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A time-dependent FEM-BEM coupling method for fluid–structure interaction in $3d$

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Abstract

We consider the well-posedness and a priori error estimates of a $3d$ FEM-BEM coupling method for fluid-structure interaction in the time domain. For an elastic body immersed in a fluid, the exterior linear wave equation for the fluid is reduced to an integral equation on the boundary involving the Poincaré-Steklov operator. The resulting problem is solved using a Galerkin boundary element method in the time domain, coupled to a finite element method for the Lamé equation inside the elastic body. Based on ideas from the time-independent coupling formulation, we obtain an a priori error estimate and discuss the implementation of the proposed method. Numerical experiments illustrate the performance of our scheme for model problems.

Keywords: Fluid-structure interaction; FEM-BEM coupling; space-time methods; a priori error estimate; wave equation.

1. Introduction

Coupled finite and boundary element procedures provide an efficient and extensively investigated tool for the numerical solution of elliptic interface and contact problems, particularly in unbounded domains [20, 31]. On the other hand, much of the current interest in boundary element procedures focuses on hyperbolic problems in the time domain, both on Galerkin methods and convo-

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lution quadrature [2, 9, 14, 26, 30].

To compute the scattering of time-dependent waves by a bounded, penetrable obstacle, the coupling of time domain finite elements (FEM) and boundary elements (BEM) becomes relevant. The recent mathematical analysis of FEM-BEM coupling in the time domain was initiated in [1], coupling discontinuous finite elements to Galerkin boundary elements for the 3d wave equation. A general analysis of the coupling between different discretizations for acoustic wave equations was provided in [6], with a focus on convolution quadrature. Since then, FEM-BEM coupling for convolution quadrature methods has been applied in a variety of applications in 2 dimensions, such as fluid-structure and fluid-thermoelastic problems, as well as nonlinear elastic problems involving piezoelectric scatterers [23, 24, 25, 29]. For time-dependent Galerkin methods and their application to 3d problems, on the other hand, much less is known. In addition to [1], previous related work includes the energy-based formulations investigated by Aimi and collaborators for wave-wave coupling in 3d multidomains and layered media [3, 4].

In this article we study a simple space-time Galerkin FEM-BEM coupling method for fluid-structure interaction, describing the transient scattering of waves in an inviscid homogeneous fluid by an elastic obstacle. Based on ideas from time-independent coupling formulations [8, 10] and the analysis in the frequency domain [23], we present a basic a priori error estimate for a space-time Galerkin approximation in anisotropic Sobolev spaces [5]. We discuss in detail the numerical implementation of our proposed coupling method in 3d. Numerical experiments for model problems illustrate the performance of the scheme.

To describe the results of this article in more detail, recall the equations for an elastic body submersed in a fluid. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and $\Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}$. The elastic deformation \mathbf{u} in Ω is described by the Lamé operator $\Delta^* \mathbf{u} = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla(\operatorname{div} \mathbf{u})$, with Lamé constants $\mu \geq 0$ and λ , such that $3\lambda + 2\mu \geq 0$. The deformation is coupled to the wave equation in Ω^c , leading to the coupled interface problem

$$\ddot{v} - c^2 \Delta v = 0, \quad (x, t) \in \Omega^c \times (0, \infty), \quad (1a)$$

$$\rho_1 \ddot{\mathbf{u}} - \Delta^* \mathbf{u} = 0, \quad (x, t) \in \Omega \times (0, \infty), \quad (1b)$$

$$\rho_2 \tilde{\sigma}(\mathbf{u}) \cdot \mathbf{n} + \dot{v} \cdot \mathbf{n} = -\dot{v}^{inc} \cdot \mathbf{n} \quad \text{on } \Gamma \times (0, \infty), \quad (1c)$$

$$\dot{\mathbf{u}} \cdot \mathbf{n} + \partial_n^+ v = -\partial_n^+ v^{inc} \quad \text{on } \Gamma \times (0, \infty), \quad (1d)$$

$$v(x, 0) = \dot{v}(x, 0) = 0 \quad \text{in } \Omega^c, \quad (1e)$$

$$\mathbf{u}(x, 0) = \dot{\mathbf{u}}(x, 0) = 0 \quad \text{in } \Omega, \quad (1f)$$

for a given incident wave v^{inc} in Ω^c . Here, the stress is given in terms of the deformation as $\tilde{\sigma}(\mathbf{u}) = (\lambda \operatorname{div} \mathbf{u})I + 2\mu \varepsilon(\mathbf{u})$, $\varepsilon(\mathbf{u}) = \frac{1}{2}((\nabla \mathbf{u}) + (\nabla \mathbf{u})^T)$, with I the

identity matrix. Time derivatives are denoted by a dot, and n is the outward-pointing unit normal vector to $\partial\Omega$. The Neumann trace on Γ from the exterior domain Ω^c is denoted by ∂_n^+ , while ∂_n^- is the corresponding Neumann trace from the interior domain Ω . We choose units in which $c = \rho_1 = \rho_2 = 1$.

To solve this interface problem numerically, we use the Poincaré-Steklov operator for the exterior wave equation to reformulate it as a coupled domain / boundary integral equation in Ω and Γ . The Poincaré-Steklov operator is expressed in terms of layer potentials for the wave equation, as known for time-independent symmetric FEM-BEM coupling methods. The resulting space-time weak formulation is approximated using finite elements in Ω and Galerkin boundary elements on Γ , based on tensor products of piecewise polynomial functions on a quasi-uniform mesh in space and a uniform mesh in time. Our a priori estimates assure convergence. We discuss a numerical implementation in detail and study the numerical performance of the method.

The article is organized as follows: Section 2 reformulates the coupled problem (1) as a domain / boundary integral equation in Ω and Γ and discusses its discretization and well-posedness. It provides the basis for the derivation of an a priori error estimate in Section 3. Numerical examples and related algorithmic considerations are the content of Section 4. Two appendices discuss boundary integral operators for the wave equation and the appropriate space-time anisotropic Sobolev spaces, as well as detailed algorithmic aspects of the proposed scheme.

Notation: To simplify notation, we will write $f \lesssim g$, if there exists a constant $C > 0$ independent of the arguments of the functions f and g such that $f \leq Cg$. We will write $f \lesssim_\sigma g$, if C may depend on a parameter σ . For a function v on $\mathbf{R}^n \setminus \Gamma$, we denote by $v^\pm|_\Gamma$ the trace of v on Γ from the exterior domain Ω^c , resp. from the interior domain Ω .

2. Weak formulation and FEM-BEM coupling

Recall that the fluid-structure interaction problem (1) is well-posed [11]:

Theorem 1. *Let $\sigma > 0$, $s \geq 0$ and assume that $v^{inc}|_\Gamma \in H_\sigma^{3+s}(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma))$, $\partial_n^+ v^{inc} \in H_\sigma^{3+s}(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$. Then the system (1) admits a unique solution $(\mathbf{u}, v) \in H_\sigma^{1+s}(\mathbb{R}^+, H^1(\Omega)) \times H_\sigma^s(\mathbb{R}^+, H^1(\Omega^c))$, which depends continuously on the data.*

See Appendix A for the definitions of the relevant space-time anisotropic Sobolev spaces, depending on a weight parameter $\sigma > 0$.

In this section we reformulate problem (1) as a coupled domain / boundary integral equation and propose a finite element / boundary element coupling method for its numerical solution.

The key ingredient to reduce equation (1a) in Ω^c to Γ is the retarded Poincaré-Steklov operator \mathcal{S} on Γ , defined as $\mathcal{S}v^+|_\Gamma = \partial_n^+ v$ for a solution v of (1a).

Using \mathcal{S} , equation (1d) becomes:

$$-\dot{\mathbf{u}} \cdot \mathbf{n} - \mathcal{S}v^+|_\Gamma = \partial_n^+ v^{inc} \quad \text{on } \Gamma. \quad (2)$$

To compute the Poincaré-Steklov operator \mathcal{S} , we use the following formulation in terms of boundary integral operators [20], see Appendix A:

$$\mathcal{S}v^+|_\Gamma = Wv^+|_\Gamma - (K^T - \frac{1}{2})V^{-1}(K - \frac{1}{2})v^+|_\Gamma.$$

In terms of $\phi = v^+|_\Gamma$ and an auxiliary variable $\lambda = V^{-1}(K - \frac{1}{2})\phi$, (2) becomes

$$-\dot{\mathbf{u}} \cdot \mathbf{n} - W\phi + (K^T - \frac{1}{2})\lambda = \partial_n^+ v^{inc}.$$

Problem (1) is therefore equivalent to the following system on Ω and Γ :

$$\ddot{\mathbf{u}} - \Delta^* \mathbf{u} = 0, \quad (x, t) \in \Omega \times (0, \infty), \quad (3a)$$

$$\tilde{\sigma}(\mathbf{u}) \cdot \mathbf{n} + \dot{\phi} \cdot \mathbf{n} = -\dot{v}^{inc} \cdot \mathbf{n} \quad \text{on } \Gamma \times (0, \infty), \quad (3b)$$

$$-\dot{\mathbf{u}} \cdot \mathbf{n} - W\phi + (K^T - \frac{1}{2})\lambda = \partial_n^+ v^{inc} \quad \text{on } \Gamma \times (0, \infty), \quad (3c)$$

$$(\frac{1}{2} - K)\phi + V\lambda = 0 \quad \text{on } \Gamma \times (0, \infty), \quad (3d)$$

$$\phi(x, 0) = \dot{\phi}(x, 0) = \lambda(x, 0) = 0 \quad \text{on } \Gamma, \quad (3e)$$

$$\mathbf{u}(x, 0) = \dot{\mathbf{u}}(x, 0) = 0 \quad \text{in } \Omega. \quad (3f)$$

The solution v in Ω^c is recovered from the representation formula $v = D\phi - S\lambda$, using the single and double layer potentials from Appendix A.

We derive a weak formulation of (3) in the weighted L^2 -Sobolev spaces from Theorem 1. Recall the weighted L^2 inner products for given $\sigma > 0$:

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\Omega \times \mathbb{R}^+, \sigma} := \int_0^\infty e^{-2\sigma t} \int_\Omega \mathbf{u} \cdot \mathbf{v} dx dt \quad \text{and} \quad \langle \mathbf{u}, \mathbf{v} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} := \int_0^\infty e^{-2\sigma t} \int_\Gamma \mathbf{u} \cdot \mathbf{v} ds_x dt.$$

Given a smooth solution $(\mathbf{u}, \phi, \lambda)$ of (3), equations (3a) and (3b) combined with Betti's formula,

$$\langle \tilde{\sigma}(\mathbf{u}) \cdot \mathbf{n}, \mathbf{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = (\tilde{\sigma}(\mathbf{u}), \varepsilon(\mathbf{w}))_{\Omega \times \mathbb{R}^+, \sigma} + (\Delta^* \mathbf{u}, \mathbf{w})_{\Omega \times \mathbb{R}^+, \sigma},$$

lead to

$$(\tilde{\sigma}(\mathbf{u}), \varepsilon(\dot{\mathbf{w}}))_{\Omega \times \mathbb{R}^+, \sigma} + (\ddot{\mathbf{u}}, \dot{\mathbf{w}})_{\Omega \times \mathbb{R}^+, \sigma} + \langle \dot{\phi}, \dot{\mathbf{w}}|_\Gamma \cdot \mathbf{n} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = -\langle \dot{v}^{inc}, \dot{\mathbf{w}}|_\Gamma \cdot \mathbf{n} \rangle_{\Gamma \times \mathbb{R}^+, \sigma}. \quad (4)$$

The first two terms on the left hand side define a bilinear form

$$a(\mathbf{u}, \dot{\mathbf{w}}) := (\tilde{\sigma}(\mathbf{u}), \varepsilon(\dot{\mathbf{w}}))_{\Omega \times \mathbb{R}^+, \sigma} + (\ddot{\mathbf{u}}, \dot{\mathbf{w}})_{\Omega \times \mathbb{R}^+, \sigma}.$$

Adding (4) and the weak formulations of (3c) and (3d), one obtains a weak formulation of (3) in $\tilde{X} := H_\sigma^1(\mathbb{R}^+, H^1(\Omega))^3 \times H_\sigma^1(\mathbb{R}^+, H^{1/2}(\Gamma)) \times H_\sigma^1(\mathbb{R}^+, H^{-1/2}(\Gamma))$:

