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ALMOST SURE GLOBAL WELL POSEDNESS FOR THE BBM EQUATION WITH INFINITE L^2 INITIAL DATA

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Abstract. We consider the probabilistic Cauchy problem for the Benjamin-Bona-Mahony equation (BBM) on the one-dimensional torus $\mathbb{T}$ with initial data below $L^2(\mathbb{T})$. With respect to random initial data of strictly negative Sobolev regularity, we prove that BBM is almost surely globally well-posed. The argument employs the I-method to obtain an a priori bound on the growth of the 'residual' part of the solution. We then discuss the stability properties of the solution map in the deterministically ill-posed regime.

Contents

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1. Introduction

We consider the Benjamin-Bona-Mahony equation (BBM):

$$\begin{cases} \partial_t u - \partial_{xxt} u + \partial_x u + \frac{1}{2}\partial_x(u^2) = 0 \\ u|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R}_+, \tag{1.1}$$

where $u : \mathbb{T} \times \mathbb{R}_+ \mapsto \mathbb{R}$ is the unknown function and $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ is the one-dimensional torus[1].

The BBM equation is a model for the propagation of long wavelength, short amplitude water waves [43, 2]. In particular, in [2], it was proposed as an alternative to the Korteweg-de Vries (KdV) equation. This is in part due to the boundedness of the dispersion relation for BBM while the dispersion relation for KdV is unbounded. Along with its more preferable analytical qualities, BBM is also known as the regularised long wave equation. For further discussion on the physical validity of the BBM model, see for example [1, 9, 7].

Our goal in this paper is to study the well-posedness of BBM (1.1) in the low regularity setting. We begin by putting (1.1) into an alternative form which is more amenable for this study. By factorising the time derivative, we rewrite (1.1) into the equivalent form:

$$\partial_t u = -(1-\partial_x^2)^{-1}\partial_x\left(u + \frac{1}{2}u^2\right). \tag{1.2}$$

With $D_x := -i\partial_x$, let $\varphi(D_x)$ be the Fourier multiplier operator with symbol $\varphi(n) := \frac{n}{1+n^2}$; that is,

$$\widehat{\varphi(D_x)f} = \varphi(n)\widehat{f}(n)$$

for every $n \in \mathbb{Z}$, where $\widehat{f}$ denotes the Fourier transform on $\mathbb{T}$ of f. Then, (1.2) reads as

$$i\partial_t u = \varphi(D_x)u + \frac{1}{2}\mathcal{N}(u). \tag{1.3}$$

Here, we specifically interpret the nonlinearity as

$$\mathcal{N}(u) := \sum_{n\neq 0} \varphi(n)e^{inx} \sum_{\substack{n_1,n_2\in\mathbb{Z} \\ n=n_1+n_2}} \widehat{u}(n_1)\widehat{u}(n_2). \tag{1.4}$$

The key point here is the *explicit* absence of the zero frequency[2]. Note that if $u \in L^2(\mathbb{T})$, $\mathcal{N}(u) = \varphi(D_x)(u^2)$ and hence (1.3) is equivalent to (1.1); see Remark 1.3. We thus consider (1.3) as the natural version of (1.1) to study below $L^2(\mathbb{T})$, and we will refer to (1.3) as the BBM equation unless otherwise stated. We have the following integral (Duhamel) formulation of (1.3):

$$u(t) = S(t)u_0 - \frac{i}{2}\int_0^t S(t-t')\mathcal{N}(u(t'))dt', \tag{1.5}$$

where $S(t) := e^{-it\varphi(D_x)}$ is the linear BBM propagator. We stress that solutions to (1.1) and (1.3) are real-valued; the presence of i in the above formulas is a side-effect of writing the multiplier $\varphi(D_x)$. We say that u is a solution to (1.3) if it satisfies the Duhamel formulation (1.5).

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[1] We could also consider the BBM equation (1.1) on $\mathbb{T}\times\mathbb{R}$ because of the time-reversal symmetry $u(x,t) \mapsto u(-x,-t)$ (viewing $\mathbb{T}$ as $[-\pi,\pi)$). However, for simplicity, we consider only positive times in the following.

[2] Notice that $\varphi(0) = 0$.

The well-posedness of the Cauchy problem for BBM (1.1) on $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$ has been well studied within the class of L^2-based Sobolev spaces $H^s(\mathcal{M})$. Benjamin, Bona and Mahony [2] obtained global well-posedness of (1.1) in $H^k(\mathbb{R})$ for all integers $k \geq 1$. When $k = 1$, globalisation of local-in-time solutions follows immediately from the conserved energy

$$E(u(t)) := \frac{1}{2}\int_{\mathcal{M}} u^2(t) + (\partial_x u(t))^2\, dx = \frac{1}{2}\|u(t)\|^2_{H^1(\mathcal{M})}. \tag{1.6}$$

Namely, if $u(t) \in H^1(\mathcal{M})$ satisfies (1.5) for all $t \in [0, T]$, then for any $t \in [0, T]$, we have

$$E(u(t)) = E(u(0)) = E(u_0). \tag{1.7}$$

Local well-posedness in $L^2(\mathbb{R})$ was obtained by Bona, Chen and Saut [5, 6]. In [10], Bona and Tzvetkov proved BBM (1.1) is globally well-posed in $H^s(\mathbb{R})$ for all $s \geq 0$. Adapting the arguments in [10], Roumégoux [44] extended this result to the periodic setting. On the other hand, the results of Panthee [42] and Bona and Dai [8] showed that the BBM equation (1.1) is ill-posed in negative Sobolev spaces. For further discussion on the ill-posedness of (1.1) in negative Sobolev spaces, we refer to Subsection 1.3 and Theorem 1.7 below. In this paper, we study the well-posedness of (1.3) below $L^2(\mathbb{T})$ with random initial data.

1.1. **Almost sure local well-posedness.** Recall that *well-posedness* in the sense of Hadamard corresponds to (i) existence of a solution, (ii) uniqueness of the solution (in some suitable sense) and (iii) continuous dependence with respect to initial data. The ill-posedness results for (1.1) below $L^2(\mathbb{T})$ that we mentioned above are all based on contradicting (iii). More precisely, they show the solution map $\Phi : u_0 \in H^s(\mathcal{M}) \mapsto u \in C([0, T]; H^s(\mathcal{M}))$ for BBM (1.1) is discontinuous when $s < 0$; see Theorem 1.7 and Corollary 1.8. Namely, they construct a smooth sequence $u_{0,n} \to 0$ in $H^s(\mathcal{M})$, such that the smooth solutions $\Phi(u_n)$ to (1.1) fail to converge to zero in $C([0, T_n]; H^s(\mathcal{M}))$. In particular, these same ill-posedness results also hold for (1.3). Note however that this does not preclude the possible existence (and even uniqueness) of solutions within the ill-posed regime. This leads us to the following question: can we still construct solutions in the ill-posed regime and if so, in what sense do we retain (iii), the continuity of the solution map?

Our goal in this paper is to address this question within the context of BBM (1.3) with random initial data below $L^2(\mathbb{T})$. Namely, we consider randomised initial data of the form[3]

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx}, \tag{1.8}$$

where $\alpha \in \mathbb{R}$, $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the reality condition $g_n = \overline{g_{-n}}$ and g_0 is real. A computation shows[4]

$$u_0^\omega \in H^{\alpha-\frac{1}{2}-}(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T}) \tag{1.9}$$

almost surely; see (1.11) below. Thus, in view of the global well-posedness of BBM (1.3) in $L^2(\mathbb{T})$ and above, we concentrate on when $\alpha \leq \frac{1}{2}$. Our first result is almost sure local

[3]We drop the factor of 2π as it plays no role in our analysis.

[4]Here, we use the notation $a-$ (respectively, $a+$) to denote $a - \varepsilon$ (respectively, $a + \varepsilon$), where $0 < \varepsilon \ll 1$ is extremely small.

well-posedness for the BBM equation (1.3) with respect to the random initial data (1.8) for $\alpha \in (\frac{1}{4}, \frac{1}{2}]$.

Theorem 1.1. *Let $\alpha \in (\frac{1}{4}, \frac{1}{2}]$ and $s \in (\frac{1}{2} - \alpha, 2\alpha)$. Then, the BBM equation (1.3) is locally well-posed almost surely with respect to the random initial data (1.8). More precisely, there exists a set $\Sigma \subset \Omega$ with $\mathbb{P}(\Sigma) = 1$, such that for every $\omega \in \Sigma$, there exist $T^\omega > 0$ and a unique solution u to (1.3) in*

$$e^{-it\varphi(D_x)}u_0^\omega + C([0, T^\omega]; H^s(\mathbb{T})) \subset C([0, T^\omega]; H^{\alpha - \frac{1}{2}-}(\mathbb{T}))$$

with initial condition u_0^ω of the form (1.8).

Uniqueness in the above statement refers to uniqueness within the space of functions $u \in C([0, T]; H^{\alpha - \frac{1}{2}-}(\mathbb{T}))$ which can be written as

$$\begin{aligned} u = e^{-it\varphi(D_x)}u_0^\omega + v \in C([0, T]; H^{\alpha - \frac{1}{2}-}(\mathbb{T})) + C([0, T]; H^s(\mathbb{T})) \\ \subset C([0, T]; H^{\alpha - \frac{1}{2}-}(\mathbb{T})). \end{aligned}$$

See also Remark 1.2.

In the proof of Theorem 1.1, we obtain solutions u of the form

$$u(t) = z^\omega(t) + v(t),$$

where $z^\omega(t) = S(t)u_0^\omega$ is the random linear solution and the remainder $v := u - z$ is almost surely smoother than z. In particular, v belongs to $H^s(\mathbb{T})$ for $s < 2\alpha$ and we construct v by a contraction mapping argument for the following *perturbed* BBM equation:

$$\begin{cases} i\partial_t v = \varphi(D_x)(v) + \frac{1}{2}(v^2 + 2zv) + \frac{1}{2}\mathcal{N}(z), \\ v|_{t=0} = 0. \end{cases} \tag{1.10}$$

Due to this expectation of additional smoothness for v, the Sobolev multiplication law will allow us to make sense of the product zv. However, as $z \notin L^2(\mathbb{T})$ almost surely, it is essential that we interpret the forcing term $\frac{1}{2}\mathcal{N}(z)$ in the sense of (1.4); indeed, see Remark 1.3. In studying the fixed point problem for v corresponding to (1.10) we crucially make use of the fact that while the random initial data (1.8) is no more regular in space as compared to the (deterministic) function

$$\sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^\alpha} e^{inx} \in H^{\alpha - \frac{1}{2}-}(\mathbb{T}),$$

it does benefit from a gain of integrability. More precisely, we have $u_0^\omega \in W^{\alpha - \frac{1}{2}-, \infty}(\mathbb{T})$ almost surely. Indeed, for any $2 \le p < \infty$, we have[5]

$$\begin{aligned} \mathbb{E}[\|u_0^\omega\|^p_{W^{\alpha - \frac{1}{2}-, p}}] &= \bigg\| \bigg\| \sum_n \frac{\langle n \rangle^{\alpha - \frac{1}{2}-} e^{inx}}{\langle n \rangle^\alpha} g_n(\omega) \bigg\|^p_{L^p(\Omega)} \bigg\|_{L^p_x(\mathbb{T})} \\ &\lesssim \sqrt{p} \| \| \langle n \rangle^{-\frac{1}{2}-} \|_{\ell^2_n} \|_{L^p_x(\mathbb{T})} < \infty. \end{aligned} \tag{1.11}$$

[5]Here we use that if $\{a_n\}_{n \in \mathbb{Z}} \in \ell^2_n(\mathbb{Z})$, then $\sum_{n \in \mathbb{Z}} a_n g_n(\omega)$ is a mean-zero complex-valued Gaussian random variable with variance $\|a_n\|^2_{\ell^2_n}$ and hence satisfies

$$\bigg\| \sum_{n \in \mathbb{Z}} a_n g_n(\omega) \bigg\|_{L^p(\Omega)} \sim \sqrt{p} \|a_n\|_{\ell^2_n},$$

for any $2 \le p < \infty$. See also Lemma 2.4.

For the endpoint $p = \infty$, we first apply the Sobolev inequality to reduce to some large but finite spatial integrability exponent and then apply Minkowski's integral inequality. In the dispersive PDE with random data literature, a perturbative expansion of the form (1.15) goes back to the works of McKean [36] and Bourgain [12] and is known as the Da Prato-Debussche trick in the context of stochastic PDEs, after [20].

Initial data of the form (1.8) correspond to typical elements belonging to the support of the infinite-dimensional Gaussian measure μ_α which formally has density

$$d\mu_\alpha = Z_\alpha^{-1} e^{-\frac{1}{2}\|u\|_{H^\alpha(\mathbb{T})}^2} du. \tag{1.12}$$

Here, du is the (non-existent) infinite-dimensional Lebesgue measure. More rigorously, given $\alpha \in \mathbb{R}$, the Gaussian measure μ_α is the induced probability measure under the map

$$\omega \in \Omega \longmapsto u^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx}.$$

From (1.9), we see that μ_α is supported on $H^{\alpha-\frac{1}{2}-}(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$. Using this perspective, we may rephrase Theorem 1.1 as almost sure local well-posedness of BBM (1.3) with respect to the Gaussian measure μ_α supported on $H^{\alpha-\frac{1}{2}-}(\mathbb{T})$ for any $\alpha > \frac{1}{4}$.

Beginning with the work of Bourgain [11, 12] on the periodic nonlinear Schrödinger equation (NLS) and Burq and Tzvetkov [13, 14] on nonlinear wave equations (NLW), there has been an intense interest on constructing solutions to nonlinear dispersive PDE in the ill-posed regime using randomised initial data. As the literature on nonlinear dispersive PDE with random initial data is by now quite vast, we will focus on those of immediate relevance for our study of the BBM equation (1.1). In studying invariance properties of certain weighted Gaussian measures (more specifically, Gibbs measures) for the cubic NLS on $\mathbb{T}^2$, Bourgain [12] first needed to construct a well-defined flow emanating from (a two dimensional version of) the initial data (1.8) with $\alpha = 1$. In [19], Colliander and Oh proved the cubic NLS on $\mathbb{T}$ is locally and globally well-posed almost surely with respect to random initial data of the form (1.8) for $\alpha > \frac{1}{6}$ and $\alpha > \frac{5}{12}$, respectively. We refer the reader to the survey paper [4] for further details on nonlinear dispersive PDE with random data.

For the context of BBM (1.1), the transport properties of Gaussian measures under the nonlinear flow of (1.1) have been well-studied [21, 22, 23, 48]. As BBM is a Hamiltonian PDE with Hamiltonian given by the energy (1.6), it has a naturally associated Gibbs measure

$$d\mu_1 = Z_1^{-1} e^{-E(u)} du.$$

We thus expect μ_1 to be invariant under the (nonlinear) flow of (1.1) and this was proved by de Suzzoni [22]. For the Gaussian measures μ_α with $\alpha \neq 1$, we no longer expect invariance. However, Tzvetkov [48] proved that the push-forward of the Gaussian measures μ_α under the BBM flow for integer $\alpha \geq 2$ are quasi-invariant (i.e. mutually absolutely continuous) with respect to μ_α. In view of the global well-posedness of the BBM equation (1.1) in $L^2(\mathbb{T})$, it would be of interest to study if the quasi-invariance of Gaussian measures persists for all $\alpha > \frac{1}{2}$. For $\alpha \leq \frac{1}{2}$, a first step is the probabilistic well-posedness of (1.3) below $L^2(\mathbb{T})$ with initial data of the form (1.8). In this paper, we establish local well-posedness (Theorem 1.1) and global well-posedness (Theorem 1.5) of BBM (1.3) below $L^2(\mathbb{T})$ with initial data of the form (1.8). We discuss our global well-posedness result in the next subsection.

We conclude this subsection with a few remarks.

Remark 1.2. The proof of Theorem 1.1 decomposes the ill-posed solution map $\Phi : u_0^\omega \in H^{\alpha-\frac{1}{2}-}(\mathbb{T}) \mapsto u \in C([-T,T]; H^{\alpha-\frac{1}{2}-}(\mathbb{T}))$ for (1.3) with data u_0^ω given by (1.8) into the following sequence of maps:

$$u_0^\omega \overset{\text{(I)}}{\longmapsto} (z^\omega, Z^\omega) \overset{\text{(II)}}{\longmapsto} v \in C([0,T]; H^s(\mathbb{T})) \overset{\text{(III)}}{\longmapsto} u = z + v \in C([0,T]; H^{\alpha-\frac{1}{2}-}(\mathbb{T})),$$

where $z^\omega = S(t)u_0^\omega$ and $Z^\omega := \mathcal{N}(z^\omega)$. Step (I) uses tools from stochastic analysis in order to construct the enhanced data set (z^ω, Z^ω). This is the result of Proposition 2.6. Step (II) is a deterministic fixed point argument for (1.10) which views (z^ω, Z^ω) as a given data set. In particular, this step implies the continuity of the map:

$$\begin{aligned} \Psi : (z^\omega, Z^\omega) \in C([0,T]; W^{\alpha-\frac{1}{2}-,\infty}(\mathbb{T})) \times C([0,T]; H^s(\mathbb{T})) \\ \longmapsto v^\omega \in C([0,T]; H^s(\mathbb{T})), \end{aligned} \tag{1.13}$$

where v solves the perturbed BBM equation (1.10). Finally, step (III) recovers u through the expansion $u = z + v$. Similar decompositions of this type for ill-posed solution maps appear prominently in the theories of stochastic PDEs [32, 31] and rough paths [25].

Remark 1.3. In this remark, we discuss the necessity for the interpretation of the non-linearity $\varphi(D_x)(u^2)$ as $\mathcal{N}(u)$ in (1.4). Let $\rho \in C(\mathbb{R}; [0,1])$ with $\operatorname{supp} \rho \subset \big(-\frac{1}{2}, \frac{1}{2}\big]$ be such that $\int_{\mathbb{R}} \rho \, dx = 1$ and we set $\rho_k(x) = k\rho(kx)$ for $k \in \mathbb{N}$. As $k \geq 1$, we see that $\{\rho_k\}_{k\in\mathbb{N}}$ is an approximate identity on $\mathbb{T}$. To motivate (1.4), we consider the following smoothed version of (1.3):

$$\begin{cases} i\partial_t u_k = \varphi(D_x)(u_k) + \frac{1}{2}\varphi(D_x)(u_k^2), \\ u_k|_{t=0} = u_0^\omega * \rho_k. \end{cases} \tag{1.14}$$

Given $k \in \mathbb{N}$, the global theory in $L^2(\mathbb{T})$ for (1.3) and a persistence-of-regularity argument shows the solution u_k to (1.14) exists globally in time and is smooth. Now, let $z_k(t) = S(t)(u_0^\omega * \rho_k)$ be the random linear solution to (1.14) and consider an expansion of u_k about z_k by setting

$$v_k := u_k - z_k, \qquad \text{so} \qquad u_k = z_k + v_k. \tag{1.15}$$

Then, v_k solves the perturbed BBM equation

$$i\partial_t v_k = \varphi(D_x)(v_k) + \frac{1}{2}\varphi(D_x)(v_k^2 + 2v_k z_k) + \frac{1}{2}\varphi(D_x)(z_k^2), \tag{1.16}$$

with $v_k|_{t=0} = 0$. The problematic term here is z_k^2. More specifically, the zero frequency mode (equivalently, the mean) of z_k^2 behaves like

$$\begin{aligned} C_k :&= \mathbb{E}\big[\mathbf{P}_0(z_k^2(x,t))\big] \\ &= \sum_{0=n_1+n_2} \frac{\widehat{\rho}(k^{-1}n_1)\widehat{\rho}(k^{-1}n_2)e^{-it\varphi(n_1)}e^{-it\varphi(n_2)}}{\langle n_1\rangle^\alpha \langle n_2\rangle^\alpha} \mathbb{E}[g_{n_1} g_{n_2}] \\ &\sim \sum_m \frac{|\widehat{\rho}(k^{-1}m)|^2}{\langle m\rangle^{2\alpha}} \sim \sum_{|m|\leq k} \frac{1}{\langle m\rangle^{2\alpha}}, \end{aligned} \tag{1.17}$$

where $\mathbf{P}_0$ denotes the projection onto the zero Fourier mode: $\widehat{\mathbf{P}_0 f}(n) = \widehat{f}(n)\mathbf{1}_{\{n=0\}}$. Hence, C_k diverges like $\log k$ if $\alpha = \frac{1}{2}$ and like $k^{1-2\alpha}$ if $\alpha < \frac{1}{2}$ as $k \to \infty$. Notice that C_k depends on the choice of mollifying kernel ρ but is independent of $(x,t) \in \mathbb{T} \times \mathbb{R}_+$. Thus, z_k^2 will not converge in the limit as $k \to \infty$ in any reasonable sense.

However, as z_k is smooth and the fact that[6] $\varphi(0) = 0$, we have

$$\varphi(D_x)(z_k^2) = \varphi(D_x)(\mathbf{P}_{\neq 0}(z_k^2)) = \sum_{n \neq 0} \varphi(n) e^{inx} \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 + n_2 = n}} \widehat{z}_k(n_1) \widehat{z}_k(n_2), \tag{1.18}$$

and we show that the right hand side of (1.18) converges almost surely to the distribution

$$\sum_{n \neq 0} \varphi(n) e^{inx} \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 + n_2 = n}} \widehat{z}(n_1) \widehat{z}(n_2) = \mathcal{N}(z), \tag{1.19}$$

where $z(t) := S(t) u_0^\omega$ is the random linear solution with data (1.8); see Proposition 2.6. Thus, we are led to consider $\mathcal{N}(u)$ in (1.4). Note that when $u \in L^2(\mathbb{T})$, $\mathcal{N}(u)$ is equal to $\varphi(D_x)(u^2)$ since

$$\mathcal{N}(u) = \varphi(D_x)\bigg(u^2 - \int_{\mathbb{T}} u^2 dx\bigg) = \varphi(D_x)(u^2) - \varphi(D_x)\bigg(\int_{\mathbb{T}} u^2 dx\bigg) = \varphi(D_x)(u^2).$$

The above computation shows, at least formally, we do not 'see' any difference at the level of the equation (1.1) between the two notions of nonlinearity $\varphi(D_x)(u^2)$ and $\mathcal{N}(u)$.

Remark 1.4. The lower bound $\alpha > \frac{1}{4}$ in Theorem 1.1 is sharp in the following sense. In [3, 41], it is was shown that one may improve upon regularity thresholds for almost sure local well-posedness of dispersive PDE with random initial data by considering a higher order perturbative expansion. In particular, a higher order expansion is actually necessary for the KdV equation with random initial data [39]. In the context of the BBM equation (1.3), this corresponds to writing

$$u = z^\omega + \tilde{Z}^\omega + w, \tag{1.20}$$

where

$$\tilde{Z}^\omega := -\frac{i}{2} \int_0^t S(t - t') \mathcal{N}(z(t')) dt'$$

is the second Picard iterate and studying the fixed point problem for w. The idea is that the expansion (1.20) has removed the term $\tilde{Z}^\omega$ which is responsible for the regularity threshold obtained from studying just a first-order expansion. However, we show in Subsection 3.2 that $\tilde{Z}^\omega$ fails to define a distribution almost surely when $\alpha \leq \frac{1}{4}$, and hence we find no improvement from considering higher order expansions.

1.2. **Almost sure global well-posedness.** Our next goal is to globalise in time the local solutions constructed in Theorem 1.1. In this direction, we establish the following:

Theorem 1.5. *The BBM equation* (1.3) *is almost surely globally well posed in* $H^{-\varepsilon}(\mathbb{T})$*, for any* $0 < \varepsilon \ll 1$*, with respect to random initial data of the form*

$$u_0^\omega = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle^{\frac{1}{2}}} e^{inx}. \tag{1.21}$$

[6] Equivalently, the operator $\varphi(D_x)$ vanishes on constants.

More precisely, for almost every $\omega \in \Omega$, there exists a unique solution u of (1.3) *in*

$$e^{-it\varphi(D_x)}u_0 + C(\mathbb{R}; H^s(\mathbb{T})) \subset C(\mathbb{R}; H^{-\varepsilon}(\mathbb{T}))$$

with initial condition u_0^ω of the form (1.21)*.*

For fixed $\alpha \in (\frac{1}{4}, \frac{1}{2}]$, our probabilistic local theory (Theorem 1.1) shows that we may extend the local-in-time solutions $u = z + v$ provided the $H^s(\mathbb{T})$-norm, with $s = 2\alpha-$, of the solutions v to the perturbed BBM equation (1.10) remains finite. That is, for any $T > 0$, we seek to establish the following bound:

$$\sup_{t\in[0,T]} \|v(t)\|_{H^s(\mathbb{T})} \leq C(T) < \infty. \tag{1.22}$$

Notice in this setting, we only know $v \in H^s(\mathbb{T})$ for $s < 1$ and hence we cannot make use of the energy $E(v(t))$ of (1.6), regardless of its non-conservation under the equation (1.10). This seems to indicate smoothing v which motivated us to apply the I-method of Colliander, Keel, Staffilani, Takaoka and Tao [17, 18] in this probabilistic context.

