by the shape of $\mathcal{S}$. This will first be discussed in a general setting (in Section 3), and then the results will be applied to the specific case where the hypotheses of [11] hold, so that $\mathcal{S}$ is S-shaped (in Section 4). To summarise the results in the S-shaped case, the solutions lying on the 'middle branch' of $\mathcal{S}$ are unstable, whereas the solutions lying on the 'upper' and 'lower' branches are stable. Both local 'linearised' (exponential) and more global stability results will be obtained.

We also consider a more general setting where we do not impose any additional conditions on f to force $\mathcal{S}$ to have a specific shape. In this general setting we describe the stability properties of the solutions lying on $\mathcal{S}$ for 'generic' values of λ. These stability properties are again related to the shape of $\mathcal{S}$, and generalise those obtained in the S-shaped case.

2. Properties of solutions of (1.1)–(1.2)

2.1. **Notation.** For any integer $j \geqslant 0$, $C^j[-1,1]$ will denote the standard Banach space of real valued, j-times continuously differentiable functions w defined on $[-1,1]$, with the norm $|w|_j = \sum_{i=0}^{j} |w^{(i)}|_0$, where $|\cdot|_0$ denotes the usual sup-norm on $C^0[-1,1]$ (throughout, all function spaces will be real). For any $r \geqslant 1$, $L^r(-1,1)$ will denote the standard Banach space of real valued functions on $[-1,1]$ whose rth power is integrable, with norm $\|\cdot\|_r$. Also, $W^{1,r}(-1,1)$, with norm $\|\cdot\|_{1,r}$, will denote the usual Sobolev space of absolutely continuous functions w on $[-1,1]$, with derivative $w' \in L^r(-1,1)$. We also let $C_0^j[-1,1]$, $W_0^{1,r}(-1,1)$ denote the set of functions w in $C^j[-1,1]$, $W^{1,r}(-1,1)$, respectively, satisfying the boundary conditions (1.2).

For any continuous function $g : \mathbb{R} \to \mathbb{R}$, and any $w \in C^0[-1,1]$, we define $g(w) \in C^0[-1,1]$ by $g(w)(x) = g(w(x))$, $x \in [-1,1]$ (that is, g will denote both a function and its corresponding Nemytskii operator).

A *solution* of (1.1)–(1.2) is a function $u \in C_0^1[-1,1]$ such that $\varphi_p(u') \in C^1[-1,1]$ and equation (1.1) holds. We define the p-*Laplacian* operator $\Delta_p : D(\Delta_p) \subset C_0^1[-1,1] \to C^0[-1,1]$ by

$$D(\Delta_p) := \{u \in C_0^1[-1,1] : \varphi_p(u') \in C^1[-1,1]\},$$
$$\Delta_p(u) := \varphi_p(u')', \quad u \in D(\Delta_p).$$

With the above notation, (1.1)–(1.2) can be rewritten as

$$-\Delta_p(u) = \lambda f(u), \quad (\lambda, u) \in [0,\infty) \times D(\Delta_p). \tag{2.1}$$

Clearly, $(\lambda, u) = (0,0)$ is a solution of (2.1), and it is easy to see that the only solution of (2.1) with $\lambda = 0$ or $u = 0$ is the trivial solution $(0,0)$. Let

$$\mathcal{S} := \{(\lambda, u) \in \mathbb{R}_+ \times D(\Delta_p) \text{ satisfying (2.1) }\}.$$

2.2. **Basic properties of solutions.**

Lemma 2.1 ([6, Lemma 2.2]). *If $(\lambda, u) \in \mathcal{S}$, then:*

(*a*) $u > 0$ *on* $(-1,1)$ *and* $\pm u'(\pm 1) < 0$;

(*b*) u *is symmetric about* $x = 0$ (*that is,* u *is even*), *and* $|u|_0 = u(0)$, $u'(0) = 0$;

(*c*) $u \in C^2(0,1]$, *and* $u' < 0$, $u'' < 0$ *on* $(0,1]$.

For v, $w \in C_0^1[-1,1]$ we will use the notation

$$v \ll w \iff v(x) < w(x), \quad x \in (-1,1), \quad v'(-1) < w'(-1), \quad v'(1) > w'(1). \tag{2.2}$$

Lemma 2.2. *If (λ, u_1), $(\lambda, u_2) \in \mathcal{S}$ satisfy $|u_1|_0 < |u_2|_0$, then $u_1 \ll u_2$.*

Proof. Let

$$F(\xi) := \int_0^{\xi} f(t)\, dt, \quad \xi \geqslant 0. \tag{2.3}$$

Now suppose that there exists $x_0 \in (-1,1)$ such that $u_c := u_1(x_0) = u_2(x_0)$. By Lemma 2.1 we may suppose that $x_0 \in (-1,0)$, and then, by [10, (4.7)],

$$x_0 + 1 = \left(\frac{p-1}{p\lambda}\right)^{1/p} \int_0^{u_c} [F(|u_i|_0) - F(\xi)]^{-1/p}\, d\xi, \quad i = 1,2.$$

This is a contradiction, since F is strictly increasing, by (1.3) and the definition of F. Hence, since $|u_1|_0 < |u_2|_0$, we must have $u_1 < u_2$ on $(-1,1)$. The required inequalities on the derivatives at ± 1 now follow from uniqueness of solutions of the ODE (1.1). □

2.3. **The general structure of the set $\mathcal{S}$.** The papers [6] and [11] consider the problem (2.1), and each shows that $\mathcal{S}$ is a C^2 curve and constructs a parametrisations of $\mathcal{S}$. However the motivation, and hypotheses on f, in each of these papers is different. Specifically:

- in [11] the 'shape' of the curve $\mathcal{S}$ is described – in particular, under suitable hypotheses on f, it is shown that $\mathcal{S}$ is 'S-shaped';
- in [6] the solutions on $\mathcal{S}$ are regarded as equilibria of an associated time-dependent, parabolic problem – in this context it is shown that the stability of these equilibria is determined by the shape of the curve.