Find $(\mathbf{u}, \phi, \lambda) \in \tilde{X}$ such that for all $(\mathbf{w}, w, m) \in \tilde{X}$

$$\begin{aligned} a(\mathbf{u}, \dot{\mathbf{w}}) + \langle \dot{\phi}, \dot{\mathbf{w}}|_\Gamma \cdot n \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \dot{\mathbf{u}}|_\Gamma \cdot n, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle W\phi, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \\ - \langle (\tfrac{1}{2}I - K^T)\lambda, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle (\tfrac{1}{2}I - K)\phi, \dot{m} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle V\lambda, \dot{m} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \quad (5) \\ = - \langle \dot{v}^{inc} \cdot n, \dot{\mathbf{w}}|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle \partial_n^+ v^{inc}, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} . \end{aligned}$$

Let $Z_{h,(\Delta t)} = V_{h,\Delta t}^{1,2}(\Omega)^3 \subset H_\sigma^1(\mathbb{R}^+, H^1(\Omega))^3$, $Y_{h,(\Delta t)} = V_{h,\Delta t}^{1,2} \subset H_\sigma^1(\mathbb{R}^+, H^{1/2}(\Gamma))$, $X_{h,(\Delta t)} = V_{h,\Delta t}^{0,1} \subset H_\sigma^1(\mathbb{R}^+, H^{-1/2}(\Gamma))$ be the conforming discretization spaces from Appendix A in Ω , resp. Γ , based on tensor products of piecewise polynomial functions on a quasi-uniform mesh in space and a uniform mesh in time. Let $\tilde{X}_{h,(\Delta t)} := Z_{h,(\Delta t)} \times Y_{h,(\Delta t)} \times X_{h,(\Delta t)}$. Note that the discretization order is higher in time than in space, in order to be conforming. This corresponds to the loss of one time derivative for the boundary integral operators in Theorem 6, a well-known sub-optimal aspect of the standard functional analytic framework of space-time anisotropic Sobolev spaces.

Then the discrete formulation reads:

Find $(\mathbf{u}, \phi, \lambda) \in \tilde{X}_{h,(\Delta t)}$ such that for all $(\mathbf{w}, w, m) \in \tilde{X}_{h,(\Delta t)}$

$$\begin{aligned} a(\mathbf{u}, \dot{\mathbf{w}}) + \langle \dot{\phi}, \dot{\mathbf{w}}|_\Gamma \cdot n \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \dot{\mathbf{u}}|_\Gamma \cdot n, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle W\phi, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \\ - \langle (\tfrac{1}{2}I - K^T)\lambda, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle (\tfrac{1}{2}I - K)\phi, \dot{m} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle V\lambda, \dot{m} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \quad (6) \\ = - \langle \dot{v}^{inc} \cdot n, \dot{\mathbf{w}}|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle \partial_n^+ v^{inc}, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} . \end{aligned}$$

Practical computations use $\sigma = 0$. See [26] for a detailed analysis of the role of the weight σ .

In order to prove the well-posedness of the discrete formulation, we show the equivalence to a coercive formulation.

Proposition 2. *Let $(\mathbf{u}, \phi, \lambda) \in \tilde{X}_{h,(\Delta t)}$ be a solution to the weak formulation (6). Then with*

$$v = D\phi - S\lambda , \quad (7)$$

$(\mathbf{u}, v) \in Z_{h,(\Delta t)} \times H_\sigma^1(\mathbb{R}^+, H^1(\mathbb{R}^3 \setminus \Gamma))$ satisfies for all $(\mathbf{w}, w, m) \in \tilde{X}_{h,(\Delta t)}$:

$$a(\mathbf{u}, \dot{\mathbf{w}}) + \langle [\dot{v}|_\Gamma] + \dot{v}^{inc}, \dot{\mathbf{w}}|_\Gamma \cdot n \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = 0 \quad (8a)$$

$$-\Delta v + \ddot{v} = 0 \quad \text{in } \mathbb{R}^3 \setminus \Gamma \quad (8b)$$

$$[v|_\Gamma] \in Y_{h,(\Delta t)}, \quad [\partial_n v] \in X_{h,(\Delta t)} \quad (8c)$$

$$-\langle \dot{\mathbf{u}}|_\Gamma \cdot n, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \partial_n^+ v, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = \langle \partial_n^+ v^{inc}, \dot{w} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \quad (8d)$$

$$\langle v^-|_\Gamma, \dot{m} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = 0 \quad \forall m \in X_{h,(\Delta t)} \quad (8e)$$

where $\llbracket v|_\Gamma \rrbracket$, $\llbracket \partial_n v \rrbracket$ denote the jump of v , resp. $\partial_n v$ across Γ .

Conversely, if $(\mathbf{u}, \phi, \lambda) = (\mathbf{u}, \llbracket v|_\Gamma \rrbracket, \llbracket \partial_n v \rrbracket) \in \tilde{X}_{h,(\Delta t)}$ satisfies (8) then the weak formulation (6) and (7) hold.

Proof. Let $(\mathbf{u}, \phi, \lambda) \in \tilde{X}_{h,(\Delta t)}$ fulfill the weak formulation (6). Setting $v = D\phi - S\lambda$, the wave equation (8b) holds outside Γ . Going onto the boundary with the jump relations (see Appendix A)

$$\begin{aligned} v^+|_\Gamma &= (D\phi)^+|_\Gamma - (S\lambda)^+|_\Gamma = (\tfrac{1}{2}I + K)\phi - V\lambda, \\ v^-|_\Gamma &= (D\phi)^-|_\Gamma - (S\lambda)^-|_\Gamma = (-\tfrac{1}{2}I + K)\phi - V\lambda, \end{aligned}$$

we obtain

$$\llbracket v|_\Gamma \rrbracket = v^+|_\Gamma - v^-|_\Gamma = \phi \in Y_{h,(\Delta t)}, \quad (9)$$

and therefore the first assertion in (8c). From (6) we see that

$$a(\mathbf{u}, \dot{\mathbf{w}}) + \langle \dot{\phi}, \dot{\mathbf{w}}|_\Gamma \cdot \mathbf{n} \rangle = -\langle \dot{v}^{inc}, \mathbf{n}, \dot{\mathbf{w}}|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \quad (10)$$

and also

$$\langle v^-|_\Gamma, \dot{m} \rangle_{\Gamma \times \mathbb{R}^+} = \langle (\tfrac{1}{2}I - K)\phi, \dot{m} \rangle_{\Gamma \times \mathbb{R}^+} + \langle V\lambda, \dot{m} \rangle_{\Gamma \times \mathbb{R}^+} = 0.$$

Equation (8e) follows. Using (9), we also obtain (8a).

The jump relations give for v in (7) :

$$\begin{aligned} \partial_n^+ v &= \partial_n^+(D\phi) - \partial_n^+(S\lambda) = W\phi - K^T\lambda + \tfrac{1}{2}I\lambda, \\ \partial_n^- v &= \partial_n^-(D\phi) - \partial_n^-(S\lambda) = W\phi - K^T\lambda - \tfrac{1}{2}I\lambda. \end{aligned}$$

Hence

$$\llbracket \partial_n v \rrbracket = \partial_n^+ v - \partial_n^- v = \lambda,$$

and thus $\llbracket \partial_n v \rrbracket \in X_{h,(\Delta t)}$, which establishes (8c).

Finally, from (6) with

$$-\partial_n^+ v = -W\phi + K^T\lambda - \tfrac{1}{2}I\lambda,$$

(8d) holds. Altogether (8) holds.

To show the converse direction, we define $(\mathbf{u}, \phi, \lambda) := (\mathbf{u}, \llbracket v|_\Gamma \rrbracket, \llbracket \partial_n v \rrbracket) \in Z_{h,(\Delta t)} \times Y_{h,(\Delta t)} \times X_{h,(\Delta t)}$, where \mathbf{u} and v fulfill (8). Since v satisfies the wave equation (8b), we get (7) from the representation formula:

$$v = D\llbracket v|_\Gamma \rrbracket - S\llbracket \partial_n v \rrbracket = D\phi - S\lambda.$$

Combining (8d) with the jump relations for $\partial_n^+ v$ and $v|_\Gamma$, as well as setting ϕ in the equation (8a), we obtain (6). \square

Proposition 3. *Let*

$$\tilde{Z}_{h,(\Delta t)} = \{v \in H_\sigma^1(\mathbb{R}_+, H^1(\mathbb{R}^3 \setminus \Gamma)) : \llbracket v|_\Gamma \rrbracket \in Y_{h,(\Delta t)}, \langle v^-|_\Gamma, \dot{m} \rangle_{\Gamma \times \mathbb{R}^+} = 0 \quad \forall m \in X_{h,(\Delta t)}\} .$$

Then Problem (8) is equivalent to:

Find $(\mathbf{u}, v) \in Z_{h,(\Delta t)} \times \tilde{Z}_{h,(\Delta t)}$ such that

$$\mathcal{A}((\mathbf{u}, v), (\dot{\mathbf{w}}, \dot{w})) = f((\dot{\mathbf{w}}, \dot{w})) \quad \forall (\mathbf{w}, w) \in Z_{h,(\Delta t)} \times \tilde{Z}_{h,(\Delta t)} , \quad (11)$$

where

$$\begin{aligned} \mathcal{A}((\mathbf{u}, v), (\mathbf{w}, w)) &:= (\tilde{\sigma}(\mathbf{u}), \varepsilon(\mathbf{w}))_{\Omega \times \mathbb{R}^+, \sigma} + (\ddot{\mathbf{u}}, \mathbf{w})_{\Omega \times \mathbb{R}^+, \sigma} + (\nabla v, \nabla w)_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} \\ &+ (\ddot{v}, w)_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} - \langle \dot{\mathbf{u}}|_\Gamma \cdot n, \llbracket w|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle \llbracket \dot{v}|_\Gamma \rrbracket, \mathbf{w}|_\Gamma \cdot n \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \end{aligned}$$

and

$$f((\mathbf{w}, w)) := -\langle \dot{v}^{inc}, \mathbf{w}|_\Gamma \cdot n \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle \partial_n^+ v^{inc}, \llbracket w|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} .$$

Proof. First, we assume that (6) holds with $(\mathbf{u}, \phi, \lambda) \in Z_{h,(\Delta t)} \times Y_{h,(\Delta t)} \times X_{h,(\Delta t)}$. Since (8c) and (8e) hold, we know that $(\mathbf{u}, v) \in Z_{h,(\Delta t)} \times \tilde{Z}_{h,(\Delta t)}$. Now for all $w \in \tilde{Z}_{h,(\Delta t)}$ using the second relation in (8c) and $\langle \llbracket \partial_n v \rrbracket, w^-|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+} = 0$, Green's formula and (8b) lead to

$$\begin{aligned} -\langle \partial_n^+ v, \llbracket \dot{w}|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} &= \langle \partial_n^+ v, \dot{w}^-|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \partial_n^+ v, \dot{w}^+|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \\ &= \langle \partial_n^- v, \dot{w}^-|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \partial_n^- v, \dot{w}^-|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle \partial_n^+ v, \dot{w}^-|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \partial_n^+ v, \dot{w}^+|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \\ &= \langle \partial_n^- v, \dot{w}^-|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \partial_n^+ v, \dot{w}^+|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + \langle \llbracket \partial_n v \rrbracket, \dot{w}^-|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \\ &= \langle \partial_n^- v, \dot{w}^-|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \partial_n^+ v, \dot{w}^+|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \\ &= (\nabla v, \nabla \dot{w})_{\Omega \times \mathbb{R}^+, \sigma} + (\Delta v, \dot{w})_{\Omega \times \mathbb{R}^+, \sigma} + (\nabla v, \nabla \dot{w})_{\Omega^c \times \mathbb{R}^+, \sigma} + (\Delta v, \dot{w})_{\Omega^c \times \mathbb{R}^+, \sigma} \\ &= (\nabla v, \nabla \dot{w})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} + (\ddot{v}, \dot{w})_{(\mathbb{R}^3 \setminus \Gamma) \times \mathbb{R}^+, \sigma} . \end{aligned}$$

Therefore testing (8d) with $\llbracket \dot{w}|_\Gamma \rrbracket$ for $w \in \tilde{Z}_{h,(\Delta t)}$,

$$-\langle \dot{\mathbf{u}}|_\Gamma \cdot n + \partial_n^+ v + \partial_n^+ v^{inc}, \llbracket \dot{w}|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = 0 ,$$

we get for all $w \in \tilde{Z}_{h,(\Delta t)}$

$$-\langle \dot{\mathbf{u}}|_\Gamma \cdot n, \llbracket \dot{w}|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} + (\nabla v, \nabla \dot{w})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} + (\ddot{v}, \dot{w})_{(\mathbb{R}^3 \setminus \Gamma) \times \mathbb{R}^+, \sigma} = \langle \partial_n^+ v^{inc}, \llbracket \dot{w}|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} . \quad (12)$$

Adding up (12) and (8a) yields $\mathcal{A}((\mathbf{u}, v), (\dot{\mathbf{w}}, \dot{w})) = f((\dot{\mathbf{w}}, \dot{w}))$.