The approach is as follows: we smooth the initial data by applying the Fourier multiplier operator I_N given by $\widehat{I_N f}(n) = m_N(n)\widehat{f}(n)$, $n \in \mathbb{Z}$, where $m_N(n)$ is the restriction to the integers of the smooth function $m : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$m_N(\xi) := m\bigg(\frac{\xi}{N}\bigg) = \begin{cases} 1 & \text{if } |\xi| \leq N, \\ \Big(\frac{N}{|\xi|}\Big)^{1-s} & \text{if } |\xi| > N. \end{cases} \tag{1.23}$$

Thus, the operator I_N is the identity on low frequencies and a fractional integral operator on high frequencies, hence the name I-method. For simplicity of presentation, we will now drop the subscript N. It is easy to see that $Iv(t) \in H^1(\mathbb{T})$ almost surely and satisfies

$$\begin{cases} \partial_t Iv = -(1-\partial_x^2)^{-1}\partial_x(v) - \frac{1}{2}(1-\partial_x^2)^{-1}\partial_x[I(v^2) + 2I(vz)] + \frac{1}{2}I(\mathcal{N}(z)), \\ Iv|_{t=0} = 0. \end{cases} \tag{1.24}$$

By defining the 'modified energy'

$$E(Iv)(t) = \frac{1}{2}\|Iv(t)\|_{H^1(\mathbb{T})}^2,$$

and observing

$$\|v(t)\|_{H^s(\mathbb{T})} \lesssim E(Iv)(t),$$

we reduce proving (1.22) to obtaining

$$\sup_{t\in[0,T]} E(Iv)(t) \leq C(T) < \infty. \tag{1.25}$$

Now, $E(Iv)$ will not be conserved under the flow of (1.24) because (i) v solves a perturbed BBM equation and (ii) I does not commute with products. However, Iv is expected to 'almost' solve the same equation as v, namely (1.10), in the sense that the error terms generated from the failure of commutation are themselves commutators. Indeed, taking a time derivative of $E(Iv)$, inserting (1.24) and making appear commutators, we schematically arrive at

$$\frac{d}{dt}E(Iv)(t) \sim \int_{\mathbb{T}}(\partial_x Iv)[I(v^2)-(Iv)^2]dx + \int_{\mathbb{T}} Iz\, Iv\, \partial_x Iv\, dx \\ + \text{lower order terms.} \tag{1.26}$$

We estimate the first expression by

$$\int_{\mathbb{T}}(\partial_x Iv)[I(v^2)-(Iv)^2]\,dx \lesssim N^{-\beta}E^{\frac{3}{2}}(Iv)$$

for some $\beta > 0$ and notice that this term in (1.26) implies that $E(Iv)$ will blow-up in a finite time T_N. However, up to time T_N, the negative power of N allows us view this term as part of the lower order corrections. For the second term, we have

$$\int_{\mathbb{T}} Iz\, Iv\, \partial_x Iv\, dx \lesssim \|Iz\|_{L^2}E(Iv).$$

For this term, placing Iz into $L^2(\mathbb{T})$ comes at the expense of a loss in N; see Lemma 4.2 (an analogue of this can be found in [29, 46]). When $\alpha = \frac{1}{2}$, we lose only a logarithm of N, and it is only this loss which is acceptable for ensuring we may take $T_N \geq 2T$ by choosing $N = N(T)$ large enough. By Gronwall's inequality, we obtain (1.25) provided $\alpha = \frac{1}{2}$ which yields Theorem 1.5.

This I-method approach has recently been applied in the context of nonlinear stochastic dispersive PDE by Gubinelli, Koch, Oh and Tolomeo [29] and Tolomeo [46]. We closely follow their arguments, although certain technical difficulties are absent for (1.3) as compared to their setting. A natural modification of the argument above would be to include the low-high splitting idea in [10]; however, this does not seem to lead to any regularity improvement over Theorem 1.5; see Remark 4.7.

Remark 1.6. We may relax the Gaussianity assumption on the random variables $\{g_n\}_{n\in\mathbb{Z}}$. More precisely, in Appendix A, we detail how our arguments allow us to establish analogous almost sure local and global existence results for BBM (1.3) with respect to initial data of the form:

$$u_0^\omega = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx},$$

where the complex-valued (not necessarily Gaussian) random variables $\{g_n\}_{n\in\mathbb{Z}}$ satisfy assumptions (i)-(v) in Appendix A. Note, however, that with such randomisations we lose the link with the Gaussian measures μ_α in (1.12).

1.3. **Norm inflation at general data in negative Sobolev spaces.** In this subsection, we study the (purely) deterministic ill-posedness of the solution map $\Phi : u_0 \in H^s(\mathcal{M}) \mapsto u \in C([0,T]; H^s(\mathcal{M}))$ to BBM (1.1). In [42], Panthee showed the failure of continuity of the solution map at the origin in $H^s(\mathcal{M})$ for any $s < 0$. This result implies that BBM (1.1) is ill-posed in negative Sobolev spaces. Bona and Dai [8] showed that for $s < 0$, the solution map exhibits the stronger phenomenon known as *norm inflation at zero*: given $s < 0$, for any $\varepsilon > 0$, there exists a smooth solution u_ε to BBM (1.1) and times $t_\varepsilon \in (0, \varepsilon)$ such that

$$\|u_\varepsilon(0)\|_{H^s(\mathcal{M})} < \varepsilon \qquad \text{and} \qquad \|u_\varepsilon(t_\varepsilon)\|_{H^s(\mathcal{M})} > \varepsilon^{-1}.$$

Norm inflation based at *general* initial data has been studied for NLW [50, 47] and NLS [38] in negative Sobolev spaces. We establish norm inflation at any $u_0 \in H^s(\mathcal{M})$, with $s < 0$, for the BBM equation (1.1). It is clear that the same result also holds for (1.3).

Theorem 1.7. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$, $s < 0$ and fix $u_0 \in H^s(\mathcal{M})$. Then, given any $\varepsilon > 0$, there exists a smooth solution u_ε to (1.1) on $\mathcal{M}$ and $t_\varepsilon \in (0, \varepsilon)$ such that*

$$\|u_\varepsilon(0) - u_0\|_{H^s(\mathcal{M})} < \varepsilon \qquad \text{and} \qquad \|u_\varepsilon(t_\varepsilon)\|_{H^s(\mathcal{M})} > \varepsilon^{-1}.$$

As an immediate corollary, we obtain the everywhere discontinuity of the solution map of (1.1) in negative Sobolev spaces.

Corollary 1.8. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$ and $s < 0$. Then, for any $T > 0$, the solution map $\Phi : u_0 \in H^s(\mathcal{M}) \mapsto u \in C([0,T]; H^s(\mathcal{M}))$ to the BBM equation (1.1) is discontinuous everywhere in $H^s(\mathcal{M})$.*

To prove Theorem 1.7, we employ the argument in Oh [38], where norm inflation based at general initial data was studied for the cubic NLS in negative Sobolev spaces. Briefly, the key idea is to write a solution u to (1.1) with $u|_{t=0} = u_0$ in terms of its power series expansion:

$$u = \sum_{j=1}^{\infty} \Xi_j(u_0),$$

which is an infinite sum of recursively defined homogeneous multilinear operators Ξ_j, in terms of u_0, of increasing order; see also [16, 33, 15, 35]. Such a power series expansion is motivated by the Picard iteration scheme. In [38], these power series expansions are indexed using trees, which simplifies their handling, both combinatorially and analytically (in terms of obtaining multilinear estimates). One then exploits a high-to-low energy transfer in the second term of the expansion in order to exhibit the instability stated in Theorem 1.7.

We stress that Theorem 1.7 and its proof are entirely deterministic. We relate this result to the solutions constructed from rough random initial data of the form (1.8) in the next subsection.

Remark 1.9. We extend our study of the (deterministic) BBM equation (1.1) to the context of the Fourier-Lebesgue spaces $\mathcal{F}L^{s,p}(\mathcal{M})$ which are defined through the norm

$$\|f\|_{\mathcal{F}L^{s,p}(\mathcal{M})} = \|\langle \xi \rangle^s \widehat{f}(\xi)\|_{L^p(\widehat{\mathcal{M}})},$$

where

$$\widehat{\mathcal{M}} = \begin{cases} \mathbb{R} & \text{if } \mathcal{M} = \mathbb{R}, \\ \mathbb{Z} & \text{if } \mathcal{M} = \mathbb{T}, \end{cases}$$

where we use the Lebesgue measure if $\widehat{\mathcal{M}} = \mathbb{R}$ and the counting measure if $\widehat{\mathcal{M}} = \mathbb{Z}$. When $p = 2$, we have $\mathcal{F}L^{s,2}(\mathcal{M}) = H^s(\mathcal{M})$ and when $p = 1$ and $s = 0$, the space $\mathcal{F}L^{0,1}(\mathcal{M})$ is the Wiener algebra. For convenience, we write $\mathcal{F}L^p(\mathcal{M})$ instead of $\mathcal{F}L^{0,p}(\mathcal{M})$. In Section 5, we show that BBM (1.1) is locally well-posed in $\mathcal{F}L^{s,p}(\mathcal{M})$ for any $s \geq 0$ when $1 \leq p \leq 2$ and for any $s > \frac{1}{2} - \frac{1}{p}$ when $p > 2$. In analogy to (1.1) in $H^s(\mathcal{M})$, we can extend the norm inflation result of Theorem 1.7 to norm inflation at general data in $\mathcal{F}L^{s,p}(\mathcal{M})$ for any $s < 0$

and $1 \leq p < \infty$; see Section 5 for details. In the periodic case, our interest in this result lies in the following observation:

$$\mathcal{F}L^{s,p}(\mathbb{T}) \subseteq H^s(\mathbb{T})$$

for any $1 \leq p \leq 2$ and any $s \in \mathbb{R}$. Namely, the instability in Theorem 1.7 persists in the stronger $\mathcal{F}L^{s,p}(\mathbb{T})$-norm. On $\mathcal{M} = \mathbb{R}$, there is no inclusion between these spaces for a fixed s.

1.4. **Stability in the ill-posed regime.** In this subsection, we discuss notions of stability for the solution map Φ of Theorem 1.1. Combining the almost-sure local well-posedness of Theorem 1.1 with the norm-inflation result of Theorem 1.7, we obtain the following 'almost sure norm inflation' phenomenon which highlights the strong instability in the map Φ. This phenomenon is known to also occur for certain NLW equations in the super-critical regime [50, 40].

Theorem 1.10 (Almost sure norm inflation). *Let $\alpha \in (\frac{1}{4}, \frac{1}{2}]$ and fix $\omega \in \Sigma$, where Σ is the set of full $\mathbb{P}$-measure from Theorem 1.1. Let u^ω be the (local) solution to the BBM equation (1.1) with $u^\omega|_{t=0} = u_0^\omega$. Then, given $k \in \mathbb{N}$, there exist u_k^ω smooth (random) solutions to (1.1) such that*

$$\lim_{k\to\infty} \|u_k^\omega(0) - u_0^\omega\|_{H^{\alpha-\frac{1}{2}-}(\mathbb{T})} = 0 \quad \textit{and} \quad \lim_{k\to\infty} \|u_k^\omega - u^\omega\|_{C([0,k^{-1}];H^{\alpha-\frac{1}{2}-}(\mathbb{T}))} = \infty.$$

The almost sure norm inflation above implies that the solution map Φ is almost surely discontinuous everywhere over $H^{\alpha-\frac{1}{2}-}(\mathbb{T})$. In other words, we can always find a (random) smooth sequence $\{u_{0,k}^\omega\}_{k\in\mathbb{N}}$ which approximates the realisation u_0^ω but whose (smooth) solutions exhibit the instability as stated in Theorem 1.10. However, this does not rule out the possibility that there is *some* class of reasonable smooth solutions which do approximate the random solutions lying below $L^2(\mathbb{T})$. Indeed, the class of smooth solutions obtained from mollified data provides a good approximation property.

Theorem 1.11. *Let $\alpha \in (\frac{1}{4}, \frac{1}{2}]$. Let $u = u^\omega$ be as in Theorem 1.10. Denote by $u_{0,k}^\omega = \rho_k * u_0^\omega$ the regularisation of u_0^ω by a smooth mollifier $\{\rho_k\}_{k\in\mathbb{N}}$ and let u_k be the solution to the BBM equation (1.3) with $u_k|_{t=0} = u_{0,k}^\omega$. Then, we have*

$$\lim_{k\to\infty} \|u_{0,k}^\omega - u_0^\omega\|_{H^{\alpha-\frac{1}{2}-}(\mathbb{T})} = 0 \qquad \textit{and} \qquad \lim_{k\to\infty} \|u_k^\omega - u^\omega\|_{C([0,T^\omega];H^{\alpha-\frac{1}{2}-}(\mathbb{T}))} = 0.$$

Moreover, the limit u is independent of the choice of mollification kernel ρ.

The proof of Theorem 1.11 is a direct corollary of Theorem 1.1, Remark 1.2 and Proposition 2.6. Namely, Proposition 2.6 implies the almost sure convergence of the enhanced data set $(z_k, \mathcal{N}(z_k))$ to $(z, \mathcal{N}(z))$ (in the appropriate topology, see (1.13)) with the limit independent of the mollification kernel. Then, continuity of the map Ψ given in (1.13) (from Theorem 1.1; see also Remark 1.2) implies $v_k \to v$ almost surely in $C([0,T]; H^s(\mathbb{T}))$ for any $s < 2\alpha$.

In Theorem 1.11, the independence of the limit on the choice of mollifying kernel provides a well-defined notion of stability for the solution map Φ. Thus, the random solutions constructed by Theorem 1.1 may be approximated by certain 'reasonable' regularisations of the initial data. This is in direct contrast to the setting of deterministic well-posedness where approximations may be completely arbitrary.

Remark 1.12. As will be clear by the proof of Proposition 2.6 below, we may also consider in Theorem 1.11 the regularisation by the (non-smooth) Dirichlet projection $\mathbf{P}_{\le N}$ onto frequencies $\{n : |n| \le N\}$. We then extend the uniqueness of the limiting solution among the class of mollifiers and the projection $\mathbf{P}_{\le N}$.

Remark 1.13. As Theorem 1.10 is a direct corollary of Theorem 1.7, the sequence of solutions $\{u_k^\omega\}_{k\in\mathbb{N}}$ can be taken with respect to a continuous index: for fixed 'good' ω, there is a sequence of smooth solutions $\{u_\varepsilon^\omega\}_{0<\varepsilon\ll 1}$ which exhibit the instability as stated in Theorem 1.10 as $\varepsilon \to 0$. With a sequence of mollifiers $\rho_\varepsilon(x) := \varepsilon^{-1}\rho(\varepsilon^{-1}x)$ on $\mathbb{T}$ and for $\alpha \in (\frac{1}{4}, \frac{1}{2}]$, we can show that the smooth solutions u_ε to BBM (1.3) with initial data $u_0^\omega * \rho_\varepsilon$, where u_0^ω is as in (1.8), converge in $C([0,T^\omega]; H^{\alpha-\frac{1}{2}-}(\mathbb{T}))$ in probability to a unique limit u as $\varepsilon \to 0$, where $T^\omega > 0$ almost surely. Moreover, the limit u is independent of the choice of the mollifier ρ.

We now provide an outline of the following paper. In Section 2, we collect some necessary results and tools of deterministic and probabilistic natures. We then carefully construct and study properties of the random linear solutions z and the nonlinear object $\mathcal{N}(z)$. In Section 3, we prove Theorem 1.1 on constructing local-in-time solutions below $L^2(\mathbb{T})$ and show that the regularity result there is sharp. We then show, in Section 4, we can globalise those random solutions (in a restricted regularity). In the purely deterministic setting, we establish norm inflation for BBM at general data in negative Sobolev spaces in Section 5. In Appendix A, we detail how we obtain almost sure existence of solutions with non-Gaussian random initial data as in Remark 1.6. Finally, we also include a brief appendix on obtaining exponential tail estimates on stochastic processes.

2. Deterministic and probabilistic tools

In this section, we collect here some useful deterministic and probabilistic results.

2.1. Deterministic tools.

First, we recall the following key bilinear estimate, due to Bona and Tzvetkov [10] (see also Rouméгoux [44]), which immediately implies the local well-posedness of BBM (1.1) in $H^s(\mathcal{M})$ for any $s \ge 0$. A proof is contained within that of a slightly more general bilinear estimate which we give in Lemma 5.4.

Lemma 2.1 ([10, 44])**.** *For any $s \ge 0$ and any $f, g \in H^s(\mathcal{M})$, we have*

$$\|\varphi(D_x)(fg)\|_{H^s(\mathcal{M})} \lesssim \|f\|_{H^s(\mathcal{M})}\|g\|_{H^s(\mathcal{M})}. \tag{2.1}$$

Furthermore, the estimate (2.1) *is false if $s < 0$.*

We next state a useful summing estimate, a proof of which can be found in, for example, [27, Lemma 4.2].

Lemma 2.2. *If $\beta \ge \gamma \ge 0$ and $\beta + \gamma > 1$, then*

$$\sum_n \frac{1}{\langle n-k_1\rangle^\beta \langle n-k_2\rangle^\gamma} \lesssim \frac{\phi_\beta(k_1-k_2)}{\langle k_1-k_2\rangle^\gamma},$$

where

$$\phi_\beta(k) := \sum_{|n|\le|k|} \frac{1}{\langle n\rangle^\beta} \sim \begin{cases} 1, & \text{if } \beta > 1, \\ \log(1+\langle k\rangle), & \text{if } \beta = 1, \\ \langle k\rangle^{1-\beta}, & \text{if } \beta < 1. \end{cases} \tag{2.2}$$

Finally, we need the following paraproduct estimate:

Lemma 2.3 ([28, Lemma 3.4])**.** *Let $0 \le s \le 1$ and suppose that $1 < p, q, r < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + s$. Then, we have*

$$\|\langle\nabla\rangle^{-s}(fg)\|_{L^r(\mathbb{T})} \lesssim \|\langle\nabla\rangle^{-s} f\|_{L^p(\mathbb{T})} \|\langle\nabla\rangle^{s} g\|_{L^q(\mathbb{T})}.$$

2.2. **Probabilistic tools.** Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the σ-algebra generated by the sequence $\{g_n\}_{n\in\mathbb{N}}$. Given $\ell \in \mathbb{N}\cup\{0\} = \mathbb{N}_0$, we define a polynomial chaos of degree k to be a polynomial of the form $\prod_{j=1}^\infty H_{\ell_j}(g_n)$, where $\{\ell_j\}_{j\in\mathbb{N}_0} \subset \mathbb{N}_0$ satisfy[7] $\ell = \sum_{j=1}^\infty \ell_j$ and H_{ℓ_j} is the Hermite polynomial of degree ℓ_j. We then define the homogeneous Wiener chaos $\mathcal{H}_\ell$ of order ℓ as the closure under $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of the linear span of polynomial chaoses of degree ℓ. We write

$$\mathcal{H}_{\le \ell} := \bigoplus_{j=0}^{\ell} \mathcal{H}_j$$

and we have the following so-called Wiener chaos estimate which we use to exploit the randomisation in the multilinear term $\mathcal{N}(z)$.

Lemma 2.4 (Wiener chaos estimate)**.** *Given $\ell \in \mathbb{N}_0$, let $X \in \mathcal{H}_{\le\ell}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \le (p-1)^{\frac{\ell}{2}} \|X\|_{L^2(\Omega)}$$

for any $2 \le p < \infty$.

The proof of Lemma 2.4 follows from the hypercontractivity of the Ornstein-Uhlenbeck semigroup due to Nelson [37]. See for instance [45, Proposition 2.4] for more details. Notice as a special case of Lemma 2.4, for any $(a_n) \in \ell^2_n$, we have

$$\Big\| \sum_{n\in\mathbb{Z}} a_n g_n(\omega) \Big\|_{L^p(\Omega)} \lesssim p^{\frac{1}{2}} \|a_n\|_{\ell^2_n}. \tag{2.3}$$

In order to study the regularity properties of random distributions we make use of the following result, a proof of which can be found in [41]. We specialise the argument in [41] to one spatial dimension and to the Sobolev spaces $W^{s,p}(\mathbb{T})$. Given $0 < \gamma < 1$, we say $f \in C^\gamma([0,T]; W^{s,p}_x(\mathbb{T}))$ if the following norm is finite:

$$\|f\|_{C^\gamma([0,T];W^{s,p}_x(\mathbb{T}))} = \|f\|_{C([0,T];W^{s,p}_x(\mathbb{T}))} + \|f\|_{\dot{C}^\gamma([0,T];W^{s,p}_x(\mathbb{T}))},$$

where we define the semi-norm $\|\cdot\|_{\dot{C}^\gamma([0,T];W^{s,p}_x(\mathbb{T}))}$ by

$$\|f\|_{\dot{C}^\gamma([0,T];W^{s,p}_x(\mathbb{T}))} := \sup_{0\le t'<t\le T} \frac{\|f(t)-f(t')\|_{W^{s,p}_x(\mathbb{T})}}{|t-t'|^\gamma}.$$

[7] Note at most finitely many ℓ_j are non-zero.

We occasionally write $C([0,T];W^{s,p}(\mathbb{T}))$ as $C_TW^{s,p}$. Given $h \in \mathbb{R}$, we denote by δ_h the difference operator:

$$\delta_h X(t) = X(t+h) - X(t).$$

Proposition 2.5 (Regularity and convergence of stochastic processes)**.** *Let $\{X_k\}_{k\in\mathbb{N}}$ be a sequence of stochastic processes on $\mathbb{R}_+$ with values in $\mathcal{S}'(\mathbb{T})$ such that $X_k(t) \in \mathcal{H}_{\le \ell}$ for each $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$. Fix $T > 0$. Suppose there exist $1 \le p \le \infty$ and $s \in \mathbb{R}$ such that the following statements hold:*

(i) *We have*

$$\mathbb{E}[\|X_k(t)\|^q_{W^{s,p}(\mathbb{T})}] < \infty, \tag{2.4}$$

for any $q \ge 1$ and uniformly in $t \in [0,T]$ and $k \in \mathbb{N}$.

(ii) *There exists $\theta > 0$ such that*

$$\mathbb{E}[\|X_{k'}(t) - X_k(t)\|^q_{W^{s,p}(\mathbb{T})}] \lesssim_T k^{-q\theta} q^{\frac{\ell q}{2}} \tag{2.5}$$

for any $q \ge 1$, $t \in [0,T]$ and $k' \ge k \ge 1$.

Then, (i) *implies for each fixed $t \in [0,T]$, $X_k(t) \in W^{s,p}(\mathbb{T})$ almost surely and moreover,* (ii) *implies there exists $X(t) \in W^{s,p}(\mathbb{T})$ such that $\{X_k(t,\cdot)\}_{k\in\mathbb{N}}$ converges to $X(t)$ in $L^q(\Omega; W^{s,p}(\mathbb{T}))$, for any $1 \le q < \infty$ and almost surely in $W^{s,p}(\mathbb{T})$.*

Suppose, in addition, the following statements hold:

(iii) *There exists $\gamma > 0$ such that*

$$\mathbb{E}[\|\delta_h X_k(t)\|^q_{W^{s,p}(\mathbb{T})}] \lesssim_{q,T} |h|^{\frac{q}{2}\gamma} \tag{2.6}$$

for any $q \ge 1$ and $h \in [-1,1]$, uniformly in $t \in [0,T]$ and $k \in \mathbb{N}$.

(iv) *There exist $\theta, \gamma > 0$, such that*

$$\mathbb{E}[\|\delta_h X_{k'}(t) - \delta_h X_k(t)\|^q_{W^{s,p}(\mathbb{T})}] \lesssim_{q,T} k^{-q\theta}|h|^{\frac{q}{2}\gamma} \tag{2.7}$$

for any $q \ge 1$ and $h \in [-1,1]$, uniformly in $t \in [0,T]$ and $k' \ge k \ge 1$.

Then, (iii) *implies $X_k, X \in C^\beta([0,T];W^{s,p}(\mathbb{T}))$ almost surely for $\beta < \frac{\gamma}{2}$ and moreover,* (iv) *implies $\{X_k\}_{k\in\mathbb{N}}$ converges to X in $L^q(\Omega; C^\beta([0,T];W^{s,p}(\mathbb{T})))$, for any $1 \le q < \infty$ and almost surely in $C^\beta([0,T];W^{s,p}(\mathbb{T}))$.*

2.3. **Properties of the stochastic objects.** In this section we study the regularity and integrability properties of the random linear solution to BBM (1.1) with initial data (1.8), which we write as $z = S(t)u_0^\omega$ and the bilinear term $\mathcal{N}(z)$ given in (1.19). We verify that both z and $\mathcal{N}(z)$ are the limit of the mollified sequences $\{z_k = S(t)(u_0^\omega * \rho_k)\}_{k\in\mathbb{N}}$ and $\{\mathcal{N}(z_k)\}_{k\in\mathbb{N}}$, independent of the choice of mollifier. Moreover, we verify

$$\mathcal{N}(z) \in C([0,T]; H^{2\alpha-}(\mathbb{T}))$$

almost surely.

Proposition 2.6. *Let $\frac14 < \alpha \le \frac12$,*

$$s_1 < \alpha - \frac12 \quad \textit{and} \quad s_2 < 2\alpha.$$

Let $\{\rho_k\}_{k\in\mathbb{N}}$ be a family of mollifiers on $\mathbb{T}$. Given $T>0$, let $z_k = S(t)(u_0^\omega * \rho_k)$ where $t\in[0,T]$. Then, $(z,\mathcal{N}(z))\in C([0,T];W^{s_1,\infty}(\mathbb{T}))\times C([0,T];W^{s_2,\infty}(\mathbb{T}))$ almost surely and

$$(z_k,\mathcal{N}(z_k)) \longrightarrow (z,\mathcal{N}(z)),$$

as $k\to\infty$ in $L^q(\Omega;C([0,T];W^{s_1,\infty}(\mathbb{T}))\times C([0,T];W^{s_2,\infty}(\mathbb{T})))$ for any $q\geq 1$ and almost surely in $C([0,T];W^{s_1,\infty}(\mathbb{T}))\times C([0,T];W^{s_2,\infty}(\mathbb{T}))$. Moreover, the limit $(z,\mathcal{N}(z))$ is independent of the choice of mollification kernel ρ, including the regularisation by the (non-smooth) Dirichlet projection $\mathbf{P}_{\leq N}$. Furthermore, there exist $C,C',c>0$ such that

$$\mathbb{P}\big(\|z\|_{C([0,T];W_x^{s_1,\infty}(\mathbb{T}))}+\|\mathcal{N}(z)\|_{C([0,T];W_x^{s_2,\infty}(\mathbb{T}))}>\lambda\big)\leq C'\big(e^{-\frac{C\lambda}{T^c}}+e^{-C\lambda}\big)$$

for any $T>0$ and $\lambda>0$.