Here, we wish to combine these results to determine the stability of the solutions on the S-shaped curve obtained in [11]. To do this we need to use properties of the parametrisations from each of these papers, so we briefly recall both constructions and then relate the required properties.

We first consider the problem

$$-\Delta_p(u) = \lambda f(u), \quad u(0) = \alpha, \quad (\lambda, u) \in \mathbb{R}_+ \times D(\Delta_p), \quad \alpha \in \mathbb{R}_+. \tag{2.4}$$

Theorem 2.3. *For any $\alpha \in \mathbb{R}_+$,* (2.4) *has a unique solution $(\lambda(\alpha), u(\alpha)) \in \mathbb{R}_+ \times D(\Delta_p)$. Hence, $\mathcal{S} = \{(\lambda(\alpha), u(\alpha)) : \alpha \in \mathbb{R}_+\}$. In addition:*

(*a*) *the mapping $\alpha \to (\lambda(\alpha), u(\alpha)) : \mathbb{R}_+ \to \mathbb{R}_+ \times C_0^1[-1, 1]$ is a C^2 parametrisation of the curve $\mathcal{S}$;*

(*b*) $\lim_{\alpha \to 0^+} (\lambda(\alpha) + |u(\alpha)|_0) = 0$;

(*c*) *if $0 < \alpha_1 < \alpha_2$ and $\lambda(\alpha_1) = \lambda(\alpha_2)$, then $u(\alpha_1) \ll u(\alpha_2)$.*

Remark 2.4. Here, 'parametrisation' means that the tangent vector $(\lambda_\alpha(\alpha), u_\alpha(\alpha)) \neq (0, 0)$, for all $\alpha \in \mathbb{R}^+$. We use the notation λ_α and u_α to denote the derivatives with respect to α, to avoid confusion with the x derivative of the function $u(\alpha) \in C_0^1[-1, 1]$, which will be denoted $u(\alpha)'$. Also, the value of $u(\alpha)$ at $x = 0$ will be denoted by $u(\alpha)|_0$.

Proof. For any $\alpha \in \mathbb{R}_+$ the value of $\lambda(\alpha)$ can be calculated explicitly, and has the form $\lambda(\alpha) = T(\alpha)^p$, where $T(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a so-called 'time map', whose explicit form is derived in the proof of [4, Lemma 2.4 (3)], and is stated in [11, (2.8)] (the time map is denoted by $g(\cdot)$ in [4]; we use the notation from [11] here). The time map, or variants on it, has been considered in many papers – see, for example, [1] for alternative constructions and applications of the map. The formula for $T(\cdot)$ can be differentiated (twice) to show that $T(\cdot)$ is C^2 (the details are described in [9, Section 2], for the case $p = 2$, but there is no significant difference in the case of general p). In fact, the explicit forms of the derivatives $T'(\alpha)$ and $T''(\alpha)$ are given in [11, (3.1)] and [11, (3.2)], respectively, although these will not be required here.

The construction of the time map in [4] does not construct the solution $u(\alpha)$, indeed, it presupposes the existence of $u(\alpha)$, so it does not seem so easy to verify that the mapping $u(\cdot)$ is C^2. Of course, this would follow immediately from (2.4) and standard ODE theory if it were not for the 'bad' behaviour of the differential equation (1.1) at $x = 0$ (due to the fact that $\varphi_p(u')' = (p-1)|u'|^{p-2}u''$, and $u'(0) = 0$). Continuity of $u(\cdot)$ could be proved as in the proofs in [5, Section 3] (with a trivial adaptation to deal with the presence of the C^2 term $\lambda(\alpha)$ in (2.4)). However, we are not aware of a reference for the desired C^2 result so we state it, in Lemma 2.5 below, and outline a proof. For now, we complete the proof of Theorem 2.3.

(*a*) By construction, for any $\alpha \in \mathbb{R}_+$ we have $u(\alpha)|_0 = \alpha$, so $u_\alpha(\alpha)|_0 = 1$ and $(\lambda_\alpha(\alpha), u_\alpha(\alpha)) \neq (0,0)$, so that $(\lambda(\cdot), u(\cdot))$ is a parametrisation of $\mathcal{S}$.

(*b*) From the form of $T(\cdot)$ in [11, (2.8)] it is easy to verify that $\lim_{\alpha\to 0^+} \lambda(\alpha) = 0$, and then it follows from (2.4) and Lemma 2.1 (*b*) that $\lim_{\alpha\to 0^+} |u(\alpha)|_0 = 0$.

(*c*) This follows immediately from Lemma 2.2. □

Lemma 2.5. *The solution mapping* $u(\cdot) : \mathbb{R}_+ \to C^1_0[-1,1]$ *in Theorem 2.3 is* C^2.

Remark 2.6. The proof is an adaptation of the standard proof of the Picard-Lindelöf theorem on existence of solutions of initial value problems for ordinary differential equations, see for example [12, Proposition 1.8] or, more pertinently here, the proof of [5, Theorem 1.8] ([5] deals with the p-Laplacian). However, to deal with the 'bad' behaviour at $x = 0$ and obtain the desired C^2 result, some non-standard calculations are required, so we outline the proof here.

