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COMBINATORIAL SOLUTIONS TO THE REFLECTION EQUATION

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Abstract. We use ring-theoretic methods and methods from the theory of skew braces to produce set-theoretic solutions to the reflection equation. We also use set-theoretic solutions to construct solutions of the parameter dependent reflection equation.

Introduction

Following Drinfeld's suggestion in [13], the study of set-theoretic solutions to the Yang–Baxter equation (YBE) started in the seminal papers of Etingof, Schedler and Soloviev [14] and Gateva–Ivanova and Van den Bergh [18]. Since then, different aspects of this combinatorial problem have been developed [16, 17, 24, 25, 30] and several interesting connections have been found. In pure mathematics, some of these connections are with braid and Garside groups [8, 11], (semi)groups of I-type [18, 22], matched pairs of groups [24, 31], Artin-Schelter regular algebras [15], Jacobson radical rings and generalizations [6, 26], regular subgroups and Hopf–Galois extensions [29], affine manifolds [27], orderability [1, 9] and factorizable groups [33]. In mathematical physics, the connections include Yang–Baxter maps [32], discrete integrable systems, cellular automata, crystals and tropical geometry (see [20] and references therein). As a result of all these relationships, there has been intensive study of set-theoretic solutions to the quantum Yang–Baxter equation.

Recall that a pair (X, r), where X is a set and $r\colon X \times X \to X \times X$ is a bijective map, is a set-theoretic solution to the quantum Yang–Baxter equation if[1]

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r).$$

The solution (X, r) is said to be non-degenerate if it is possible to write

$$r(x, y) = (\sigma_x(y), \tau_y(x))$$

for bijective maps $\sigma_x, \tau_x\colon X \to X$ for all $x \in X$. As usual, the solution (X, r) will be called involutive if $r^2 = \mathrm{id}_{X\times X}$.

If R is an associative ring, the operation $x \circ y = x + y + xy$ is always associative with neutral element 0_R. In the case when this operation $\circ$ turns R into a group, then R is a Jacobson radical ring [21]. As it was observed by

[1]In the physics literature this r is usually denoted by $\check{R}$.

Rump [26], Jacobson radical rings produce highly non-trivial set-theoretic solutions to the Yang–Baxter equation. More precisely, if R is a Jacobson radical ring, then

$$r\colon R\times R\to R\times R,\qquad r(x,y)=(xy+y,(xy+y)'xy),$$

is a non-degenerate involutive solution to the Yang–Baxter equation, where $(xy+y)'$ denotes the inverse of the element $xy+y$ with respect to the Jacobson circle operation $\circ$.

Rump observed that Jacobson radical rings can be generalized to braces. With braces one produces non-degenerate involutive solutions very similar to those coming from radical rings. These new solutions are universal in the sense that each non-degenerate involutive solution is isomorphic to the restriction of a solution constructed from a brace. To study non-involutive solutions one replaces braces by skew braces [19]. Skew braces still share several properties with braces and of course with Jacobson radical rings, so techniques and tools from ring theory are available to study arbitrary set-theoretic solutions.

Recently, there has been considerable interest in the reflection equation, which first appeared in the study of quantum scattering on the half-line by Cherednik [7]. The role of parameter dependent solutions to the reflection equation in the description of quantum-integrable systems with open boundaries was formulated by Sklyanin [28]. Just as for the theory of the Yang–Baxter equation, it also turns out to be interesting to study a combinatorial version of the reflection equation. For a map $k:X\to X$, and (X,r) as above, this combinatorial reflection equation is

$$r(\mathrm{id}\times k)r(\mathrm{id}\times k)=(\mathrm{id}\times k)r(\mathrm{id}\times k)r.$$

This set-theoretical equation together with the first examples of solutions first appeared in the work of Caudrelier and Zhang [4]. A more systematic study and a classification in a slightly different setting (i.e. not inspired by soliton interactions but by maps appearing in integrable discrete systems) appeared in [3]. Other solutions were also considered and used by Kuniba, Okado and Yamada within the context of cellular automata [23].

In this work we use ring-theoretic methods, and more generally methods coming from the theory of braces, to produce families of new solutions to the reflection equation. Our purely combinatorial approach is different to that of [10], where actions of skew braces are used to produce reflections.

The paper is organized as follows. In Section 1 we give the main definitions and examples and then prove that for finite, non-degenerate, involutive set-theoretic solutions to the YBE one only needs to check one of the coordinates to prove that a certain map is a reflection. Theorem 1.9 shows that each map invariant under the action of the permutation group of a finite non-degenerate involutive solution yields a reflection; this result easily produces several reflections. Section 3 contains several reflections constructed from the theory of left braces and Jacobson radical rings. Section 4 explores reflections associated with the solutions of the YBE constructed by Weinstein and Xu from factorizable groups. In Section 5 we show a way of introducing parameter dependence in both r and k to yield solutions of the respective parameter dependent quantum Yang–Baxter and reflection equations.