Conversely, assume (11) holds. Using (11) for a test function $w \in \tilde{Z}_{h,(\Delta t)}$, with compact support in $\mathbb{R}^3 \setminus \Gamma$ we obtain the equation (12):

$$(\ddot{v}, \dot{w})_{\mathbb{R}^3 \times \mathbb{R}^+, \sigma} + (\nabla v, \nabla \dot{w})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} - \langle \dot{\mathbf{u}}|_\Gamma \cdot n, \llbracket \dot{w}|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = \langle \partial_n^+ v^{inc}, \llbracket \dot{w}|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} .$$

Using integration by parts:

$$\begin{aligned} (\ddot{v}, \dot{w})_{\mathbb{R}^3 \times \mathbb{R}^+, \sigma} + (\partial_n^- v, \dot{w}^-|_\Gamma)_{\Gamma \times \mathbb{R}^+, \sigma} - (\Delta v, \dot{w})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} \\ - \langle \partial_n^+ v, \dot{w}^+|_\Gamma \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \dot{\mathbf{u}}|_\Gamma \cdot n, \llbracket \dot{w}|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = \langle \partial_n^+ v^{inc}, \llbracket \dot{w}|_\Gamma \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} . \end{aligned}$$

Since v satisfies the wave equation on the support of w in $\mathbb{R}^3 \setminus \Gamma$.

$$(\ddot{v} - \Delta v, \dot{w})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} = 0 .$$

Equation (8b) follows. Next

$$\langle \partial_n^- v, \dot{w}^- |_{\Gamma} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \partial_n^+ v, \dot{w}^+ |_{\Gamma} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \dot{\mathbf{u}}|_{\Gamma} \cdot n, \llbracket \dot{w}|_{\Gamma} \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = \langle \partial_n^+ v^{inc}, \llbracket \dot{w}|_{\Gamma} \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} .$$

Hence, for all $w \in \tilde{Z}_{h, \Delta}$

$$-\langle \partial_n^+ v + \dot{\mathbf{u}}|_{\Gamma} \cdot n + \partial_n^+ v^{inc}, \llbracket \dot{w}|_{\Gamma} \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \llbracket \partial_n v \rrbracket, \dot{w}|_{\bar{\Gamma}} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = 0 .$$

Choosing $\llbracket w|_{\Gamma} \rrbracket = 0$, we get the second relation in (8c) because

$$\langle \llbracket \partial_n v \rrbracket, \dot{w}|_{\bar{\Gamma}} \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = 0 .$$

Second, choose $w \in \tilde{Z}_{h, (\Delta t)}$ such that $w|_{\bar{\Gamma}} = 0$ yields

$$-\langle \partial_n^+ v + \dot{\mathbf{u}}|_{\Gamma} \cdot n + \partial_n^+ v^{inc}, \llbracket \dot{w}|_{\Gamma} \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} = 0 .$$

and hence (8d), since $\llbracket w|_{\Gamma} \rrbracket \in Y_{h, (\Delta t)}$. From the definition of $\tilde{Z}_{h, (\Delta t)}$ we already get (8e) and (8c).

Finally, we obtain the equation (8a) from the remaining terms in (11). \square

An analogous assertion to Propositions 2 and 3 holds for the continuous problem, instead of the finite element discretization.

We now aim to prove coercivity of the problem (11) for (\mathbf{u}, v) in a suitable norm, defined as:

$$\|(\mathbf{u}, v)\|^2 = (\tilde{\sigma}(\mathbf{u}), \varepsilon(\mathbf{u}))_{\Omega \times \mathbb{R}^+, \sigma} + (\dot{\mathbf{u}}, \dot{\mathbf{u}})_{\Omega \times \mathbb{R}^+, \sigma} + (\nabla v, \nabla v)_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} + (\dot{v}, \dot{v})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} .$$

Note that

$$\begin{aligned} \mathcal{A}((\mathbf{u}, v), (\dot{\mathbf{u}}, \dot{v})) &= (\tilde{\sigma}(\mathbf{u}), \varepsilon(\dot{\mathbf{u}}))_{\Omega \times \mathbb{R}^+, \sigma} + (\dot{\mathbf{u}}, \dot{\mathbf{u}})_{\Omega \times \mathbb{R}^+, \sigma} + (\nabla v, \nabla \dot{v})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} \\ &+ (\dot{v}, \dot{v})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} + \langle \dot{\mathbf{u}}|_{\Gamma} \cdot n, \llbracket \gamma \dot{v} \rrbracket \rangle_{\Gamma \times \mathbb{R}^+, \sigma} - \langle \llbracket \dot{v}|_{\Gamma} \rrbracket, \dot{\mathbf{u}}|_{\Gamma} \cdot n \rangle_{\Gamma \times \mathbb{R}^+, \sigma} \\ &= (\tilde{\sigma}(\mathbf{u}), \varepsilon(\dot{\mathbf{u}}))_{\Omega \times \mathbb{R}^+, \sigma} + (\dot{\mathbf{u}}, \dot{\mathbf{u}})_{\Omega \times \mathbb{R}^+, \sigma} + (\nabla v, \nabla \dot{v})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} + (\dot{v}, \dot{v})_{\mathbb{R}^3 \setminus \Gamma \times \mathbb{R}^+, \sigma} \\ &= \int_0^\infty \int_{\Omega} \tilde{\sigma}(\mathbf{u}) : \varepsilon(\dot{\mathbf{u}}) dx e^{-2\sigma t} dt + \int_0^\infty \int_{\Omega} \dot{\mathbf{u}} \dot{\mathbf{u}} dx e^{-2\sigma t} dt \\ &+ \int_0^\infty \int_{\mathbb{R}^3 \setminus \Gamma} \nabla v \nabla \dot{v} dx e^{-2\sigma t} dt + \int_0^\infty \int_{\mathbb{R}^3 \setminus \Gamma} \dot{v} \dot{v} dx e^{-2\sigma t} dt \\ &= \int_0^\infty \frac{1}{2} \partial_t \left(\int_{\Omega} \tilde{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{u}) dx \right) e^{-2\sigma t} dt + \int_0^\infty \frac{1}{2} \partial_t \left(\int_{\Omega} \dot{\mathbf{u}}^2 dx \right) e^{-2\sigma t} dt \\ &+ \int_0^\infty \frac{1}{2} \partial_t \left(\int_{\mathbb{R}^3 \setminus \Gamma} (\nabla v)^2 dx \right) e^{-2\sigma t} dt + \int_0^\infty \frac{1}{2} \partial_t \left(\int_{\mathbb{R}^3 \setminus \Gamma} \dot{v}^2 dx \right) e^{-2\sigma t} dt . \end{aligned}$$

Using integration by parts in time, the zero initial condition and $\sigma > 0$, we obtain the coercivity estimate:

$$\begin{aligned}
\mathcal{A}((\mathbf{u}, v), (\dot{\mathbf{u}}, \dot{v})) &= -\frac{1}{2} \int_0^\infty \left(\int_\Omega \tilde{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{u}) dx \right) \partial_t (e^{-2\sigma t}) dt - \frac{1}{2} \int_0^\infty \left(\int_\Omega \dot{\mathbf{u}}^2 dx \right) \partial_t (e^{-2\sigma t}) dt \\
&\quad - \frac{1}{2} \int_0^\infty \left(\int_{\mathbb{R}^3 \setminus \Gamma} (\nabla v)^2 dx \right) \partial_t (e^{-2\sigma t}) dt - \frac{1}{2} \int_0^\infty \left(\int_{\mathbb{R}^3 \setminus \Gamma} \dot{v}^2 dx \right) \partial_t (e^{-2\sigma t}) dt \\
&= \sigma \left(\int_0^\infty \left(\int_\Omega \tilde{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{u}) dx \right) e^{-2\sigma t} dt + \int_0^\infty \left(\int_\Omega \dot{\mathbf{u}}^2 dx \right) e^{-2\sigma t} dt \right. \\
&\quad \left. + \int_0^\infty \left(\int_{\mathbb{R}^3 \setminus \Gamma} (\nabla v)^2 dx \right) e^{-2\sigma t} dt + \int_0^\infty \left(\int_{\mathbb{R}^3 \setminus \Gamma} \dot{v}^2 dx \right) e^{-2\sigma t} dt \right) \\
&= \sigma \|(\mathbf{u}, v)\|_{0,1,\Omega}^2 \gtrsim_\sigma \|\mathbf{u}\|_{0,1,\Omega}^2 + \|v\|_{0,1,\Omega^c}^2.
\end{aligned}$$

This implies, in particular, uniqueness of the solution (11) and therefore also the solutions to (5) and (6).

3. A priori error estimate

We state an a priori error estimate:

Theorem 4. *Let $(\mathbf{u}, \phi, \lambda) \in \tilde{X}$ satisfy (5) and $(\mathbf{u}_h, \phi_h, \lambda_h) \in \tilde{X}_{h,\Delta t}$ satisfy (6). Then*

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|_{0,1,\Omega}^2 + \|\phi - \phi_h\|_{0,1/2,\Gamma}^2 + \|\lambda - \lambda_h\|_{0,-1/2,\Gamma}^2 \lesssim_\sigma \\
&\quad \inf_{(\mathbf{w}_h, \psi_h, \mu_h) \in \tilde{X}_{h,\Delta t}} \left(1 + \frac{1}{(\Delta t)^2}\right) \|\mathbf{u} - \mathbf{w}_h\|_{1,1,\Omega}^2 + \left(1 + \frac{1}{(\Delta t)^2}\right) \|\phi - \psi_h\|_{1,1/2,\Gamma}^2 + \left(1 + \frac{1}{(\Delta t)^2}\right) \|\lambda - \mu_h\|_{1,-1/2,\Gamma}^2.
\end{aligned}$$

Proof. Let $(\mathbf{u}, \phi, \lambda) \in \tilde{X}$ satisfy (5) and $(\mathbf{u}_h, \phi_h, \lambda_h) \in \tilde{X}_{h,\Delta t}$ satisfy (6). Then for all $(\tilde{\mathbf{w}}, \tilde{\phi}, \tilde{\lambda}) \in \tilde{X}_{h,\Delta t}$

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}_h\|_{0,1,\Omega}^2 + \|\phi - \phi_h\|_{0,1/2,\Gamma}^2 + \|\lambda - \lambda_h\|_{0,-1/2,\Gamma}^2 \\
&\lesssim \|\mathbf{u} - \tilde{\mathbf{w}}\|_{0,1,\Omega}^2 + \|\tilde{\mathbf{w}} - \mathbf{u}_h\|_{0,1,\Omega}^2 + \|\phi - \tilde{\phi}\|_{0,1/2,\Gamma}^2 + \|\tilde{\phi} - \phi_h\|_{0,1/2,\Gamma}^2 \\
&\quad + \|\lambda - \tilde{\lambda}\|_{0,-1/2,\Gamma}^2 + \|\tilde{\lambda} - \lambda_h\|_{0,-1/2,\Gamma}^2.
\end{aligned}$$