Proof. We first verify the claims made for the random linear solution z. Clearly, $\{z_k\}_{k\in\mathbb{N}}\subset\mathcal{H}_{\leq 1}$ for every fixed $t\in[0,T]$. We have

$$\widehat{z_k}(n,t)=\frac{g_n(\omega)\widehat{\rho_k}(n)e^{-it\varphi(n)}}{\langle n\rangle^\alpha}.$$

Then, by Sobolev embedding, Minkowski's integral inequality (for q sufficiently large) and (2.4), we have

$$\begin{aligned}\mathbb{E}[\|z_k(t)\|^q_{W^{s_1,\infty}(\mathbb{T})}] &\leq \Big\|\big\|\langle\partial_x\rangle^{s_1-\varepsilon}z_k(t)\big\|_{L^q(\Omega)}\Big\|^q_{L^p(\mathbb{T})}\\ &\lesssim q^{\frac{q}{2}}\Big\|\|\langle\partial_x\rangle^{s_1-\varepsilon}z_k(t)\|_{L^2(\Omega)}\Big\|^q_{L^p(\mathbb{T})}\\ &\lesssim q^{\frac{q}{2}}\bigg\|\bigg(\sum_{n\in\mathbb{Z}}\frac{\langle n\rangle^{2s_1-2\varepsilon}|\widehat{\rho_k}(n)|^2}{\langle n\rangle^{2\alpha}}\bigg)^{\frac12}\bigg\|^q_{L^p(\mathbb{T})},\end{aligned}\tag{2.8}$$

where $0<\varepsilon:=\varepsilon(p,q)\ll 1$. As $|\widehat{\rho_k}(n)|\lesssim 1$ uniformly in both $n\in\mathbb{Z}$ and $k\in\mathbb{N}$ and $s_1<\alpha-\frac12$, we verfiy (2.4) and hence $z_k(t)\in W^{s_1,\infty}(\mathbb{T})$ almost surely. From the same computations as in (2.8), we have

$$\mathbb{E}[\|z_{k'}(t)-z_k(t)\|^q_{W^{s_1,\infty}(\mathbb{T})}]\lesssim q^{\frac{q}{2}}\bigg(\sum_{n\in\mathbb{Z}}\frac{\langle n\rangle^{2s_1-2\varepsilon}|\widehat{\rho_{k'}}(n)-\widehat{\rho_k}(n)|^2}{\langle n\rangle^{2\alpha}}\bigg)^{\frac{q}{2}}.\tag{2.9}$$

Now, given $k'\geq k>0$, the mean value theorem implies

$$|\widehat{\rho_{k'}}(n)-\widehat{\rho_k}(n)|\lesssim|n||(k')^{-1}-k^{-1}|\leq 2|n|k^{-1}.$$

Interpolating this with the trivial bound $|\widehat{\rho_k}(n)-\widehat{\rho_{k'}}(n)|\lesssim 2$, we obtain

$$|\widehat{\rho_{k'}}(n)-\widehat{\rho_k}(n)|\lesssim|n|^\theta k^{-\theta}\tag{2.10}$$

for $0\leq\theta\leq 1$. Inserting (2.10) into (2.9) we get

$$\mathbb{E}[\|z_{k'}(t)-z_k(t)\|^q_{W^{s_1,\infty}(\mathbb{T})}]\lesssim k^{-q\theta}q^{\frac{q}{2}},$$

provided $s_1<\alpha-\frac12-\theta$. As $\theta>0$ was arbitrary, we have verified (i) and (ii) of Proposition 2.5 with $s_1<\alpha-\frac12$. We now move onto establishing the temporal regularity of z_k.

We verify the appropriate analogue of (2.7) since the same ideas will apply to obtain (2.6). Analogously to (2.8), we have

$$\begin{aligned}\mathbb{E}[\|\delta_h z_{k'}(t)-\delta_h z_k(t)\|^q_{W^{s_1,\infty}(\mathbb{T})}]\\ \lesssim q^{\frac{q}{2}}\bigg(\sum_{n\in\mathbb{Z}}\frac{\langle n\rangle^{2s_1-2\varepsilon}|\widehat{\rho_{k'}}(n)-\widehat{\rho_k}(n)|^2|e^{-ih\varphi(n)}-1|^2}{\langle n\rangle^{2\alpha}}\bigg)^{\frac{q}{2}}.\end{aligned}$$

Using (2.10) and $|e^{-ih\varphi(n)}-1|\lesssim|h||\varphi(n)|$, we then obtain

$$\mathbb{E}[\|\delta_h z_{k'}(t)-\delta_h z_k(t)\|^q_{W^{s_1,\infty}(\mathbb{T})}]\lesssim q^{\frac{q}{2}}k^{-q\theta}|h|^{\frac{q}{2}}.$$

Thus by Proposition 2.5, z_k converges almost surely to z in $C_TW^{s_1,\infty}(\mathbb{T})$ and in $L^q(\Omega;C_TW^{s_1,\infty}(\mathbb{T}))$ for any $q\geq 1$. We verify the independence of z on the mollifier ρ later in this proof; see (2.25). Taking a limit in the analogue of (2.6) as $k\to\infty$ gives

$$\mathbb{E}[\|z(t)-z(t')\|^q_{W^{s_1,\infty}(\mathbb{T})}]\lesssim q^{\frac{q}{2}}|t-t'|^q. \tag{2.11}$$

With $\gamma<1-\frac{1}{q}$, we have

$$\|z\|_{C_TW^{s_1,\infty}_x}\leq T^\gamma\|z\|_{\dot{C}^\gamma([0,T];W^{s_1,\infty}_x(\mathbb{T}))}+\|z(0)\|_{W^{s_1,\infty}_x}. \tag{2.12}$$

Therefore by (B.5) in Appendix B, we have

$$\begin{aligned}\mathbb{P}\big(\|z\|_{C([0,T];W^{s_1,\infty}_x(\mathbb{T}))}>\lambda\big)&\leq\mathbb{P}\big(\|z\|_{\dot{C}^\gamma([0,T];W^{s_1,\infty}_x(\mathbb{T}))}>\tfrac{\lambda}{2}\big)\\&\quad+\mathbb{P}\big(\|z(0)\|_{W^{s_1,\infty}_x(\mathbb{T})}>\tfrac{\lambda}{2}\big)\\&\leq e^{-C\frac{\lambda^2}{T^c}}+e^{-c\lambda^2}.\end{aligned} \tag{2.13}$$

We now consider the object $\mathcal{N}(z)$. For any $k\in\mathbb{N}$ and fixed $t>0$, $\mathcal{N}(z_k)\in\mathcal{H}_{\leq 2}$. We verify appropriate versions of (2.5) and (2.6), which themselves contain the necessary calculations required to also obtain versions of (2.4) and (2.7). We write

$$\langle\partial_x\rangle^{s_2-\varepsilon}\mathcal{N}(z_k)=\sum_{n_1,n_2\in\mathbb{Z}}\mathcal{R}_k(n_1,n_2;t)g_{n_1}g_{n_2},$$

where

$$\begin{aligned}\mathcal{R}_k(n_1,n_2;t)&:=\mathbf{1}_{\{n_1+n_2\neq 0\}}\langle n_1+n_2\rangle^{s_2-\varepsilon}\varphi(n_1+n_2)e^{i(n_1+n_2)x}\prod_{j=1}^2 a_k(n_j;t),\\ a_k(n;t)&:=\frac{e^{-it\varphi(n)}}{\langle n\rangle^\alpha}\widehat{\rho_k}(n).\end{aligned} \tag{2.14}$$

By Sobolev embedding, Minkowski's integral inequality (for q sufficiently large) and Lemma 2.4, we have

$$\begin{aligned}\mathbb{E}[\|\mathcal{N}(z_{k'})(t)-\mathcal{N}(z_k)(t)\|^q_{W^{s_2,\infty}(\mathbb{T})}]\\ \leq q^q\bigg\|\bigg\|\sum_{n_1,n_2\in\mathbb{Z}}\big[\mathcal{R}_{k'}-\mathcal{R}_k\big](n_1,n_2;t)g_{n_1}g_{n_2}\bigg\|_{L^2(\Omega)}\bigg\|^q_{L^p(\mathbb{T})}.\end{aligned}$$

It suffices to show

$$\bigg\|\sum_{n_1,n_2\in\mathbb{Z}}\big[\mathcal{R}_{k'}-\mathcal{R}_k\big](n_1,n_2;t)g_{n_1}g_{n_2}\bigg\|_{L^2(\Omega)}\lesssim k^{-\theta} \tag{2.15}$$

for some $\theta > 0$, any $t \in [0, T]$ and $k' \geq k \geq 1$. Now

$$\begin{aligned}
&\Big\| \sum_{n_1, n_2 \in \mathbb{Z}} \big[\mathcal{R}_{k'} - \mathcal{R}_k\big](n_1, n_2; t) g_{n_1} g_{n_2} \Big\|^2_{L^2(\Omega)} \\
&\quad = \sum_{\substack{n_1 . n_2 \in \mathbb{Z} \\ m_1, m_2 \in \mathbb{Z}}} \big[\mathcal{R}_{k'} - \mathcal{R}_k\big](n_1, n_2; t) \overline{\big[\mathcal{R}_{k'} - \mathcal{R}_k\big](m_1, m_2; t)} \mathbb{E}[g_{n_1} g_{n_2} \overline{g_{m_1} g_{m_2}}].
\end{aligned} \tag{2.16}$$

First, we assume all of n_1, n_2, m_1 and m_2 are non-zero and not all equal. Then, Wick's theorem implies

$$\begin{aligned}
\mathbb{E}[g_{n_1} g_{n_2} \overline{g_{m_1} g_{m_2}}] = \mathbb{E}[g_{n_1} \overline{g_{m_1}}] \mathbb{E}[g_{n_2} \overline{g_{m_2}}] + \mathbb{E}[g_{n_1} \overline{g_{m_2}}] \mathbb{E}[g_{n_2} \overline{g_{m_1}}] \\
+ \mathbb{E}[g_{n_1} g_{n_2}] \mathbb{E}[\overline{g_{m_1} g_{m_2}}].
\end{aligned}$$

The first two terms are non-zero if and only if $n_j = m_{\sigma(j)}$, where σ is a permutation of $\{1, 2\}$. The third term vanishes identically since $n_1 + n_2 \neq 0$ and $m_1 + m_2 \neq 0$. This is precisely where we use the definition of the product (1.4) which does not contain the zero frequency. Therefore, we have

$$\begin{aligned}
\text{LHS of (2.16)} &= \sum_{n_1, n_2 \in \mathbb{Z}} |\mathcal{R}_{k'}(n_1, n_2; t) - \mathcal{R}_k(n_1, n_2; t)|^2 \\
&\sim \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s_2 - 2\varepsilon - 2} \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n = n_1 + n_2}} \frac{|\widehat{\rho_{k'}}(n_1) \widehat{\rho_{k'}}(n_2) - \widehat{\rho_k}(n_1) \widehat{\rho_k}(n_2)|^2}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}}.
\end{aligned} \tag{2.17}$$

By the triangle inequality, we bound the inner summation in (2.17) by

$$\sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n = n_1 + n_2}} \frac{|\widehat{\rho_k}(n_1)|^2 |\widehat{\rho_{k'}}(n_2) - \widehat{\rho_k}(n_2)|^2}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}} + \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n = n_1 + n_2}} \frac{|\widehat{\rho_k}(n_1)|^2 |\widehat{\rho_k}(n_2) - \widehat{\rho_{k'}}(n_2)|^2}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}}.$$

It suffices to estimate just the first sum, with the same ideas applying for the second. By (2.10) and Lemma 2.2, we get

$$\begin{aligned}
\sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n = n_1 + n_2}} \frac{|\widehat{\rho_k}(n_1)|^2 |\widehat{\rho_{k'}}(n_2) - \widehat{\rho_k}(n_2)|^2}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}} &\lesssim \frac{1}{k^{2\theta}} \sum_{n_2 \in \mathbb{Z}} \frac{1}{\langle n - n_2 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha - 2\theta}} \\
&\lesssim k^{-2\theta} \langle n \rangle^{-2\alpha + 2\theta} \phi_{2\alpha}(n),
\end{aligned}$$

provided $4\alpha - 2\theta > 1$, where $\phi_{2\alpha}$ is given in (2.2). From this contribution, we arrive at

$$\text{LHS of (2.17)} \lesssim k^{-2\theta} \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s_2 - 2\varepsilon - 2 - 2\alpha + 2\theta} \phi_{2\alpha}(n) \lesssim k^{-2\theta}$$

provided $s_2 < 2\alpha - \theta$. Now we consider the case when at least one of n_1, n_2, m_1 or m_2 are zero. Noting that $|\mathcal{R}_k(n_1, n_2; t)| = |\mathcal{R}_k(n_2, n_1; t)|$ and $n_1 + n_2 \neq 0$, we may assume $n_1 = 0$. Then, the only non-zero contribution comes from when $m_1 = 0$ and $m_2 = n_2$ (using the symmetry in $|\mathcal{R}_k(m_1, m_2; t)|$). Using $\widehat{\rho_k}(0) = 1$ and (2.10), we have

$$\begin{aligned}
\text{LHS of (2.16)} &= \sum_{n_2} |\mathcal{R}_{k'}(0, n_2; t) - \mathcal{R}_k(0, n_2; t)|^2 \\
&\sim \sum_{n_2} \langle n_2 \rangle^{2s_2 - 2\varepsilon - 2 - 2\alpha} |\widehat{\rho_{k'}}(n_2) - \widehat{\rho_k}(n_2)|^2 \lesssim k^{-2\theta},
\end{aligned}$$

provided $s_2 < \frac{1}{2} + \alpha - \theta$. Finally, we have the contribution when $n_1 = n_2 = m_1 = m_2$, which occurs only if $n_1 + n_2, m_1 + m_2 \in 2\mathbb{Z}$. We then have

$$\text{LHS of (2.16)} \sim \sum_{n \in \mathbb{Z}} \big|\mathcal{R}_{k'}\big(\tfrac{n}{2}, \tfrac{n}{2}; t\big) - \mathcal{R}_k\big(\tfrac{n}{2}, \tfrac{n}{2}; t\big)\big|^2 \lesssim k^{-2\theta},$$

provided $s_2 < \frac{1}{2} + 2\alpha - \theta$. This completes the proof of (2.15). As $\theta > 0$ is arbitrary, we conclude the limit $\mathcal{N}(z)(t) \in W^{s_2,\infty}(\mathbb{T})$ for any $s_2 < 2\alpha$ and for every fixed $t \in [0,T]$, provided $\alpha > \frac{1}{4}$.

We now show

$$\mathbb{E}[\|\delta_h \mathcal{N}(z_k)(t)\|_{W^{s_2,\infty}}^q] \lesssim q^q |h|^q. \tag{2.18}$$

As before, this reduces to proving

$$\Big\| \sum_{n_1,n_2 \in \mathbb{Z}} \delta_h \mathcal{R}_k(n_1,n_2;t) g_{n_1} g_{n_2} \Big\|_{L^2(\Omega)} \lesssim |h|. \tag{2.19}$$

for $h \in [-1,1]$, uniformly in $t \in [0,T]$ and $k \in \mathbb{N}$. Expanding, we have

$$(\text{LHS of (2.19)})^2 = \sum_{\substack{n_1.n_2 \in \mathbb{Z} \\ m_1,m_2 \in \mathbb{Z}}} \delta_h \mathcal{R}_k(n_1,n_2;t) \overline{\delta_h \mathcal{R}_k(m_1,m_2;t)} \mathbb{E}[g_{n_1} g_{n_2} \overline{g_{m_1} g_{m_2}}].$$

Assuming all of n_1, n_2, m_1 and m_2 are non-zero and not equal, the expectation is non-zero only if $n_j = m_{\sigma(j)}$, where σ is a permutation of $\{1,2\}$. In this case, we get

$$\begin{aligned}
(\text{LHS of (2.19)})^2 &= \sum_{n_1,n_2 \in \mathbb{Z}} |\delta_h \mathcal{R}_k(n_1,n_2;t)|^2 \\
&\sim \sum_n \langle n \rangle^{2s_2 - 2\varepsilon - 2} \sum_{\substack{n_1,n_2 \in \mathbb{Z} \\ n = n_1 + n_2}} \frac{|\widehat{\rho_k}(n_1)\widehat{\rho_k}(n_2)|^2}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}} |e^{-ih[\varphi(n_1)+\varphi(n_2)]} - 1|^2 \\
&\lesssim |h|^2 \sum_n \langle n \rangle^{2s_2 - 2\varepsilon - 2} \sum_{\substack{n_1,n_2 \in \mathbb{Z} \\ n = n_1 + n_2}} \frac{|\varphi(n_1)|^2 + |\varphi(n_2)|^2}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}} \\
&\lesssim |h|^2,
\end{aligned}$$

provided $s < \frac{1}{2} + \alpha$. The remaining contributions to the expectation can be estimated in a similar fashion. We conclude by Proposition 2.5 that $\mathcal{N}(z_k)$ converges almost surely in $C_T W^{s_2,\infty}(\mathbb{T})$ to $\mathcal{N}(z)$ as long as $\alpha > \frac{1}{4}$. Taking a limit as $k \to \infty$ in (2.18) implies

$$\mathbb{E}[\|\delta_h \mathcal{N}(z)(t)\|_{W^{s_2,\infty}}^q] \lesssim q^q |h|^q.$$

Applying the arguments in Appendix B and a similar analysis as in (2.12) and (2.13), we obtain

$$\mathbb{P}\big(\|\mathcal{N}(z)\|_{C([0,T];W_x^{s_2,\infty}(\mathbb{T}))} > \lambda\big) \le e^{-C\frac{\lambda}{T^c}} + e^{-c\lambda}.$$

We now show that $\mathcal{N}(z)$ is independent of the chosen mollifying kernel. Given two mollifying kernels $\{\rho_k\}_{k \in \mathbb{N}}$ and $\{\eta_\ell\}_{\ell \in \mathbb{N}}$, let

$$\begin{aligned}
\mathcal{N}_\rho(z_k) &= \mathcal{N}(\rho_k * z), \\
\mathcal{N}_\eta(z_\ell) &= \mathcal{N}(\eta_\ell * z).
\end{aligned}$$

Additionally, suppose there exist events of full probability Ω_ρ and Ω_η and random variables $\mathcal{N}_\rho(z), \mathcal{N}_\eta(z) \in C([0,T]; W^{s_2,\infty}(\mathbb{T}))$ almost surely, such that

$$\begin{aligned} \mathcal{N}_\rho(z_k) \to \mathcal{N}_\rho(z) \ \text{ in } C_T W^{s_2,\infty}(\mathbb{T}) \text{ for every } \omega \in \Omega_\rho \text{ and} \\ \mathcal{N}_\eta(z_\ell) \to \mathcal{N}_\eta(z) \ \text{ in } C_T W^{s_2,\infty}(\mathbb{T}) \text{ for every } \omega \in \Omega_\delta, \end{aligned} \tag{2.20}$$

as $k, \ell \to \infty$. The goal is to show $\mathcal{N}_\rho(z) \equiv \mathcal{N}_\eta(z)$ almost surely (at least on $\Omega_\rho \cap \Omega_\delta$). We claim it suffices to establish the following difference estimate: there exists $\theta > 0$ sufficiently small such that

$$\mathbb{E}[\|\mathcal{N}_\rho(z_k)(t) - \mathcal{N}_\eta(z_\ell)(t)\|^q_{W^{s_2,\infty}(\mathbb{T})}] \lesssim q^q (k^{-\theta} + \ell^{-\theta})^q, \tag{2.21}$$

uniformly in $t \in [0,T]$. By (2.20), taking $k \to \infty$ gives

$$\mathbb{E}[\|\mathcal{N}_\rho(z)(t) - \mathcal{N}_\eta(z_\ell)(t)\|^q_{W^{s_2,\infty}(\mathbb{T})}] \lesssim q^q \ell^{-\theta q}.$$

Now taking $\ell \to \infty$ implies $\mathcal{N}_\eta(z_\delta)(t) \to \mathcal{N}_\rho(z)(t)$ in $L^q(\Omega; W^{s_2,\infty}(\mathbb{T}))$ and hence $\mathcal{N}_\rho(z)(t) = \mathcal{N}_\eta(z)(t)$ for each $t \in [0,T]$. As we have seen multiple times before, in order to prove (2.21), it suffices to prove

$$\Big\| \sum_{n_1,n_2 \in \mathbb{Z}} \big[\mathcal{R}^\rho_k - \mathcal{R}^\eta_\ell\big](n_1,n_2;t) g_{n_1} g_{n_2} \Big\|_{L^2(\Omega)} \lesssim k^{-\theta} + \ell^{-\theta}. \tag{2.22}$$

uniformly in $t \in [0,T]$, where $\mathcal{R}^\rho_k$ and $\mathcal{R}^\eta_\ell$ are defined as in (2.14) with the mollifiers ρ and η inserted appropriately. We expand out to get

$$\begin{aligned} (\text{LHS of (2.22)})^2 = \sum_{\substack{n_1.n_2 \in \mathbb{Z} \\ m_1,m_2 \in \mathbb{Z}}} \big[\mathcal{R}^\rho_k - \mathcal{R}^\eta_\ell\big](n_1,n_2;t) \overline{\big[\mathcal{R}^\rho_k - \mathcal{R}^\eta_\ell\big](m_1,m_2;t)} \\ \times \mathbb{E}[g_{n_1} g_{n_2} \overline{g_{m_1} g_{m_2}}]. \end{aligned}$$

We will just consider the case when $n_1 = m_1$ and $n_2 = m_2$ with remaining cases following by either symmetry or similar calculations. We have

$$(\text{LHS of (2.22)})^2 = \sum_n \langle n \rangle^{2s_2 - 2\varepsilon - 2} \sum_{\substack{n_1,n_2 \in \mathbb{Z} \\ n = n_1 + n_2}} \frac{|\widehat{\rho_k}(n_1)\widehat{\rho_k}(n_2) - \widehat{\eta_\ell}(n_1)\widehat{\eta_\ell}(n_2)|^2}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}}. \tag{2.23}$$

For fixed $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, $\widehat{\rho}(0) = 1$ and the mean value theorem imply $|1 - \widehat{\rho_k}(n_2)| \lesssim |n_2| k^{-1}$. Interpolating this with the trivial bound $|1 - \widehat{\rho_k}(n_2)| \lesssim 2$, gives

$$|1 - \widehat{\rho_k}(n_2)| \lesssim |n_2|^\theta k^{-\theta}, \tag{2.24}$$

for any $0 \le \theta \le 1$. A similar bound to (2.24) is also true for the mollifier η replacing the mollifier ρ. The triangle inequality and (2.24) imply

$$\begin{aligned} |\widehat{\rho_k}(n_1)\widehat{\rho_k}(n_2) - \widehat{\eta_\ell}(n_1)\widehat{\eta_\ell}(n_2)|^2 &\lesssim \sum_{j=1}^2 |\widehat{\rho_k}(n_j) - 1|^2 + |\widehat{\eta_\ell}(n_j) - 1|^2 \\ &\lesssim (|n_1|^{2\theta} + |n_2|^{2\theta})(k^{-2\theta} + \ell^{-2\theta}). \end{aligned}$$

Inserting this into (2.23) and applying Lemma 2.2 yields (2.22). Finally, as described above, to show z is independent of the choice of mollifier ρ, it suffices to obtain the estimate

$$\mathbb{E}[\|z_{k,\rho}(t) - z_{\ell,\eta}(t)\|^q_{W^{s_1,\infty}(\mathbb{T})}] \lesssim q^q (k^{-\theta} + \ell^{-\theta})^q, \tag{2.25}$$

for any $0 < \theta \le 1$. This follows easily using similar analysis as above and is thus omitted.

□

3. Probabilistic local theory on $\mathbb{T}$

3.1. **Proof of Theorem 1.1.** Our goal in this section will be to obtain the local theory as in Theorem 1.1. For the purposes of iteration of the probabilistic local theory to a global result, we consider first the deterministic perturbed initial value problem:

$$\begin{cases} i\partial_t v = \varphi(D_x)(v + \frac{1}{2}v^2 + z_1 v) + \frac{1}{2}z_2, \\ v|_{t=t_0} = v_0, \end{cases} \tag{3.1}$$

with initial data $v_0 \in H^s(\mathbb{T})$, $s \in (0,1)$ and under the following assumptions on the forcings (z_1, z_2):

$$z_1 \in C_{t,\text{loc}} W_x^{s_1,\infty}(\mathbb{R} \times \mathbb{T}) \qquad \text{and} \qquad z_2 \in C_{t,\text{loc}} H_x^s(\mathbb{R} \times \mathbb{T}), \tag{3.2}$$

where $s_1 < 0$.

Proposition 3.1. *Fix $-\frac{1}{2} < s_1 < 0$ and let*

$$\text{(i) } s \geq -s_1 \text{ when } s \in \big(0, \tfrac{1}{2}\big] \quad \text{or} \quad \text{(ii) } s \leq 1 + s_1 \text{ when } s \in \big(\tfrac{1}{2}, 1\big). \tag{3.3}$$

Then, there exists a constant $C > 0$ such that for every time interval $I = [t_0, t_1]$ of size 1, every $L \geq 1$, every $v_0 \in H^s(\mathbb{T})$, and every pair of forcings (z_1, z_2) satisfying (3.2) and such that

$$\|v_0\|_{H_x^s} + \|z_1\|_{C_t W_x^{s_1,\infty}(I \times \mathbb{T})} + \|z_2\|_{C_t H_x^s(I \times \mathbb{T})} \leq L, \tag{3.4}$$

there exists a unique solution $v \in C_t H_x^s([t_0, t_0 + C^{-1}L^{-1}] \times \mathbb{T})$ to (3.1).