1. Preliminaries

A set-theoretic solution to the YBE is a pair (X,r), where X is a set and $r\colon X\times X\to X\times X$ is a bijective map such that

$$r_1r_2r_1=r_2r_1r_2,$$

where $r_1=r\times\mathrm{id}$ and $r_2=\mathrm{id}\times r$. By convention, we write

$$r(x,y)=(\sigma_x(y),\tau_y(x)).$$

If the solutions is said to be non-degenerate, then $\sigma_x,\tau_x\colon X\to X$ are assumed to be bijective. The solution (X,r) is finite if X is finite and it is involutive if $r^2=\mathrm{id}$.

Remark 1.1. *If (X,r) is a non-degenerate involutive solution, then*

$$\sigma_{\sigma_x(y)}(\tau_y(x))=x,\quad \tau_{\tau_y(x)}(\sigma_x(y))=y.$$

for all $x,y\in X$. In particular, if furthermore (X,r) is non-degenerate, then

$$\tau_y(x)=\sigma^{-1}_{\sigma_x(y)}(x),\quad \sigma_x(y)=\tau^{-1}_{\tau_y(x)}(y).$$

for all $x,y\in X$.

Let us first recall the basic definitions from [3].

Definition 1.2. *Let (X,r) be a non-degenerate solution. We say that a map $k\colon X\to X$ is a* reflection *of (X,r) if*

$$r(\mathrm{id}\times k)r(\mathrm{id}\times k)=(\mathrm{id}\times k)r(\mathrm{id}\times k)r.$$

The reflection k is said to be involutive *if $k^2=\mathrm{id}$.*

If $X=\{1,\dots,n\}$ and $k\colon X\to X$, we use the one-line notation for k which is simply the string $k(1)k(2)\cdots k(n)$. For example, the identity would be $\mathrm{id}=123\cdots n$.

Example 1.3. *Let $X=\{1,2,3\}$ and $r(x,y)=(\varphi_x(y),\varphi_y(x))$, where $\varphi_1=\varphi_2=\mathrm{id}$ and $\varphi_3=(12)$. There are five reflection maps:*

$$k_1=123=\mathrm{id},\quad k_2=113,\quad k_3=213,\quad k_4=223,\quad k_5=333.$$

Moreover, k_j is involutive if and only if $j\in\{1,3\}$.

Example 1.4. *Let X be a set and $\sigma,\tau\colon X\to X$ be permutations such that $\sigma\tau=\tau\sigma$. Then (X,r), where $r(x,y)=(\sigma(y),\tau(x))$ is a solution to the YBE. Each map $k\colon X\to X$ that commutes with $\sigma\tau$ is a reflection map.*

Example 1.5. *Let $X = \{1,2,3,4\}$ and $r(x,y) = (\sigma_x(y), \tau_y(x))$, where*

$$\sigma_1 = (34), \qquad \sigma_2 = (1324), \qquad \sigma_3 = (1423), \qquad \sigma_4 = (12),$$
$$\tau_1 = (24), \qquad \tau_2 = (1423), \qquad \tau_3 = (1234), \qquad \tau_4 = (13).$$

There are ten reflection maps:

$$k_1 = 1144, \quad k_2 = 1212, \quad k_3 = 1234, \quad k_4 = 1331, \quad k_5 = 2143,$$
$$k_6 = 2233, \quad k_7 = 3412, \quad k_8 = 3434, \quad k_9 = 4224, \quad k_{10} = 4321.$$

Moreover, k_j is involutive if and only if $j \in \{3,5,7,10\}$.

Notation 1.6. *Let (X,r) be a solution. For each $x, y \in X$ let*

$$t(x,y) = \sigma_{\sigma_x(y)} k(\tau_y(x)), \qquad u(x,y) = \tau_{k(\tau_y(x))}\sigma_x(y).$$

Lemma 1.7. *Let (X,r) be a non-degenerate solution to the YBE. The map $k\colon X \to X$ is a reflection of (X,r) if and only if*

$$t(x,k(y)) = t(x,y) \quad \text{and} \quad u(x,k(y)) = k(u(x,y))$$

for all $x,y \in X$.

Proof. It is straightforward. □

The following theorem shows that for involutive solutions one needs to check only one of the formulas of Lemma 1.7.

Theorem 1.8. *Let (X,r) be a non-degenerate involutive solution to the YBE and let $k\colon X \to X$ be a map. Then k is a reflection of (X,r) if and only if*

$$t(x,y) = t(x,k(y)) \tag{1.1}$$

for all $x,y, \in X$.