We therefore focus on estimates for $\|\tilde{\mathbf{w}} - \mathbf{u}_h\|_{0,1,\Omega}^2 + \|\tilde{\phi} - \phi_h\|_{0,1/2,\Gamma}^2 + \|\tilde{\lambda} - \lambda_h\|_{0,-1/2,\Gamma}^2$. Now using coercivity and Galerkin orthogonality with $v := D\tilde{\phi} - S\tilde{\lambda}$, $v_h := D\phi_h - S\lambda_h$ and $\tilde{r} := D\tilde{\phi} - S\tilde{\lambda}$

$$\begin{aligned}
&\|\tilde{\mathbf{w}} - \mathbf{u}_h\|_{0,1,\Omega}^2 + \|\tilde{\phi} - \phi_h\|_{0,1/2,\Gamma}^2 + \|\tilde{\lambda} - \lambda_h\|_{0,-1/2,\Gamma}^2 \\
&\lesssim \|\tilde{\mathbf{w}} - \mathbf{u}_h\|_{0,1,\Omega}^2 + \|\tilde{r} - v_h\|_{0,1,\mathbb{R}^3 \setminus \Gamma}^2 \\
&\lesssim_\sigma \|(\tilde{\mathbf{w}} - \mathbf{u}_h, \tilde{r} - v_h)\|^2 = \mathcal{A}\left(\begin{pmatrix} \tilde{\mathbf{w}} - \mathbf{u}_h \\ \tilde{r} - v_h \end{pmatrix}^\top, \begin{pmatrix} \dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h \\ \dot{\tilde{r}} - \dot{v}_h \end{pmatrix}^\top\right) \\
&= \mathcal{A}\left(\begin{pmatrix} \tilde{\mathbf{w}} - \mathbf{u} \\ \tilde{r} - v \end{pmatrix}^\top, \begin{pmatrix} \dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h \\ \dot{\tilde{r}} - \dot{v}_h \end{pmatrix}^\top\right) + \mathcal{A}\left(\begin{pmatrix} \mathbf{u} - \mathbf{u}_h \\ v - v_h \end{pmatrix}^\top, \begin{pmatrix} \dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h \\ \dot{\tilde{r}} - \dot{v}_h \end{pmatrix}^\top\right) \\
&= \mathcal{A}\left(\begin{pmatrix} \tilde{\mathbf{w}} - \mathbf{u} \\ \tilde{r} - v \end{pmatrix}^\top, \begin{pmatrix} \dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h \\ \dot{\tilde{r}} - \dot{v}_h \end{pmatrix}^\top\right).
\end{aligned}$$

By definition of the single and double layer potentials S and D , the functions v, v_h, \tilde{r} all satisfy the wave equation in $\mathbb{R}^3 \setminus \Gamma$. Further, from Proposition 2, $[[v_h]_\Gamma], [[\tilde{r}]_\Gamma] \in Y_{h,\Delta t}^0$, $[[\partial_n v_h]_\Gamma], [[\partial_n \tilde{r}]_\Gamma] \in X_{h,\Delta t}^0$ and $\langle v_h^-|_\Gamma, m \rangle_{\Gamma \times \mathbb{R}^+} = 0$, $\langle \tilde{r}^-|_\Gamma, m \rangle_{\Gamma \times \mathbb{R}^+} = 0$. Using the definition of \mathcal{A} and Green's theorem, we find

$$\begin{aligned}
& \|\tilde{\mathbf{w}} - \mathbf{u}_h\|_{0,1,\Omega}^2 + \|\tilde{\phi} - \phi_h\|_{0,1/2,\Gamma}^2 + \|\tilde{\lambda} - \lambda_h\|_{0,-1/2,\Gamma}^2 \\
& \lesssim_\sigma \int_0^\infty e^{-2\sigma t} \left\{ \int_\Omega \tilde{\sigma}(\tilde{\mathbf{w}} - \mathbf{u}) : \varepsilon(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h) dx + \int_\Omega (\ddot{\tilde{\mathbf{w}}} - \ddot{\mathbf{u}})(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h) dx \right. \\
& + \int_{\mathbb{R}^3 \setminus \Gamma} \nabla(\tilde{r} - v) \nabla(\dot{\tilde{r}} - \dot{v}_h) dx + \int_{\mathbb{R}^3 \setminus \Gamma} (\ddot{\tilde{r}} - \ddot{v})(\dot{\tilde{r}} - \dot{v}_h) dx \\
& - \int_\Gamma (\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}})|_\Gamma \cdot n [[\dot{\tilde{r}} - \dot{v}_h]_\Gamma] ds_x + \int_\Gamma [[(\dot{\tilde{r}} - \dot{v})|_\Gamma] (\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h)|_\Gamma \cdot n ds_x \Big\} dt \\
& = \int_0^\infty e^{-2\sigma t} \left\{ \int_\Omega \tilde{\sigma}(\tilde{\mathbf{w}} - \mathbf{u}) : \varepsilon(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h) dx + \int_\Omega (\ddot{\tilde{\mathbf{w}}} - \ddot{\mathbf{u}})(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h) dx \right. \\
& + \int_\Gamma \partial_n^+(\tilde{r} - v) [[\dot{\tilde{r}} - \dot{v}_h]_\Gamma] ds_x - \int_\Gamma (\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}})|_\Gamma \cdot n [[\dot{\tilde{r}} - \dot{v}_h]_\Gamma] ds_x \\
& + \left. \int_\Gamma [[(\dot{\tilde{r}} - \dot{v})|_\Gamma] (\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h)|_\Gamma \cdot n ds_x \right\} dt \\
& = \int_0^\infty e^{-2\sigma t} \left\{ \int_\Omega \tilde{\sigma}(\tilde{\mathbf{w}} - \mathbf{u}) : \varepsilon(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h) dx + \int_\Omega (\ddot{\tilde{\mathbf{w}}} - \ddot{\mathbf{u}})(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h) dx \right. \\
& + \int_\Gamma (W(\tilde{\phi} - \phi) - (K^T - \frac{1}{2}I)(\tilde{\lambda} - \lambda))(\dot{\tilde{\phi}} - \dot{\phi}_h) ds_x \\
& - \left. \int_\Gamma (\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}})|_\Gamma \cdot n(\dot{\tilde{\phi}} - \dot{\phi}_h) ds_x + \int_\Gamma (\dot{\tilde{\phi}} - \dot{\phi})(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h)|_\Gamma \cdot n ds_x \right\} dt .
\end{aligned}$$

We estimate the individual terms. Using Young's inequality we have for $\epsilon > 0$

$$\begin{aligned}
& \int_0^\infty e^{-2\sigma t} \left\{ \int_\Omega \tilde{\sigma}(\tilde{\mathbf{w}} - \mathbf{u}) : \varepsilon(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h) dx + \int_\Omega (\ddot{\tilde{\mathbf{w}}} - \ddot{\mathbf{u}})(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h) dx \right\} dt \\
& \lesssim_\sigma \frac{1}{\epsilon(\Delta t)^2} \|\tilde{\mathbf{w}} - \mathbf{u}\|_{1,1,\Omega}^2 + \epsilon \|\tilde{\mathbf{w}} - \mathbf{u}_h\|_{0,1,\Omega}^2 .
\end{aligned}$$

Next we estimate

$$\begin{aligned}
& \int_0^\infty e^{-2\sigma t} \int_\Gamma (W(\tilde{\phi} - \phi) - (K^T - \frac{1}{2}I)(\tilde{\lambda} - \lambda))(\dot{\tilde{\phi}} - \dot{\phi}_h) ds_x dt \\
& \lesssim (\|W(\tilde{\phi} - \phi)\|_{0,-1/2,\Gamma} + \|(K^T - \frac{1}{2}I)(\tilde{\lambda} - \lambda)\|_{0,-1/2,\Gamma}) \|\tilde{\phi} - \phi_h\|_{1,1/2,\Gamma} .
\end{aligned}$$

Using the inverse estimate as in (3.182) in [19]

$$\|\tilde{\phi}\|_{1,1/2,\Gamma} \lesssim \frac{1}{\Delta t} \|\tilde{\phi}\|_{0,1/2,\Gamma} ,$$

we further estimate with the mapping properties of the integral operators

$$\begin{aligned} &\lesssim_{\sigma} (\|W(\tilde{\phi} - \phi)\|_{0,-1/2,\Gamma} + \|(K^T - \frac{1}{2}I)(\tilde{\lambda} - \lambda)\|_{0,-1/2,\Gamma}) \frac{1}{\Delta t} \|\tilde{\phi} - \phi_h\|_{0,1/2,\Gamma} \\ &\lesssim \frac{1}{\epsilon(\Delta t)^2} \|\tilde{\phi} - \phi\|_{1,1/2,\Gamma}^2 + \frac{1}{\epsilon(\Delta t)^2} \|\tilde{\lambda} - \lambda\|_{1,-1/2,\Gamma}^2 + \epsilon \|\tilde{\phi} - \phi_h\|_{0,1/2,\Gamma}^2 . \end{aligned}$$

For the fourth term, we get:

$$\begin{aligned} &\int_0^{\infty} e^{-2\sigma t} \int_{\Gamma} (\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}})|_{\Gamma} \cdot n(\dot{\tilde{\phi}} - \dot{\phi}_h) ds_x dt \\ &\lesssim_{\sigma} \|(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}})|_{\Gamma} \cdot n\|_{0,-1/2,\Gamma} \|(\dot{\tilde{\phi}} - \dot{\phi}_h)\|_{0,1/2,\Gamma} \\ &\lesssim \|\tilde{\mathbf{w}} - \mathbf{u}\|_{1,1/2,\Gamma} \|\tilde{\phi} - \phi_h\|_{1,1/2,\Gamma} \lesssim \frac{1}{\epsilon(\Delta t)^2} \|\tilde{\mathbf{w}} - \mathbf{u}\|_{1,1,\Omega}^2 + \epsilon \|\tilde{\phi} - \phi_h\|_{0,1/2,\Gamma}^2 . \end{aligned}$$

For the last term, analogously the trace theorem and the inverse estimate show:

$$\begin{aligned} &\int_0^{\infty} e^{-2\sigma t} \int_{\Gamma} (\dot{\tilde{\phi}} - \dot{\phi})(\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h)|_{\Gamma} \cdot n ds_x dt \lesssim_{\sigma} \|\dot{\tilde{\phi}} - \dot{\phi}\|_{0,1/2,\Gamma} \|\dot{\tilde{\mathbf{w}}} - \dot{\mathbf{u}}_h|_{\Gamma} \cdot n\|_{0,-1/2,\Gamma} \\ &\lesssim \frac{1}{\Delta t} \|\tilde{\phi} - \phi\|_{1,1/2,\Gamma} \|\tilde{\mathbf{w}} - \mathbf{u}_h\|_{0,1,\Omega} \lesssim \frac{1}{\epsilon(\Delta t)^2} \|\tilde{\phi} - \phi\|_{1,1/2,\Gamma}^2 + \epsilon \|\tilde{\mathbf{w}} - \mathbf{u}_h\|_{0,1,\Omega}^2 . \end{aligned}$$

Moving the terms with positive powers of ϵ to the left hand side **and choosing a fixed, sufficiently small $\epsilon > 0$ depending on σ , we conclude:**

$$\begin{aligned} &\|\tilde{\mathbf{w}} - \mathbf{u}_h\|_{0,1,\Omega}^2 + \|\tilde{\phi} - \phi_h\|_{0,1/2,\Gamma}^2 + \|\tilde{\lambda} - \lambda_h\|_{0,-1/2,\Gamma}^2 \\ &\lesssim_{\sigma} \frac{1}{(\Delta t)^2} \|\tilde{\mathbf{w}} - \mathbf{u}\|_{1,1,\Omega}^2 + \frac{1}{(\Delta t)^2} \|\tilde{\phi} - \phi\|_{1,1/2,\Gamma}^2 + \frac{1}{(\Delta t)^2} \|\tilde{\lambda} - \lambda\|_{1,-1/2,\Gamma}^2 , \end{aligned}$$

and therefore

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{0,1,\Omega}^2 + \|\phi - \phi_h\|_{0,1/2,\Gamma}^2 + \|\lambda - \lambda_h\|_{0,-1/2,\Gamma}^2 \\ &\lesssim_{\sigma} \left(1 + \frac{1}{(\Delta t)^2}\right) \left(\|\tilde{\mathbf{w}} - \mathbf{u}\|_{1,1,\Omega}^2 + \|\tilde{\phi} - \phi\|_{1,1/2,\Gamma}^2 + \|\tilde{\lambda} - \lambda\|_{1,-1/2,\Gamma}^2\right) . \end{aligned}$$

This proves the assertion. \square

4. Numerical results

This section presents numerical results for the fluid structure-interaction problem given by (5), in 3d. While finite element discretizations of fluid-structure interaction have attracted significant recent interest, coupled finite and boundary element procedures in the time domain are only beginning to be explored. In 2d, numerical results have been presented in [23], based on time discretization by convolution quadrature of the boundary integral operators. A similar approach has been demonstrated for the interaction of waves with a

thermoelastic solid in 2d [25]. The authors are not aware of any related numerical results in the mathematical literature based on time-domain Galerkin boundary element methods, as presented in this work. On the other hand, such methods are now actively being studied for wave-wave interaction in 2d and 3d, as in [1] and [3].