Proof. For any $\beta \in (0,1]$, let

$$P^{0,\beta} := \{h \in C^0[0,\beta] : h > 0 \text{ on } [0,\beta]\}$$

and, for any $h \in P^{0,\beta}$, consider the problem

$$\begin{aligned} -\varphi_p(w')' = h, &\quad \text{on } [0,\beta], \\ w(0) = \alpha, &\quad w'(0) = 0. \end{aligned} \tag{2.5}$$

It can be verified that (2.5) has a unique solution w given by $w = S^\beta_p(h)$, where

$$S^\beta_p(h)(x) := \alpha - \int_0^x \Big\{ \int_0^y h(t)\,dt \Big\}^{p^*} dy, \quad x \in [0,\beta], \tag{2.6}$$

and $p^* := 1/(p-1) > 0$. The operator $S^\beta_p : P^{0,\beta} \to C^1[0,\beta]$ is similar to the operator S_p considered in [6] (which solved (2.5) with the boundary value conditions $w'(0) = 0$, $w(1) = 0$) and, by following the proof of [6, Theorem 4.2], it can be shown that $S^\beta_p : P^{0,\beta} \to C^1[0,\beta]$ is C^2.

Next, we define $H^\beta : \mathbb{R}_+ \times C^0[0,\beta] \to C^0[0,\beta]$ by

$$H^\beta(\alpha, w) := w - S^\beta_p(\lambda(\alpha) f(w)), \quad (\alpha, w) \in \mathbb{R}_+ \times C^0[0,\beta].$$

In this definition $\lambda(\alpha) = T(\alpha)^p$, where $T(\cdot)$ is the C^2 function explicitly defined in [11, (2.8)] in terms of the function F, so $\lambda(\alpha)$ does not depend on $u(\alpha)$, and at this stage in the argument we only use the fact that $\lambda(\cdot)$ is C^2. Hence, by the preceding remarks, H^β is C^2, with derivative $D_w H^\beta$ given by

$$D_w H^\beta(\alpha, w)\bar{w} = \bar{w} - DS^\beta_p(\lambda(\alpha) f(w))(\lambda(\alpha) f'(w)\bar{w}), \quad (\alpha, w) \in \mathbb{R}_+ \times C^0[0,\beta],\ \bar{w} \in C^0[0,\beta].$$

We now consider the equation

$$H^\beta(\alpha, w) = 0, \quad (\alpha, w) \in \mathbb{R}_+ \times C^0[0,\beta]. \tag{2.7}$$

For arbitrary $\alpha_0 \in \mathbb{R}_+$, if $\beta > 0$ is sufficiently small then (2.7) has a unique solution $u^\beta(\alpha_0)$, and the derivative $D_u H^\beta(\alpha_0, u^\beta(\alpha_0))$ is nonsingular, so applying the implicit function theorem to (2.7), at $(\alpha, w) = (\alpha_0, u^\beta(\alpha_0))$, shows that $u^\beta(\cdot) : \mathbb{R}_+ \to C^0[0,\beta]$ is C^2 in a neighbourhood N_{α_0} of α_0. Then, by standard ODE theory (which holds away from the point $x = 0$), for each $\alpha \in N_{\alpha_0}$ the solution $u^\beta(\alpha)$ on $[0,\beta]$ extends, uniquely, to a solution $u^1(\alpha)$ on $[0,1]$, and $u^1(\cdot) : N_{\alpha_0} \to C^0[0,1]$ is C^2. Hence, since α_0 was arbitrary, the mapping $u^1(\cdot) : \mathbb{R}_+ \to C^0[0,1]$ is C^2.

Next, substituting $u^1(\cdot)$ into equation (2.7) (with $\beta = 1$), and using the properties of S_p^β, shows that $u^1(\cdot) : \mathbb{R}_+ \to C^1[0,1]$ is C^2. Finally, for each $\alpha \in \mathbb{R}_+$, let $u(\alpha)$ denote the extension, by symmetry, of $u^1(\alpha)$ from $[0,1]$ to $[-1,1]$. From these constructions it is clear that $u(\cdot) : \mathbb{R}_+ \to C^1[-1,1]$ is C^2 and, for each $\alpha \in \mathbb{R}_+$, $u(\alpha)$ satisfies the differential equation (1.1) on $(-1,1)$ and the initial conditions in (2.5). Given this, the calculations in the proof of [4, Lemma 2.4] now show that $u(\alpha)$ also satisfies the boundary conditions (1.2), and hence $u(\alpha)$ satisfies (2.4). This completes the proof of Lemma 2.5. □

As mentioned above, a C^2 parametrisation of $\mathcal{S}$ was also constructed in [6, Theorem 3.1], which we will here denote by $s \to (\tilde\lambda(s), \tilde u(s)) : \mathbb{R}_+ \to \mathbb{R}_+ \times C_0^1[-1,1]$, such that

$$\lim_{\alpha \to 0^+} (\tilde\lambda(s) + |\tilde u(s)|_0) = 0. \tag{2.8}$$

We note that various conditions were imposed on f in [6] to obtain the shape of the curve $\mathcal{S}$, but only the positivity condition (1.3) was used to construct the parametrisation of $\mathcal{S}$.

Of course, the construction of the parametrisation of $\mathcal{S}$ in [6] need not lead to the same parametrisation as in Theorem 2.3. However, the two parametrisations will be diffeomorphic, and will have the same orientation (it follows from Theorem 2.3 and (2.8) that $s \to 0^+$ corresponds to $\alpha \to 0^+$), so there exists a C^2 diffeomorphism $\tilde s : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$(\lambda(\alpha), u(\alpha)) = (\tilde\lambda(\tilde s(\alpha)), \tilde u(\tilde s(\alpha))), \quad \tilde s_\alpha(\alpha) > 0, \quad \alpha \in \mathbb{R}_+.$$

It follows from this that

$$\lambda_\alpha(\alpha) = \tilde s_\alpha(\alpha) \tilde\lambda_s(\tilde s(\alpha)), \quad \operatorname{sgn} \lambda_\alpha(\alpha) = \operatorname{sgn} \tilde\lambda_s(\tilde s(\alpha)), \quad \alpha \in \mathbb{R}_+, \tag{2.9}$$

which will be useful below when we use the results of [6] to discuss the stability of the solutions on $\mathcal{S}$.