Proof. For simplicity, we assume $I = [0,1]$. We fix $0 < T \leq 1$ to be chosen later. We will construct v as a fixed point of the operator

$$\Gamma v(t) := S(t)v_0$$

$$- \frac{i}{2}\int_0^t S(t-t')\varphi(D_x)v^2(t')\,dt' \tag{3.5}$$

$$- i\int_0^t S(t-t')\varphi(D_x)(z_1 v)(t')\,dt' \tag{3.6}$$

$$- \frac{i}{2}\int_0^t S(t-t')z_2(t')\,dt'. \tag{3.7}$$

in the ball

$$B_R := \{v \in L^\infty([0,T]; H^s(\mathbb{T})) \,:\, \|v\|_{L^\infty([0,T];H^s)} \leq R\},$$

with $R > 0$ also to be chosen later. By the unitarity of the linear operator $S(t)$ on H^s we have

$$\|S(t)v_0\|_{L_T^\infty H^s} = \|v_0\|_{H^s}.$$

We estimate each of (3.5), (3.6) and (3.7) separately.

(3.5): By Minkowski's inequality, unitarity of $S(t)$ on H^s and Lemma 2.1, we have

$$\|(3.5)\|_{L_T^\infty H_x^s} \lesssim T\|v\|_{L_T^\infty H^s}^2.$$

(3.7): From (3.4), we have

$$\left\| \int_0^t S(t-t')z_2(t')dt' \right\|_{L^\infty_T H^s_x} \lesssim T\|z_2\|_{L^\infty_T H^s} \lesssim TL.$$

(3.6): Consider first when $s \in (0, \frac{1}{2}]$. By Lemma 2.3, we have

$$\begin{aligned} \|\varphi(D_x)(z_1 v)\|_{H^s(\mathbb{T})} &= \|\langle \partial_x \rangle^s \varphi(D_x)(z_1 v)\|_{L^2(\mathbb{T})} \\ &\lesssim \|\langle \partial_x \rangle^{-(1-s)}(z_1 v)\|_{L^2(\mathbb{T})} \\ &\lesssim \|\langle \partial_x \rangle^{-s}(z_1 v)\|_{L^2(\mathbb{T})} \\ &\lesssim \|\langle \partial_x \rangle^{-s} z_1\|_{L^{\frac{1}{s}}(\mathbb{T})} \|\langle \partial_x \rangle^s v\|_{L^2(\mathbb{T})}. \end{aligned}$$

Now, provided $s \geq -s_1$, (3.4) implies $\|\varphi(D_x)(z_1 v)\|_{L^\infty_T H^s(\mathbb{T})} \lesssim L\|v\|_{L^\infty_T H^s(\mathbb{T})}$.

For $s \in (\frac{1}{2}, 1)$, we apply Lemma 2.3 as follows:

$$\begin{aligned} \|\varphi(D_x)(z_1 v)\|_{H^s(\mathbb{T})} &\lesssim \|\langle \partial_x \rangle^{-(1-s)}(z_1 v)\|_{L^2(\mathbb{T})} \\ &\lesssim \|\langle \partial_x \rangle^{-(1-s)} z_1\|_{L^{\frac{1}{1-s}}(\mathbb{T})} \|\langle \partial_x \rangle^{1-s} v\|_{L^2(\mathbb{T})} \\ &\lesssim L\|v\|_{H^s(\mathbb{T})} \end{aligned}$$

provided $s \leq 1 + s_1$.

With s satisfying (3.3), we have shown

$$\|\Gamma v\|_{L^\infty_T H^s} \leq C\|v_0\|_{H^s} + CTL + CTLR + CTR^2, \tag{3.8}$$

for any $v \in B_R$. Choosing $R = 4CL$ and $T \leq \min(\tilde{C}^{-1}K^{-1}, 1)$, (3.8) implies Γ maps B_R into itself for sufficiently small $T > 0$. Similarly, given $v_1, v_2 \in B_R$ with $v_1|_{t=0} = v_2|_{t=0} = v_0$, in the same way as we estimated the terms (3.5) and (3.6) above, we obtain

$$\begin{aligned} \|\varphi(D_x)\left[v_1^2 + 2z_1 v_1 - 2z_1 v_2 - v_2^2\right]\|_{H^s} &\lesssim \|\varphi(D_x)\left[(v_1 - v_2)(v_1 + v_2)\right]\|_{H^s} \\ &\quad + \|\varphi(D_x)\left[(v_1 - v_2)z_1\right]\|_{H^s} \\ &\lesssim (\|v_1\|_{H^s} + \|v_2\|_{H^s} + 1)\|v_1 - v_2\|_{H^s}, \end{aligned}$$

and by reducing $T > 0$ if necessary, this implies

$$\|\Gamma v_1 - \Gamma v_2\|_{L^\infty_T H^s} \leq \frac{1}{2}\|v_1 - v_2\|_{L^\infty_T H^s}.$$

Combining this with (3.8) shows Γ is a strict contraction from B_R into itself and hence has a unique fixed point $v^\omega \in L^\infty_T H^s_x(\mathbb{T})$ where $T \sim L^{-1}$. The continuity in time of v now follows by the continuity in time of z_1 and z_2 and that if $v \in L^\infty_T H^s_x$, then

$$\int_0^t S(t-t')\varphi(D_x)(v^2(t'))dt' \in C_T H^s_x.$$

We omit details. □

We now apply Proposition 3.1 to prove Theorem 1.1.

Proof of Theorem 1.1. Let $z_1 = z^\omega$, where z^ω solves the linear problem

$$\begin{cases} i\partial_t z^\omega = \varphi(D_x)(z^\omega) \\ z^\omega|_{t=0} = u_0^\omega, \end{cases}$$

with u_0^ω given by (1.8), and $z_2 = \mathcal{N}(z^\omega)$ defined in (1.4). From Proposition 2.6, we have

$$z^\omega \in C_{t,\text{loc}} W_x^{\alpha-\frac{1}{2}-,\infty}(\mathbb{R}\times\mathbb{T}) \qquad \text{and} \qquad \mathcal{N}(z^\omega) \in C_{t,\text{loc}} H_x^{2\alpha-}(\mathbb{R}\times\mathbb{T}),$$

almost surely, provided $\alpha > \frac{1}{4}$. So now we fix $\alpha \in (\frac{1}{4}, \frac{1}{2}]$ and let $s \in (\frac{1}{2}-\alpha, 2\alpha)$. For $0 < T \le 1$, we define $\Omega_T \subset \Omega$ by

$$\Omega_T := \{\omega \in \Omega : \|z^\omega\|_{C([0,1];W^{\alpha-\frac{1}{2}-,q(s)})} + \|\mathcal{N}(z^\omega)\|_{C([0,1];H^{2\alpha-})} \le C^{-1}T^{-1}\},$$

where

$$q(s) := \begin{cases} \frac{1}{s} & \text{if } s \in (0, \frac{1}{2}], \\ \frac{1}{1-s} & \text{if } s \in (\frac{1}{2}, 1). \end{cases} \tag{3.9}$$

By Proposition 2.6, we have $\mathbb{P}(\Omega_T^c) \le Ce^{-\frac{c}{T}}$. Then, for each $\omega \in \Omega_T$, we apply Proposition 3.1 to obtain a unique solution $v^\omega \in C([0,T]; H^s(\mathbb{T}))$ to (1.10). It follows, that for each $\omega \in \Omega_T$, there exists a solution $u^\omega = z^\omega + v^\omega$ to (1.3) on $[0,T]$. To obtain the almost sure existence, we set $\Sigma = \cup_{n=1}^\infty \Omega_{1/n}$ and note that $\mathbb{P}(\Sigma) = 1$. Hence, for every $\omega \in \Sigma$, there exists $T^\omega > 0$ and a unique solution $v^\omega \in C([0,T^\omega]; H^s(\mathbb{T}))$ to (1.10). □

3.2. **Sharpness of Theorem 1.1.** The limiting restriction $\alpha > \frac{1}{4}$ of the above argument arises from the nonlinear term of the second order Picard expansion:

$$\tilde{Z}^\omega(t) := -\frac{i}{2}\int_0^t S(t-t')\mathcal{N}(z(t'))\,dt'.$$

We show in this subsection that $\mathcal{N}(z)$ fails to be a distribution almost surely when $\alpha \le \frac{1}{4}$.

If Z were in fact a distribution almost surely at least for some $\alpha \ge \alpha_0$ where $\alpha_0 \le \frac{1}{4}$, then we may hope to lower the regularity restriction in Theorem 1.1 by considering the higher order perturbative expansion:

$$u = z^\omega + \tilde{Z}^\omega + w,$$

where we now solve the fixed point problem for the remainder w. The idea here is that the equation for w does not contain the term $\tilde{Z}$ which was responsible for the regularity restriction in solving (1.10) for the first order perturbative expansion v (see (3.7)). We thus expect w to be almost surely smoother than v. We show that for BBM, no improvement occurs because $\mathcal{N}(z)$ fails to be a distribution almost surely when $\alpha \le \frac{1}{4}$. We adapt an argument in [30] for the stochastic Burgers equation. For fixed α, let f belong to the support of the Gaussian measure μ_α in (1.12). For simplicity, we will show the Dirichlet projected regularisation $\mathcal{N}(f_N)$, where $f_N = \mathbf{P}_{\le N} f$, fails to define a distribution almost surely as $N \to \infty$. To this end, let $\phi \in C^\infty(\mathbb{T})$ be such that $\widehat{\phi}(0) = 0$ and $\phi \not\equiv 0$; for instance, we will assume $\widehat{\phi}(1) \ne 0$. Let $N \ge M \gg 1$ be dyadic and define

$$X_N(\phi) := \langle \mathcal{N}(f_N), \phi\rangle.$$

We will show $X_N(\phi)$ fails to converge almost everywhere with respect to μ_α. This implies $X_N^\omega(\phi) = \langle \mathcal{N}(f_N^\omega), \phi\rangle$, where f^ω is given by (1.8), fails to converge almost surely. We begin by showing the sequence $\{X_N(\phi)\}_N$ fails to converge in the Gaussian Hilbert space $L^2(\mu_\alpha)$.

Indeed, we have

$$\begin{aligned}\|X_N(\phi) - X_M(\phi)\|^2_{L^2(\mu_\alpha)} &= \mathbb{E}[|X_N^\omega(\phi) - X_M^\omega(\phi)|^2] \\ &= \sum_{n,m\neq 0} \varphi(n)\widehat{\phi}(n)\varphi(m)\widehat{\phi}(m) \sum_{(*)} \frac{\mathbb{E}[g_{n_1}g_{n_2}\overline{g_{m_1}g_{m_2}}]}{\langle n_1\rangle^\alpha \langle n_2\rangle^\alpha \langle m_1\rangle^\alpha \langle m_2\rangle^\alpha},\end{aligned}$$

where the inner summation above is restricted to the set of $(n_1, n_2, m_1, m_2) \in \mathbb{Z}^4$ satisfying:

$$n = n_1 + n_2, \quad m = m_1 + m_2, \quad M < \max(|n_1|, |n_2|) \le N, \quad M < \max(|m_1|, |m_2|) \le N.$$

Hence, we have

$$\begin{aligned}\mathbb{E}[|X_N^\omega(\phi) - X_M^\omega(\phi)|^2] &\gtrsim \sum_{n\neq 0} |\varphi(n)|^2 |\widehat{\phi}(n)|^2 \sum_{\substack{n=n_1+n_2 \\ M<|n_1|\le N}} \frac{1}{\langle n_1\rangle^{2\alpha}\langle n_2\rangle^{2\alpha}} \\ &\gtrsim |\varphi(1)|^2 |\widehat{\phi}(1)|^2 \sum_{M<|n_1|\le N} \frac{1}{\langle n_1\rangle^{2\alpha}\langle n_1 - 1\rangle^{2\alpha}} \\ &\sim N^{1-4\alpha}.\end{aligned}$$

Thus, if $\alpha \le \frac{1}{4}$, $X_N(\phi)$ fails to be a Cauchy sequence in $L^2(\mu_\alpha)$ and hence fails to converge in $L^2(\mu_\alpha)$. Now for every N, we have

$$\begin{aligned}X_N(\phi) = \langle \mathcal{N}(f_N), \phi\rangle &= \Big\langle \varphi(D_x)\Big(f_N^2 - \int_{\mathbb{T}} f_N^2 dx\Big), \phi \Big\rangle \\ &= \langle \varphi(D_x)(f_N^2 - \|f_N\|^2_{L^2(\mu_\alpha)}), \phi\rangle + \Big\langle \varphi(D_x)\Big(\|f_N\|^2_{L^2(\mu_\alpha)} - \int_{\mathbb{T}} f_N^2 dx\Big), \phi\Big\rangle \\ &= \langle \varphi(D_x)(f_N^2 - \|f_N\|^2_{L^2(\mu_\alpha)}), \phi\rangle \\ &=: Y_N(\phi).\end{aligned}$$

and for every N, $Y_N(\phi) \in \mathcal{H}_2$, the homogeneous Wiener chaos of order 2; see [34, Chapter II]. For elements in a fixed homogeneous Wiener chaos, convergence in L^2 is equivalent to convergence in probability; see [34, Theorem 3.50]. Therefore, $Y_N(\phi) = X_N(\phi)$ fails to converge in probability and hence $X_N^\omega(\phi)$ fails to converge almost surely when $\alpha \le \frac{1}{4}$. Applying the above with $f^\omega = z^\omega(t) = S(t)u_0^\omega$, we obtain the same conclusion, for each fixed t.

4. Probabilistic global theory on $\mathbb{T}$

In this section we show that when $\alpha = \frac{1}{2}$, we can extend the local-in-time (random) solutions constructed in Theorem 1.1 globally in time. Notice that when $\alpha = \frac{1}{2}$, the local solution $u(t) \in H^s(\mathbb{T})$ for $s < 0$ for each fixed t and thus almost surely does not belong to $L^2(\mathbb{T})$. From the general local theory in Proposition 3.1, we can extend the local-in-time solutions if we have an a priori bound on the remainder $v := u - z$ of the form (1.22) with $s = 2\alpha-$. Ultimately, we were able to obtain such a bound only when $\alpha = \frac{1}{2}$. We view this heuristically as overcoming the logarithmic divergence in (1.17). In the following though we will keep $\alpha \in (\frac{1}{4}, \frac{1}{2}]$ general.

In order to make the following computations secure, we consider the smoothed initial value problem (1.16) for v_k:

$$\begin{cases} i\partial_t v_k = \varphi(D_x)\big(v_k + \frac{1}{2}v_k^2 + z_k v_k\big) + \frac{1}{2}\mathcal{N}(z_k) \\ v_k|_{t=0} = 0, \end{cases} \tag{4.1}$$

where $z_k := \rho_k * z$ for some smooth mollifier $\{\rho_k\}_{k\in\mathbb{N}}$ and $k \in \mathbb{N}$. Notice that

$$\mathcal{N}(z_k) = -(1-\partial_x^2)^{-1}\partial_x(\mathbf{P}_{\neq 0}(z_k^2)). \tag{4.2}$$

As z_k is smooth, there is a unique smooth global-in-time solution v_k to (4.1) for every $k \in \mathbb{N}$. The brunt of the work will be to establish the following *uniform* in k bound on solutions v_k to (4.1). This is the content of Subsections 4.1 and 4.2.

Proposition 4.1. *Let $\alpha = \frac{1}{2}$ and $s < 1$ sufficiently close to one. Given $T, \varepsilon > 0$, there exists $\tilde{\Omega}_{T,\varepsilon} \subset \Omega$ such that*

$$\mathbb{P}((\tilde{\Omega}_{T,\varepsilon})^c) < \varepsilon,$$

a sufficiently large integer $k_0 = k_0(T,\varepsilon)$ and a finite constant $C(T,\varepsilon) > 0$ such that the following bound holds:

$$\sup_{k\geq k_0}\sup_{t\in[0,T]} \|v_k^\omega(t)\|_{H^s(\mathbb{T})} \leq C(T,\varepsilon), \tag{4.3}$$

for every solution v_k^ω to (4.1) with $\omega \in \tilde{\Omega}_{T,\varepsilon}$.

From this result, we iterate the probabilistic local theory in Subsection 3.1 to conclude Theorem 1.5; see Subsection 4.3.

To obtain the bound (4.3), we will apply the I-method in this probabilistic context, which we now describe. Given $\alpha \in (\frac{1}{4}, \frac{1}{2}]$, we fix $s := 2\alpha - \delta$, where $\delta > 0$ is to be sufficiently small. Given $N \geq 1$, let $I_N = I$ be the Fourier multiplier operator defined by $\widehat{If}(n) = m_N(n)\widehat{f}(n)$, where m_N is defined in (1.23). The operator I is smoothing of order $(1-s)$ and by the Littlewood-Payley square function theorem, for any $1 < p < \infty$, $s_0 \in \mathbb{R}$ and $0 \leq a \leq 1-s$, we have

$$\|If\|_{W^{s_0+a,p}(\mathbb{T})} \lesssim N^a \|f\|_{W^{s_0,p}(\mathbb{T})}. \tag{4.4}$$

We also have

$$\|f\|_{H^s} \lesssim \|If\|_{H^1} \tag{4.5}$$

and hence, in order to obtain (4.3), it suffices to obtain a uniform in k (sufficiently large) bound on Iv_k in H^1 to which we turn to in the next subsection.

The following probabilistic lemma quantifies the growth rate of the smoothed random linear solution Iz.

Lemma 4.2. *For any $p \geq 2$ and any fixed $t \in \mathbb{R}$, we have*

$$\mathbb{E}\Big[\|Iz(t)\|_{L_x^p}^p\Big]^{\frac{1}{p}} \leq C_s p^{\frac{1}{2}} \phi_{2\alpha}^{\frac{1}{2}}(N), \tag{4.6}$$

where $\phi_{2\alpha}$ is defined in (2.2). Furthermore, we have

$$\mathbb{P}\left(\frac{\|Iz(t)\|_{L_x^p}}{p^{\frac{1}{2}}\phi_{2\alpha}^{\frac{1}{2}}(N)} > \lambda\right) \leq \frac{C_s^p}{\lambda^p}. \tag{4.7}$$

Proof. We split $z = \mathbf{P}_{\le N} z + \mathbf{P}_{>N} z$ and consider each piece separately. For the low frequency one, (2.3) implies

$$\begin{aligned}\mathbb{E}\Big[\|I\mathbf{P}_{\le N} z\|_{L^p_x}^p\Big] &\lesssim \int_{\mathbb{T}} \mathbb{E}\Bigg[\Bigg|\sum_{|n|\le N} \frac{e^{it\varphi(n)} g_n(\omega)}{\langle n\rangle^{\alpha}}\Bigg|^p\Bigg]\,dx \\ &\lesssim \int_{\mathbb{T}} C^p p^{\frac{p}{2}} \Bigg(\sum_{|n|\le N} \frac{1}{\langle n\rangle^{2\alpha}}\Bigg)^{\frac{p}{2}} dx \lesssim p^{\frac{p}{2}} \phi_{2\alpha}^{\frac{p}{2}}(N).\end{aligned}$$

For the high frequency piece, (2.3) again implies

$$\begin{aligned}\mathbb{E}\Big[\|I\mathbf{P}_{>N} z\|_{L^p_x}^p\Big] = \int_{\mathbb{T}} \mathbb{E}\left[|I\mathbf{P}_{>N} z|^p\right] dx &\lesssim \int_{\mathbb{T}} C^p p^{\frac{p}{2}} \Bigg(\sum_{|n|>N} \frac{m_N(n)^2}{\langle n\rangle^{2\alpha}}\Bigg)^{\frac{p}{2}} dx \\ &\lesssim C^p p^{\frac{p}{2}} \Bigg(\sum_{|n|>N} \frac{N^{2(1-s)}}{\langle n\rangle^{1+(1-2s+2\alpha)}}\Bigg)^{\frac{p}{2}} \\ &\lesssim p^{\frac{p}{2}} N^{\frac{p}{2}(1-2\alpha)},\end{aligned}$$

where we note $1 - 2s + 2\alpha = 1 - 2\alpha + 2\delta > 0$. Then (4.7) follows from (4.6) and the Chebyshev inequality. □

We will actually only ever use Lemma 4.2 when $p = 2$. In this case, the set in (4.7) no longer depends on $t \in \mathbb{R}$ because the operators I and $S(t)$ commute and $S(t)$ is unitary on $L^2(\mathbb{T})$.

4.1. **Modified energy estimate.** Applying the I-operator to (4.1) and noting (4.2), we see that Iv_k satisfies

$$\begin{cases}\partial_t I v_k = -(1-\partial_x^2)^{-1}\partial_x\left[Iv_k + \frac{1}{2}I(v_k^2) + I(v_k z_k) + \frac{1}{2}I(\mathbf{P}_{\neq 0}(z_k^2))\right] \\ Iv_k|_{t=0} = 0,\end{cases} \tag{4.8}$$

We define the modified energy functional $E(Iv_k)(t) := \frac{1}{2}\|Iv_k(t)\|_{H^1}^2$. Using (4.8), we compute

$$\begin{aligned}E(Iv_k)(t) - E(Iv_k)(0) &= \int_0^t \int_{\mathbb{T}} (\partial_t I v_k)(Iv_k) + (\partial_t\partial_x I v_k)(\partial_x I v_k)\,dxdt' \\ &= \int_0^t \int_{\mathbb{T}} (Iv_k)(1-\partial_x^2)\partial_t(Iv_k)\,dxdt' \\ &= \frac{1}{2}\int_0^t \int_{\mathbb{T}} (\partial_x I v_k)[I(v_k^2) - (Iv_k)^2]\,dxdt' &\text{(I)} \\ &\quad + \frac{1}{2}\int_0^t \int_{\mathbb{T}} (\partial_x I v_k) I(\mathbf{P}_{\neq 0}(z_k^2))\,dxdt' &\text{(II)} \\ &\quad + \int_0^t \int_{\mathbb{T}} (\partial_x I v_k) I(v_k z_k)\,dxdt'. &\text{(III)}\end{aligned}$$

We now estimate each of (I) through (III) in the following section. Note that all implicit constants in these estimate will be independent of $k \in \mathbb{N}$. We also write $E_k(t) := E(Iv_k)(t)$.

• **Estimate for (I):** We begin with the following lemma which arranges for a negative power of N from the commutator.

Lemma 4.3. *Let $s > \frac{1}{2}$ and $w \in H^s(\mathbb{T})$. Then, we have*

$$\left| \int_{\mathbb{T}} (\partial_x Iw)[I(w^2) - (Iw)^2]\, dx \right| \lesssim N^{-\frac{3}{2}+} \|Iw\|^3_{H^1(\mathbb{T})}.$$

Proof. The argument here is similar to that in [49, Lemma 3.4]. By Plancherel, we have

$$\int_{\mathbb{T}} (\partial_x Iw)[I(w^2) - (Iw)^2]\, dx = \sum_{n_1+n_2+n_3=0} i n_3 m(n_3)(m(n_1+n_2) - m(n_1)m(n_2))\widehat{w}(n_1)\widehat{w}(n_2)\widehat{w}(n_3).$$

We symmetrise this to obtain

$$\sum_{n_1+n_2+n_3=0} M(n_1,n_2,n_3)\widehat{w}(n_1)\widehat{w}(n_2)\widehat{w}(n_3),$$

where M is defined to be the symmetric multiplier

$$\begin{aligned} M(n_1,n_2,n_3) = \frac{i}{3}[& n_1 m(n_1)(m(n_2+n_3) - m(n_2)m(n_3)) \\ & + n_2 m(n_2)(m(n_1+n_3) - m(n_1)m(n_3)) \\ & + n_3 m(n_3)(m(n_1+n_2) - m(n_1)m(n_2))]. \end{aligned}$$

By symmetry, we assume $|n_3| \le |n_2| \le |n_1|$. Furthermore, we assume $|n_1| > N$, since otherwise $m(n_j) = 1$ for all $j = 1,2,3$, which implies $M(n_1,n_2,n_3) = 0$ on $n_1+n_2+n_3 = 0$. In addition, we also assume $|n_2| \gtrsim N$ since if $|n_2| \ll N$ we obtain a contradiction to the conditions $n_1+n_2+n_3 = 0$, $|n_1| > N$ and $|n_3| \le |n_2|$. For shorthand, we define

$$\begin{aligned} \Lambda_N(\overline{n}) := \{(n_1,n_2,n_3) \in \mathbb{Z}^3 : n_1+n_2+n_3 = 0,\ & |n_3| \le |n_2| \le |n_1|, \\ & |n_1| > N,\ |n_2| \gtrsim N\}. \end{aligned}$$

Using the condition $n_1+n_2+n_3 = 0$, it is easy to verify

$$M(n_1,n_2,n_3) = \frac{i}{3}\left[n_1 m^2(n_1) + n_2 m^2(n_2) + n_3 m^2(n_3)\right],$$

and hence on $\Lambda_N(\overline{n})$, we have

$$|M(n_1,n_2,n_3)| \lesssim |n_3| m^2(n_3). \tag{4.9}$$

Setting $y(n) = \langle n \rangle m(n)\widehat{w}(n)$ and using (4.9), we have thus reduced to showing

$$\sum_{\Lambda_N(\overline{n})} \frac{|y(n_1)||y(n_2)||y(n_3)|}{\langle n_1\rangle\langle n_2\rangle m(n_1)m(n_2)} \lesssim N^{-\frac{3}{2}+}\|y(n)\|^3_{\ell^2_n}. \tag{4.10}$$

Now we note that on $\Lambda_N(\overline{n})$, we have for any $a \ge 1-s$,

$$\langle n_j\rangle^a m(n_j) \gtrsim N^a, \qquad j = 1,2.$$

Applying this with $a = 1$ and $a = \frac{1}{2}-$ (as $s > \frac{1}{2}$) and using Young's inequality, the left hand side of (4.10) is bounded by

$$\begin{aligned} CN^{-\frac{3}{2}+} \sum_{\Lambda_N(\overline{n})} \frac{|y(n_1)||y(n_2)||y(n_3)|}{\langle n_1 \rangle^{\frac{1}{2}+}} &\lesssim CN^{-\frac{3}{2}+} \|y(n)\|_{\ell_n^2}^2 \|\langle n \rangle^{-\frac{1}{2}-} y(n)\|_{\ell_n^1} \\ &\lesssim N^{-\frac{3}{2}+} \|y(n)\|_{\ell_n^2}^3, \end{aligned}$$

as required. □

As $\alpha > \frac{1}{4}$, we have $s > \frac{1}{2}$ and hence Lemma 4.3, implies

$$|(\mathrm{I})| = \left| \frac{1}{2} \int_0^t \int_{\mathbb{T}} (\partial_x I v_k)[I(v_k^2) - (I v_k)^2] \, dx dt' \right| \lesssim N^{-\frac{3}{2}+} \int_0^t E_k^{\frac{3}{2}}(t') dt'. \tag{4.11}$$

• **Estimate for (II):** By Cauchy-Schwarz and (4.4), we have

$$\begin{aligned} |(\mathrm{II})| &= \left| \int_0^t \int_{\mathbb{T}} (\partial_x I v_k) I(\mathbf{P}_{\neq 0}(z_k^2)) \, dx dt' \right| \\ &\leq \|I(\mathbf{P}_{\neq 0}(z_k^2))\|_{L^\infty([0,t];L^2)} \int_0^t E_k^{\frac{1}{2}}(t') dt' \\ &\lesssim N^{1-2\alpha+} \|\mathbf{P}_{\neq 0}(z_k^2)\|_{L^\infty([0,t];H^{2\alpha-1-})} \int_0^t E_k^{\frac{1}{2}}(t') dt'. \end{aligned} \tag{4.12}$$

Notice that $\|\mathbf{P}_{\neq 0}(z_k^2)\|_{L^\infty([0,t];H^{2\alpha-1-})} \sim \|\mathcal{N}(z_k)\|_{L^\infty([0,t];H^{2\alpha-})}$.