In the numerical experiments for Problem (1), the variational formulation (5) is solved, for $\sigma = 0$, by choosing as ansatz function in the interior domain Ω

$$\mathbf{u}_{h,\Delta t}(x,t) = \sum_{k=1}^{N_t} \sum_{\nu=1}^3 \sum_{i=1}^{N_o} u_{\nu,i}^k \beta_{\Delta t}^k(t) \mathbf{e}_{\nu} \eta_h^i(x), \quad (13)$$

where $\{\eta_h^i\}$ denotes the basis of piecewise linear hat functions for $V_h^1(\Omega)$ and

$$\beta_{\Delta t}^m(t) = (\Delta t)^{-1}((t - t_{m-1})\gamma_{\Delta t}^m(t) - (t - t_{m+1})\gamma_{\Delta t}^{m+1}(t)).$$

Here $\gamma_{\Delta t}^m(t) = H(t - t_{m-1}) - H(t - t_m)$, with H the Heaviside function. The test functions are given by

$$\dot{\mathbf{w}}_{h,\Delta t} = \eta_h^l(x) \gamma_{\Delta t}^n(t) \mathbf{e}_{\mu}, \quad (14)$$

for $l = 1, \dots, N_o$, $n = 1, \dots, N_t$ and $\mu = 1, 2, 3$.

The ansatz functions on Γ are taken as

$$\phi_{h,\Delta t}(x,t) = \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \varphi_i^m \beta_{\Delta t}^m(t) \xi_h^i(x) \quad (15)$$

and

$$\lambda_{h,\Delta t}(x,t) = \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \lambda_i^m \beta_{\Delta t}^m(t) \xi_h^i(x), \quad (16)$$

where $\{\xi_h^i\}$ denotes the basis of piecewise linear hat functions for V_h^1 . The corresponding test functions are chosen as

$$\dot{w}_{h,\Delta t} = \gamma_{\Delta t}^n(t) \xi_h^j(x), \quad (17)$$

$$m_{h,\Delta t} = \gamma_{\Delta t}^n(t) \xi_h^j(x), \quad (18)$$

for $1 \leq n \leq N_t$ and $1 \leq j \leq N_{s'}$. The resulting discretization of the Poincaré-Steklov operator has been tested in [15, 16], and corresponding results are obtained for more natural discretizations with piecewise constant $\lambda_{h,\Delta t}$.

As shown in Appendix B, this discretization of (5) leads to a time-stepping scheme, which solves a system of the following structure in each time step ≥ 3 :

$$\begin{pmatrix} \frac{(\Delta t)}{2}A + \frac{1}{(\Delta t)}M & [0, n_x R I]^T & 0 \\ [0, -R I n_x] & -W^0 & K^T 0 - \frac{1}{2} \frac{(\Delta t)}{2} I \\ 0 & -K^0 - \frac{1}{2} I & V^0 \end{pmatrix} \begin{pmatrix} u^n \\ \varphi^n \\ \lambda^n \end{pmatrix} \\ = \begin{pmatrix} H^n + H^{n-1} - A \frac{(\Delta t)}{2} u^{n-1} + M \frac{2}{(\Delta t)} u^{n-1} - M \frac{1}{(\Delta t)} u^{n-2} + n_x R I \varphi^{n-1} \\ G^n + G^{n-1} + R I n_x u_{\Gamma}^{n-1} + \sum_{m=1}^{n-1} W^{n-m} \varphi^m - \sum_{m=1}^{n-1} K^T{}^{n-m} \lambda^m + \frac{1}{2} \frac{(\Delta t)}{2} I \lambda^{n-1} \\ \sum_{m=1}^{n-1} K^{n-m} \varphi^m - \frac{1}{2} I \varphi^{n-1} - \sum_{m=1}^{n-1} V^{n-m} \lambda^m \end{pmatrix}.$$

The system in the first two time steps is similar, see Appendix B. We solve this system repeatedly until our desired time step N_t is reached.

Example. Let $\Omega = [-1, 1]^3$. Using the discretization described above, we compute the solutions to the discrete system (5) up to time $T = 4$ for data v^{inc} corresponding to the exact solution

$$\mathbf{u}(x, t) = \begin{pmatrix} (\sin(\pi(t - \frac{x_1}{2})))^5 (H(-1 + t - \frac{x_1}{2}) - H(-3 + t - \frac{x_1}{2})) \\ 0 \\ 0 \end{pmatrix}, \quad (19)$$

$$v(x, t) = \frac{|x| - t}{2|x|} (1 + \cos(\frac{\pi(|x| - t)}{0.9})) H(0.9 - ||x| - t|). \quad (20)$$

We use uniform discretizations by tetrahedra as depicted in Figure 1 and a time step Δt such that $\frac{\Delta t}{h} \simeq 0.1414$. We denote the number of grid points on an edge of the cube by $n + 1$. The finest mesh is then given by $n = 24$ and consists of 69120 tetrahedra, corresponding to $\Delta t = 0.01667$. The convergence of the numerical solution to the exact solution is studied as the mesh is refined, and we measure the error in terms of the L^2 -norm in space, resp. space-time.

Figure 2 shows the first component \mathbf{u}_1 of the numerical and exact solutions at the corner point $\mathbf{x}_0 = (-1, -1, -1)$ as a function of time for $n = 2, 4, 8$. The behaviour of the solution in Ω , resp. Ω^c , is illustrated in Figure 3, which plots the L^2 -norms of \mathbf{u} in Ω , resp. of $v|_{\Gamma} = \varphi$ on Γ , as a function of time for $n = 2, 8, 24$. These norms are approximated from the solution vectors u^n and φ^n of the discrete system using a trapezoidal rule for the integrals. The L^2 error as a function of time is shown in Figure 4, corresponding to the numerical solutions depicted in Figure 3. All plots show excellent approximation of the simple behavior of the solution for short times and a monotonous convergence on the whole time interval. Note that error in Figure 4 does not seem to grow with time, as expected for a variational method. Figure 5 considers the convergence of the numerical solutions up to $n = 24$ in terms of the mesh size h . It depicts the $L^2(\Omega \times [0, T])$ -norm of the error in \mathbf{u} , as well as the $L^2(\Gamma \times [0, T])$ -norm of the error in φ . Similar convergence rates are obtained for \mathbf{u} and φ in these L^2 -norms: 0.76 for \mathbf{u} , 0.80 for φ . Note that based on Theorem 4 and the trace theorem for Sobolev spaces, one might naively expect slower convergence in φ than in \mathbf{u} (by a difference of the rates 0.5) in the norms used here. However, as shown in [27] for time-independent FEM-BEM coupling, under mild regularity assumptions the BEM solution φ converges at a rate 0.5 faster than predicted from a joint estimate for (\mathbf{u}, φ) as in Theorem 4. This exactly cancels the above difference of rates and leads to identical convergences rates for \mathbf{u} , φ in the space-time L^2 -norms on Ω , resp. Γ . The identical observed convergence rates for \mathbf{u} , φ are therefore expected and in line with those known for FEM-BEM coupling in time-independent problems [27].

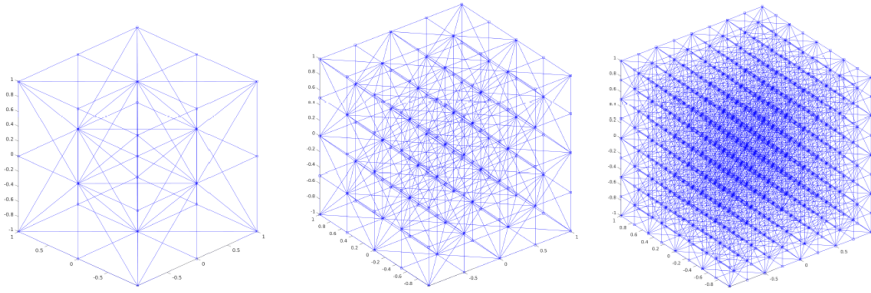


Figure 1: Meshes for $[-1, 1]^3$ with 27 ($n=2$), 125 ($n=4$) and 729 ($n=8$) nodes.

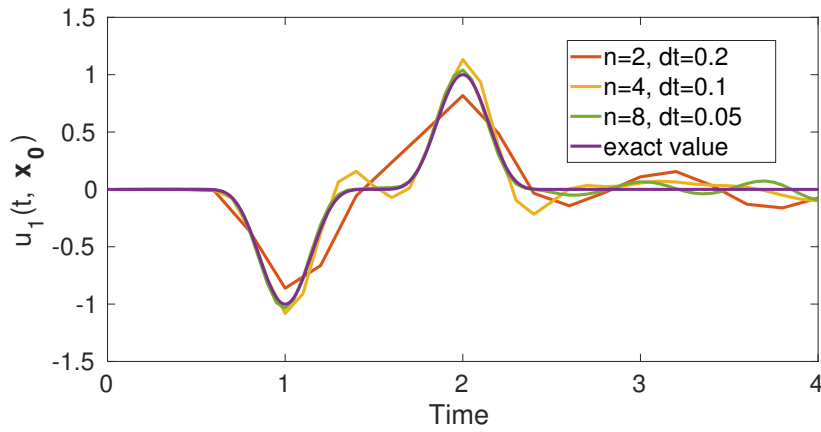


Figure 2: Numerical and exact solutions $u_1(t, \mathbf{x}_0)$, $\mathbf{x}_0 = (-1, -1, -1)$.

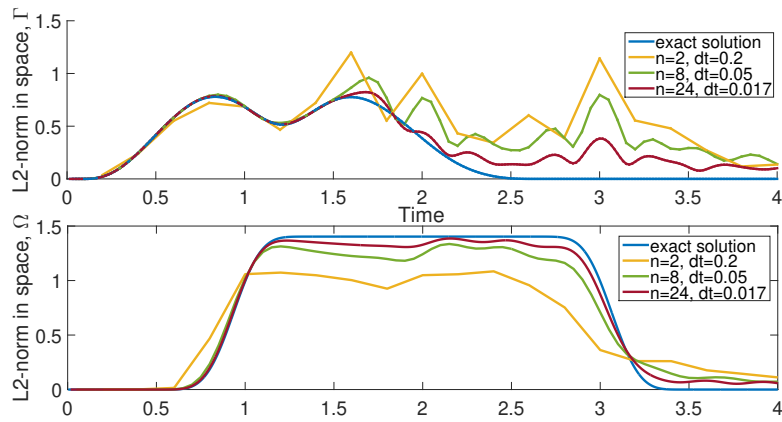


Figure 3: L^2 -norm in space of the exact and numerical solutions for \mathbf{u} , resp. φ .

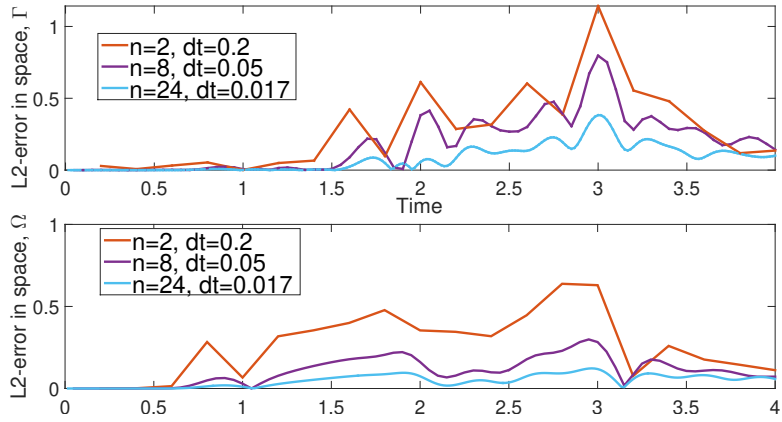


Figure 4: L^2 -error in space for \mathbf{u} , resp. φ .