3. Stability of equilibria of a related parabolic problem

We now consider the following time-dependent, parabolic, initial-boundary value problem

$$\frac{dv}{dt} = \Delta_p(v) + \lambda f(v), \quad v(0) = v_0 \in C_0^0[-1,1], \tag{3.1}$$

for fixed $\lambda > 0$. Clearly, $(\lambda, u) \in \mathcal{S}$ iff u is an equilibrium (constant in time) solution of (3.1); by a slight abuse of terminology we will say that (λ, u) is an equilibrium solution of (3.1). In this section we wish to discuss the local and global stability properties of these equilibria, regarded as solutions of the time-dependent problem (3.1).

3.1. **Existence and uniqueness of solutions of** (3.1)**.** We first define, briefly, what we mean by a solution of (3.1), and then state a standard result on the existence and uniqueness of such solutions. Further details are given in [7], but these results are based on many preceding publications, see [3] for an overview of these. In this context we need to extend the domain of the operator Δ_p to an $L^2(-1,1)$ setting by defining

$$D(\Delta_p) := \{u \in C_0^1[-1,1] : \varphi_p(u') \in W^{1,2}(-1,1)\},$$

so we now have $\Delta_p(u) = \varphi_p(u')' \in L^2(-1,1)$, for $u \in D(\Delta_p)$.

Definition 3.1. For $0 < T \leqslant \infty$, let

$$\Sigma_T := C([0,T), C_0^0[-1,1]) \cap C((0,T), W_0^{1,p}(-1,1)) \cap W_{\rm loc}^{1,2}((0,T), L^2(-1,1)).$$

A *solution* of (3.1) is a function $v \in \Sigma_T$, for some $T > 0$, such that $v(0) = v_0$ and for a.e. $t \in [0,T)$:

(*a*) $v(t) \in D(\Delta_p)$;

(*b*) the function $v : [0,T) \to L^2(-1,1)$ is differentiable at t;

(*c*) $\dfrac{dv}{dt}(t) = \Delta_p(v(t)) + \lambda f(v(t))$ (this equation holds in the $L^2(-1,1)$ sense).

Theorem 3.2. (*a*) *Equation* (3.1) *has a unique solution* $v_{\lambda,v_0} \in \Sigma_{T_{\lambda,v_0}}$*, defined on a maximal time* $T_{\lambda,v_0} > 0$*; the time* T_{λ,v_0} *is maximal in the sense that*

$$T_{\lambda,v_0} < \infty \implies \lim_{t \nearrow T_{\lambda,v_0}} |v_{\lambda,v_0}(t)|_0 = \infty. \tag{3.2}$$

(*b*) *Suppose that* $T_{\lambda,v_0} = \infty$*, and there exists a sequence* (t_n) *in* $\mathbb{R}_+$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and the set* $\{|v_{\lambda,v_0}(t_n)|_0 : n = 1,2,\dots\}$ *is bounded. Then, after choosing a subsequence if necessary, the limit* $v_\infty := \lim_{n\to\infty} v_{\lambda,v_0}(t_n)$ *exists in* $W_0^{1,p}(-1,1)$*, and* v_∞ *is an equilibrium of* (3.1).

Part (*a*) of Theorem 3.2 is proved in [7, Theorem 3.1] (it is assumed in [7] that f has the form $f(u) = g(u)\varphi_p(u)$, $u \in \mathbb{R}$, but this assumption is not required for the basic existence theorem). Part (*b*) is proved in the proof of [7, Theorem 4.1] (based on an argument in the proof of [3, Lemma 3.1]).

In view of (3.2), and to facilitate our discussion of stability below, a criterion for the boundedness of solutions of (3.1) will be useful. We consider the hypothesis:

$(H1)$ the limit $f_\infty := \lim_{\xi\to\infty} f(\xi)/\xi^{p-1}$ exists and is finite.

Let

$$\lambda_{0,p} := \Big(\frac{\pi/p}{\sin \pi/p}\Big)^p \quad \text{and} \quad \lambda_\infty := \begin{cases} \lambda_{0,p}/f_\infty, & \text{if } f_\infty > 0, \\ \infty, & \text{if } f_\infty = 0. \end{cases}$$

The number $\lambda_{0,p}$ is the principal eigenvalue of the problem $-\Delta_p(u) = \lambda\varphi_p(u)$ (see, for example, [2, p. 377]).

Lemma 3.3. *Suppose that the hypotheses of Theorem 3.2 hold and* f *satisfies* $(H1)$*. If* $0 < \lambda < \lambda_\infty$ *then there exists a constant* M_{λ,v_0} *such that* $|v_{\lambda,v_0}(t)|_0 \leqslant M_{\lambda,v_0}$*, for* $t \geqslant 0$*, and hence, by* (3.2)*,* $T_{\lambda,v_0} = \infty$.

Proof. We first note that, by the comparison theorem [8, Theorem 4.4],

$$v_{\lambda,v_0}(t) \geqslant \min\{v_0(x) : x \in [-1,1]\}, \quad t \geqslant 0. \tag{3.3}$$

The result now follows from a slight adaptation of the proof of [7, Lemma 2.2] together with [7, Theorem 3.1 (*c*)]. A 'p-concavity' condition on f was assumed in [7], but this is not needed for the proof of either of these results. □

Remark 3.4. Positive solutions of (3.1) are of particular significance. The inequality (3.3) shows that if the initial condition v_0 is positive ($v_0 \geqslant 0$) then v_{λ,v_0} is positive ($v_{\lambda,v_0}(t) \geqslant 0$ for all $t \geqslant 0$).

3.2. **Local stability of equilibria of** (3.1)**.** We now consider the local exponential stability, or instability, of the equilibria of (3.1). The following definition describes what we mean by this. It encapsulates the results obtained in [8, Theorem 5.1].

Definition 3.5. Suppose that (λ, u_0) is an equilibrium of (3.1).