Proof. One of the implications follows directly from Lemma 1.7. Let us then assume that (1.1) holds for all $x,y \in X$. Let $x,y \in X$. We need to prove that $u(x,k(y)) = k(u(x,y))$. Let

$$a = \sigma_x(y), \qquad b = \tau_y(x), \qquad d = \sigma_a(k(b)), \qquad e = k(\tau_{k(b)}(a)). \tag{1.2}$$

A straightforward calculation shows that $\sigma_d(e) = t(a,k(b))$. Since (X,r) is involutive, $\sigma_a(b) = x$ and $\tau_b(a) = y$. These facts and Equality (1.1) imply that

$$\sigma_d(e) = t(a,k(b)) = t(a,b) = \sigma_x(k(y)).$$

Since (X,r) is non-degenerate, $e = \sigma_d^{-1}\sigma_x(k(y))$ and hence

$$k(u(x,y)) = k\left(\tau_{k(\tau_y(x))}\sigma_x(y)\right) = \sigma_d^{-1}\sigma_x(k(y))$$

by (1.2) and the definition of $u(x,y)$. Now let

$$g = u(x,k(y)) = \tau_{k(\tau_y(x))}\sigma_x(k(y)), \quad f = \sigma_{\sigma_x(k(y))}k(\tau_{k(y)}(x)).$$

Applying $r^2 = \mathrm{id}$ to the pair $(\sigma_x(k(y)), k(\tau_y(x)))$ one gets

$$\sigma_f(g) = \sigma_x(k(y)), \quad \tau_g(f) = k(\tau_y(x)).$$

Since $d = t(x, y) = t(x, k(y)) = f$, it follows that

$$u(x, k(y)) = g = \sigma_f^{-1}\sigma_x(k(y)) = \sigma_d^{-1}\sigma_x(k(y)).$$

Then Lemma 1.7 implies the claim. □

The *involutive Yang–Baxter group* of an involutive non-degenerate solution (X, r) is the group $\mathcal{G}(X, r)$ generated by $\{\sigma_x : x \in X\}$. Clearly this group acts on X by evaluation. A map $k\colon X \to X$ is said to be $\mathcal{G}(X, r)$-equivariant if $k(gx) = gk(x)$ for all $g \in \mathcal{G}(X, r)$ and $x \in X$. Thus k is $\mathcal{G}(X, r)$-equivariant if and only if $k\sigma_x = \sigma_x k$ for all $x \in X$.

Theorem 1.9. *Let (X, r) be an involutive non-degenerate solution to the YBE. Each $\mathcal{G}(X, r)$-equivariant map $k\colon X \to X$ is a reflection of (X, r).*

Proof. Let $x, y \in X$ and $u = \sigma_x(k(y))$ and $v = \tau_{k(y)}(x)$. Using Remark 1.1 and that k is $\mathcal{G}(X, r)$-equivariant, we write

$$rk_2rk_2(x, y) = (\sigma_u(k(v)), \beta) = (k(\sigma_u(v)), \beta) = (k(x), \beta)$$

for some $\beta \in X$. Similarly, we write

$$k_2rk_2r(x, y) = k_2rk_2(\sigma_x(y), \tau_y(x)) = (\sigma_{\sigma_x(y)}k(\tau_y(x)), \gamma) = (k(x), \gamma)$$

for some $\gamma \in X$. Hence the claim follows from Theorem 1.8. □

The converse of Theorem 1.9 does not hold:

Example 1.10. *Let $X = \{1, 2, 3, 4\}$ and $r(x, y) = (\sigma_x(y), \tau_y(x))$, where*

$$\begin{array}{llll} \sigma_1 = (12), & \sigma_2 = (1324), & \sigma_3 = (34), & \sigma_4 = (1423), \\ \tau_1 = (14), & \tau_2 = (1243), & \tau_3 = (23), & \tau_4 = (1342). \end{array}$$

There are then ten reflection maps:

$$\begin{array}{lllll} k_1 = 1133, & k_2 = 1221, & k_3 = 1234, & k_4 = 1414, & k_5 = 2143, \\ k_6 = 2244, & k_7 = 3232, & k_8 = 3412, & k_9 = 4321, & k_{10} = 4334. \end{array}$$

A direct calculation shows that $\mathcal{G}(X, r)$ is isomorphic to the dihedral group of eight elements. Moreover, k_j is $\mathcal{G}(X, r)$-equivariant if and only if $j \in \{3, 5\}$.

2. Reflections and left braces

A left brace is a triple $(A, +, \circ)$ such that $(A, +)$ is an abelian group, $(A, \circ)$ is a group and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds for all $a, b, c \in A$. If A is a left brace, the multiplicative group $(A, \circ)$ of A acts by automorphism on the additive group $(A, +)$, i.e. the map $\lambda\colon (A, \circ) \to \mathrm{Aut}(A, +)$ given by $a \mapsto \lambda_a$, where $\lambda_a(b) = -a + a \circ b$, is a group homomorphism. For $a, b \in A$ one defines the operation

$$a * b = \lambda_a(b) - b = -a + a \circ b - b.$$

Left braces produce solutions to the YBE. The map

$$r_A\colon A\times A\to A\times A,\quad r_A(a,b)=(\lambda_a(b),\mu_b(a)),\tag{2.1}$$

where

$$\mu_b(a)=\lambda_a(b)'\circ a\circ b=\lambda_a(b)'*a+a,$$

where x' denotes the inverse of the element x with respect to the circle operation, is an involutive non-degenerate solution to the YBE. We refer to [5] for an introduction to the theory of left braces.