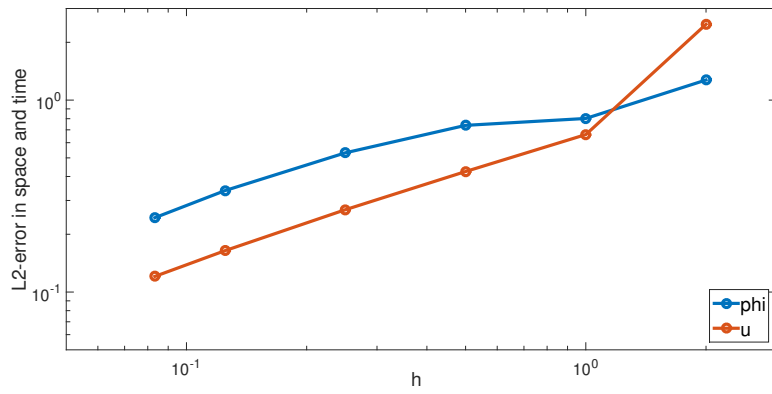


Figure 5: L^2 -error in space-time for \mathbf{u} , φ in terms of h .

Appendix A. Integral operators and finite-boundary elements

We recall basic definitions and properties of boundary integral operators for the wave equation from [30], as well as from [9, 21].

Let Γ be the boundary of a polyhedral domain Ω in \mathbb{R}^3 , consisting of curved, polygonal boundary faces. In $\mathbb{R}^3 \setminus \Gamma$, a solution v to the homogeneous wave equation may be represented in terms of the jump of the Dirichlet and Neumann data across Γ : $v = D\phi - S\lambda$. Here for $x \in \mathbb{R}^3 \setminus \Gamma$ and $t \geq 0$

$$S\lambda(x, t) = \int_0^\infty \int_\Gamma G(t - \tau, x, y) \lambda(y, \tau) dy d\tau, \quad (\text{A.1})$$

$$D\phi(x, t) = \int_0^\infty \int_\Gamma \frac{\partial G}{\partial n_y}(t - \tau, x, y) \phi(y, \tau) dy d\tau, \quad (\text{A.2})$$

are the single, resp. double layer potential for the wave equation defined from the fundamental solution $G(\tau, x, y) = \frac{\delta(\tau - |x - y|)}{4\pi|x - y|}$.

The coupling method presented in this article relies on the resulting boundary integral operators on $\Gamma \times (0, \infty)$. For $(x, t) \in \Gamma \times (0, \infty)$ we define

$$\begin{aligned} V\phi(x, t) &= \int_0^\infty \int_\Gamma G(t - \tau, x, y) \phi(y, \tau) dy d\tau, \\ K\phi(x, t) &= \int_0^\infty \int_\Gamma \frac{\partial G}{\partial n_y}(t - \tau, x, y) \phi(y, \tau) dy d\tau, \\ K^T\phi(x, t) = K'\phi(x, t) &= \int_0^\infty \int_\Gamma \frac{\partial G}{\partial n_x}(t - \tau, x, y) \phi(y, \tau) dy d\tau, \\ W\phi(x, t) &= \int_0^\infty \int_\Gamma \frac{\partial^2 G}{\partial n_x \partial n_y}(t - \tau, x, y) \phi(y, \tau) dy d\tau. \end{aligned} \quad (\text{A.3})$$

They are studied in space-time anisotropic Sobolev spaces $H_\sigma^r(\mathbb{R}^+, \tilde{H}^s(\Gamma))$ [21].

To define an explicit scale of Sobolev norms, fix a partition of unity α_i subordinate to a covering of $\tilde{\Gamma}$ by open sets B_i and diffeomorphisms ϕ_i mapping each B_i into the unit cube $\subset \mathbb{R}^n$. They induce a family of norms from \mathbb{R}^d :

$$\|u\|_{s, \Gamma} = \left(\sum_{i=1}^p \int_{\mathbb{R}^n} (|\omega|^2 + |\xi|^2)^s |\mathcal{F}\{(\alpha_i u) \circ \phi_i^{-1}\}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

\mathcal{F} here denotes the Fourier transform. The norms for different $\omega \in \mathbb{C} \setminus \{0\}$ are equivalent.

Weighted Sobolev spaces in time for $r \in \mathbb{R}$ and $\sigma > 0$: are defined as

$$H_\sigma^r(\mathbb{R}^+) = \{u \in \mathcal{D}'_+ : e^{-\sigma t} u \in \mathcal{S}'_+ \text{ and } \|u\|_{H_\sigma^r(\mathbb{R}^+)} < \infty\}.$$

Here, \mathcal{D}'_+ denotes the space of distributions on \mathbb{R} with support in $[0, \infty)$, and \mathcal{S}'_+ the subspace of tempered distributions. The Sobolev spaces are Hilbert spaces endowed with the norm

$$\|u\|_{H_\sigma^r(\mathbb{R}^+)} = \left(\int_{-\infty + i\sigma}^{+\infty + i\sigma} |\omega|^{2r} |\hat{u}(\omega)|^2 d\omega \right)^{\frac{1}{2}}.$$

The scale of space-time anisotropic Sobolev spaces on Γ combines the Sobolev norms in space and time:

Definition 5. For $r, s \in \mathbb{R}$ and $\sigma > 0$ define

$$H_\sigma^r(\mathbb{R}^+, H^s(\Gamma)) = \{u \in \mathcal{D}'_+(H^s(\Gamma)) : e^{-\sigma t} u \in \mathcal{S}'_+(H^s(\Gamma)) \text{ and } \|u\|_{r,s,\Gamma} < \infty\} .$$

$\mathcal{D}'_+(E)$ denotes the space of distributions on \mathbb{R} with support in $[0, \infty)$, taking values in $E = H^s(\Gamma), \tilde{H}^s(\Gamma)$, and $\mathcal{S}'_+(E)$ the subspace of tempered distributions. These Sobolev spaces are Hilbert spaces endowed with the norm

$$\|u\|_{r,s,\Gamma} = \left(\int_{-\infty+i\sigma}^{+\infty+i\sigma} |\omega|^{2r} \|\hat{u}(\omega)\|_{s,\Gamma}^2 d\omega \right)^{\frac{1}{2}} .$$

When $|s| \leq 1$ one can show that the spaces are independent of the choice of α_i and ϕ_i .

In a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, we define space-time anisotropic Sobolev spaces $H_\sigma^r(\mathbb{R}^+, H^s(\Omega))$ analogously to above, starting from the standard Sobolev spaces $H^s(\Omega)$ with norm $\|u\|_{s,\Omega} = \inf_v \left(\int_{\mathbb{R}^d} (|\omega|^2 + |\xi|^2)^s |\mathcal{F}v(\xi)|^2 d\xi \right)^{\frac{1}{2}}$. Here the infimum extends over all extensions $v \in H^s(\mathbb{R}^d)$ of $u \in H^s(\Omega)$, i.e. all v with $v|_\Omega = u$.

We state the mapping properties of the boundary integral operators, see e.g. [9, 21]:

Theorem 6. The following operators are continuous for $r \in \mathbb{R}, \sigma > 0$:

$$\begin{aligned} V &: H_\sigma^{r+1}(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) , \\ K' &: H_\sigma^{r+1}(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) , \\ K &: H_\sigma^{r+1}(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) , \\ W &: H_\sigma^{r+1}(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma)) \rightarrow H_\sigma^r(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) . \end{aligned}$$

By a fundamental observation of Bamberger and Ha-Duong [5], $V\partial_t$ satisfies a coercivity estimate in the norm of $H_\sigma^0(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$, provided $\sigma > 0$: $\|\psi\|_{0,-\frac{1}{2},\Gamma}^2 \lesssim_\sigma \langle V\psi, \dot{\psi} \rangle$. From the mapping properties of Theorem 6 one also has the continuity of the bilinear form associated to $V\partial_t$ in a bigger norm: $\langle V\psi, \dot{\psi} \rangle \lesssim \|\psi\|_{1,-\frac{1}{2},\Gamma}^2$. Similar estimates hold for $W\partial_t$: $\|\phi\|_{0,\frac{1}{2},\Gamma}^2 \lesssim_\sigma \langle W\phi, \dot{\phi} \rangle \lesssim \|\phi\|_{1,\frac{1}{2},\Gamma}^2$. Proofs and further information may be found in [21].

For sufficiently regular ϕ and λ , the following jump relations hold [21]:

$$\begin{aligned} (S\lambda)^-|_\Gamma &= (S\lambda)^+|_\Gamma = V\lambda, \quad \partial_n^-(D\phi) = \partial_n^+(D\phi) = W\phi, \\ \partial_n^+(S\lambda) &= \left(-\frac{1}{2}I + K'\right)\lambda, \quad \partial_n^-(S\lambda) = \left(\frac{1}{2}I + K'\right)\lambda, \end{aligned}$$

$$(D\phi)^+|_\Gamma = \left(\frac{1}{2}I + K\right)\phi, \quad (D\phi)^-|_\Gamma = \left(-\frac{1}{2}I + K\right)\phi.$$

We consider space-time discretizations based on tensor products of piecewise polynomials:

For simplicity, we assume that Ω is a polygonal domain, with a quasi-uniform triangulation $\mathcal{T}_\Omega = \{T_1, \dots, T_{N_o}\}$ by N_o tetrahedra. The induced quasi-uniform triangulation of the boundary Γ , $\mathcal{T}_S = \{\Delta_1, \dots, \Delta_{N_{s'}}\}$, should consist of closed triangular faces Δ_i , such that each Δ_i is a face of one T_j and at most one face of T_j is contained in Γ .

We consider the space $V_h^q(\Omega)$ of piecewise polynomial functions on \mathcal{T}_Ω of degree $q \geq 0$ in space (continuous if $q \geq 1$). V_h^q consists of traces on Γ of functions in $V_h^q(\Omega)$. The parameter h denotes the maximal diameter of an element in \mathcal{T}_Ω .

We choose an equidistant temporal mesh on the positive half-line $\mathcal{T}_T = \{[0, t_1), [t_1, t_2), \dots\}$, where $t_n = n(\Delta t)$. $V_{\Delta t}^p$ is the space of piecewise polynomial functions of degree p on \mathcal{T}_T (continuous and vanishing at $t = 0$ if $p \geq 1$, C^1 if $p \geq 2$).

The space-time approximation spaces are given by tensor products of the approximation spaces in space and time, V_h^q and $V_{\Delta t}^p$, associated to the space-time meshes $\mathcal{T}_{\Omega, T} = \mathcal{T}_\Omega \times \mathcal{T}_T$, respectively $\mathcal{T}_{S, T} = \mathcal{T}_S \times \mathcal{T}_T$. We write

$$V_{\Delta t, h}^{p, q}(\Omega) := V_{\Delta t}^p \otimes V_h^q(\Omega), \quad V_{\Delta t, h}^{p, q} := V_{\Delta t}^p \otimes V_h^q. \quad (\text{A.5})$$

Appendix B. Discretization and MOT-Algorithm

This appendix discusses the details of the discretization (6), where we set $\sigma = 0$. The resulting formulas for the entries of the Galerkin matrices reduce their assembly to numerical quadratures over certain light cones E_l below. This structure is crucial for the practical implementation in standard time-domain boundary element codes, see [14, 32], as well as the recent Ph.D. thesis [28] of the second author.

We choose the finite element ansatz and test functions as in (13) - (18).

For the discretization of the boundary integral operators, we begin with the retarded hypersingular operator.