- u_0 is *locally exponentially stable, with exponent* $\kappa > 0$, if there exists δ, $C > 0$ such that
$$|v_0 - u_0|_0 < \delta \implies |v_{\lambda,v_0}(t) - u_0|_0 < Ce^{-\kappa t}, \quad t \geqslant 0.$$
- u_0 is *locally exponentially unstable, with exponent* $\kappa > 0$, if there exists δ_0, $C > 0$ such that, for any $\delta \in (0, C\delta_0)$, there exists $v_{0,\delta} \in C_0^1[-1,1]$ such that
$$|v_{0,\delta} - u_0|_1 < \delta \quad \text{and} \quad |v_{v_{0,\delta}}(t_\delta) - u_0|_0 \geqslant \delta_0, \quad \text{where} \quad e^{\kappa t_\delta} = \delta_0/\delta.$$

The following result now relates the local stability of the equilibria on $\mathcal{S}$ to the shape of $\mathcal{S}$.

Theorem 3.6 (Local exponential stability)**.** *For any* $\alpha \in \mathbb{R}_+$*, the equilibrium* $(\lambda(\alpha), u(\alpha))$ *of* (3.1) *is locally exponentially stable if* $\lambda_\alpha(\alpha) > 0$*, and is locally exponentially unstable if* $\lambda_\alpha(\alpha) < 0$*.*

Proof. The proof is based on the 'linearised stability' results in [8], so we first describe these (briefly). A more general p-Laplacian problem is considered in [8] and, given an equilibrium solution u_0 of this general problem, a linearisation of the problem is constructed at u_0. Then, denoting the principal eigenvalue of this linearisation by $\sigma(u_0)$, it is shown in [8, Theorem 5.1] that if $\sigma(u_0) < 0$ then, for any $\kappa \in (0, |\sigma(u_0)|)$, u_0 is locally exponentially stable with exponent κ, while if $\sigma(u_0) > 0$ then, for any $\kappa \in (0, \sigma(u_0))$, u_0 is locally exponentially unstable with exponent κ.

Next, [6, (5.7)] shows that for any $s \in \mathbb{R}_+$, and corresponding solution $(\tilde{\lambda}(s), \tilde{u}(s)) \in \mathcal{S}$, we have $\operatorname{sgn} \sigma(\tilde{u}(s)) = -\operatorname{sgn} \tilde{\lambda}_s(s)$. Combining this with (2.9) and the linearised stability results of [8] described above then yields the result. □

Remark 3.7. By Theorem 3.6, for any $\alpha \in \mathbb{R}_+$ the local stability, or instability, of the equilibrium $(\lambda(\alpha), u(\alpha))$ is determined by the sign of $\lambda_\alpha(\alpha)$, that is, by the shape of $\mathcal{S}$. This will be illustrated in the next section, where we consider an S-shaped curve $\mathcal{S}$. However, the exponent κ is determined by the magnitude of $\sigma(u(\alpha))$, and this is not determined by the value of $\lambda_\alpha(\alpha)$, so we cannot determine the value of κ from the shape of $\mathcal{S}$.

4. S-shaped solution curves

The above results on the structure of $\mathcal{S}$, and on the stability of solutions $(\lambda, u) \in \mathcal{S}$, were based on our standing assumption that f is C^2 and satisfies the positivity condition (1.3). However, under additional conditions on f the shape of $\mathcal{S}$ can be described. This was done in [6], using a convexity condition on f which ensured that the curve $\mathcal{S}$ had a single bend. Here, we will use certain conditions, taken from [11], which ensure that $\mathcal{S}$ has two bends, that is, it is 'S-shaped' (see Theorem 4.1, and the remarks following this theorem, for a precise statement, and Fig. 1 for some illustrations).

Let
$$\theta(\xi) := pF(\xi) - \xi f(\xi), \quad \xi \geqslant 0 \tag{4.1}$$
(recall that F was defined in (2.3)), and suppose that the following hypotheses hold (see [11]).

$(H2)$ There exists numbers $0 < C_1 < D_1 < \gamma < C_2 < D_2$ such that

$$\theta(D_1) = \theta(D_2) = \theta'(C_1) = \theta'(C_1) = 0,$$

and either of the following conditions hold:

(*i*)

$$\theta''(\xi) = (p-2)f'(\xi) - \xi f''(\xi) \begin{cases} < 0 & \xi \in (0,\gamma), \\ = 0 & \xi = \gamma, \\ > 0 & \xi \in (\gamma,\infty); \end{cases}$$

(*ii*) there exists $\tilde{\gamma} \in (0, C_1)$ such that

$$\theta''(\xi) = (p-2)f'(\xi) - \xi f''(\xi) \begin{cases} > 0 & \xi \in (0,\tilde{\gamma}), \\ = 0 & \xi = \tilde{\gamma}, \\ < 0 & \xi \in (\tilde{\gamma},\gamma), \\ = 0 & \xi = \gamma, \\ > 0 & \xi \in (\gamma,\infty), \end{cases}$$

and $\theta(\tilde{\gamma}) - \tilde{\gamma}\theta'(\tilde{\gamma}) - \theta(C_2) \geqslant 0$.

$(H3)$ $\xi f'(\xi)/f(\xi) \geqslant -1/(p+1)$ on $(0, C_1)$ and $\xi f'(\xi)/f(\xi)$ is increasing on (C_1, D_1).

The following result shows that under these conditions, the curve $\mathcal{S}$ is 'S-shaped'. It was proved in [11, Lemma 2.2], in the case $f_\infty = 0$.