A left ideal of a left brace A is a subgroup X of the additive group of A such that $A*X\subseteq X$. An ideal I of A is a left ideal I of A such that $a+I=I+a$ and $a\circ I=I\circ a$ for all $a\in A$. The *socle* of a left brace A is defined as

$$\operatorname{Soc}(A)=\{a\in A:\lambda_a=\mathrm{id}\}=\{a\in A:a+b=a\circ b\text{ for all }b\in A\}$$

and it is an ideal of A.

Theorem 2.1. *Let A be a left brace, $X\subseteq A$ be a subset such that the restriction $r=r_A|_{X\times X}$ is a solution (X,r) to the YBE and $g\colon X\to Y$ be a map for some subset $Y\subseteq A$. Assume that there exists a map $X\times Y\to X$, $(x,y)\mapsto x\odot y$, such that*

$$\lambda_x(y\odot g(z))=\lambda_x(y)\odot g(z)\tag{2.2}$$

for all $x,y,z\in X$. Let $f\colon X\to X$ be $\mathcal{G}(X,r)$-equivariant and $k\colon X\to X$ be given by $k(x)=f(x)\odot g(x)$. Then k is a reflection of (X,r) if and only if

$$f(x)\odot g(\mu_{k(y)}(x))=f(x)\odot g(\mu_y(x))$$

for all $x,y\in X$.

Proof. Let $x,y\in X$ and

$$(u,v)=rk_2(x,y)=(\lambda_x(k(y)),\mu_{k(v)}(x)).$$

Then by Remark 1.1 and using that (2.2) holds and f is $\mathcal{G}(X,r)$-equivariant,

$$\begin{aligned}\lambda_u k(v)&=\lambda_u(f(v)\odot g(v))\\&=\lambda_u(f(v))\odot g(v)=f(\lambda_u(v))\odot g(v)=f(x)\odot g(v)\end{aligned}$$

as $r^2(x,k(y))=r(u,v)=(\lambda_u(v),\mu_v(u))=(x,k(y))$. Then

$$rk_2rk_2(x,y)=(f(x)\odot g(v),\beta)\tag{2.3}$$

for some $\beta\in X$. Let $a,b\in X$ be such that $k_2r(x,y)=k_2(a,b)=(a,k(b))$. As we did before,

$$\begin{aligned}\lambda_a(k(b))&=\lambda_a(f(b)\odot g(b))\\&=\lambda_a(f(b))\odot g(b)=f(\lambda_a(b))\odot g(b)=f(x)\odot g(b)\end{aligned}$$

Hence

$$k_2rk_2r(x,y)=(f(x)\odot g(b),\gamma)\tag{2.4}$$

for some $\gamma \in X$. Using (2.2),

$$f(x) \odot g(v) = f(x) \odot g(\mu_{k(y)}(x)) = f(x) \odot g(\mu_y(x)) = f(x) \odot g(b)$$

and therefore the first coordinate of (2.3) is equal to the first coordinate of (2.4). Now the claim follows from Theorem 1.8. $\square$

As an application of Theorem 1.9 we construct reflections using two-sided braces and left ideals. As was mentioned before, Rump observed that two-sided braces are equivalent to Jacobson radical rings. This equivalence is based on the following lemma of [26]. We provide a proof for completeness.

Lemma 2.2. *Let A be a left brace. Then*

$$(a \circ b) * c = (a \circ b \circ a') * \lambda_a(c) + a * c \tag{2.5}$$

for all $a, b, c \in A$. Furthermore, if A is two-sided, then

1. $(a + b) * c = a * c + b * c$ *for all $a, b, c \in A$,*
2. $(-a) * b = -a * b$ *for all $a, b \in A$, and*
3. $\lambda_a(b * c) = \lambda_a(b) * c$ *for all $a, b, c \in A$.*

Proof. Let $a, b, c \in A$. To prove (2.5) we compute

$$\begin{aligned}(a \circ b) * c &= \lambda_{a\circ b}(c) - c = \lambda_{a\circ b\circ a'}\lambda_a(c) - c \\ &= (a \circ b \circ a') * \lambda_a(c) + \lambda_a(c) - c = (a \circ b \circ a') * \lambda_a(c) + a * c.\end{aligned}$$

We now prove (1). Using the commutativity of the additive group of A,

$$(a + b) * c = \lambda_{a+b}(c) - c = -a - b + a \circ c - c + b \circ c - c = a * c + b * c$$

for all $a, b, c \in A$. To prove (2) we use (1) to obtain

$$0 = 0 * b = (a + (-a)) * b = a * b + (-a) * b,$$

from where (2) follows. Now we prove (3). On the one hand,

$$\lambda_a(b * c) = \lambda_a(\lambda_b(c) - c) = \lambda_{a\circ b\circ a'}\lambda_a(c) - \lambda_a(c) = (a \circ b \circ a') * \lambda_a(c). \tag{2.6}$$