• **Estimate for (III):**

Lemma 4.4. *Let $\frac{1}{4} < \alpha \leq \frac{1}{2}$, $w \in H^s(\mathbb{T})$ and $k \in \mathbb{N} \cup \{\infty\}$. Then, there exists $p = p(\alpha, s) \gg 2$ sufficiently large, so that*

$$\|I(wz_k) - (Iw)(Iz_k)\|_{L^2} \lesssim N^{-\left(s - \frac{1}{2} - \frac{1}{p} +\right)} \|Iw\|_{H^1} \|z_k\|_{W^{\alpha - \frac{1}{2} -, p}}, \tag{4.13}$$

where we understand $z_\infty := z$.

Proof. We may suppose $k = \infty$. Let $\sigma := \frac{1}{2} - \alpha + \frac{1}{10}\delta$. Split $w = w_{N^{\frac{1}{2}}} + w^{N^{\frac{1}{2}}}$, and $z = z_N + z^N$, where $f_N := \mathcal{F}^{-1}\{\mathbb{1}_{|n| \leq N/2} \widehat{f}(n)\}$ and $f^N := \mathcal{F}^{-1}\{\mathbb{1}_{|n| > N/2} \widehat{f}(n)\}$. Then

$$\begin{aligned} I(wz) - (Iw)(Iz) =& I(w_{N^{\frac{1}{2}}} \cdot z_N) - I(w_{N^{\frac{1}{2}}}) I(z_N) && \text{(A)} \\ &+ I(w_{N^{\frac{1}{2}}} \cdot z^N) - (I w_{N^{\frac{1}{2}}})(I z^N) && \text{(B)} \\ &- I(w^{N^{\frac{1}{2}}})(Iz) && \text{(C)} \\ &+ I(w^{N^{\frac{1}{2}}} \cdot z). && \text{(D)} \end{aligned}$$

We estimate each piece above separately.

• **(A):** Since I is the identity on frequencies $\{|n| \leq N\}$, we have $I(w_{N^{\frac{1}{2}}}) = w_{N^{\frac{1}{2}}}$ and $I(z_N) = z_N$. Next, notice that $\operatorname{supp} \mathcal{F}\{w_{N^{\frac{1}{2}}} \cdot z_N\} \subset \{|n| \leq N\}$ and hence $I(w_{N^{\frac{1}{2}}} \cdot z_N) = w_{N^{\frac{1}{2}}} \cdot z_N$. Combining these two observations we see that (A) $\equiv 0$.

• **(B):** In this case, we argue by duality:

$$\|(\mathrm{B})\|_{L^2(\mathbb{T})} = \sup_{\|h\|_{L^2(\mathbb{T})}=1} \left| \int_{\mathbb{T}} h\,(\mathrm{B}) dx \right|$$

Denote $f(n_1) := \widehat{w_{N^{\frac{1}{2}}}}(n_1)$ and $g(n_2) := \widehat{z^N}(n_2)$. By Parseval, we have

$$\left| \int_{\mathbb{T}} h\,(\mathrm{B}) dx \right| = \sum_{\substack{|n_1|<N^{1/2}/2 \\ |n_2|>N/2}} |m(n_1)m(n_2) - m(n_1+n_2)||f(n_1)||g(n_2)||\widehat{h}(n_1+n_2)|. \quad (4.14)$$

In this regime, $m(n_1) \equiv 1$. The mean value theorem implies $|m(n_2) - m(n_1+n_2)| \lesssim N^{1-s}|n_1||n_2|^{-2+s}$. Thus,

$$\begin{aligned}
(4.14) &\lesssim N^{1-s} \sum_{\substack{|n_1|<N^{1/2}/2 \\ |n_2|>N/2}} \frac{|n_1|^{\frac{1}{2}+\frac{1}{10}\delta}}{|n_2|^{2-s-\sigma}} \frac{|n_1 m(n_1) f(n_1)|}{|n_1|^{\frac{1}{2}+\frac{1}{10}\delta}} \frac{|g(n_2)|}{|n_2|^{\sigma}} |\widehat{h}(n_1+n_2)| \\
&\lesssim N^{1-s+\frac{1}{4}+\frac{\delta}{20}-2+s+\sigma} \left\| \frac{|n_1 m(n_1) f(n_1)|}{|n_1|^{\frac{1}{2}+\frac{1}{10}\delta}} \right\|_{\ell^1_{n_1}} \left\| \frac{|g(n_2)|}{|n_2|^{\sigma}} \right\|_{\ell^2_{n_2}} \|\widehat{h}\|_{\ell^2} \\
&\lesssim N^{-\left(\frac{1}{4}+\frac{s}{2}+\right)} \|Iv\|_{H^1} \|z\|_{H^{-\sigma}} \|h\|_{L^2}.
\end{aligned}$$

We thus have $\|(B)\|_{L^2} \lesssim N^{-\left(\frac{1}{4}+\frac{s}{2}+\right)} \|Iv\|_{H^1} \|z\|_{H^{-\sigma}}$.

• **(C):** By the Sobolev embedding theorem and the mapping property of I (4.4), we have

$$\begin{aligned}
\|(\mathrm{C})\|_{L^2} = \|(Iv^{N^{\frac{1}{2}}})(Iz)\|_{L^2} &\lesssim \|Iv^{N^{\frac{1}{2}}}\|_{L^2} \|Iz\|_{L^\infty} \\
&\lesssim N^{-\frac{s}{2}} \|v^{N^{\frac{1}{2}}}\|_{H^s} \|Iz\|_{W^{\frac{1}{p}+\frac{1}{10}\delta,p}} \\
&\lesssim N^{-\left(\frac{s}{2}-\frac{1}{p}-\sigma-\frac{1}{10}\delta\right)} \|Iv\|_{H^1} \|z\|_{W^{-\sigma,p}} \\
&\lesssim N^{-\left(s-\frac{1}{2}-\frac{1}{p}+\right)} \|Iv\|_{H^1} \|z\|_{W^{-\sigma,p}},
\end{aligned}$$

provided that

$$\frac{1}{4} + \frac{1}{2p} + \frac{1}{2}\left(\frac{\delta}{2} + \frac{2}{10}\delta\right) < \alpha < \frac{1}{2} - \frac{1}{p} + \delta - \frac{2}{10}\delta.$$

The lower bound above appears to ensure we have a negative power of N while the upper bound is due to the mapping property of I (4.4). We can afford these conditions on α if we choose $p \gg \frac{1}{\delta}$.

• **(D):** Once again, we argue by duality writing

$$\|(\mathrm{D})\|_{L^2(\mathbb{T})} = \sup_{\|h\|_{L^2(\mathbb{T})}=1} \left| \int_{\mathbb{T}} h(\mathrm{D}) dx \right|.$$

Now by the fractional Leibniz rule, we have

$$\begin{aligned}\left|\int_{\mathbb{T}} hI(v^{N^{\frac12}}\cdot z)\,dx\right| &= \left|\int_{\mathbb{T}} I(h)v^{N^{\frac12}} z\,dx\right| \\ &\lesssim \|I(h)v^{N^{\frac12}}\|_{W^{\sigma,\frac{p}{p-1}}}\|z\|_{W^{-\sigma,p}} \\ &\lesssim \|z\|_{W^{-\sigma,p}}\left(\|I(h)\|_{L^{\frac{2p}{p-2}}}\|v^{N^{\frac12}}\|_{H^\sigma}+\|I(h)\|_{H^\sigma}\|v^{N^{\frac12}}\|_{L^{\frac{2p}{p-2}}}\right).\end{aligned}$$

For each of these four terms, we use:

- By the Sobolev inequality and (4.4), $\|Ih\|_{L^{\frac{2p}{p-2}}}\lesssim\|Ih\|_{H^{\frac1p}}\lesssim N^{\frac1p}$, since $\frac1p<1-s$ provided we choose $p\gg\frac1\delta$,
- $\|Ih\|_{H^\sigma}\lesssim N^\sigma$, since $\sigma<1-s$ which is true as $\alpha\le\frac12$,
- $\|v^{N^{\frac12}}\|_{H^\sigma}\lesssim N^{-\frac12(s-\sigma)}\|Iv\|_{H^1}$,
- $\|v^{N^{\frac12}}\|_{L^{\frac{2p}{p-2}}}\lesssim N^{-\frac12\left(s-\frac1p\right)}\|Iv\|_{H^1}$.

With these, we obtain

$$\begin{aligned}\|(\mathrm{D})\|_{L^2} &\lesssim \left(N^{-\left(\frac s2-\frac\sigma2-\frac1p\right)}+N^{-\left(\frac s2-\sigma-\frac1{2p}\right)}\right)\|Iv\|_{H^1}\|z\|_{W^{-\sigma,p}} \\ &\lesssim N^{-\left(s-\frac12+\right)}\|Iv\|_{H^1}\|z\|_{W^{-\sigma,p}}.\end{aligned}$$

Finally, combining the results of (A) through (D) we obtain (4.13) with $p=p(\alpha,s)=\frac{100}{2\alpha-s}$. ☐

Lemma 4.5. *Let $s>\frac12$ and $k\in\mathbb{N}\cup\{\infty\}$. Then, we have*

$$\left|\int_0^t\int_{\mathbb{T}}(\partial_x Iv_k)(Iv_k)(Iz_k)dxdt'\right|\lesssim\|Iu_0^\omega\|_{L^2_x}\left(\int_0^t E_k(Iv_k)dt'\right).$$

Proof. By the algebra property of $H^s(\mathbb{T})$ and Cauchy-Schwarz, we compute

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{T}}(\partial_x Iv_k)(Iv_k)(Iz_k)dxdt'\right| &= \frac12\left|\int_0^t\int_{\mathbb{T}}\partial_x[(Iv_k)^2](Iz_k)dxdt'\right| \\ &\lesssim \int_0^t\|(Iv_k)^2\|_{H^1_x}\|Iz_k\|_{L^2_x}dt' \\ &\lesssim \int_0^t\|Iv_k\|_{H^1}^2\|IS(t')u_{0,k}^\omega\|_{L^2_x}dt' \\ &\lesssim \|Iu_0^\omega\|_{L^2_x}\left(\int_0^t E(Iv_k)(t')dt'\right).\end{aligned}$$

☐

Writing

$$\begin{aligned}\int_{\mathbb{T}}(\partial_x Iv_k)I(v_kz_k) &= \int_{\mathbb{T}}(\partial_x Iv_k)[I(v_kz_k)-(Iv_k)(Iz_k)]dx \\ &\quad+\int_{\mathbb{T}}(\partial_x Iv_k)(Iv_k)(Iz_k)dx,\end{aligned}$$

we have from Lemmas 4.4 and 4.5,

$$|(\mathrm{III})| \lesssim N^{-\left(s-\frac{1}{2}+\right)} \|z_k\|_{L^\infty([0,t];W^{\alpha-\frac{1}{2}-,p})} \int_0^t E_k(t')dt' + \|Iu_0^\omega\|_{L_x^2} \int_0^t E_k(t')dt'. \tag{4.15}$$

Combining (4.11), (4.12) and (4.15), we have shown the following energy estimate:

$$\begin{aligned} E(Iv_k)(t) \leq & E(Iv_k)(0) + C_{s,\alpha} N^{-\frac{3}{2}+} \int_0^t E^{\frac{3}{2}}(Iv_k)(t')dt' \\ & + C_{s,\alpha} N^{1-2\alpha+} \|\mathcal{N}(z_k)\|_{L^\infty([0,t];H^{2\alpha-})} \int_0^t E^{\frac{1}{2}}(Iv_k)(t')dt' \\ & + C_{s,\alpha} N^{-\left(s-\frac{1}{2}+\right)} \|z_k\|_{L^\infty([0,t];W^{\alpha-\frac{1}{2}-,p})} \int_0^t E(Iv_k)(t')dt' \\ & + C_{s,\alpha} \|Iu_0^\omega\|_{L_x^2} \int_0^t E(Iv_k)(t')dt'. \end{aligned} \tag{4.16}$$

4.2. **Proof of Propostion 4.1.** In this subsection, we complete the proof of Proposition 4.1 by turning the inequality (4.16) into a bound on Iv_k in H^1 and hence a bound on v_k in H^s. To begin, we first deal with a technical point: we must ensure that the set $\tilde{\Omega}_{T,\varepsilon}$ we construct in Proposition 4.1 is independent of all $k \in \mathbb{N}$ large enough. The point here is that while Young's inequality ensures

$$\|z_k\|_{C([0,2T];W^{\alpha-\frac{1}{2}-,p})} \leq \|z\|_{C([0,2T];W^{\alpha-\frac{1}{2}-,p})},$$

for every $k \in \mathbb{N}$ and $p > 1$, we cannot conclude a similar statement for the nonlinearities $\mathcal{N}(z_k)$ and $\mathcal{N}(z)$. To get around this, we appeal to the almost sure convergence of their norms from Proposition 2.6.

With $T > 0$ fixed, we define

$$\Sigma_{\mathrm{conv},T} := \{\omega \in \Omega \,:\, (z_k^\omega, \mathcal{N}(z_k^\omega)) \to (z^\omega, \mathcal{N}(z^\omega)) \text{ in } C_{2T}W^{\alpha-\frac{1}{2}-,r} \times C_{2T}H^s \text{ as } k \to \infty\},$$

where $r(s,\alpha) := \max\{p(s,\alpha), q(s)\}$ with $p(s,\alpha)$ given by Lemma 4.4 and $q(s)$ given by (3.9). Put simply, $\Sigma_{\mathrm{conv},T}$ is the set on which we have almost sure convergence of the mollified enhanced data set (in the appropriate norms). That this set is of full probability is a direct consequence of Proposition 2.6. With $K > 0$ fixed, we also define

$$\Omega_{K,T,\alpha} = \{\omega \in \Sigma_{\mathrm{conv},T} \,:\, \|z\|_{C_{2T}W^{\alpha-\frac{1}{2}-,r}} + \|\mathcal{N}(z)\|_{C_{2T}H^s} \leq K\}.$$

By Egorov's theorem, for any $\varepsilon > 0$, there exists a measurable set $\Omega_\varepsilon \subset \Sigma_{\mathrm{conv},T}$ with $\mathbb{P}(\Sigma_{\mathrm{conv},T} \setminus \Omega_\varepsilon) < \frac{\varepsilon}{3}$ such that $\mathcal{N}(z_k^\omega)$ converges uniformly to $\mathcal{N}(z^\omega)$ as $k \to \infty$ in $C_{2T}H_x^s$ for every $\omega \in \Omega_\varepsilon$. Hence, there exists $k_0 = k_0(T,\varepsilon)$ such that for every $k \geq k_0$, we have

$$\|\mathcal{N}(z_k^\omega)\|_{C_{2T}H_x^s} \leq 1 + K \tag{4.17}$$

for every $\omega \in \Omega_{K,T,\alpha,\varepsilon} := \Omega_\varepsilon \cap \Omega_{K,T,\alpha}$. Now, Proposition 2.6 implies

$$\mathbb{P}(\Omega_{K,T,\alpha,\varepsilon}^c) < e^{-C\frac{K}{T^c}} + \frac{\varepsilon}{4}. \tag{4.18}$$

We will need the following nonlinear Gronwall inequality which follows from [24, Theorem 21]:

Lemma 4.6. *Given $T > 0$, let f be a non-negative function on $[0, T]$ satisfying*

$$f(t) \leq c + a \int_0^t f(t')\, dt' + b \int_0^t f^\gamma(t')\, dt', \tag{4.19}$$

where $a, b, c \geq 0$, $0 \leq \gamma < 1$ and for $t \in [0, T]$. Then, we have

$$f^{1-\gamma}(t) \leq c^{1-\gamma} e^{(1-\gamma)at} + \frac{b}{a}(e^{(1-\gamma)at} - 1)$$

for $t \in [0, T]$.

Proof of Proposition 4.1. Fix $T, \varepsilon > 0$. For $\Lambda, K > 0$ to be determined later, we set

$$\Omega_{\Lambda,N} = \{\omega \in \Omega \,:\, A(N) \leq \Lambda\}, \tag{4.20}$$

where we have defined

$$A(N) := \frac{\|Iu_0^\omega\|_{L^2}}{2^{\frac{1}{2}} \phi_{2\alpha}^{\frac{1}{2}}(N)},$$

and let $\Omega_{K,T,\alpha,\varepsilon}$ be defined as above. We now fix $\omega \in \Omega_{K,T,\alpha,\varepsilon} \cap \Omega_{\Lambda,N}$ and $k \geq k_0$. From (4.16) and (4.17), we have

$$\begin{aligned} E_k(t) \leq\, & N^{-\frac{3}{2}+} \int_0^t E_k^{\frac{3}{2}}(t')dt' && \text{(I)} \\ & + N^{1-2\alpha+} K \int_0^t E_k^{\frac{1}{2}}(t')dt' && \text{(II)} \\ & + \big(N^{-(s-\frac{1}{2}+)}K + A(N)\phi_{2\alpha}^{\frac{1}{2}}(N)\big) \int_0^t E_k(t')dt'. && \text{(III)} \end{aligned}$$

Let

$$\overline{T}_k = \sup\{t > 0 \,:\, E(Iv_k)(t) \leq C_{K,T,s,\alpha} N^{2-}\},$$

where we stress that $C_{K,T,s,\alpha}$ is independent of any $k \geq k_0$. From $E(Iv_k)(0) = 0$ and continuity in time of $E_k(t)$ (since $v_k \in C_T H_x^1$, at least), $\overline{T}_k > 0$. For $t \in [0, \overline{T}_k]$, (I) is dominated by (II) and hence

$$E_k(t) \leq C_{K,T,s,\alpha} \Lambda \phi_{2\alpha}^{\frac{1}{2}}(N) \int_0^t E_k(t')dt' + C_{K,T,s,\alpha} N^{1-2\alpha+} \int_0^t E_k^{\frac{1}{2}}(t')dt' \tag{4.21}$$

By Lemma 4.6, this implies

$$E_k^{\frac{1}{2}}(t) \leq \frac{N^{1-2\alpha+}}{\phi_{2\alpha}^{\frac{1}{2}}(N)\Lambda} \big(e^{\frac{1}{2}C_{K,T,s,\alpha}\phi_{2\alpha}^{\frac{1}{2}}(N)\Lambda t} - 1\big).$$

Now by continuity $E_k(\overline{T}_k) = C_{K,T,s,\alpha} N^{2-}$ and therefore the above inequality implies

$$\overline{T}_k \geq \frac{2 \log \big(1 + C_{K,T,s,\alpha}^{\frac{1}{2}} N^{2\alpha-} \phi_{2\alpha}^{\frac{1}{2}}(N)\Lambda\big)}{C_{K,T,s,\alpha}\phi_{2\alpha}^{\frac{1}{2}}(N)\Lambda}. \tag{4.22}$$

Notice that this lower bound is independent of $k \geq k_0$ and as our k was arbitrary, $\overline{T} := \inf_{k \geq k_0} \overline{T}_k$ is bounded below by the same quantity. Now given $\varepsilon > 0$, Proposition 2.6, Lemma 4.2 and (4.18) allow us to choose $K = K(\varepsilon, T)$ and $\Lambda = \Lambda(\varepsilon, s)$ large enough so that

$$\mathbb{P}(\Omega^c_{K,T,\frac{1}{2},\varepsilon}) + \mathbb{P}(\Omega^c_{\Lambda,N}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus when $\alpha = \frac{1}{2}$, (2.2) and (4.22) imply

$$\overline{T} \geq C_{\varepsilon,T,s} \log^{\frac{1}{2}}(N).$$

We now choose

$$N = N(\varepsilon, T, s) = \exp\left(\frac{4T^2}{C^2_{\varepsilon,T,s}}\right) \tag{4.23}$$

so that $\overline{T} \geq 2T$ and hence

$$\sup_{0 \leq t \leq T} E(Iv_k)(t) \leq C(\varepsilon, T), \qquad \text{on} \qquad \tilde{\Omega}_{T,\varepsilon} := \Omega_{K(\varepsilon,T),T,\frac{1}{2},\varepsilon} \cap \Omega_{\Lambda(\varepsilon,s),N(\varepsilon,T)},$$

for any $k \geq k_0(T, \varepsilon)$. By (4.5), the definition of the modified energy $E(Iv_k)$ and (4.23) we obtain (4.3). This completes the proof of Proposition 4.1.

□

4.3. **Proof of Theorem 1.5.** Our aim here is to iterate the local theory for solutions v to (1.10) by showing that the bound (4.3) also holds for v in place of v_k. To do this, we decompose the whole interval $[0, T]$ into $\lceil \frac{T}{\delta} \rceil$-many subintervals $I_j := [j\delta, (j + 1)\delta] \cap [0, T]$ where δ is to be determined. Let

$$\Omega_{\text{lwp}} := \cap_{j=0}^{\lceil \frac{T}{\delta} \rceil} \{\omega \in \Omega \,:\, \|z\|_{C_{I_j} W^{\alpha - \frac{1}{2}-, r(s)}} + \|\mathcal{N}(z)\|_{C_{I_j} H^s} \leq L\}$$

With $\delta = \delta(L) \lesssim L^{-1}$ and for $\omega \in \Omega_{T,\varepsilon} := \Omega_{\text{lwp}} \cap \tilde{\Omega}_{T,\frac{\varepsilon}{2}}$, we have by Proposition 3.1, that v exists on $[0, \delta]$ and solves (1.10). By Theorem 1.11, we may take the limit $k \to \infty$ in (4.3) to obtain

$$\sup_{t \in [0,\delta]} \|v(t)\|_{H^s(\mathbb{T})} \leq C(T, \varepsilon) < \infty,$$

Then by reducing δ further so that

$$\delta \sim (C(T, \varepsilon) + L)^{-1},$$

we conclude v now exists on $[0, 2\delta]$ and using Theorem 1.11 we may take the limit $k \to \infty$ in (4.3) to obtain

$$\sup_{t \in [0,2\delta]} \|v(t)\|_{H^s(\mathbb{T})} \leq C(T, \varepsilon) < \infty.$$

Iterating in this manner finitely many times shows v exists on $[0, T]$ and satisfies the bound

$$\sup_{t \in [0,T]} \|v(t)\|_{H^s(\mathbb{T})} \leq C(T, \varepsilon) < \infty.$$

It remains to check that Ω_{lwp}^c stays small. By Proposition 2.6, we have

$$\begin{aligned}
\mathbb{P}(\Omega_{\text{lwp}}^c) &\leq \sum_{j=0}^{\lceil \frac{T}{\delta} \rceil} \mathbb{P}(\|z\|_{C_{I_j} W^{\alpha-\frac{1}{2}-,r(s)}} + \|\mathcal{N}(z)\|_{C_{I_j} H^s} > L) \\
&\lesssim \frac{T}{\delta} e^{-C\frac{L}{\delta^c}} \\
&\lesssim T(L + C(T,\varepsilon)) e^{-CL(L+C(T,\varepsilon))^c} \\
&\lesssim T(L + C(T,\varepsilon)) e^{-CL} < \frac{\varepsilon}{2},
\end{aligned}$$

by choosing $L = L(T,\varepsilon)$ sufficiently large. Thus

$$\mathbb{P}(\Omega_{T,\varepsilon}^c) \leq \mathbb{P}(\Omega_{\text{lwp}}^c) + \mathbb{P}(\tilde{\Omega}_{T,\frac{\varepsilon}{2}}^c) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

To obtain the almost sure existence is a standard argument. We detail it for the convenience of the reader. The set $\Omega_{T,\varepsilon}$ depends on K, Λ and N which in turn depend on T and ε. Given $\varepsilon > 0$, let $T_j = 2^j$ and set $\varepsilon_j = \varepsilon / T_j$. From the above, we obtain sets $\Omega_{T_j,\varepsilon_j}$ by choosing $K_j = K(T_j, \varepsilon_j)$, $\Lambda_j = \Lambda(\varepsilon_j, s)$ large enough so that $\mathbb{P}(\Omega_{T_j,\varepsilon_j}^c) < \varepsilon_j$ and then $N_j = N(T_j, \varepsilon_j)$ as in (4.23) (with T and ε replaced by T_j and ε_j) which implies, as above, v exists on $[0, T_j]$ and satisfies

$$\sup_{t \in [0,T_j]} \|v(t)\|_{H^s(\mathbb{T})} \leq C(T_j, \varepsilon_j) < \infty.$$

Then the set $\Omega_\varepsilon = \cap_{j=1}^\infty \Omega_{T_j,\varepsilon_j}$ has measure $\mathbb{P}(\Omega_\varepsilon^c) < \varepsilon$ with the property that for any $\omega \in \Omega_\varepsilon$, there exists a unique solution $v^\omega \in C([0,\infty); H_x^s(\mathbb{T}))$ to (1.10), and hence $u^\omega = z^\omega + v^\omega$ solves (1.3) on $[0,\infty)$. Then, the same property is true on $\Sigma := \cup_{\varepsilon>0} \Omega_\varepsilon$ and $\mathbb{P}(\Sigma) = 1$.