Theorem 4.1 ([11, Lemma 2.2])**.** *Suppose that* $(H1)$-$(H3)$ *hold. Then*:

(*a*) $\lim_{\alpha\to\infty} \lambda(\alpha) = \lambda_\infty$;

(*b*) *the function* $\lambda(\cdot)$ *has exactly two critical points* $\alpha^* < \alpha_*$ *on* $\mathbb{R}_+$*, and,*

- $\lambda^* := \lambda(\alpha^*)$ *is a strict local maximum on* $\mathbb{R}_+$*;*
- $\lambda_* := \lambda(\alpha_*)$ *is a strict local minimum on* $\mathbb{R}_+$*.*

Proof. Part (*a*). Only the case $\lambda_\infty = \infty$ is mentioned in [11], and the proof is not described. Similar result have been proved before, but for completeness we briefly sketch a proof here – the proof explains the relationship between f_∞, $\lambda_{0,p}$ and λ_∞. Let (α_n) be a sequence in $\mathbb{R}_+$ such that $\alpha_n \to \infty$ and $\lambda(\alpha_n) \to L < \infty$. Then it can be shown that $\alpha_n^{-1}u(\alpha_n) \to w_\infty \in D(\Delta_p)$, with $w_\infty \geqslant 0$, $|w_\infty|_0 = 1$ and $-\Delta_p(w_\infty) = Lf_\infty\varphi_p(w_\infty)$ (details, in a more general, PDE setting, are given in the proof of part (*c*) of [7, Theorem 2.3]). Hence, if $f_\infty > 0$ then $Lf_\infty = \lambda_{0,p}$. On the other hand, if $f_\infty = 0$ then the sequence $(\lambda(\alpha_n))$ cannot converge to a finite limit. These results prove part (*a*).

Part (*b*). This is proved in [11, Lemma 2.2] in the case that $f_\infty = 0$, but the proof is the same for $f_\infty > 0$. □

The curve $\mathcal{S}$ is sketched in Fig. 1, for various values of λ_∞. In each case we see that $\mathcal{S}$ has two bends, and three 'branches' over the λ-axis. We will denote these branches as the 'lower', 'middle' and 'upper' branches, in the obvious manner. These branches have parametrisations of the form $\lambda \to (\lambda, e(\lambda))$, with parameter λ, which we will denote by:

$$e^l(\cdot) : (0, \lambda^*) \to C_0^1[-1,1], \quad e^m(\cdot) : (\lambda_*, \lambda^*) \to C_0^1[-1,1], \quad e^u(\cdot) : (\lambda_*, \lambda_\infty) \to C_0^1[-1,1].$$

It follows from Theorem 2.3 (c) that

$$\begin{aligned} \lambda \in (\lambda_*, \lambda^*) &\implies e^l(\lambda) \ll e^m(\lambda), \\ \lambda \in (\lambda_*, \lambda_r) &\implies e^m(\lambda) \ll e^u(\lambda), \end{aligned} \tag{4.2}$$

where $\lambda_r := \min\{\lambda^*, \lambda_\infty\}$. Theorem 3.6 now shows that the equilibria on the upper and lower branches of $\mathcal{S}$ are locally exponentially stable, while those on the middle branch are locally exponentially unstable. We state this in the following theorem.

Theorem 4.2 (Local exponential stability)**.** *Suppose that* $(H1)$-$(H3)$ *hold.*

(a) *If* $\lambda \in (0, \lambda_*)$ *then* $e^l(\lambda)$ *is locally exponentially stable.*

(b) *If* $\lambda \in (\lambda_*, \lambda^*)$, *then* $e^m(\lambda)$ *is locally exponentially unstable.*

(c) *If* $\lambda \in (\lambda_*, \lambda_\infty)$, *then* $e^u(\lambda)$ *is locally exponentially stable.*

Remark 4.3. As noted above, the paper [6] gives more details of this argument. In [6] the curve $\mathcal{S}$ has only two branches rather than three, but the local stability arguments for the solutions on these branches are similar in both cases.

Next, we will show that the local stability properties of the branches of equilibria described in Theorem 4.2 extend to more global stability properties. We recall the ordering property of the solutions $e^l(\lambda)$, $e^m(\lambda)$, $e^u(\lambda)$, described in (4.2).

4.1. **The case** $\lambda_\infty = \infty$**.** For notational simplicity we first consider the case $\lambda_\infty = \infty$ (see Fig. 1 (a)).

Theorem 4.4 (Global stability)**.** *Suppose that* $(H1)$-$(H3)$ *hold, and* $\lambda_\infty = \infty$.

(a) *If* $\lambda \in (0, \lambda_*)$, *then, for any* $v_0 \in C^0_0[-1, 1]$,

$$\lim_{t\to\infty} \|v_{\lambda,v_0}(t) - e^l(\lambda)\|_{1,p} = 0.$$

(b) *If* $\lambda \in (\lambda_*, \lambda^*)$, *then*:

(i) *if* $v_0 \ll e^m(\lambda)$ *then* $v_{\lambda,v_0}(t) \ll e^m(\lambda)$ *for all* $t \geqslant 0$, *and*

$$\lim_{t\to\infty} \|v_{\lambda,v_0}(t) - e^l(\lambda)\|_{1,p} = 0;$$

(ii) *if* $e^m(\lambda) \ll v_0$ *then* $e^m(\lambda) \ll v_{\lambda,v_0}(t)$ *for all* $t \geqslant 0$, *and*

$$\lim_{t\to\infty} \|v_{\lambda,v_0}(t) - e^u(\lambda)\|_{1,p} = 0.$$

(c) *If* $\lambda \in (\lambda^*, \infty)$, *then, for any* $v_0 \in C^0_0[-1, 1]$,

$$\lim_{t\to\infty} \|v_{\lambda,v_0}(t) - e^u(\lambda)\|_{1,p} = 0.$$

Proof. We first note that, by Lemma 3.3, $|v_{\lambda,v_0}(\cdot)|_0$ is bounded and $T_{\lambda,v_0} = \infty$, in each of the cases (a)-(c).