On the other, using (2) and (2.5),

$$\begin{aligned}\lambda_a(b) * c &= (-a + a \circ b) * c = (-a) * c + (a \circ b) * c \\ &= -a * c + (a \circ b \circ a') * \lambda_a(c) + a * c = (a \circ b \circ a') * \lambda_a(c).\end{aligned}$$ $\square$

Lemma 2.3. *Let A be a left brace and X be a left ideal of A. The map $r = r_A|_{X\times X}$ is a solution (X, r) to the YBE.*

Proof. Since X is a left ideal, $(X, +)$ is a subgroup of $(A, +)$ and $(X, \circ)$ is a subgroup of $(A, \circ)$. Then $r_A(x, y) = (\lambda_x(y), \lambda_x(y)' \circ x \circ y) \in X \times X$ if $x, y \in X$. $\square$

Now we obtain several corollaries of Theorem 2.1 in the case of Jacobson radical rings.

Corollary 2.4. *Let A be a two-sided brace, X be a left ideal of A and $f,g\colon X\to X$ be maps such that f is $\mathcal{G}(X,r)$-equivariant and*

$$f(x)*g(\mu_{k(y)}(x))=f(x)*g(\mu_y(x)) \tag{2.7}$$

for all $x,y\in X$. Then $k\colon X\to X$, $k(x)=f(x)*g(x)$, is a reflection of (X,r_X).

Proof. It follows from Theorem 2.1 with $x\odot y=x*y$. $\square$

Corollary 2.5. *Let A be a two-sided brace, X be a left ideal of A and $g\colon X\to X$ be a map such that $g(\mu_{k(y)}(x))=g(\mu_y(x))$ for all $x,y\in X$. Then $k\colon X\to X$, $k(x)=x*g(x)$ is a reflection of (X,r_X).*

Proof. It follows from Corollary 2.4 with $f=\mathrm{id}$. $\square$

Corollary 2.6. *Let A be a two-sided brace and $a\in A$. Then $k(x)=x*a$ is a reflection of (A,r_A).*

Proof. Use Corollary 2.4 with $f=\mathrm{id}$ and $g(x)=a$ for all $x\in A$. $\square$

Reflections from Corollaries 2.5 and 2.6 are $\mathcal{G}(X,r)$-equivariant.

Corollary 2.7. *Let A be a two-sided brace, X be a left ideal of A and $g\colon X\to X$ be a map such that $g(\mu_{k(y)}(x))=g(\mu_y(x))$ for all $x,y\in X$. Then $k\colon X\to X$, $k(x)=x+x*g(x)$, is a reflection of (X,r_X).*

Proof. It follows from Theorem 2.1 with $f=\mathrm{id}$ and $x\odot y=x+x*y$. $\square$

Corollary 2.8. *Let A be a two-sided brace, X be an ideal of A and $g\colon A\to A$ be such that $g(A)\subseteq X$ and $g(a+x)=g(a)$ for all $a\in A$ and $x\in X$. Then $k(x)=x+x*g(x)$ is a reflection of (A,r_A).*

Proof. Note that $k(x)-x=x*g(x)\in X$ for all $x\in X$. Since

$$\mu_{k(y)}(x)=\lambda^{-1}_{\lambda_x(k(y))}(x)=\lambda^{-1}_{\lambda_x(y)}(x)=\mu_y(x)$$

in the quotient A/X, it follows that $\lambda_x(k(y))=\lambda_x(y)+z$ for some $z\in X$. Hence

$$g(\mu_{k(y)}(x))=g(\mu_y(x)+z)=g(\mu_y(x))$$

and the claim follows from Theorem 2.1 with $f=\mathrm{id}$ and $a\odot b=a+a*b$. $\square$

Reflections from Corollaries 2.7 and 2.8 do not seem to be necessarily $\mathcal{G}(X,r)$-equivariant.

Corollary 2.9. *Let A be a two-sided brace, X be an ideal of A, $f\colon A\to A$ be $\mathcal{G}(X,r)$-equivariant and such that $f(x)-x\in X$ for all $x\in A$ and $g\colon A\to A$ be such that $g(A)\subseteq X$ and $g(a+x)=g(a)$ for all $a\in A$ and $x\in X$. Then $k(x)=f(x)+f(x)*g(x)$ is a reflection of (A,r_A).*

Proof. It is similar to the proof of Corollary 2.8. We only need to write in the first line $k(x)-x=f(x)*g(x)\in X$ for all $x\in X$. $\square$

For some type of solutions (X,r) all the reflection maps are as those of the previous corollaries as the following example shows:

Example 2.10. *Let $(A, +, *)$ be a nilpotent algebra over a field of characteristic two and let $z \in A$. Then*

$$S = \{z * r, z * r + z : r \in A\}$$

is a nilpotent ring. Let

$$X = \{\lambda_s(z) : s \in S\}$$

and let (X, r) be the associated solution to the YBE. We will show that if $a, b \in X$ then $b = \lambda_a(c)$ for some $c \in A$. Indeed, if $a = \lambda_s(z)$, for $s \in S$ then $b = \lambda_s(\lambda_{s^{-1}}(b))$ and $\lambda_{s^{-1}}(b) \in X$, thus $\lambda_{s^{-1}}(b) = z * r + z$ for some $r \in R$. It follows that $b = \lambda_s(z * d + z) = a * d + a$, consequently any function $k : X \to X$ can be written as $k(x) = x * g(x) + x$ for some $g(x) \in Az$. Therefore any reflection k on (X, r) satisfies assumptions of Theorem 2.1 with $x \odot y = x * y + x$ for $x, y \in X$ and $Y = Az$. In particular $k(x)$ is a solution of the reflection equation on (X, r) if and only if $k(x) = x * g(x) + x$ for some $g : X \to Az$ such that $g(\tau_y(x)) = g(\tau_{k(y)}(x))$ for all $x, y \in X$.

3. Other reflections

The following modification of Theorem 2.1 is sometimes useful for describing solutions of the reflection equation related to two-sided braces.

Theorem 3.1. *Let A be a two-sided brace, $X \subseteq A$ be a subset such that the restriction $r = r_A|_{X \times X}$ is a solution (X, r) to the YBE, $g_1, \ldots, g_n \colon X \to X$, $f_1, \ldots, f_n \colon X \to X$ be $\mathcal{G}(X, r)$-equivariant and $k \colon X \to X$ be given by*

$$k(x) = x + \sum_{i=1}^{n} f_i(x) \odot g_i(x),$$

where the operation $\odot$ is as in Theorem 2.1. Then k is a reflection of (X, r) if and only if

$$\sum_{i=1}^{n} f_i(x) \odot g_i(\mu_{k(y)}(x)) = \sum_{i=1}^{n} f_i(x) \odot g_i(\mu_y(x))$$

for all $x, y \in X$.

Proof. It is similar to the proof of Theorem 2.1. □

For a map $k \colon X \to X$, write $k_1 = k \times \mathrm{id}$ and $k_2 = \mathrm{id} \times k$.

Proposition 3.2. *Let (X, r) be a non-degenerate involutive solution. Then $k_2 r = r k_2$ if and only if $k = \mathrm{id}$.*

Proof. We only need to prove the non-trivial implication. The assumption is equivalent to $\sigma_x(y) = \sigma_x(k(y))$ and $k(\tau_y(x)) = \tau_{k(y)}(x)$ for all $x, y \in X$. Then

$$k(\tau_y(x)) = \tau_{k(y)}(x) = \sigma^{-1}_{\sigma(k(y))}(x) = \sigma^{-1}_{\sigma(y)}(x) = \tau_y(x)$$

and the claim follows since (X, r) is non-degenerate. □

Proposition 3.3. *Let (X,r) be a non-degenerate solution and $k\colon X\to X$ be such that either $k_2r=r^{-1}k_1$ or $k_1r^{-1}=rk_2$. Then k is a reflection of (X,r).*

Proof. Let us first assume that $k_2r=r^{-1}k_1$. Then $rk_2rk_2=rr^{-1}=k_1k_2$ and $k_2rk_2r=k_2rr^{-1}k_1=k_2k_1$. Since k_1 and k_2 commute, the claim follows. Similarly one proves the case where $k_1r^{-1}=rk_2$. □

Proposition 3.3 of course applies in the particular case of involutive solutions.

Proposition 3.4. *Let (X,r) be a non-degenerate solution and $k\colon X\to X$ be such that $k_2r=rk_1$ and $k_1r=rk_2$. Then k is a reflection of (X,r).*

Proof. We compute $k_2rk_2r=rk_1k_2r=r^2k_2k_1=r^2k_1k_2$ and similarly $rk_2rk_2=r^2k_1k_2$. □

Let (X,r) be a non-degenerate involutive solution to the YBE; it is known that X is a subset of some brace A such that X generates A as an additive group and restriction $r=r_A|_{X\times X}$ is a solution (X,r) to the YBE. This statement is implicit in [26] and explicit in Theorem 4.4 of [6]. In particular A can be taken to be the structure group of (X,r) (see [6] for more information). Moreover, if X is finite, A can be assumed to be finite.