Remark 4.7. As mentioned earlier, the conservation of the energy (1.7) yields global well-posedness of the BBM equation (1.1) in $H^1(\mathbb{T})$. For data in[8] $L^2(\mathbb{T})$, Bona and Tzvetkov [10] employed a low-high (or long wave-short wave) splitting argument to globalise solutions to (1.1). Their idea was to split the data $u_0 \in L^2(\mathbb{T})$ as

$$u_0 = \mathbf{P}_{\leq N} u_0 + \mathbf{P}_{>N} u_0$$

and to write the (local) solution as

$$u = u_{\text{low}} + u_{\text{high}},$$

where the high part u_{high} solves the original (nonlinear) BBM equation (1.3) with $u_{\text{high}}|_{t=0} = \mathbf{P}_{>N} u_0$ and the low part u_{low} solves the difference equation

$$\partial_t u_{\text{low}} - \partial_{xxt} u_{\text{low}} + \partial_x u_{\text{low}} + u_{\text{low}} \partial_x u_{\text{low}} + u_{\text{low}} \partial_x u_{\text{low}} + \partial_x (u_{\text{low}} u_{\text{high}}) = 0, \tag{4.24}$$

with $u_{\text{low}}|_{t=0} = \mathbf{P}_{\leq N} u_0$. Then given any $T > 0$, by choosing $N = N(T)$ sufficiently large, we ensure that $\mathbf{P}_{>N} u_0$ is so small in $L^2(\mathbb{T})$ that by local theory, $u_{\text{high}}(t) \in L^2(\mathbb{T})$ exists on $[0,T]$. Meanwhile, since $\mathbf{P}_{\leq N} u_0$ is smooth, we can solve (4.24) locally in time within H^1.

[8]Their argument works on both $\mathbb{T}$ and $\mathbb{R}$; see also [44].

To show that u_{low} exists up to time T, a computation shows

$$\begin{aligned}\left|\frac{d}{dt}\left(\frac{1}{2}\|u_{\text{low}}(t)\|_{H^1}^2\right)\right| &= \left|\int u_{\text{high}} u_{\text{low}} \partial_x u_{\text{low}}\, dx\right| \\ &\leq \|u_{\text{high}}(t)\|_{L^2} \|u_{\text{low}}(t)\|_{L^\infty} \|\partial_x u_N(t)\|_{L^2} \\ &\leq C\|u_{\text{low}}(t)\|_{L^2} \|u_{\text{low}}(t)\|_{H^1}^2.\end{aligned}$$

By Gronwall's inequality, we get the a priori bound

$$\begin{aligned}\sup_{t\in[0,T]} \|u_{\text{low}}(t)\|_{H^1} &\leq \|\mathbf{P}_{\leq N} u_0\|_{H^1} \exp\left(C\int_0^T \|u_{\text{high}}(t')\|_{L^2}\, dt'\right) \\ &\lesssim \|\mathbf{P}_{\leq N} u_0\|_{H^1} \exp(CT\|\mathbf{P}_{>N} u_0\|_{L^2}).\end{aligned}$$

Thus u_{low} lives up to time $t = T$ which completes the argument. Let us emphasise here that u_{low} does not solve (1.3), but rather the perturbed equation (4.24) with the additional linear term $\partial_x(u_{\text{low}} u_{\text{high}})$. So whilst the 'energy' $E(u_{\text{low}}(t)) = \frac{1}{2}\|u_{\text{low}}(t)\|_{H^1}^2$ is no longer conserved, its growth can still be controlled.

A natural modification of the I-method based argument above would be to include the low-high splitting idea in [10]. With $M > 0$ to be fixed later, we would set

$$u_0^\omega = \mathbf{P}_{\leq M} u_0 + \mathbf{P}_{>M} u_0^\omega,$$

and write

$$u = S(t)\mathbf{P}_{>M} u_0^\omega + S(t)\mathbf{P}_{\leq M} u_0 + v_{\text{high}} + v_{\text{low}},$$

where v_{high} solves (1.10) with $v_{\text{high}}|_{t=0} = \mathbf{P}_{>M} u_0^\omega$ and v_{low} solves a difference equation with $v_{\text{low}}|_{t=0} = \mathbf{P}_{\leq M} u_0$. Modifying the proof of Theorem 1.1 shows v_{high} exists almost surely up to any time $T > 0$ by choosing $M = M(T)$ large enough. We then try to show v_{low} is global in-time by applying the I-method with the modified energy

$$E(Iv_{\text{low}})(t) = \frac{1}{2}\|Iv_{\text{low}}(t)\|_{H^1(\mathbb{T})}^2.$$

To bound the growth of $E(Iv_{\text{low}})$, we must deal with the term

$$\int_{\mathbb{T}} (\partial_x I v_{\text{low}}) \cdot I v_{\text{low}} \cdot I\mathbf{P}_{>M} S(t) u_0^\omega. \tag{4.25}$$

With $2N < M$, the proof of Lemma 4.2 gives

$$\mathbb{P}\big(M^{\alpha+\frac{1}{2}-s} N^{-(1-s)} \|I\mathbf{P}_{>M} S(t) u_0^\omega\|_{L_x^2} > \lambda\big) \leq \frac{C}{\lambda^2}.$$

Choosing N such that $M \sim N^k$ we obtain a non-positive power of N in estimating (4.25) provided

$$k \geq 2\left(1 - \frac{\delta}{1 - 2\alpha + 2\delta}\right). \tag{4.26}$$

Applying Gronwall's inequality, we get a blow-up time

$$T^*(N) \sim \frac{\log\left(1 + \frac{B(N)^2}{\sqrt{E(Iv_{\text{low}})(0)} N^{-\beta}}\right)}{B(N)},$$

where $B(N)$ is almost surely bounded when $k \geq 2$ and grows polynomially in N otherwise. We conclude provided there exists $\rho \geq 0$ such that

$$\sqrt{E(Iv_{\text{low}})(0)}N^{-\beta} \lesssim N^{1-s}M^{s+\frac{1}{2}-\alpha+}N^{-\beta}\|u_0^\omega\|_{H^{\alpha-\frac{1}{2}-}(\mathbb{T})} \lesssim N^{-\rho},$$

but as $\beta = \frac{3}{2}-$ (see Lemma 4.3), we require

$$k \leq 1 + \frac{2\alpha}{2\alpha+1-2\delta}.$$

This final condition fails to agree with (4.26) unless $\alpha = \frac{1}{2}$. Thus, we elected to present the simpler argument in this paper.

5. Norm inflation at arbitrary data

We establish in this section norm inflation at arbitrary data for the BBM equation (1.1) in negative regularity spaces.

Theorem 5.1. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$, $1 \leq p < \infty$, $s < 0$ and fix $u_0 \in \mathcal{F}L^{s,p}(\mathcal{M})$. Then, given any $\varepsilon > 0$, there exists a smooth solution u_ε to* (1.1) *on $\mathcal{M}$ and $t_\varepsilon \in (0, \varepsilon)$ such that*

$$\|u_\varepsilon(0) - u_0\|_{\mathcal{F}L^{s,p}(\mathcal{M})} < \varepsilon, \qquad \text{and} \qquad \|u_\varepsilon(t_\varepsilon)\|_{\mathcal{F}L^{s,p}(\mathcal{M})} > \varepsilon^{-1}.$$

Then, Theorem 1.7 follows from Theorem 5.1 upon putting $p = 2$. In this section, we present the proof of Theorem 5.1. It suffices to establish the following result. We denote by $\mathcal{C}(\mathbb{T}) = C^\infty(\mathbb{T})$ and $\mathcal{C}(\mathbb{R}) = \mathcal{S}(\mathbb{R})$.

Proposition 5.2. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$, $1 \leq p < \infty$, $s < 0$, and fix $u_0 \in \mathcal{C}(\mathcal{M})$. Then given any $n \in \mathbb{N}$, there exists a smooth solution u_n to the BBM equation* (1.1) *and $t_n \in (0, \frac{1}{n})$ such that*

$$\|u_n(0) - u_0\|_{\mathcal{F}L^{s,p}(\mathcal{M})} < \frac{1}{n}, \quad \text{and} \quad \|u_n(t_n)\|_{\mathcal{F}L^{s,p}(\mathcal{M})} > n. \tag{5.1}$$

To see how Theorem 5.1 follows from Proposition 5.2, fix $u_0 \in \mathcal{F}L^{s,p}(\mathcal{M})$ and $s < 0$. By density, we can find a sequence $\{u_{0,k}\}_{k\in\mathbb{N}} \in \mathcal{C}(\mathcal{M})$ such that, for k sufficiently large, we have

$$\|u_{0,k} - u_0\|_{\mathcal{F}L^{s,p}(\mathcal{M})} < \frac{1}{k}. \tag{5.2}$$

For each fixed k, Proposition 5.2 implies there exists solutions $\{u_{n,k}\}_{n\in\mathbb{N}}$ to (1.1) such that

$$\|u_{n,k}(0) - u_{0,k}\|_{\mathcal{F}L^{s,p}(\mathcal{M})} < \frac{1}{n} \qquad \text{and} \qquad \|u_{n,k}(t_n)\|_{\mathcal{F}L^{s,p}(\mathcal{M})} > n. \tag{5.3}$$

Now given $\varepsilon > 0$, set $u_\varepsilon = u_{n,n}$ where $n \in \mathbb{N}$ is fixed such that $n \geq \frac{1}{2\varepsilon}$. Combining (5.2) and (5.3) we obtain (5.1), completing the proof.

In order to prove Proposition 5.2, we follow the argument in [38] which we set up in the next subsection and complete in Subsection 5.2.

5.1. Binary trees, power series expansions and multilinear estimates.

In this section, we will briefly describe the power series expansion indexed by binary trees, arising in the works [38, 16]. We then establish multilinear estimates controlling the terms in the power series. We begin by establishing the following local well-posedness result for BBM (1.1) in the Fourier-Lebesgue spaces.

Lemma 5.3. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$, $1 \le p \le \infty$ and $s \ge \max\{0, \frac{1}{2} - \frac{1}{p}\}$, with strict inequality when $p > 2$. Then for each $u_0 \in \mathcal{F}L^{s,p}(\mathcal{M})$, there exists a time $T \sim \|u_0\|_{\mathcal{F}L^{s,p}(\mathcal{M})}^{-1} > 0$ and a unique solution $u \in C([0,T]; \mathcal{F}L^{s,p}(\mathcal{M}))$ to the BBM equation (1.1) with $u|_{t=0} = u_0$.*

The proof of this result follows by a fixed point argument using the following bilinear estimate.

Lemma 5.4. *Let $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$, $1 \le p \le \infty$ and $s \ge \max\{0, \frac{1}{2} - \frac{1}{p}\}$, with strict inequality when $p > 2$. Suppose that $u, v \in \mathcal{F}L^{s,p}(\mathcal{M})$. Then $\varphi(D_x)(uv) \in \mathcal{F}L^{s,p}(\mathcal{M})$ and*

$$\|\varphi(D_x)(uv)\|_{\mathcal{F}L^{s,p}(\mathcal{M})} \lesssim \|u\|_{\mathcal{F}L^{s,p}(\mathcal{M})} \|v\|_{\mathcal{F}L^{s,p}(\mathcal{M})}. \tag{5.4}$$

Proof. Consider first the case when $1 \le p \le 2$. By duality, it suffices to show that

$$\left| \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{\langle \xi \rangle^s}{\langle \xi - \xi_1 \rangle^s \langle \xi_1 \rangle^s} \varphi(\xi) w(\xi) \widehat{u}(\xi - \xi_1) \widehat{v}(\xi_1)\, d\xi_1 d\xi \right| \lesssim \|\widehat{u}\|_{L^p} \|\widehat{v}\|_{L^p}, \tag{5.5}$$

where $w \in L^{p'}$ satisfying $\|w\|_{L^{p'}} = 1$ and $1/p + 1/p' = 1$. As $s \ge 0$, the trivial inequality $\langle \xi \rangle^s \lesssim \langle \xi - \xi_1 \rangle^s \langle \xi_1 \rangle^s$ implies

$$(5.5) \lesssim \left| \int_{\mathcal{M}} \int_{\mathcal{M}} \varphi(\xi) w(\xi) \widehat{u}(\xi - \xi_1) \widehat{v}(\xi_1)\, d\xi_1 d\xi \right|.$$

By Young's and Hölder's inequalities, we have

$$\begin{aligned} \left| \int_{\mathcal{M}} \int_{\mathcal{M}} \varphi(\xi) w(\xi) \widehat{u}(\xi - \xi_1) \widehat{v}(\xi_1)\, d\xi_1 d\xi \right| &\lesssim \|\varphi(\xi) w(\xi)\|_{L^{\frac{p'}{2}}} \|\widehat{u} * \widehat{v}\|_{L^{\frac{p}{2-p}}} \\ &\lesssim \|\widehat{u}\|_{L^p} \|\widehat{v}\|_{L^p} \|w\|_{L^{p'}} \|\varphi(\xi)\|_{L^{p'}}. \end{aligned}$$

Finally, $\|\varphi(\xi)\|_{L^{p'}} < \infty$ as $p' > 1$, verifying (5.5) when $1 \le p \le 2$. The estimate (5.4) can also be deduced by appealing to multilinear interpolation between the trivial inequality when $p = 1$ and the result of Lemma 2.1 when $p = 2$.

For the case when $2 < p < \infty$, first notice that the above arguments lead to the following stronger estimate when $p = 2$: for any $s \ge 0$,

$$\|\langle \partial_x \rangle^{-\frac{1}{2}+}(uv)\|_{\mathcal{F}L^{s,2}} \lesssim \|u\|_{\mathcal{F}L^{s,2}} \|v\|_{\mathcal{F}L^{s,2}}. \tag{5.6}$$

Fix $2 < p < \infty$ and let $s = \frac{1}{2} - \frac{1}{p} + s_0$ for some $s_0 > 0$. Then by the embeddings $\ell^2 \subset \ell^p$ for any $p > 2$, Hölder's inequality and with some $\varepsilon = \varepsilon(p)$ sufficiently small, we have using (5.6),

$$\begin{aligned} \|\varphi(D_x)(uv)\|_{\mathcal{F}L^{s,p}} &\lesssim \|\langle \partial_x \rangle^{-\left(\frac{1}{2} + \frac{1}{p} - \varepsilon\right)}(uv)\|_{\mathcal{F}L^{s_0 - \varepsilon, 2}} \\ &\lesssim \|u\|_{\mathcal{F}L^{s_0-\varepsilon,2}} \|v\|_{\mathcal{F}L^{s_0-\varepsilon,2}} \\ &\lesssim \|u\|_{\mathcal{F}L^{s,p}} \|v\|_{\mathcal{F}L^{s,p}}. \end{aligned}$$

The case when $p = \infty$ follows from Hölder's inequality. □

Remark 5.5. The estimate (5.5) is false if $p > 2$ and $s < \frac{1}{2} - \frac{1}{p}$. To see this, it suffices to show that (5.5) fails. For this, let $A \gg 1$ and take $\widehat{u}(\xi) = \widehat{v}(\xi) = \mathbf{1}_{[-A,A]}(\xi)$ and $w(\xi) = \mathbf{1}_{[0,1]}(\xi)$. Then the left hand side of (5.5) behaves like A^{1-2s} while the right hand side behaves like $A^{\frac{2}{p}}$. Thus we obtain a contradiction when p and s are as above and A becomes large.

Given $u_0 \in \mathcal{F}L^p(\mathcal{M})$, Lemma 5.3 gives the existence of a unique solution u to BBM (1.1) in the sense that there exists $T \sim \|u_0\|^{-1}_{\mathcal{F}L^p(\mathcal{M})}$ such that for each $t \in [0, T]$, u satisfies

$$u(t) = S(t)u_0 + \mathcal{I}^2[u](t),$$

where $\mathcal{I}^2[u] := \mathcal{I}[u, u]$ and $\mathcal{I}$ is the bilinear Duhamel integral operator

$$\mathcal{I}[u_1, u_2](t) := -\frac{i}{2}\int_0^t S(t - t')\varphi(D_x)(u_1(t')u_2(t'))dt'. \tag{5.7}$$

In order to set-up the necessary notation for the power series expansion of u indexed by trees, we first restate the terminology used in [38] for the binary trees we work with.

Definition 5.6. (i) Given a partially ordered set $\mathcal{T}$ with partial order $\leq$, we say that $b \in \mathcal{T}$ with $b \leq a$ and $b \neq a$ is a child of $a \in \mathcal{T}$, if $b \leq c \leq a$ implies either $c = a$ or $c = b$. If the latter condition holds, we also say that a is the parent of b.

(ii) A tree $\mathcal{T}$ is a finite partially ordered set, satisfying the following properties:

- Let $a_1, a_2, a_3, a_4 \in \mathcal{T}$. If $a_4 \leq a_2 \leq a_1$ and $a_4 \leq a_3 \leq a_1$, then we have $a_2 \leq a_3$ or $a_3 \leq a_2$,
- A node $a \in \mathcal{T}$ is called terminal, if it has no child. A non-terminal node $a \in \mathcal{T}$ is a node with exactly two children,
- There exists a maximal element $r \in \mathcal{T}$ (called the root node) such that $a \leq r$ for all $a \in \mathcal{T}$,
- $\mathcal{T}$ consists of the disjoint union of $\mathcal{T}^0$ and $\mathcal{T}^\infty$, where $\mathcal{T}^0$ and $\mathcal{T}^\infty$ denote the collections of non-terminal and terminal nodes, respectively.

We recall some basic combinatorial properties of binary trees.

Lemma 5.7. *Let $\mathcal{T}$ be a binary tree. The number of non-terminal $|\mathcal{T}^0|$ and terminal $|\mathcal{T}^\infty|$ nodes in $\mathcal{T}$ are j and $j+1$ respectively, where $j \in \mathbb{N} \cup \{0\}$. Consequently, $|\mathcal{T}| = 2j+1$. Let $\boldsymbol{T}(j)$ denote the set of all trees with j parent nodes. Then there exists a constant $C_0 > 0$ such that*

$$|\boldsymbol{T}(j)| \leq C_0^j. \tag{5.8}$$

For a proof of (5.8), we refer to the argument in [38, Lemma 2.3] which can be adapted easily for binary trees.

We have an injective map

$$\Psi_\phi : \bigcup_{j=1}^\infty \boldsymbol{T}(j) \mapsto \mathcal{D}'(\mathcal{M} \times [0, T]),$$

which encodes the nodes of a given tree $\mathcal{T} \in \boldsymbol{T}(j)$ as j-times iterated Duhamel operators acting on inputs $S(t)\phi$. More precisely, given a binary tree, we replace the non-terminal nodes by the bilinear Duhamel operator (5.7) with its children as arguments u_1 and u_2. Then, each terminal node is replaced by the linear solution $S(t)\phi$. Set

$$\Xi_j(\phi)(t) := \sum_{\mathcal{T} \in \boldsymbol{T}(j)} \Psi_\phi(\mathcal{T}).$$

Then we have the following multilinear estimates.

Lemma 5.8. *There exists $C > 0$ such that for all $j \in \mathbb{N}$, we have the following: Given any $\phi \in \mathcal{F}L^1(\mathcal{M})$ and $\psi \in L^2(\mathcal{M})$, we have*

$$\|\Xi_j(\phi)(t)\|_{\mathcal{F}L^1} \le C^j t^j \|\phi\|_{\mathcal{F}L^1}^{j+1} \tag{5.9}$$

$$\|\Xi_j(\psi)(t)\|_{\mathcal{F}L^\infty} \le C^j t^j \|\psi\|_{L^2}^{j+1}. \tag{5.10}$$

Furthermore, for any $1 \le q \le \infty$ and $u_0 \in \mathcal{F}L^q(\mathcal{M}) \cap \mathcal{F}L^1(\mathcal{M})$,

$$\|\Xi_j(u_0+\phi)(t) - \Xi_j(\phi)(t)\|_{\mathcal{F}L^q} \le C^j t^j \|u_0\|_{\mathcal{F}L^q}(\|u_0\|_{\mathcal{F}L^1}^j + \|\phi\|_{\mathcal{F}L^1}^j). \tag{5.11}$$

Proof. Estimates (5.9) and (5.11) are proved exactly as in [38, Lemma 2.5 and Lemma 2.6] by using the unitarity of $S(t)$ in $\mathcal{F}L^1$ and Young's inequality. For (5.10), we notice that Hölder's inequality implies

$$\|\mathcal{I}[u_1,u_2]\|_{\mathcal{F}L^\infty} \lesssim |t| \|u_1\|_{L^2} \|u_2\|_{L^2}.$$

now for a given $\mathcal{T} \in T(\mathcal{T})$, $\Psi_\psi(\mathcal{T})$ is essentially $j = |\mathcal{T}^0|$-many iterative compositions of the Duhamel integral operator $\mathcal{I}^2[\psi]$. Thus we first apply the above estimate followed by successive applications of Lemma 2.1, namely

$$\|\mathcal{I}[u_1,u_2]\|_{L^2} \lesssim |t| \|u_1\|_{L^2} \|u_2\|_{L^2}.$$

□

Remark 5.9. The estimate (5.10) differs from that in [38, (2.14) in Lemma 2.5] because we have made use of the explicit smoothing of the Duhamel operator for BBM.

The estimate (5.9) and Lemma 5.7 imply the power series expansion

$$\sum_{j=0}^{\infty} \Xi_j(u_0) = \sum_{j=0}^{\infty} \sum_{\mathcal{T} \in \boldsymbol{T}(j)} \Psi_{u_0}(\mathcal{T}),$$

is absolutely convergent in $C([0,T];\mathcal{F}L^1(\mathcal{M}))$ provided $T \lesssim \|u_0\|_{\mathcal{F}L^1}^{-1}$ and that the solution $u \in C([0,T];\mathcal{F}L^1(\mathcal{M}))$ with $u|_{t=0} = u_0$ can be represented as

$$u = \sum_{j=0}^{\infty} \Xi_j(u_0).$$

This is the power series representation of u indexed by trees.

We now begin to proceed towards the construction of the smooth solutions u_n stated in Proposition 5.2. Define ϕ_n through its Fourier transform by

$$\widehat{\phi}_n(\xi) := R\big\{\mathbf{1}_{-N+Q_A}(\xi) + \mathbf{1}_{N+Q_A}(\xi)\big\} \tag{5.12}$$

where $Q_A = [-2A, 2A]$, $R = R(N) \ge 1$, and $A = A(N) \ge 1$ satisfying

$$\|u_0\|_{\mathcal{F}L^1} \ll RA, \qquad \text{and} \qquad A \ll N, \tag{5.13}$$

For fixed $u_0 \in \mathcal{C}(\mathcal{M})$, set

$$u_{0,n} = u_0 + \phi_n. \tag{5.14}$$

For each n, $\widehat{\phi}_n$ is even and real-valued and hence $u_{0,n}$ is also real-valued. Let u_n be the solution of (1.1) with initial data $u_n|_{t=0} = u_{0,n}$. We have the following series expansion

$$u_n = \sum_{j=0}^{\infty} \Xi_j(u_{0,n}), \tag{5.15}$$

on $[-T,T]$ as long as

$$T \lesssim (\|u_0\|_{\mathcal{F}L^1} + RA)^{-1} \sim (RA)^{-1}. \tag{5.16}$$

We now state some further multilinear estimates that exploit the explicit expression of ϕ_n.