We choose the ansatz function as in (15) and the test function as in (17).

$$\begin{aligned} \langle W\phi_{h, \Delta t}, \dot{w}_{h, \Delta t} \rangle_{\Gamma \times \mathbb{R}_+} &= \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \varphi_i^m \left[- \iint_{E_{n-m}} \frac{(n_x \cdot n_y) \xi_h^i(y) \xi_h^j(x)}{(\Delta t)|x-y|4\pi} ds_y ds_x \right. \\ &+ 2 \iint_{E_{n-m-1}} \frac{(n_x \cdot n_y) \xi_h^i(y) \xi_h^j(x)}{(\Delta t)|x-y|4\pi} ds_y ds_x - \iint_{E_{n-m-2}} \frac{(n_x \cdot n_y) \xi_h^i(y) \xi_h^j(x)}{(\Delta t)|x-y|4\pi} ds_y ds_x \left. \right] \\ &+ \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \varphi_i^m \iint_{\Gamma \times \Gamma} \frac{(\text{curl}_\Gamma \xi_h^i)(y) \cdot (\text{curl}_\Gamma \xi_h^j)(x)}{4\pi|x-y|} \mathcal{Y}^{n-m} ds_y ds_x \\ &=: \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} W_{j,i}^{n-m} \varphi_i^m := \sum_{m=1}^{N_t} W^{n-m} \varphi^m, \end{aligned}$$

where [28]

$$\begin{aligned}
\mathcal{Y}^{n-m}(x, y) &= (2(\Delta t))^{-1}(|x-y|^2 - 2|x-y|(n-m+1)(\Delta t) + ((n-m+1)(\Delta t))^2)\chi_{E_{n-m}} \\
&+ (2(\Delta t))^{-1}(|x-y|^2 - 2|x-y|(n-m-2)(\Delta t) + ((n-m-2)(\Delta t))^2)\chi_{E_{n-m-2}} \\
&+ (2(\Delta t))^{-1}(-2|x-y|^2 + 2|x-y|((n-m-1)(\Delta t) + (n-m)(\Delta t)) \\
&- (((n-m-1)(\Delta t))^2 + ((n-m)(\Delta t))^2) + 2(\Delta t)^2)\chi_{E_{n-m-1}}.
\end{aligned}$$

Here, for $l \in \mathbb{N}_0$ we define the light cone $E_l = \{(x, y) \in \Gamma \times \Gamma : t_l \leq |x-y| \leq t_{l+1}\} \subset \Gamma \times \Gamma$, and $\chi_{E_l}(x, y) = 1$ if $(x, y) \in E_l$, and $= 0$ otherwise.

The matrix W^{n-m} is therefore a sum of integrals over the [three](#) light cones E_{n-m}, E_{n-m-1} and E_{n-m-2} .

We now consider the discretization of the single layer potential. For the ansatz function we choose (16) and as test function we choose (18).

After some computations, we obtain

$$\begin{aligned}
\langle V\lambda_{h,\Delta t}, \dot{m}_{h,\Delta t} \rangle_{\Gamma \times \mathbb{R}_+} &= \int_0^\infty \int_\Gamma V\lambda_{h,\Delta t}(x, t) \cdot \dot{m}_{h,\Delta t}(x, t) ds_x dt \\
&= \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \lambda_i^m \left[\iint_{E_{n-m}} \left(-(n-m+1) \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi|x-y|} + \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi(\Delta t)} \right) ds_y ds_x \right. \\
&+ \iint_{E_{n-m-1}} \left((2(n-m)-1) \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi|x-y|} - 2 \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi(\Delta t)} \right) ds_y ds_x \\
&+ \left. \iint_{E_{n-m-2}} \left(-(n-m-2) \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi|x-y|} + \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi(\Delta t)} \right) ds_y ds_x \right] \\
&=: \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} V_{j,i}^{n-m} \lambda_i^m =: \sum_{m=1}^{N_t} V^{n-m} \lambda^m.
\end{aligned}$$

We next consider the retarded adjoint double layer potential:

$$\begin{aligned}
\langle K^T \lambda_{h,\Delta t}, \dot{w}_{h,\Delta t} \rangle_{\Gamma \times \mathbb{R}_+} &= \int_0^\infty \int_\Gamma K^T \lambda_{h,\Delta t} \dot{w}_{h,\Delta t} ds_x dt \\
&= \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \lambda_i^m \iint_{\Gamma \times \Gamma} \frac{n_x \cdot (x-y)}{4\pi|x-y|^3} \xi_h^i(y) \xi_h^j(x) \mathcal{Y}^{n-m}(x,y) ds_y ds_x \\
&+ \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \lambda_i^m \left[\iint_{E_{n-m}} n_x \cdot (x-y) \left((n-m+1) \frac{\xi_h^i(y) \xi_h^j(x)}{4\pi|x-y|^2} - \frac{\xi_h^i(y) \xi_h^j(x)}{4\pi(\Delta t)|x-y|} \right) ds_y ds_x \right. \\
&+ \iint_{E_{n-m-1}} n_x \cdot (x-y) \left(-(2(n-m)-1) \frac{\xi_h^i(y) \xi_h^j(x)}{4\pi|x-y|^2} + 2 \frac{\xi_h^i(y) \xi_h^j(x)}{4\pi(\Delta t)|x-y|} \right) ds_y ds_x \\
&+ \left. \iint_{E_{n-m-2}} n_x \cdot (x-y) \left((n-m-2) \frac{\xi_h^i(y) \xi_h^j(x)}{4\pi|x-y|^2} - \frac{\xi_h^i(y) \xi_h^j(x)}{4\pi(\Delta t)|x-y|} \right) ds_y ds_x \right] \\
&= \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} (K^T)_{j,i}^{n-m} \lambda_i^m = \sum_{m=1}^{N_t} K^{T^{n-m}} \lambda^m .
\end{aligned}$$

The related term for the mass matrix is given by

$$\begin{aligned}
\langle \frac{1}{2} \lambda_{h,\Delta t}, \dot{w}_{h,\Delta t} \rangle_{\Gamma \times \mathbb{R}_+} &= \frac{1}{2} \int_0^\infty \int_\Gamma \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \lambda_i^m \beta_{\Delta t}^m(t) \xi_h^i(x) \gamma_{\Delta t}^n(t) \xi_h^j(x) ds_x dt \\
&= \frac{1}{2} \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \lambda_i^m \left(\int_\Gamma \xi_h^i(x) \xi_h^j(x) ds_x \right) \left(\int_0^\infty \beta_{\Delta t}^m(t) \gamma_{\Delta t}^n dt \right) \\
&= \frac{1}{2} \sum_{i=1}^{N_{s'}} \lambda_i^m \left(\int_\Gamma \xi_h^i(x) \xi_h^j(x) ds_x \right) \frac{(\Delta t)}{2} \begin{cases} \lambda_i^1 & , n = 1 \\ \lambda_i^n + \lambda_i^{n-1} & , n \geq 2 \end{cases} \\
&=: \frac{1}{2} \sum_{i=1}^{N_{s'}} I_{j,i} \lambda_i = \frac{1}{2} I \lambda_I ,
\end{aligned}$$

where

$$\lambda_I = \frac{(\Delta t)}{2} \begin{cases} \lambda^1 & , n = 1 \\ \lambda^n + \lambda^{n-1} & , n \geq 2 \end{cases} .$$

Furthermore

$$\begin{aligned}
\langle K\phi_{h,\Delta t}, \dot{m} \rangle_{\Gamma \times \mathbb{R}_+} &= \int_0^\infty \int_\Gamma K\phi_{h,\Delta t} \dot{m}_{h,\Delta t} ds_x dt \\
&= \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \varphi_i^m \left[\iint_{E_{n-m}} n_y \cdot (x-y) \left(-(n-m+1) \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi|x-y|^3} + \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi(\Delta t)|x-y|^2} \right) ds_y ds_x \right. \\
&+ \iint_{E_{n-m-1}} n_y \cdot (x-y) \left((2(n-m)-1) \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi|x-y|^3} - 2 \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi(\Delta t)|x-y|^2} \right) ds_y ds_x \\
&+ \left. \iint_{E_{n-m-2}} n_y \cdot (x-y) \left(-(n-m-2) \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi|x-y|^3} + \frac{\xi_h^i(y)\xi_h^j(x)}{4\pi(\Delta t)|x-y|^2} \right) ds_y ds_x \right] \\
&+ \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \varphi_i^m \left[\iint_{E_{n-m}} \frac{-n_y \cdot (x-y)}{4\pi(\Delta t)|x-y|^2} \xi_h^i(y)\xi_h^j(x) ds_y ds_x + \iint_{E_{n-m-1}} \frac{2n_y \cdot (x-y)}{4\pi(\Delta t)|x-y|^2} \xi_h^i(y)\xi_h^j(x) ds_y ds_x \right. \\
&+ \left. \iint_{E_{n-m-2}} -\frac{n_y \cdot (x-y)}{4\pi(\Delta t)|x-y|^2} \xi_h^i(y)\xi_h^j(x) ds_y ds_x \right] = \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} K_{j,i}^{n-m} \varphi_i^m = \sum_{m=1}^{N_t} K^{n-m} \varphi^m
\end{aligned}$$

and

$$\begin{aligned}
\langle \frac{1}{2}\phi_{h,\Delta t}, \dot{m}_{h,\Delta t} \rangle_{\Gamma \times \mathbb{R}_+} &= \frac{1}{2} \int_0^\infty \int_\Gamma \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \varphi_i^m \beta_{\Delta t}^m(t) \xi_h^i(x) \dot{\gamma}_{\Delta t}^n(t) \xi_h^j(x) ds_x dt \\
&= \frac{1}{2} \sum_{i=1}^{N_{s'}} \varphi_i^m \left(\int_\Gamma \xi_h^i(x) \xi_h^j(x) ds_x \right) \left(\int_0^\infty \beta_{\Delta t}^m(t) \dot{\gamma}_{\Delta t}^n dt \right) \\
&= \frac{1}{2} \sum_{m=1}^{N_t} \sum_{i=1}^{N_{s'}} \varphi_i^m \left(\int_\Gamma \xi_h^i(x) \xi_h^j(x) ds_x \right) \begin{cases} -\varphi_i^1 & , n=1 \\ -(\varphi_i^n - \varphi_i^{n-1}) & , n \geq 2 \end{cases} \\
&= \frac{1}{2} \sum_{i=1}^{N_{s'}} I_{j,i} \varphi_I = \frac{1}{2} I \varphi_I ,
\end{aligned}$$

with

$$\varphi_I = \begin{cases} -\varphi^1 & , n=1 \\ -\varphi^n + \varphi^{n-1} & , n \geq 2 \end{cases} .$$

It remains to consider the coupling contributions. For $j = 1, \dots, N_{s'}$

$$\begin{aligned}
\langle \dot{\mathbf{u}}_{h,\Delta t} | \Gamma \cdot n, \dot{w}_{h,\Delta t} \rangle_{\Gamma \times \mathbb{R}_+} &= \int_0^\infty \int_\Gamma \dot{\mathbf{u}}_{h,\Delta t} | \Gamma \cdot n_x \dot{w}_{h,\Delta t} ds_x dt \\
&= \sum_{\nu=1}^3 \sum_{i=1}^{N_o} \left(\int_\Gamma \eta_h^i | \Gamma(x) \tilde{e}_\nu \cdot n_x \xi_h^j(x) ds_x \right) \begin{cases} u_{\nu,i}^1 & , n=1 \\ u_{\nu,i}^n - u_{\nu,i}^{n-1} & , n \geq 2 \end{cases} \\
&=: \sum_{\nu=1}^3 \sum_{i=1}^{N_o} (RIn_x)_{(i,\nu),j} u_T = RIn_x u_T ,
\end{aligned}$$

with

$$u_T = \begin{cases} u_\Gamma^1 & , n = 1 \\ u_\Gamma^n - u_\Gamma^{n-1} & , n \geq 2 \end{cases} .$$

For the second coupling term:

$$\begin{aligned} \langle \dot{\phi}_{h,\Delta t} \cdot n, \dot{\mathbf{w}}_{h,\Delta t}|_\Gamma \rangle_{\Gamma \times \mathbb{R}_+} &= \int_0^\infty \int_\Gamma \dot{\phi}_{h,\Delta t} \cdot n_x \dot{\mathbf{w}}_{h,\Delta t}|_\Gamma ds_x dt \\ &= \sum_{i=1}^{N_{s'}} \left(\int_\Gamma \xi_h^i(x) n_x \cdot \eta_h^j|_\Gamma(x) \bar{e}_\mu ds_x \right) \begin{cases} \varphi_i^1 & , n = 1 \\ \varphi_i^n - \varphi_i^{n-1} & , n \geq 2 \end{cases} \\ &=: \sum_{i=1}^{N_{s'}} (n_x RI)_{i,(j,\eta)} \varphi_T = n_x RI \varphi_T \end{aligned}$$

with

$$\varphi_T = \begin{cases} \varphi^1 & , n = 1 \\ \varphi^n - \varphi^{n-1} & , n \geq 2 \end{cases} .$$

For completeness we mention the right hand side: Set $h = \dot{v}^{inc} n$ and $g = \partial_n^+ v^{inc}$. We approximate the time integral by the trapezoidal rule, so that:

$$\begin{aligned} -\langle \dot{v}^{inc} n, \dot{\mathbf{w}}|_\Gamma \rangle_{\Gamma \times \mathbb{R}_+} &= -\langle h, \dot{\mathbf{w}}|_\Gamma \rangle_{\Gamma \times \mathbb{R}_+} = -\frac{(\Delta t)}{2} \int_\Gamma (h^n + h^{n-1}) \eta_h^j(x) e_\mu ds_x =: H^n + H^{n-1} \\ \langle \partial_n^+ v^{inc}, \dot{w} \rangle_\Gamma &= \langle g, \dot{w} \rangle_\Gamma = \frac{(\Delta t)}{2} \int_\Gamma (g^n + g^{n-1}) \xi_h^j(x) ds_x =: G^n + G^{n-1} , \end{aligned}$$

where $h^n = h(x, t_n)$ and $g^n = g(x, t_n)$.