Theorem 3.5. *Let A be a left brace and $X\subseteq A$ be a subset such that the restriction $r=r_A|_{X\times X}$ is a solution (X,r) to the YBE. Assume that X generates the additive group of A. Let $k\colon X\to X$ be a map. The following statements are equivalent:*

1. $k_2r=rk_1$.
2. $k_1r=rk_2$.
3. $k(x)-x\in\operatorname{Soc}(A)$ *for all* $x\in X$ *and* k *is* $\mathcal{G}(X,r)$*-equivariant.*

Proof. Let us first prove that (1) and (2) are equivalent. Assume first that (2) holds. This means that $\lambda_x(k(y))=k(\lambda_x(y))$ and $\mu_{k(y)}(x)=\mu_y(x)$ hold for all $x,y\in X$. Since

$$\lambda^{-1}_{k(\lambda_x(y))}(x)=\lambda^{-1}_{\lambda_x(k(y))}(x)=\mu_{k(y)}(x)=\mu_y(x)=\lambda^{-1}_{\lambda_x(y)}(x),$$

holds for all $x,y\in X$, by writing $y=\lambda_x^{-1}(z)$ one concludes that $\lambda^{-1}_{k(z)}(x)=\lambda_z^{-1}(x)$ for all $x,z\in X$. Now

$$\mu_y(k(x))=\lambda^{-1}_{\lambda_{k(x)}(y)}(k(x))=\lambda^{-1}_{\lambda_x(y)}(k(x))=k(\lambda^{-1}_{\lambda_x(y)}(x))=k(\mu_y(x))$$

and hence (1) holds. The implication (1) $\implies$ (2) is similar.

Now we prove that (1) and (2) imply (3). Let $x,z\in X$ and $y\in X$ be such that $z=\lambda_x(y)$. By (2), $\lambda_z(y)=\lambda_{k(z)}(y)$ for every $z,y\in X$. Since $z*x=k(z)*x$ and X generates additively the group A, it follows that $z*g=k(z)*g$ for all $g\in A$. Thus $\lambda_z=\lambda_{k(z)}$ and hence, in the quotient

$G(X,r)/Soc(G(X,r))$, one has $z = k(z)$. Notice that this implies $z' = k(z)'$ in $G(X,r)/Soc(G(X,r))$. Moreover,

$$\begin{aligned} k(z*x+x) &= k\mu_y(x) = \mu_y(k(x)) \\ &= \lambda_{k(x)}(y)' * k(x) + k(x) \\ &= \lambda_x(y)' * k(x) + k(x) = z * k(x) + k(x). \end{aligned}$$

Finally we prove that (3) $\implies$ (2). We need to show that both

$$\lambda_x(k(y)) = k(\lambda_x(y)), \quad \mu_{k(y)}(x) = \mu_y(x)$$

hold for all $x, y \in X$.

Notice that $\mu_{k(y)}(x) = \mu_y(x)$ holds because $k(y) - y \in \mathrm{Soc}(A)$ (by the definition of μ). By the definition of k we have $k(\lambda_x(y)) = \lambda_x(k(y))$, which concludes the proof. $\square$

The following example fits in the context of Theorem 3.5.

Example 3.6. *Let $(A,+,*)$ be a nilpotent ring and $(A,+,\circ)$ be the corresponding left brace, so $a \circ b = a*b+a+b$ for $a,b \in A$. Let (A, r_A) be the associated solution of the YBE and $k(x) = x + x^{n-1}$. Assume that $A^n = 0$. Then*

$$\begin{aligned} &k(x) - x = x^{n-1} \in Soc(A), \\ &k(\lambda_x(y)) = (x*y+y) + (x*y+y)^{n-1} = x*y+y+y^{n-1}, \\ &\lambda_x(k(y)) = \lambda_x(y+y^{n-1}) = x*y+y+y^{n-1}. \end{aligned}$$

and therefore $r_A k_2 = k_2 r_A$ by Theorem 3.5.

The following example shows that there are reflections that are *not* of the type covered by Theorem 3.5.

Example 3.7. *Let A be a ring generated by one generator b subject to relation $b^4 = 0$, so A is a nilpotent ring. Suppose that $x + x = 0$ for every $x \in A$. Let $X = \{b + bf : f \in A\}$, and let r be a restriction of the Yang–Baxter map associated to A. Observe that any element $x \in X$ can be written as $x = b + i \cdot b^2 + j \cdot b^3 + \langle b^4 \rangle$ for some $i, j \in \mathbb{Z}_2$ - for simplicity we will write $x = b + i \cdot b^2 + j \cdot b^3$. Let $x, y \in X$; then $x = b + ib^2 + jb^3$ and $y = b + mb^2 + nb^3$ for some $i, j, m, n \in \mathbb{Z}_2$. By using formula (2.1) for the map r associated to the brace A we get $r(x,y) = (xy + y, zx + x)$ where $z(xy+y) + z + (xy+y) = 0$. Therefore,*

$$r(x,y) = (b + (m+1)b^2 + (i+m+n)b^3, b + (i+1)b^2 + (m+i+j)b^3).$$

Define $k(x) = x + x^2$ for every $x \in X$. It follows that

$$k(b + ib^2 + jb^3) = b + (i+1)b^2 + jb^3.$$

Then

$$\begin{aligned} k_2 r k_2 r(x,y) &= (b + (i+1)b^2 + (j+1)b^3, b + (m+1)b^2 + (n+1)b^3) \\ &= r k_2 r k_2(x,y). \end{aligned}$$

Note that $k(x) - x = x^2 \notin \mathrm{Soc}(A)$.