(a) Suppose that the result is false, so that there exists a sequence (t_n) in $(0, \infty)$ and $\epsilon > 0$ such that $t_n \to \infty$ and $\|v_{\lambda,v_0}(t_n) - e^l(\lambda)\|_{1,p} > \epsilon$, for $n = 1, 2, \ldots$. Then, by Theorem 3.2 (b), we may suppose that $v_{\lambda,v_0}(t_n) \to v_\infty$ in $W_0^{1,p}(-1,1)$, where v_∞ is an equilibrium of (3.1). However, the only equilibrium of (3.1) is $e^l(\lambda)$, but the construction of v_∞ precludes $v_\infty = e^l(\lambda)$. This contradiction proves part (*a*).

(*b*) Suppose that $\lambda \in (\lambda_*, \lambda^*)$ and $\delta > 0$. By combining the exponential instability of $e^m(\lambda)$ (as obtained in Theorem 4.2) with the proof of [8, Theorem 5.1], we can construct functions $\underline{S}_{\lambda,\delta}$, $\overline{S}_{\lambda,\delta} \in C_0^1[-1,1]$ having the following properties:

- $\underline{S}_{\lambda,\delta} < e^m(\lambda) < \overline{S}_{\lambda,\delta}$ on $(-1,1)$, and $|\overline{S}_{\lambda,\delta} - \underline{S}_{\lambda,\delta}|_1 \leqslant \delta$;
- $v_0 \leqslant \underline{S}_{\lambda,\delta} \implies v_{\lambda,v_0}(t) \leqslant \underline{S}_{\lambda,\delta}$ for all $t \geqslant 0$;
- $v_0 \geqslant \overline{S}_{\lambda,\delta} \implies v_{\lambda,v_0}(t) \geqslant \overline{S}_{\lambda,\delta}$ for all $t \geqslant 0$

(the notation $S_\delta^\pm$ is used in [8]). Specifically, to construct $\overline{S}_{\lambda,\delta}$ we follow the construction of S_δ^+ in the proof of [8, Theorem 5.1 (*b*)], setting the parameter κ there to be zero; $\underline{S}_{\lambda,\delta}$ can be constructed in the same manner.

We now consider case (*i*), and suppose that $v_0 \ll e^m(\lambda)$. Then, by the above properties of $\underline{S}_{\lambda,\delta}$, we can choose $\delta > 0$ sufficiently small that $v_0 \leqslant \underline{S}_{\lambda,\delta}$ and, by (3.3),

$$-M_{\lambda,v_0} \leqslant v_{\lambda,v_0}(t) \leqslant \underline{S}_{\lambda,\delta} < e^m(\lambda), \quad t \geqslant 0, \tag{4.3}$$

where M_{λ,v_0} is as in Lemma 3.3. If the result is false then there exists a sequence (t_n) in $(0,\infty)$ and $\epsilon > 0$ such that $t_n \to \infty$ and $\|v_{\lambda,v_0}(t_n) - e^l(\lambda)\|_{1,p} > \epsilon$, $n = 1, 2, \ldots$. Then, as in the proof of part (*a*), we may suppose that $v_{\lambda,v_0}(t_n) \to v_\infty$ in $W_0^{1,p}(-1,1)$, where v_∞ is an equilibrium of (3.1). In this case the only equilibria of (3.1) are $e^l(\lambda)$, $e^m(\lambda)$, $e^u(\lambda)$, but the construction of v_∞ precludes $v_\infty = e^l(\lambda)$, while (4.3) precludes $v_\infty = e^m(\lambda)$ or $v_\infty = e^u(\lambda)$. This contradiction proves part (*i*). The proof of part (*ii*) is similar, using the following analogue of (4.3)

$$\overline{S}_{\lambda,\delta} \leqslant v_{\lambda,v_0}(t) \leqslant M_{\lambda,v_0}, \quad t \geqslant 0.$$

(*c*) The proof is, essentially, identical to that of part (*a*). □

The results of Theorem 4.4 on the behaviour of the solutions v_{λ,v_0} are illustrated in Fig. 1 (a), and can be summarised as follows.

- If $\lambda \in (0, \lambda_*) \cup (\lambda^*, \infty)$ then the single equilibrium, say $e(\lambda)$, is globally stable, that is, for any $v_0 \in C_0^0[-1,1]$ the solution $v_{\lambda,v_0}(\cdot)$ converges to $e(\lambda)$.
- If $\lambda \in (\lambda_*, \lambda^*)$ then the equilibria $e^l(\lambda)$ and $e^u(\lambda)$ are stable, and $e^m(\lambda)$ is unstable. Heuristically, if the initial condition v_0 is below the middle equilibrium $e^m(\lambda)$ then $v_{\lambda,v_0}(\cdot)$ converges to the lower equilibrium $e^l(\lambda)$, whereas if v_0 is above $e^m(\lambda)$ then $v_{\lambda,v_0}(\cdot)$ converges to the upper equilibrium $e^u(\lambda)$.

That is, in the λ intervals where there is only a single equilibrium, this equilibrium is globally stable, but when three equilibria exist the middle equilibrium is unstable, while the upper and lower ones are stable. Strictly speaking, the result in part (*b*) of Theorem 4.4, while not a 'local' result is not 'global', since there exist initial values v_0 which 'cross' the unstable, middle equilibrium $e^m(\lambda)$, and so are not considered in this result.

4.2. **The case** $0 < \lambda_\infty < \infty$**.** Theorem 4.4 assumed that $f_\infty = 0$ (so that $\lambda_\infty = \infty$). We now consider the case where $0 < f_\infty < \infty$. In this case the bifurcation diagram has one of the forms shown in Fig. 1 (b) or (c), depending on whether $\lambda^* < \lambda_\infty$ or $\lambda_\infty < \lambda^*$. The stability properties of the solutions in these two cases are also illustrated in these figures, and are similar to those described in Theorem 4.4 and shown in Fig. 1 (a). These figures clearly and concisely display the solution behaviour, whereas a formal statement of this behaviour, detailing all the alternatives, would be quite long-winded (see the statement of Theorem 4.4, in the simpler case $\lambda_\infty = \infty$), so we will omit such a formal statement of the results for this case. However, there is one type of solution behaviour shown in Fig. 1 that was not discussed in the proof of Theorem 4.4. For completeness, this is described in the following theorem.