Proposition 5.10. *Let $1 \le p < \infty$, $s < 0$, $u_0 \in \mathcal{C}(\mathcal{M})$ satisfying* (5.13) *and ϕ_n and $u_{0,n}$ as in* (5.12) *and* (5.14) *respectively. For any $j \in \mathbb{N}$, the following estimates hold:*

$$\|u_{0,n} - u_0\|_{\mathcal{F}L^{s,p}} \lesssim RA^{\frac{1}{p}} N^s, \tag{5.17}$$

$$\|\Xi_0(u_{0,n})(t)\|_{\mathcal{F}L^{s,p}} \lesssim 1 + RA^{\frac{1}{p}} N^s \tag{5.18}$$

$$\|\Xi_1(u_{0,n})(t) - \Xi_1(\phi_n)(t)\|_{\mathcal{F}L^{s,p}} \lesssim t\|u_0\|_{\mathcal{F}L^p} RA^{\frac{1}{p}}. \tag{5.19}$$

$$\|\Xi_j(u_{0,n})(t)\|_{\mathcal{F}L^{s,p}} \lesssim C^j t^j (RA)^j (\|u_0\|_{\mathcal{F}L^p} + Rf_p(A)), \tag{5.20}$$

where

$$f_p(A) := \begin{cases} 1 & \text{if } s < -\frac{1}{p}, \\ (\log A)^{\frac{1}{p}} & \text{if } s = -\frac{1}{p}, \\ A^{\frac{1}{p}+s} & \text{if } s > -\frac{1}{p}. \end{cases} \tag{5.21}$$

Proof. The estimates (5.17), (5.18) and (5.19) are easy consequences of Lemma 5.8, (5.12) and (5.13). For (5.20), we write

$$\|\Xi_j(u_{0,n})(t)\|_{\mathcal{F}L^{s,p}} \le \|\Xi_j(u_{0,n})(t) - \Xi_j(\phi_n)(t)\|_{\mathcal{F}L^p} + \|\Xi_j(\phi_n)(t)\|_{\mathcal{F}L^{s,p}}. \tag{5.22}$$

Now (5.11), (5.12) and (5.13) imply

$$\begin{aligned}\|\Xi_j(u_{0,n})(t) - \Xi_j(\phi_n)(t)\|_{\mathcal{F}L^p} &\le C^j t^j \|u_0\|_{\mathcal{F}L^p} (\|u_0\|^j_{\mathcal{F}L^1} + \|\phi_n\|^j_{\mathcal{F}L^1}) \\ &\lesssim C^j t^j \|u_0\|_{\mathcal{F}L^p} (RA)^j.\end{aligned}$$

Meanwhile, as the support of $\widehat{\phi_n}$ is two disjoint intervals of width approximately A, we see that for fixed $\mathcal{T} \in T(j)$, the iterated convolution structure of $\mathcal{F}\{\Psi_{\phi_n}(\mathcal{T})\}$ implies $\operatorname{supp} \mathcal{F}\{\Xi_j(\phi_n)\}$ is contained within at most 2^{j+1} intervals of width approximately A. As $s < 0$, $\langle \xi \rangle^s$ is decreasing in $|\xi|$ and hence by (5.10), we have

$$\begin{aligned}\|\Xi_j(\phi_n)(t)\|_{\mathcal{F}L^{s,p}} &\le \|\langle \xi \rangle^s\|_{L^p_\xi(\operatorname{supp} \mathcal{F}\{\Xi_j(\phi_n)\})} \|\Xi_j(\phi_n)\|_{\mathcal{F}L^\infty} \\ &\lesssim \|\langle \xi \rangle^s\|_{L^p_\xi(C^j Q_A)} t^j (RA^{\frac{1}{2}})^{j+1} \\ &\lesssim C^j t^j f_p(A) (RA^{\frac{1}{2}})^{j+1},\end{aligned}$$

Returning to (5.22) we have shown

$$\|\Xi_j(u_{0,n})(t)\|_{\mathcal{F}L^{s,p}} \lesssim C^j t^j (RA)^j (\|u_0\|_{\mathcal{F}L^p} + Rf_p(A)),$$

which is (5.20). □

The following estimate shows the term $\Xi_1[\phi_n]$ is culpable for the norm-inflation phenomenon. The argument given below is essentially the same as similar arguments in [10, 42]. For completeness we include it here but adapted to the data (5.12).

Proposition 5.11. *Let ϕ_n be as in* (5.12) *and $s < 0$. Then, for $0 < t \ll A$, we have*

$$\|\Xi_1(\phi_n)(t)\|_{\mathcal{F}L^{s,p}} \gtrsim tR^2 A.$$

Proof. We have

$$\Xi_1(\phi_n)(t) = -\frac{i}{2}\int_0^t S(t-t')\varphi(D_x)(S(t')\phi_n S(t')\phi_n)\,dt'.$$

Taking the Fourier transform, we obtain

$$\begin{aligned}\mathcal{F}_{x\to\xi}[\Xi_1(\phi_n)(t)](\xi) &= -\frac{i}{2}\int_0^t \mathcal{F}_{x\to\xi}\left[S(t-t')\varphi(D_x)(S(t')\phi_n S(t')\phi_n)\right](\xi)\,dt' \\ &= -\frac{i}{2}\int_0^t e^{-i(t-t')\varphi(\xi)}\varphi(\xi)\mathcal{F}_{x\to\xi}\left[(S(t')\phi_n S(t')\phi_n)\right]\,dt' \\ &= -\frac{i}{2}e^{-it\varphi(\xi)}\varphi(\xi)\int_0^t e^{it'\varphi(\xi)}\int_{\mathbb{R}}\widehat{\phi}_n(\xi_1)\widehat{\phi}_n(\xi-\xi_1)e^{-it'\varphi(\xi_1)}e^{-it'\varphi(\xi_1)}d\xi_1\,dt' \\ &= -\frac{i}{2}e^{-it\varphi(\xi)}\varphi(\xi)\int_{\mathbb{R}}\widehat{\phi}_n(\xi_1)\widehat{\phi}_n(\xi-\xi_1)\int_0^t e^{-it'\theta(\xi,\xi_1)}dt'\,d\xi_1,\end{aligned}$$

where

$$\theta(\xi,\xi_1) := \varphi(\xi_1)+\varphi(\xi-\xi_1)-\varphi(\xi) = \frac{\xi\xi_1(\xi-\xi_1)(\xi^2-\xi\xi_1+\xi_1^2+3)}{(1+\xi_1^2)[1+(\xi-\xi_1)^2](1+\xi^2)}. \tag{5.23}$$

Integrating over t' yields

$$\mathcal{F}_{x\to\xi}[\Xi_1(\phi_n)(t)](\xi) = \frac{1}{2}e^{-it\varphi(\xi)}\varphi(\xi)\int_{\mathbb{R}}\widehat{\phi}_n(\xi_1)\widehat{\phi}_n(\xi-\xi_1)\frac{e^{-it'\theta(\xi,\xi_1)}-1}{\theta(\xi,\xi_1)}d\xi_1$$

Writing $I_1 := -N+Q_A$, $I_2 := N+Q_A$ and in view of (5.12),

$$\begin{aligned}\mathcal{F}_{x\to\xi}[\Xi_1(\phi_n)(t)](\xi) &= \frac{R^2}{2}e^{-it\varphi(\xi)}\varphi(\xi)\int_{\substack{\xi_1\in I_1\cup I_2\\ \xi-\xi_1\in I_1\cup I_2}}\frac{e^{-it\theta(\xi,\xi_1)}-1}{\theta(\xi,\xi_1)}d\xi_1 \\ &= \frac{R^2}{2}e^{-it\varphi(\xi)}\varphi(\xi)\left\{\int_{A_1(\xi)}\cdot+\int_{A_2(\xi)}\cdot\right\} \\ &=: g_1(t,\xi)+g_2(t,\xi),\end{aligned}$$

where

$$\begin{aligned}A_1(\xi) &:= \{\xi_1 : \xi_1\in I_1,\, \xi-\xi_1\in I_1 \text{ or } \xi_1\in I_2,\, \xi-\xi_1\in I_2\}, \\ A_2(\xi) &:= \{\xi_1 : \xi_1\in I_1,\, \xi-\xi_1\in I_2 \text{ or } \xi_1\in I_2,\, \xi-\xi_1\in I_1\}.\end{aligned}$$

With $f_j(x,t) := \mathcal{F}^{-1}_{\xi\to x}(g_j),\ j=1,2$, we have

$$\Xi_1(\phi_n)(t) = f_1+f_2.$$

If $\xi_1\in A_1(\xi)$, then $\xi\in 2I_1$ or $\xi\in 2I_2$, while if $\xi_1\in A_2(\xi)$, then $\xi\in Q_{2A}$. As $A\ll N$, the supports of the g_j are disjoint which implies

$$\|\Xi_1(\phi_n)(t)\|_{\mathcal{F}L^{s,p}} \sim \|f_1(t,\cdot)\|_{\mathcal{F}L^{s,p}}+\|f_2(t,\cdot)\|_{\mathcal{F}L^{s,p}}.$$

The dominant contribution to the H^s norm of $\Xi_1(\phi_n)$ arises from that of f_2. Indeed, we have

$$\|f_1\|_{\mathcal{F}L^{s,p}} \lesssim \frac{tR^2A^{1+\frac{1}{p}}}{N^{1-s}} \ll tR^2A,$$

where second inequality follows from $A \ll N$ and $s < 0$. In the region $A_2(\xi)$, $|\xi_1| \sim |\xi - \xi_1| \sim N$ and $\xi \in Q_{2A}$. From (5.23), we find $|\theta(\xi, \xi_1)| \lesssim A^{-1}$, and hence for $0 < t \ll A$, we have

$$\operatorname{Im} \frac{e^{it\theta(\xi,\xi_1)} - 1}{\theta(\xi, \xi_1)} \geq \frac{1}{2}t.$$

Furthermore, for $\xi \in Q_A \setminus Q_{1/4}$ we have $|\xi||\text{meas}(A_2(\xi))| \gtrsim A$ and hence

$$\begin{aligned} \|f_2(t,\cdot)\|_{\mathcal{F}L^{s,p}} &\sim R^2 \left(\int_{\mathbb{R}} \langle \xi \rangle^{ps} |\varphi(\xi)|^p \left| \int_{A_2(\xi)} \frac{e^{-it\theta(\xi,\xi_1)} - 1}{\theta(\xi,\xi_1)} d\xi_1 \right|^p d\xi \right)^{1/p} \\ &\gtrsim |t| R^2 \left(\int_{Q_{2A}} \langle \xi \rangle^{-p(2-s)} |\xi|^p |\text{meas}(A_2(\xi))|^p d\xi \right)^{1/p} \\ &\gtrsim |t| R^2 \left(\int_{Q_A \setminus Q_{1/4}} \langle \xi \rangle^{-p(2-s)} |\xi|^p |\text{meas}(A_2(\xi))|^p d\xi \right)^{1/p} \\ &\gtrsim |t| R^2 A \left(\int_{Q_1 \setminus Q_{1/4}} \langle \xi \rangle^{-p(2-s)} d\xi \right)^{1/p} \\ &\gtrsim |t| R^2 A, \end{aligned}$$

as $A \geq 1$. □

Remark 5.12. Although the proof of Proposition 5.11 was stated for $\mathbb{R}$, it also holds on $\mathbb{T}$ using the same ideas and the obvious modifications.

5.2. **Proof of Proposition 5.11.** Fix $p \in [1, \infty)$ and $s < 0$. In order to prove Proposition 5.2, it suffices to show, given $n \in \mathbb{N}$, the following properties hold:

$$\begin{aligned} &\text{(i)} \quad RA^{\frac{1}{p}} N^s \ll \tfrac{1}{n}, \\ &\text{(ii)} \quad TRA \ll 1, \\ &\text{(iii)} \quad TR^2 A \gg n, \\ &\text{(iv)} \quad TR^2 A \gg T^2 R^3 A^2 f_p(A), \\ &\text{(v)} \quad \|u_0\|_{\mathcal{F}L^p} \ll R f_p(A), \\ &\text{(vi)} \quad T \ll A, \quad \text{and} \quad A \ll N. \end{aligned}$$

for some particular choices of A, R, T and N all depending on n.

To see why this is true, we have that condition (i) ensures by (5.17) that the approximating data $u_{0,n}$ is close to u_0 which is the first part of (5.1). Condition (ii) combined with (5.16) implies the power series expansion (5.15) converges in $C([0,T]; \mathcal{F}L^1(\mathbb{T}))$. Proposition 5.11 and conditions (iii) and (vi) are responsible for the required growth to conclude norm inflation, while (iv) and (v) ensure that the first term of the expansion $\Xi_1(u_{0,n})$ dominates all other terms. We detail these last two deductions now. Namely, assuming (ii) and

(v) hold, (5.20) implies

$$\begin{aligned}\Big\|\sum_{j=2}^{\infty}\Xi_j(u_{0,n})(T)\Big\|_{\mathcal{F}L^{s,p}} &\lesssim \sum_{j=2}^{\infty}\Big\|\Xi_j(u_{0,n})(T)\Big\|_{\mathcal{F}L^{s,p}}\\ &\lesssim \sum_{j=2}^{\infty}(CTRA)^j(\|u_0\|_{\mathcal{F}L^p}+Rf_p(A))\\ &\lesssim T^2R^2A^2Rf_p(A)\sim T^2R^3A^2f_p(A).\end{aligned}$$

Then, assuming (i), (ii), (iii), (iv) and (v) hold and using Propositions 5.10 and 5.11, we have

$$\begin{aligned}\|u_n(T)\|_{\mathcal{F}L^{s,p}} &\geq \|\Xi_1(\phi_n)(T)\|_{\mathcal{F}L^{s,p}}-\|\Xi_0(u_{0,n})\|_{\mathcal{F}L^{s,p}}\\ &\quad -\|\Xi_1(u_{0,n})(T)-\Xi_1(\phi_n)(T)\|_{\mathcal{F}L^{s,p}}-\Big\|\sum_{j=2}^{\infty}\Xi_j(u_{0,n})(T)\Big\|_{\mathcal{F}L^{s,p}}\\ &\gtrsim TR^2A-(1+RA^{\frac{1}{p}}N^s)-T\|u_0\|_{\mathcal{F}L^p}RA^{\frac{1}{p}}-T^2R^3A^2f_p(A)\\ &\sim TR^2A\gg n.\end{aligned}$$

This verifies the second estimate of (5.1) at time $t_n := T$. Finally choosing $N = N(n)$ sufficiently large, we conclude the proof of Proposition 5.2.

It remains to verify (i)-(vi) hold. Notice that (iv) follows if we obtain $TRAf_p(A)\ll 1$. This is stronger than (ii), so we focus on obtaining (i) and (iii) through (vi). Recalling the definition of $f_p(A)$ from (5.21), it is natural to consider the following three cases:

- **Case 1:** $s<-\frac{1}{p}$

For $\delta>0$ small enough so that

$$\frac{1}{p}+\Big(2-\frac{1}{p}\Big)\delta<-s,$$

we choose

$$A=N^{1-\delta},\quad R=N^{2\delta},\quad \text{and}\quad T=N^{-1-2\delta}.$$

Then we check

$$\begin{aligned}&RA^{\frac{1}{p}}N^s=N^{\frac{1}{p}+\left(2-\frac{1}{p}\right)\delta+s}\ll\frac{1}{n},\\ &TR^2A=N^{\delta}\gg n,\\ &TRAf_p(A)\sim TRA=N^{-\delta}\ll 1.\end{aligned}$$

Clearly (v) and (vi) are also satisfied.

- **Case 2:** $s=-\frac{1}{p}$

We choose

$$A=\Big(\frac{N}{\log N}\Big)^{\frac{1}{2}},\quad R=\Big(\frac{N}{\log N}\Big)^{\frac{1}{2p}},\quad\text{and}\quad T=\frac{1}{N^{\frac{1+p}{2p}}(\log N)^{\frac{3-p}{2p}}}.$$

Thus, by choosing $N = N(n)$ sufficiently large, we ensure

$$RA^{\frac{1}{p}}N^s = (\log N)^{-\frac{1}{p}} \ll \frac{1}{n},$$
$$TR^2A = \frac{N^{\frac{1}{2p}}}{(\log N)^{\frac{5}{2p}}} \gg n,$$
$$TRAf_p(A) \sim TRA(\log A)^{\frac{1}{p}} \sim (\log N)^{-\frac{2}{p}}(\log N - \log\log N)^{\frac{1}{p}} \sim (\log N)^{-\frac{1}{p}} \ll 1.$$

Furthermore, both (v) and (vi) are also satisfied.

- **Case 3:** $-\frac{1}{p} < s < 0$

  We choose

$$A = N^{\delta p}, \quad R = N^{-s-\delta-\theta}, \quad \text{and} \quad T = N^{2s+3\theta+(2-p)\delta},$$

where $0 < \theta \ll \delta \le \frac{1}{3p}$ are sufficiently small so that

$$-s > \max\left(\frac{2(\theta+\delta)}{1+\delta p}, \frac{3\theta}{2} - (p-1)\delta\right). \tag{5.24}$$

Then

$$RA^{\frac{1}{p}}N^s = N^{-\theta} \ll \frac{1}{n},$$
$$TR^2A = N^{\theta} \gg n,$$
$$TRAf_p(A) \sim TRA^{1+\frac{1}{p}+s} = N^{(1+\delta p)s+2(\theta+\delta)} \ll 1.$$

The second condition in (5.24) ensures (vi) holds. Meanwhile, we also satisfy (v) because when $\delta \le \frac{1}{3p}$, we have $\frac{1}{1-\delta p} \le \frac{2}{1+\delta p}$.

Appendix A. On well-posedness of BBM below $L^2(\mathbb{T})$ with non-Gaussian randomised initial data

In this appendix, we discuss how we may extend the local and global well-posedness results of Theorems 1.1 and 1.5 for BBM (1.3) to more general random initial data of the form:

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx}, \tag{A.1}$$

where the family of (not necessarily Gaussian) complex-valued $\{g_n\}_{n\in\mathbb{Z}}$ random variables satisfy the following assumptions:

(i) the random variables $\{g_n\}_{n\in\mathbb{N}\cup\{0\}}$ are independent,

(ii) $g_{-n} := \overline{g_n}$ for $n \in \mathbb{N}\cup\{0\}$ and g_0 is real,

(iii) $\mathbb{E}[g_n] = 0$ and $\mathbb{E}[|g_n|^2] = 1$,

(iv) there exist $C_0, C_1 > 0$ such that for all $\gamma \in \mathbb{R}$ and for all $n \in \mathbb{Z}$, we have

$$\mathbb{E}[e^{\gamma|g_n|}] \le C_0 e^{C_1\gamma^2},$$

(v) there exists an angle θ satisfying[9] $840\,\theta \ne 0 \mod 2\pi$, such that the law of $e^{i\theta}g_n$ is the same as the law of g_n.

[9] Note $840 = \mathrm{lcm}(2,3,5,7,8)$.

A computation shows the random distribution u_0^ω given in (A.1) belongs to $H^{\alpha-\frac{1}{2}-}(\mathbb{T})$ almost surely. If we additionally impose the following non-degeneracy condition:

$$\text{(vi)}\;\; \text{there exists}\, c>0 \text{ such that } \limsup_{n\to\infty}\mathbb{P}(|g_n|\le c)<1,$$

then the argument in [13, Lemma B.1] shows u_0^ω does not belong to $H^{\alpha-\frac{1}{2}}(\mathbb{T})$ almost surely. In view of the global well-posedness of BBM (1.3) in $L^2(\mathbb{T})$, we focus on $\alpha\le\frac{1}{2}$.

For simplicity, in the following we make the additional assumption

$$\text{(vii)}\;\; g_0\equiv 0.$$

We stress that this assumption only reduces the number of cases we must consider in the proof of Lemma A.2 below. We handle the remaining cases coming from removing assumption (vii) in that proof using similar analysis and they yield the same (overall) restrictions on α and s as stated in Lemma A.2.

We have three main points to discuss:

(I) the regularity and integrability properties of the stochastic objects: $z^\omega(t)=S(t)u_0^\omega$, the random linear solution to BBM (1.1) with initial data (A.1), and $\mathcal{N}(z^\omega)$, where $\mathcal{N}$ is defined in (1.4),

(II) almost sure local well-posedness for BBM (1.3) with initial data (A.1),

(III) almost sure global well-posedness for BBM (1.3) with initial data (A.1).

• **(I):** We begin with the random linear solution z^ω. By assumption (iv), the argument in [13, Lemma 3.1] implies, for any $(a_n)\in\ell^2$, we have

$$\Big\|\sum_{n\in\mathbb{Z}}a_ng_n\Big\|_{L^p(\Omega)}\lesssim p^{\frac{1}{2}}\|a_n\|_{\ell^2_n}\sim p^{\frac{1}{2}}\Big\|\sum_{n\in\mathbb{Z}}a_ng_n\Big\|_{L^2(\Omega)} \tag{A.2}$$

for any $p\ge 2$. Now we note that, in Proposition 2.5, we may weaken the assumption that the stochastic process $X_k(t)\in\mathcal{H}_{\le\ell}$ for each $t\in\mathbb{R}_+$ to the following moment control: there exists $C,k>0$ such that

$$\|X_k(t)\|_{L^p(\Omega)}\le Cp^{\frac{\ell}{2}}\|X_k(t)\|_{L^2(\Omega)}, \tag{A.3}$$

for any $p\ge 2$, for every $k\in\mathbb{N}$ and for each $t\in\mathbb{R}_+$. In particular, stochastic processes belonging to $\mathcal{H}_{\le\ell}$ for some $\ell\in\mathbb{N}$ satisfy (A.3) because of the Wiener chaos estimate (Lemma 2.4). Thus by (A.2), z^ω satisfies (A.3) and then applying the same arguments as in the proof of Proposition 2.6 for the Gaussian random linear solution, for any $T>0$ we obtain

$$z\in C([0,T];W^{\alpha-\frac{1}{2}-,\infty}(\mathbb{T}))$$

almost surely. Moreover, if $\{\rho_k\}_{k\in\mathbb{N}}$ is a family of mollifiers on $\mathbb{T}$, then we have

$$z_k=S(t)(u_0^\omega*\rho_k)\to z \tag{A.4}$$

as $\varepsilon\to 0$ in $L^q(\Omega;C([0,T];W^{\alpha-\frac{1}{2}-,\infty}(\mathbb{T}))$, for any $1\le q<\infty$ and almost surely in $C([0,T];W^{\alpha-\frac{1}{2}-,\infty}(\mathbb{T})$. Furthermore, the limit is independent of the mollification kernel ρ.

For the stochastic object $\mathcal{N}(z)$, it is not at all obvious if (A.3) is satisfied (we may not even have a version of Wick's theorem). However, it will suffice for our purposes to show the fourth moment of $\|\mathcal{N}(z)\|_{L^4_T H^{2\alpha-}_x}$ is finite; see Lemma A.2 below. In this case, we argue directly using the following lemma. This lemma appears in [23, Lemma 4.3] up to considering fourth-order moments. In this appendix, we require knowledge up to the eighth-order moments. Given $n \in \mathbb{N}$, we denote by $\{\overline{n}\}$ the set $\{k \in \mathbb{N} : k \leq n\}$.

Lemma A.1. *Let $\{g_n\}_{n\in\mathbb{Z}}$ be a family of complex-valued random variables satisfying assumptions* (i) *through* (v) *and* (vii) *above. Given $n \in \mathbb{N}$, we denote by S^n the set of permutations of $\{\overline{n}\}$. Then, we have $\mathbb{E}[g_{n_1} g_{n_2}] = \delta_{n_1,-n_2}$ and*

$$\mathbb{E}\bigg[\prod_{j=1}^{k} g_{n_k}\bigg] = 0 \quad \text{for every odd } k \in \{\overline{8}\}.$$

Furthermore, we have

$$\mathbb{E}\bigg[\prod_{j=1}^{4} g_{n_j}\bigg] = \begin{cases} \mathbb{E}[|g_n|^4] & \text{if } \exists\, \sigma \in S^4 \text{ such that } n_{\sigma(1)} = n_{\sigma(3)} = -n_{\sigma(2)} = -n_{\sigma(4)} \\ \mathbb{E}[|g_n|^2]^2 & \text{if } \exists\, \sigma \in S^4 \text{ such that } n_{\sigma(1)} = -n_{\sigma(3)},\ \ n_{\sigma(2)} = -n_{\sigma(4)} \\ & \text{and } |n_{\sigma(1)}| \neq |n_{\sigma(2)}|, \\ 0 & \text{if } \exists\, \sigma \in S^4 \text{ such that } |n_{\sigma(j)}| \neq |n_{\sigma(j')}| \text{ for every } j \neq j', \\ & \text{where } j, j' \in \{\overline{4}\} \end{cases}$$

and

$$\mathbb{E}\bigg[\prod_{j=1}^{8} g_{n_j}\bigg] = \begin{cases} \mathbb{E}[|g_n|^8] & \text{if } \exists\, \sigma \in S^8 \text{ such that } n_{\sigma(1)} = n_{\sigma(3)} = n_{\sigma(5)} = n_{\sigma(7)} \\ & \qquad = -n_{\sigma(2)} = -n_{\sigma(4)} = -n_{\sigma(6)} = -n_{\sigma(8)}, \\ \mathbb{E}[|g_n|^6]\,\mathbb{E}[|g_n|^2] & \text{if } \exists\, \sigma \in S^8 \text{ such that } n_{\sigma(1)} = n_{\sigma(3)} = n_{\sigma(5)} = -n_{\sigma(2)} \\ & = -n_{\sigma(4)} = -n_{\sigma(6)},\ n_{\sigma(7)} = -n_{\sigma(8)} \text{ and } |n_{\sigma(1)}| \neq |n_{\sigma(7)}|, \\ \mathbb{E}[|g_n|^4]^2 & \text{if } \exists\, \sigma \in S^8 \text{ such that } n_{\sigma(1)} = n_{\sigma(3)} = -n_{\sigma(2)} = -n_{\sigma(4)}, \\ & n_{\sigma(5)} = n_{\sigma(7)} = -n_{\sigma(6)} = -n_{\sigma(8)} \text{ and } |n_{\sigma(1)}| \neq |n_{\sigma(5)}|, \\ \mathbb{E}[|g_n|^2]^4 & \text{if } \exists\, \sigma \in S^8 \text{ such that for each odd } j \in \{\overline{8}\}, \text{ we have} \\ & n_{\sigma(j)} = -n_{\sigma(j+1)} \text{ and } |n_{\sigma(j)}| = |n_{\sigma(j')}| \text{ for } j \neq j' \\ & \text{and } j, j' \in \{\overline{8}\} \text{ odd} \\ 0 & \text{if } \exists\, \sigma \in S^8 \text{ such that } |n_{\sigma(j)}| \neq |n_{\sigma(j')}| \text{ for every } j \neq j', \\ & \text{where } j, j' \in \{\overline{8}\}. \end{cases}$$

Proof. The proof follows by a long case-by-case analysis using the assumptions (i), (ii), (iii) and the following consequence of assumption (v): for any non-negative integers k and ℓ

satisfying $k+\ell \le 8$, we have

$$\mathbb{E}[g_n^k \overline{g_n}^\ell] = \mathbb{E}[|g_n|^{2k}]\delta_{k,\ell}. \tag{A.5}$$

To observe (A.5), we may assume $k<\ell$. Then by assumption (v), we have

$$\mathbb{E}[g_n^k \overline{g_n}^\ell] = \mathbb{E}[|g_n|^{2k}\overline{g_n}^{\ell-k}] = e^{i\theta(k-\ell)}\mathbb{E}[|g_n|^{2k}\overline{g_n}^{\ell-k}],$$

but now the second equality and assumption (v) imply $\mathbb{E}[|g_n|^{2k}\overline{g_n}^{\ell-k}] = 0$. □

Lemma A.2. *Given $\alpha \in (\frac{1}{4}, \frac{1}{2}]$, let $s < 2\alpha$ and fix $T>0$. Then, there exists $C_{s,\alpha} > 0$ such that*

$$\mathbb{E}[\|\mathcal{N}(z)\|^4_{L^4_T H^s_x}] \le C_{s,\alpha} T < \infty. \tag{A.6}$$

Moreover, if $\{\rho_k\}_{k\in\mathbb{N}}$ is a family of mollifiers on $\mathbb{T}$, then $\mathcal{N}(z_k)$ converges to $\mathcal{N}(z)$ as $k\to\infty$ in $L^4(\Omega; L^4_T H^s_x)$. In particular, $\mathcal{N}(z_M)$ converges to $\mathcal{N}(z)$ as $M\to\infty$, for M dyadic, almost surely in $L^4_T H^s_x$.