Defining

$$u_A = (\Delta t) \cdot \begin{cases} \frac{1}{2} u^1 & , n = 1 \\ \frac{u^n + u^{n-1}}{2} & , n \geq 2 \end{cases} \quad \text{and} \quad u_M = \frac{1}{(\Delta t)} \cdot \begin{cases} u^1 & , n = 1 \\ u^2 - 2u^1 & , n = 2 \\ u^n - 2u^{n-1} + u^{n-2} & , n \geq 3 \end{cases} ,$$

the resulting system of equations therefore becomes

$$\begin{aligned} Au_A + Mu_M - RIn_x u_T + n_x RI \varphi_T - \sum_{m=1}^{N_t} W^{n-m} \varphi^m + \sum_{m=1}^{N_t} K^{T^{n-m}} \lambda^m - \frac{1}{2} I \lambda_I \\ + \frac{1}{2} I \varphi_I - \sum_{m=1}^{N_t} K^{n-m} \varphi^m + \sum_{m=1}^{N_t} V^{n-m} \lambda^m = H^n + H^{n-1} + G^n + G^{n-1} . \end{aligned} \quad (\text{B.1})$$

The matrices W^k, K^k, K^{T^k}, V^k vanish if the index k is negative. Therefore we get for (B.1) in the first time step ($n = 1$):

$$\begin{pmatrix} \frac{(\Delta t)}{2} A + \frac{1}{(\Delta t)} M & [0, n_x RI]^T & 0 \\ [0, -RIn_x] & -W^0 & K^{T^0} - \frac{1}{2} \frac{(\Delta t)}{2} I \\ 0 & -K^0 - \frac{1}{2} I & V^0 \end{pmatrix} \begin{pmatrix} u^1 \\ \varphi^1 \\ \lambda^1 \end{pmatrix} = \begin{pmatrix} H^1 + H^0 \\ G^1 + G^0 \\ 0 \end{pmatrix} .$$

Note the zero block in $[0, -RI n_x]$, as $\mathbf{u}_{h,\Delta t}|_\Gamma$ only depends on the values in the nodes on the boundary Γ . Similarly, one obtains a zero block in $[0, n_x RI]^T$, corresponding to the vanishing contribution of nodes in the interior of Ω to the trace of the test function $\mathbf{w}_{h,\Delta t}|_\Gamma$.

For the second time step ($n = 2$) we obtain

$$\begin{pmatrix} \frac{(\Delta t)}{2}A + \frac{1}{(\Delta t)}M & [0, n_x RI]^T & 0 \\ [0, -RI n_x] & -W^0 & K^{T0} - \frac{(\Delta t)}{4}I \\ 0 & -K^0 - \frac{1}{2}I & V^0 \end{pmatrix} \begin{pmatrix} u^2 \\ \varphi^2 \\ \lambda^2 \end{pmatrix} = \begin{pmatrix} H^2 + H^1 - A \frac{(\Delta t)}{2}u^1 + M \frac{2}{(\Delta t)}u^1 + n_x RI \varphi^1 \\ G^2 + G^1 + RI n_x u_\Gamma^1 + W \varphi^1 + K^{T1} \lambda^1 + \frac{(\Delta t)}{4}I \lambda^1 \\ K^1 \varphi^1 - \frac{1}{2}I \varphi^1 - V^1 \lambda^1 \end{pmatrix},$$

using u^1, φ^1 and λ^1 from above. For later time steps $n \geq 3$ we conclude:

$$\begin{pmatrix} \frac{(\Delta t)}{2}A + \frac{1}{(\Delta t)}M & [0, n_x RI]^T & 0 \\ [0, -RI n_x] & -W^0 & K^{T0} - \frac{1}{2} \frac{(\Delta t)}{2}I \\ 0 & -K^0 - \frac{1}{2}I & V^0 \end{pmatrix} \begin{pmatrix} u^n \\ \varphi^n \\ \lambda^n \end{pmatrix} \\ = \begin{pmatrix} H^n + H^{n-1} - A \frac{(\Delta t)}{2}u^{n-1} + M \frac{2}{(\Delta t)}u^{n-1} - M \frac{1}{(\Delta t)}u^{n-2} + n_x RI \varphi^{n-1} \\ G^n + G^{n-1} + RI n_x u_\Gamma^{n-1} + \sum_{m=1}^{n-1} W^{n-m} \varphi^m - \sum_{m=1}^{n-1} K^{T^{n-m}} \lambda^m + \frac{1}{2} \frac{(\Delta t)}{2}I \lambda^{n-1} \\ \sum_{m=1}^{n-1} K^{n-m} \varphi^m - \frac{1}{2}I \varphi^{n-1} - \sum_{m=1}^{n-1} V^{n-m} \lambda^m \end{pmatrix}.$$

This system is solved repeatedly until reaching time step $N_t \geq 3$.

References

- [1] T. Abboud, P. Joly, J. Rodriguez, I. Terrasse, *Coupling discontinuous Galerkin methods and retarded potentials for transient wave propagation on unbounded domains*, J. Comp. Phys. **230** (2011), 5877–5907.
- [2] A. Aimi, M. Diligenti, C. Guardasoni, I. Mazzieri, S. Panizzi, *An energy approach to space-time Galerkin BEM for wave propagation problems*, Internat. J. Numer. Methods Engrg. **80** (2009), 1196–1240.
- [3] A. Aimi, M. Diligenti, A. Frangi, C. Guardasoni, *Energetic BEM-FEM coupling for wave propagation in 3D multidomains*, Internat. J. Numer. Methods Engrg. **97** (2014), 377–394.
- [4] A. Aimi, M. Diligenti, C. Guardasoni, S. Panizzi, *Energetic BEM-FEM coupling for wave propagation in layered media*, Commun. Appl. Ind. Math. **3** (2012), 418–438.
- [5] A. Bamberger, T. Ha Duong, *Formulation variationnelle espace-temps pour le calcul par potentiel retarde d'une onde acoustique*, Math. Meth. Appl. Sci. **8** (1986), 405–435 and 598–608.
- [6] L. Banjai, C. Lubich, F.-J. Sayas, *Stable numerical coupling of exterior and interior problems for the wave equation*, Numer. Math. **129** (2015), 611–646.

- [7] L. Banz, H. Gimperlein, Z. Nezhi, E. P. Stephan, *Time domain BEM for sound radiation of tires*, Computational Mechanics 58 (2016), 45–57.
- [8] J. Bielak, R. C. MacCamy, X. Zeng, *Stable coupling method for interface scattering problems by combined integral equations and finite elements*, J. Comput. Phys. 119 (1995), 374–384.
- [9] M. Costabel, F.-J. Sayas, *Time-dependent problems with the boundary integral equation method*. In Encyclopedia of Computational Mechanics (second edition), E. Stein, R. de Borst, and J. R. Hughes (editors), John Wiley & Sons, Chichester, 2017, pp. 1–24.
- [10] C. Dominguez, E. P. Stephan, M. Maischak, *FE/BE coupling for an acoustic fluid-structure interaction problem. Residual a posteriori error estimates*, Int. J. Numer. Meth. Engng. 89 (2012), 299–322.
- [11] M. Filipe, *Etude mathématique et numérique d’un problème d’interaction fluide – structure dépendant du temps par la méthode de couplage éléments finis – équations intégrales*, PhD thesis, Ecole Polytechnique, 1994.
- [12] Y. Gao, P. Li and B. Zhang, *Analysis of transient acoustic-elastic interaction in an unbounded structure*, SIAM J. Math. Anal. 49 (2017), 3951–3972.
- [13] H. Gimperlein, Z. Nezhi and E. P. Stephan, *A priori error estimates for a time-dependent boundary element method for the acoustic wave equation in a half-space*, Math. Methods Appl. Sci. 40 (2017), 448–462.
- [14] H. Gimperlein, M. Maischak and E. P. Stephan, *Adaptive time domain boundary element methods and engineering applications*, J. Integral Equations Appl. 29 (2017), 75–105.
- [15] H. Gimperlein, F. Meyer, C. Özdemir and E. P. Stephan, *Time domain boundary elements for dynamic contact problems*, Computer Methods in Applied Mechanics and Engineering 333 (2018), 147–175.
- [16] H. Gimperlein, F. Meyer, C. Özdemir, D. Stark and E. P. Stephan, *Boundary elements with mesh refinements for the wave equation*, Numer. Math. 139 (2018), 867–912.
- [17] H. Gimperlein, C. Özdemir and E. P. Stephan, *Time domain boundary element methods for the Neumann problem and sound radiation of tires: Error estimates and acoustic problems*, J. Comp. Mathematics, 36 (2018), 70–89.
- [18] H. Gimperlein, C. Özdemir, D. Stark and E. P. Stephan, *hp-version time domain boundary elements for the wave equation on quasi-uniform meshes*, Computer Methods in Applied Mechanics and Engineering 356 (2019), 145–174.

- [19] M. Gläfke. *Adaptive Methods for Time Domain Boundary Integral Equations*. Ph.D. thesis, Brunel University, London, 2012.
- [20] J. Gwinner and E. P. Stephan, *Advanced boundary element methods: Treatment of boundary value, transmission and contact problems*, volume 52 of *Springer Series in Computational Mathematics*, Springer, 2018.
- [21] T. Ha-Duong, On retarded potential boundary integral equations and their discretisation. In *Topics in computational wave propagation*, volume 31 of *Lect. Notes Comput. Sci. Eng.*, pages 301–336. Springer, Berlin, 2003.
- [22] G. D. Hatzigeorgiou, D. E. Beskos, *Dynamic inelastic structural analysis by the BEM: A review*, *Engineering Analysis with Boundary Elements* **35** (2011), 159–169.
- [23] G. C. Hsiao, T. Sánchez-Vizuet and F.-J. Sayas, *Boundary and coupled boundary-finite element methods for transient wave-structure interaction*, *IMA J. Numer. Anal.* **37** (2017), 237–265.
- [24] G. C. Hsiao, F.-J. Sayas and R. J. Weinacht, *Time-dependent fluid-structure interaction*, *Math. Methods Appl. Sci.* **40** (2017), 486–500.
- [25] G. C. Hsiao, T. Sánchez-Vizuet, F.-J. Sayas and R. J. Weinacht, *A time-dependent wave-thermoelastic solid interaction*, *IMA J. Numer. Anal.* **39** (2019), 924–9565.
- [26] P. Joly, J. Rodriguez, *Mathematical aspects of variational boundary integral equations for time dependent wave propagation*, *J. Integral Equations Appl.* **29** (2017), 137–187.
- [27] J. M. Melenk, D. Praetorius, B. Wohlmuth, *Simultaneous quasi-optimal convergence rates in FEM-BEM coupling*, *Math. Meth. Appl. Sci.* **40** (2017), 463–485.
- [28] C. Özdemir, *Finite elements boundary elements – coupling in time domain*. Ph.D. thesis, Leibniz University Hannover, 2019.
- [29] T. Sanchez-Vizuet, F.-J. Sayas, *Symmetric boundary-finite element discretization of time dependent acoustic scattering by elastic obstacles with piezoelectric behavior*, *J. Sci. Comput.* **70** (2017), 1290–1315.
- [30] F.-J. Sayas, *Retarded potentials and time domain boundary integral equations: A road map*, volume 50 of *Springer Series in Computational Mathematics*. Springer, 2016.
- [31] E. P. Stephan, Coupling of boundary element methods and finite element methods, *Encyclopedia of Computational Mechanics, Fundamentals*, E. Stein, R. de Borst, T. J. R. Hughes (eds.), vol. I : 375–412, 2004.
- [32] I. Terrasse, *Résolution mathématique et numérique des équations de Maxwell instationnaires par une méthode de potentiels retardés*, Ph.D. thesis, École Polytechnique, Palaiseau, 1993.