Theorem 4.5. *Suppose that* $(H1)$-$(H3)$ *hold, and one of the following conditions holds*:

(*i*) $\lambda > \max\{\lambda^*, \lambda_\infty\}$, *and* $v_0 \in C_0^0[-1,1]$;

(*ii*) $\lambda_\infty < \lambda < \lambda^*$, *and* $e^m(\lambda) \ll v_0$.

Then either $T_{\lambda,v_0} < \infty$ *or* $T_{\lambda,v_0} = \infty$ *and* $\lim_{t\to\infty} \|v_{\lambda,v_0}(t)\|_{1,p} = \infty$.

Proof. Suppose that one of the conditions (*i*) or (*ii*) holds, but the conclusion does not, that is, there exists a sequence of times (t_n) such that $t_n \to \infty$ and $\|v_{\lambda,v_0}(t_n)\|_{1,p} \leqslant M$, for some constant $M > 0$. Then, by Theorem 3.2 (*b*), we may suppose that $v_{\lambda,v_0}(t_n)$ converges to an equilibrium. However, in case (*i*) there is no equilibrium, so this is clearly contradictory. In case (*ii*) there are two equilibria $e^l(\lambda) \ll e^m(\lambda) \ll v_0$, but the argument in the proof of Theorem 4.4 (*b*) (*ii*) shows that $v_{\lambda,v_0}(t_n)$ cannot converge to either of these equilibria, so again we have a contradiction. This completes the proof. □

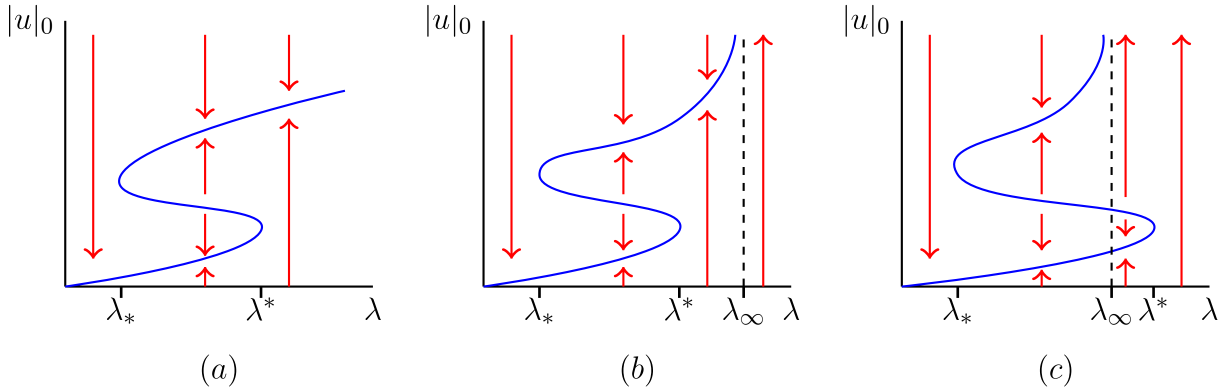

Figure 1: The solution curve $\mathcal{S}$, and its stability properties, for the cases: (a) $\lambda_\infty = \infty$; (b) $\lambda^* < \lambda_\infty < \infty$; (c) $\lambda_\infty < \lambda^*$.

5. Generic stability properties

It is clear from the proof of [11, Lemma 2.2] that obtaining the exact shape of $\mathcal{S}$ is non-trivial. However, even when the exact shape of $\mathcal{S}$ cannot be determined, the pattern of alternating stable and unstable equilibria seen in Fig. 1 holds for 'generic' values of λ. To make this statement more precise, let $\mathcal{R}$ denote the complement, in $\mathbb{R}_+$, of the set of critical values of the function $\lambda(\cdot)$. By Sard's theorem ([12, Proposition 4.55]) the set $\mathcal{R}$ has full measure in $\mathbb{R}_+$. The preceding arguments can be combined to yield the following theorem.

Theorem 5.1. *Suppose that* $(H1)$ *holds, and* $\lambda \in \mathcal{R} \setminus \{\lambda_\infty\}$. *Then one of the following alternatives holds.*

(*i*) *Equation* (3.1) *has no equilibria. In this case, for any* $v_0 \in C_0^0[-1,1]$ *the conclusion of Theorem 4.5 holds.*

(ii) Equation (3.1) *has finitely many equilibria* $e_1(\lambda) \ll e_2(\lambda) \ll \cdots \ll e_k(\lambda)$*, with the following stability properties.*

- *For* $i = 1, \dots, k$*, the equilibrium* $e_i(\lambda)$ *is locally exponentially stable if* i *is odd, and locally exponentially unstable if* i *is even.*
- *If* $e_{2i}(\lambda) \ll v_0 \ll e_{2i+2}(\lambda)$*, for some integer* i *with* $0 \leqslant i \leqslant (k-1)/2$*, then*

$$\lim_{t\to\infty} \|v_{\lambda,v_0}(t) - e_{2i+1}(\lambda)\|_{1,p} = 0$$

  (*here we write, formally,* $e_0(\lambda) = -\infty$*,* $e_{k+1}(\lambda) = \infty$).
- *If* k *is even and* $e_k(\lambda) \ll v_0$ *then the conclusion of Theorem 4.5 holds.*

The results of Theorem 5.1 are illustrated in Fig. 1, in the cases of no equilibria and $k = 1, 2, 3$, at suitable values of λ.

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Department of Mathematics and the Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Edinburgh EH14 4AS, Scotland

Email address: B.P.Rynne@hw.ac.uk