Proof. Using (1.4) and (A.1), we write

$$\langle \partial_x\rangle^s \mathcal{N}(z) = \sum_{n_1,n_2\in\mathbb{Z}} \mathcal{B}(n_1,n_2) g_{n_1} g_{n_2},$$

where

$$\begin{aligned}\mathcal{B}(n_1,n_2) &:= \mathbf{1}_{\{n_1+n_2\neq 0\}} \langle n_1+n_2\rangle^s \varphi(n_1+n_2) e^{i(n_1+n_2)x} a(n_1)a(n_2),\\ a(n) &:= \frac{e^{-it\varphi(n)}}{\langle n\rangle^\alpha}.\end{aligned}$$

Notice $\mathcal{B}(n_1,n_2) = \mathcal{B}(n_2,n_1)$. With this notation, we have

$$\mathbb{E}[\|\mathcal{N}(z)\|^4_{L^4_T H^s_x}] \le \Bigg\| \Big\| \sum_{n_1,n_2\in\mathbb{Z}} \mathcal{B}(n_1,n_2) g_{n_1} g_{n_2} \Big\|_{L^4(\Omega)} \Bigg\|^4_{L^4_T L^2_x}. \tag{A.7}$$

Clearly, if we show

$$\Big\| \sum_{n_1,n_2\in\mathbb{Z}} \mathcal{B}(n_1,n_2) g_{n_1} g_{n_2} \Big\|_{L^4(\Omega)} \le C < \infty,$$

where C above is independent of $(x,t)\in\mathbb{T}\times\mathbb{R}_+$, (A.7) will imply (A.6). By expanding, we have

$$\begin{aligned}&\Big\| \sum_{n_1,n_2\in\mathbb{Z}} \mathcal{B}(n_1,n_2) g_{n_1} g_{n_2} \Big\|^4_{L^4(\Omega)}\\ &\quad = \sum_{\substack{n_1,n_2,k_1,k_2\in\mathbb{Z}\\ m_1,m_2,\ell_1,\ell_2\in\mathbb{Z}}} \mathcal{B}(n_1,n_2)\overline{\mathcal{B}(m_1,m_2)}\mathcal{B}(k_1,k_2)\overline{\mathcal{B}(\ell_1,\ell_2)}\\ &\qquad\qquad \times \mathbb{E}[g_{n_1}g_{n_2}\overline{g_{m_1}g_{m_2}}g_{k_1}g_{k_2}\overline{g_{\ell_1}g_{\ell_2}}]\end{aligned} \tag{A.8}$$

We now use Lemma A.1 to handle the expectation above. This naturally requires a case-by-case analysis. We first fix some terminology. We say we have a pair if there exist $j\in\{n_1,n_2,k_1,k_2\}$ and $j'\in\{m_1,m_2,\ell_1,\ell_2\}$ such that $j=j'$. Let $j_1,j_2\in\{n_1,n_2,k_1,k_2\}$ be distinct and $j_3,j_4\in\{m_1,m_2,\ell_1,\ell_2\}$ be distinct. We say we have a 2-pair if, in fact,

$j_1 = j_2 = j_3 = j_4$. Similarly, we also define 3-pairs and 4-pairs in the obvious way. Hence, Lemma A.1 implies the right hand side of (A.8) is non-zero if we have:

- **Case 1:** a 4-pair
  In this case, we have $n_1 = n_2 = k_1 = k_2 = m_1 = m_2 = \ell_1 = \ell_2$. Hence,

$$\text{RHS of (A.8)} \sim \sum_n |\mathcal{B}(n,n)|^4 \sim \sum_n \frac{1}{\langle n\rangle^{8\alpha-4s+4}},$$

which is summable provided $s < \frac{3}{4} + 2\alpha$.

- **Case 2:** a 3-pair and a pair
  By the symmetry in $\mathcal{B}$, we may assume

$$k_1 = k_2 = n_1 = m_1 = m_2 = \ell_1 \text{ and } n_2 = \ell_2,$$

but $n_1 \neq n_2$. Since $s < 1$ and using Lemma 2.2, we have

$$\begin{aligned}
\text{RHS of (A.8)} &\sim \sum_{n_1} |\mathcal{B}(n_1,n_1)|^2 \sum_{n_2} |\mathcal{B}(n_1,n_2)|^2 \\
&\lesssim \sum_{n_1} |\mathcal{B}(n_1,n_1)|^2 \frac{1}{\langle n_1\rangle^{2\alpha}} \sum_{n_2} \frac{1}{\langle n_1+n_2\rangle^{2(1-s)}} \frac{1}{\langle n_2\rangle^{2\alpha}} \\
&\lesssim \sum_{n_1} |\mathcal{B}(n_1,n_1)|^2 \frac{1}{\langle n_1\rangle^{2\alpha}} \frac{1}{\langle n_1\rangle^{2(1-s)+2\alpha-1}} \\
&\lesssim \sum_{n_1} \frac{1}{\langle n_1\rangle^{3-4s+8\alpha}} < \infty
\end{aligned}$$

provided $s < \frac{1}{2} + \alpha$.

- **Case 3:** a 2-pair and a 2-pair
  Again by the symmetry in $\mathcal{B}$, we have two further subcases.

- **Subcase 3.1:** $n_1 = m_1 = n_2 = m_2$ and $k_1 = k_2 = \ell_1 = \ell_2$
  We have

$$\text{RHS of (A.8)} \sim \bigg(\sum_{n_1} |\mathcal{B}(n_1,n_1)|^2\bigg)^2 \sim \bigg(\sum_{n_1} \frac{1}{\langle n_1\rangle^{4\alpha}} \frac{1}{\langle n_1\rangle^{2(1-s)}}\bigg)^2 < \infty,$$

provided $s < \frac{1}{2} + 2\alpha$.

- **Subcase 3.2:** $n_1 = m_1 = k_1 = \ell_1$ and $n_2 = m_2 = k_2 = \ell_2$
  Using Lemma 2.2, we have

$$\begin{aligned}
\text{RHS of (A.8)} &\sim \sum_{n_1,n_2} |\mathcal{B}(n_1,n_2)|^4 \\
&\sim \sum_n \frac{1}{\langle n\rangle^{4-4s}} \sum_{n=n_1+n_2} \frac{1}{\langle n_1\rangle^{4\alpha}\langle n_2\rangle^{4\alpha}} \\
&\lesssim \sum_n \frac{1}{\langle n\rangle^{4-4s+4\alpha}} < \infty
\end{aligned}$$

provided $s < \frac{3}{4} + \alpha$.

- **Case 4:** four pairs

We reduce to three further subcases.

- **Subcase 4.1:** $n_1 = n_2$, $k_1 = k_2$, $m_1 = m_2$, $\ell_1 = \ell_2$
Using Lemma 2.2, we get

$$\text{RHS of (A.8)} \sim \bigg(\sum_{n_1} |\mathcal{B}(n_1, n_1)| \bigg)^4 \sim \bigg(\sum_{n_1} \frac{1}{\langle n_1 \rangle^{1-s}} \frac{1}{\langle n_1 \rangle^{2\alpha}} \bigg)^4 < \infty,$$

provided $s < 2\alpha$.

- **Subcase 4.2:** $n_1 = m_1$, $n_2 = m_2$, $k_1 = \ell_1$, $k_2 = \ell_2$
Using Lemma 2.2, we have

$$\begin{aligned} \text{RHS of (A.8)} &\sim \bigg(\sum_{n_1, n_2} |\mathcal{B}(n_1, n_2)|^2 \bigg)^2 \\ &\lesssim \bigg(\sum_{n} \frac{1}{\langle n \rangle^{2(1-s)}} \sum_{n = n_1 + n_2} \frac{1}{\langle n_1 \rangle^{2\alpha}} \frac{1}{\langle n_2 \rangle^{2\alpha}} \bigg)^2 \\ &\lesssim \bigg(\sum_{n} \frac{1}{\langle n \rangle^{2(1-s)}} \frac{1}{\langle n \rangle^{4\alpha - 1}} \bigg)^2 < \infty, \end{aligned}$$

provided $\alpha > \frac{1}{4}$ and $s < 2\alpha$.

- **Subcase 4.3:** $n_1 = m_1$, $n_2 = \ell_2$, $k_1 = m_2$, $k_2 = \ell_1$
In this case, we have

$$\begin{aligned} \text{RHS of (A.8)} \sim \sum_{n_1, n_2} \frac{1}{\langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha} \langle n_1 + n_2 \rangle^{1-s}} \sum_{k_2} \frac{1}{\langle k_2 \rangle^{2\alpha}} \frac{1}{\langle k_2 + n_2 \rangle^{1-s}} \\ \times \sum_{k_1} \frac{1}{\langle k_1 \rangle^{2\alpha} \langle k_1 + k_2 \rangle^{1-s} \langle k_1 + n_1 \rangle^{1-s}}. \end{aligned}$$

For the innermost summation, we apply Cauchy-Schwarz and Lemma 2.2 twice to get

$$\sum_{k_1} \frac{1}{\langle k_1 \rangle^{2\alpha} \langle k_1 + k_2 \rangle^{1-s} \langle k_1 + n_1 \rangle^{1-s}} \lesssim \frac{1}{\langle k_2 \rangle^{\frac{1}{2} - s + \alpha}} \frac{1}{\langle n_1 \rangle^{\frac{1}{2} - s + \alpha}},$$

provided $s < \frac{1}{2} + \alpha$. Inserting this bound back into the above and using Lemma 2.2 to sum in k_2 provided $s < \frac{1}{4} + \frac{3}{2}\alpha$, we have

$$\begin{aligned} \sum_{n_1, n_2} \frac{1}{\langle n_1 \rangle^{\frac{1}{2} - s + 3\alpha} \langle n_2 \rangle^{2\alpha} \langle n_1 + n_2 \rangle^{1-s}} \sum_{k_2} \frac{1}{\langle k_2 \rangle^{\frac{1}{2} - s + 3\alpha}} \frac{1}{\langle k_2 + n_2 \rangle^{1-s}} \\ \lesssim \sum_{n_1} \frac{1}{\langle n_1 \rangle^{\frac{1}{2} - s + 3\alpha}} \sum_{n_2} \frac{1}{\langle n_2 \rangle^{\frac{1}{2} - 2s + 5\alpha}} \frac{1}{\langle n_1 + n_2 \rangle^{1-s}}. \end{aligned}$$

Using Lemma 2.2 to sum in n_2 provided $s < \frac{1}{6} + \frac{5}{3}\alpha$, we bound the above by

$$\sum_{n_1} \frac{1}{\langle n_1 \rangle^{\frac{1}{2} - s + 3\alpha}} \frac{1}{\langle n_1 \rangle^{\frac{1}{2} - 3s + 5\alpha}} = \sum_{n_1} \frac{1}{\langle n_1 \rangle^{8\alpha + 1 - 4s}} < \infty$$

as long as $s < 2\alpha$.

Collating all the cases, we see that for $\frac{1}{4} < \alpha \leq \frac{1}{2}$, the worst regularity restriction is indeed $s < 2\alpha$.

By slightly modifying the above arguments and using (2.10) and the uniform (in k and n) of $\widehat{\rho_k}(n)$, we also obtain

$$\mathbb{E}[\|\mathcal{N}(z_k) - \mathcal{N}(z_{k'})\|_{L_T^4 H_x^s}^4] \lesssim C_{T,s,\alpha} k^{-4\theta} \tag{A.9}$$

for some small $\theta > 0$. From (A.9), we have $\mathcal{N}(z_k)$ converges to $\mathcal{N}(z)$ as $k \to \infty$ in $L^4(\Omega; L_T^4 H_x^s)$. In particular, taking $k' \to 0$ in (A.9), we get

$$\mathbb{E}[\|\mathcal{N}(z_M) - \mathcal{N}(z)\|_{L_T^4 H_x^s}^4] \lesssim M^{-4\theta}, \tag{A.10}$$

where $M \geq 1$ is dyadic. Thus by a Borel-Cantelli argument with (A.10), we have $\mathcal{N}(z_M)$ converges to $\mathcal{N}(z)$ as $M \to \infty$, for M dyadic, almost surely in $L_T^4 H_x^s$. □

At this stage, we do not know how to show the almost sure convergence of $\mathcal{N}(z_k)$ along $k \in \mathbb{N}$. If we did have convergence of the full sequence, we would then obtain analogues of the results in Theorem 1.10 and Theorem 1.11 for random initial data of the form (A.1). We note that this does not cause an issue for the global well-posedness argument in (iii) below as we only require the almost sure convergence of a subsequence (which we take to be dyadic).

• **(II):** The key difference here is we weaken the assumption (3.2) to $z_2 \in L_t^2 H_x^s([0,T] \times \mathbb{T})$. Then, for the fixed point argument in the proof of Proposition 3.1, we apply Cauchy-Schwarz to bound the z_2 term as follows:

$$\left\| \int_0^t S(t-t') z_2(t') dt' \right\|_{L_T^\infty H_x^s} \lesssim T^{\frac{1}{2}} \|z_2\|_{L_T^2 H_x^s}.$$

Hence, for the analogue of the almost sure local well-posedness result of Theorem 1.1 for random initial data of the form (A.1), we put $z_2 = \mathcal{N}(z)$ and use Lemma A.2. This implies almost sure existence of solutions below $L^2(\mathbb{T})$ for BBM (1.3) with random initial data of the form (A.1), provided $\alpha > \frac{1}{4}$.

• **(III):** We now describe the analogue of the almost sure global well-posedness result of Theorem 1.5. All that is necessary is to obtain the following analogue of Proposition 4.1, except here we now replace the smoothed initial value problem (4.1) by the smoothed initial value problem with dyadic $M \geq 1$:

$$\begin{cases} i\partial_t v_M = \varphi(D_x)\big(v_M + \frac{1}{2}v_M^2 + z_M v_M\big) + \frac{1}{2}\mathcal{N}(z_M) \\ v_M|_{t=0} = 0, \end{cases} \tag{A.11}$$

Proposition A.3. *Let $\alpha = \frac{1}{2}$ and $s < 1$ sufficiently close to one. Given $T, \varepsilon > 0$, there exist $\tilde{\Omega}_{T,\varepsilon} \subset \Omega$ such that*

$$\mathbb{P}((\tilde{\Omega}_{T,\varepsilon})^c) < \varepsilon,$$

a dyadic integer $M_0 = M_0(T, \varepsilon)$ and a finite constant $C(T, \varepsilon) > 0$ such that the following bound holds:

$$\sup_{\substack{M \geq M_0 \\ M \text{dyadic}}} \sup_{t \in [0,T]} \|v_M(t)\|_{H^s(\mathbb{T})} \leq C(T, \varepsilon),$$

for every solution v_M^ω to (A.11) *with $\omega \in \tilde{\Omega}_{T,\varepsilon}$.*

Proof. With $T>0$ fixed, we define

$$\Sigma_{\mathrm{conv},T} = \{\omega\in\Omega\,:\,(z_M^\omega,\mathcal{N}(z_M^\omega))\to(z^\omega,\mathcal{N}(z^\omega)) \text{ in } C_{2T}W^{\alpha-\frac12-,r}\times L^4_{2T}H^s \\ \text{as } M\to\infty,\ M \text{ dyadic}\},$$

where $r=r(s,\alpha)$ is as in Subsection 4.2. Note that (A.4) and Lemma A.2 imply $\mathbb{P}(\Sigma_{\mathrm{conv},T})=1$. With $K>0$ fixed, we define

$$\Omega_{K,T,\alpha} = \{\omega\in\Sigma_{\mathrm{conv},T}\,:\,\|z\|_{C_{2T}W_x^{\alpha-\frac12-,p}}+\|\mathcal{N}(z)\|_{L^4_{2T}H^s_x}\le K\}.$$

By Egoroff's theorem, for any $\varepsilon>0$, there exists $\Omega_\varepsilon\subset\Sigma_{\mathrm{conv},T}$ with $\mathbb{P}(\Sigma_{\mathrm{conv},T}\setminus\Omega_\varepsilon)<\frac{\varepsilon}{3}$, such that $\mathcal{N}(z_M^\omega)$ converges uniformly to $\mathcal{N}(z^\omega)$ as $M\to\infty$ in $L^4_{2T}H^s_x$ for every $\omega\in\Omega_\varepsilon$. Hence, there exists $M_0=M_0(T,\varepsilon)$ such that for every $M\ge M_0$, we have

$$\|\mathcal{N}(z_M^\omega)\|_{L^4_{2T}H^s_x}\le 1+K$$

for every $\omega\in\Omega_{K,T,\alpha,\varepsilon}:=\Omega_\varepsilon\cap\Omega_{K,T,\alpha}$. Lemma A.2 implies

$$\mathbb{P}(\Omega^c_{K,T,\alpha,\varepsilon})\le\frac{C_{T,s,\alpha}}{K^4}+\frac{\varepsilon}{3},$$

and hence with $K=K(\varepsilon,T,s,\alpha)$ large enough, we have $\mathbb{P}(\Omega^c_{K,T,\alpha,\varepsilon})<\frac{2\varepsilon}{3}$.

We now obtain an analogue of (4.16) for the growth of the modified energy $E(Iv_M)(t)=:E_M(t)$. We first note that the result of Lemma 4.2 holds for the initial data (A.1) in view of (A.2). The only modification we make is in estimating the term (II):

$$\int_0^t\int_{\mathbb{T}}(\partial_x Iv_M)I(\mathbf{P}_{\neq0}(z_M^2))dxdt'.$$

Namely, by Cauchy-Schwarz and Hölder's inequalities, we have

$$\begin{aligned}\bigg|\int_0^t\int_{\mathbb{T}}&(\partial_x Iv_M)I(\mathbf{P}_{\neq0}(z_M^2))dxdt'\bigg|\\ &\le\int_0^t\|I(\mathbf{P}_{\neq0}(z_M^2))\|_{L^2_x}E^{\frac12}(t')\,dt'\\ &\le\|I(\mathbf{P}_{\neq0}(z_M^2))\|_{L^4_TL^2_x}\bigg(\int_0^tE^{\frac23}(t')\,dt'\bigg)^{\frac34}\\ &\lesssim N^{1-2\alpha+}\|\mathbf{P}_{\neq0}(z_M^2)\|_{L^4_TH^{s-1-}_x}\bigg[1+\int_0^tE^{\frac23}(t')\,dt'\bigg].\end{aligned}$$

Then, applying the same ideas as in the proof of Proposition 4.1 we obtain an inequality of the form (4.19) (the analogue of (4.21)) with $\gamma=\frac23$ and $c\sim N^{1-2\alpha+}K$. We then apply Lemma 4.6 and we complete the argument as in the proof of Proposition 4.1 provided $\alpha=\frac12$, with $\tilde\Omega_{T,\varepsilon}:=\tilde\Omega_{T,\varepsilon}:=\Omega_{K(\varepsilon,T),T,\frac12,\varepsilon}\cap\Omega_{\Lambda,N}$ and $\Omega_{\Lambda,N}$ defined in (4.20).

□

Appendix B. Tail estimates on random variables

In this appendix, we state some standard results which allows us to prove tail estimates of random variables using estimates on the moments of differences as in, for example, (2.6). Our reference for this appendix is [26, Appendix A.2 and A.3].

Theorem B.1 (Garsia-Rudemich-Rumsey inequality, [26, Theorem A.1])**.** *Let (E,d) be a metric space and $f \in C([0,T];E)$. Let Ψ and P be continuous strictly increasing functions on $[0,\infty)$ with $P(0)=\Psi(0)=0$ and $\Psi(x)\to\infty$ as $x\to\infty$. Suppose*

$$\int_0^T\int_0^T \Psi\bigg(\frac{d(f(t'),f(t))}{P(|t-s|)}\bigg)dt'dt \le F.$$

Then for any $0\le s<t\le T$, we have

$$d(f(t'),f(t)) \le 8\int_0^{t-t'} \Psi^{-1}\bigg(\frac{4F}{x^2}\bigg)dP(x).$$

Putting $\Psi(x)=x^q$ and $P(x)=x^{\beta+\frac{1}{q}}$ for $\beta>\frac{1}{q}$, we obtain the following useful corollary.

Corollary B.2. *Suppose $q>1$ and $\beta>\frac{1}{q}$. Then for $0\le t'<t\le T$, we have*

$$d(f(t'),f(t))^q \le C(\beta,q)^q|t-t'|^{q\beta-1}\iint_{[t',t]}\frac{d(f(u),f(v))^q}{|u-v|^{q\beta+1}}dudv,$$

where

$$C(\beta,q) := 32^q\bigg(\frac{q\beta+1}{q\beta-1}\bigg)^q.$$

In particular, with $\gamma=\beta-\frac{1}{q}$, we have

$$\begin{aligned}\|f\|_{\dot C^\gamma([0,T];E)} &:= \sup_{0\le t'<t\le T}\frac{d(f(t),f(t'))}{|t-t'|^\gamma}\\ &\le C(\beta,q)^{\frac{1}{q}}\bigg(\int_0^T\int_0^T\frac{d(f(u),f(v))^q}{|u-v|^{q\beta+1}}dudv\bigg)^{\frac{1}{q}}.\end{aligned}\tag{B.1}$$

We use (B.1) to help estimate the moments of $C^\gamma([0,T];E)$-norms of random variables. In applying Kolmogorov's continuity criterion, one shows a difference estimate like

$$\mathbb{E}[d(f(t),f(t'))^q] \le K(\eta,q)|t-t'|^{1+\eta},\tag{B.2}$$

where $\eta>0$.

We then conclude from (B.2) that the process f belongs to $C^{\frac{\eta}{q}-\gamma}([0,T];E)$ for any $\gamma<\frac{\eta}{q}$ almost surely. Using (B.2) in (B.1), we have

$$\begin{aligned}\mathbb{E}[\|f\|^q_{\dot C^\gamma([0,T];E)}] &\le K(\eta,q)C(\beta,q)\int_0^T\int_0^T\frac{|u-v|^{1+\eta}}{|u-v|^{q\beta+1}}dudv\\ &= K(\eta,q)C(\beta,q)\frac{2T^{2+\eta-q\beta}}{(1+\eta-q\beta)(2+\eta-q\beta)},\end{aligned}\tag{B.3}$$

provided $q\beta-\eta<1$. Thus we require:

$$\beta\in\bigg(\frac{1}{q},\frac{\eta}{q}+\frac{1}{q}\bigg)\qquad\text{and}\qquad\gamma<\frac{\eta}{q}.$$

We then use (B.3) to obtain appropriate tail estimates.

As an example, in (2.11) we obtained

$$\mathbb{E}[\|z(t)-z(t')\|^q_{W^{s_1,p}(\mathbb{T})}] \lesssim q^{\frac{q}{2}}|t-t'|^q$$

for $q \geq 2$. Thus from (B.3) we get

$$\mathbb{E}[\|z\|^q_{\dot{C}^\gamma([0,T];W^{s_1,p}(\mathbb{T}))}] \leq \frac{2C(\beta,q)q^{\frac{q}{2}}}{q(1-\beta)[1+q(1-\beta)]}T^{q(1-\beta)+1},$$

provided $\gamma < 1 - \frac{1}{q}$ and $\beta \in \left(\frac{1}{q}, 1\right)$. To convert this into a tail estimate on $\|z\|_{\dot{C}^\gamma([0,T];W^{s_1,p}(\mathbb{T}))}$, Chebyshev's inequality implies

$$\mathbb{P}(\|z\|_{\dot{C}^\gamma([0,T];W^{s_1,p}(\mathbb{T}))} > \lambda) \leq \frac{D(\beta,q)}{\lambda^q}T^{q(1-\beta)+1}, \tag{B.4}$$

where

$$D(\beta,q) = \frac{32^q\left(\frac{q\beta+1}{q\beta-1}\right)^q q^{\frac{q}{2}}}{q(1-\beta)[1+q(1-\beta)]} \leq \frac{32^q\left(\frac{q\beta+1}{q\beta-1}\right)^q q^{\frac{q}{2}}}{q(1-\beta)}$$

We notice that since $q \geq 2$, we can choose $\beta = \frac{1}{4} + \frac{1}{q}$, which allows for the loose bound

$$D(\beta,q) \leq (Cq)^{\frac{q}{2}}.$$

Now, we optimise in q the bound on the right hand side (B.4) and we obtain the exponential tail estimate

$$\mathbb{P}(\|z\|_{\dot{C}^\gamma([0,T];W^{s_1,p}(\mathbb{T}))} > \lambda) \leq \exp(-c\lambda^2 T^{-\frac{3}{4}}). \tag{B.5}$$

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