4. Reflections from factorizable groups

An obvious question is how the solutions of the YBE and reflection equation that come from braces relate to previously known solutions. In this section we consider the skew brace description of the YBE solutions of Weinstein and Xu, and consider associated solutions of the reflection equation. These YBE solutions are also related to the triangular Hopf algebras obtained by Beggs, Gould and Majid [2]

In [33] Weinstein and Xu produced set-theoretic solutions to the YBE by using factorizable groups. Using the language of skew braces we use group factorizations to construct reflections.

A skew left brace is a non-abelian generalization of a left brace [19]. More precisely it is a set with two compatible group structures. One of these groups is known as the multiplicative group; the other as the additive group. The terminology used in the theory of Hopf-Galois extensions suggests that the additive group determines the type of the skew left brace. For example, skew left braces of abelian type are Rump braces, that is, braces with an abelian additive group. A skew left brace is a triple $(A,+,\circ)$, where $(A,+)$ and $(A,\circ)$ are (not necessarily abelian) groups such that the compatibility $a\circ(b+c) = a\circ b - a + a\circ c$ holds for all $a,b,c\in A$. For a skew left brace A, the map $r_A : A\times A\to A\times A$, $r_A(a,b) = (-a+a\circ b, (-a+a\circ b)'\circ a\circ b)$ is a non-degenerate set-theoretic solution of the Yang–Baxter equation. We write a' to denote the inverse of a with respect to the circle operation $\circ$.

We say that an additive (and not necessarily abelian) group G admits an exact factorization through subgroups A and B if

$$G = A + B = \{a + b : a\in A, b\in B\}$$

and $A\cap B = \{0\}$. This means that for $x\in G$ there are unique $a\in A$ and $b\in B$ such that $x = a+b$.

By [29, Theorem 3.3] the group G with circle operation $x\circ y = a+y+b$ whenever $x = a+b\in AB$ is a skew left brace. By [19], the pair (G, r_G), where

$$r_G\colon G\times G\to G\times G,\quad r_G(x,y) = (\lambda_x(y), \mu_y(x))$$

is a non-degenerate solution to the YBE. Note that $x\circ y = \lambda_x(y)\circ\mu_y(x)$ for all $x,y\in G$. Moreover, if $x = a+b$, then

$$\lambda_x(y) = -x + x\circ y = -b - a + a + y + b = -b + y + b$$

for all $y\in G$. We collect some useful formulas in the following lemma:

Lemma 4.1. *Let G be a group that admits an exact factorization through subgroups A and B and $z = cd\in AB$ for $c\in A$ and $d\in B$ central elements* *of G. Then*

$$\lambda_{xz}(y) = \lambda_x(y), \qquad \lambda_x(yz) = \lambda_x(y)z, \tag{4.1}$$

$$x \circ (yz) = (a \circ b)z, \qquad (xz) \circ y = (x \circ y)z \tag{4.2}$$

for all $x, y \in G$.

Proof. Write $x = ab$ with $a \in A$ and $b \in B$. Then

$$\lambda_{xz}(y) = (bd)^{-1}y(bd) = b^{-1}yb = \lambda_x(y).$$

The other formulas are proved similarly. □

Now we prove the main result in this section:

Theorem 4.2. *Let G be a group that admits an exact factorization through subgroups A and B and let (G, r_G) be its associated solution to the YBE. For elements $c \in A$ and $d \in B$ which are central in G and such that cd is central in G, the map $k(x) = x(cd)$ is a reflection of (G, r_G).*

Proof. Let $x = ab \in G$ with $a \in A$ and $b \in B$ and $y \in G$. We claim that $x \circ y = \lambda_x(y) \circ \mu_{yz}(x)$. Indeed,

$$\begin{aligned}(x \circ y)z &= x \circ (yz) = x \circ k(y)\\ &= \lambda_x(yz) \circ \mu_{yz}(x) = (\lambda_x(y)z) \circ \mu_{yz}(x) = (\lambda_x(y) \circ \mu_{yz}(x))z.\end{aligned}$$

Therefore $\mu_{yz}(x) = \mu_y(x)$. This implies that

$$\begin{aligned}rk_2(x, y) &= r(x, k(y)) = r(x, yz)\\ &= (\lambda_x(yz), \mu_{yz}(x)) = (\lambda_x(y)z, \mu_y(x)) = k_1 r(x, y).\end{aligned}$$

By Theorem 3.5, $rk_1 = k_2 r$. Hence k is a reflection by Proposition 3.4. □

5. Parameter-dependent solutions

In the application to quantum integrable systems, the interest is usually in solutions to the following parameter-dependent quantum Yang–Baxter and reflections equations

$$\begin{aligned}&\big(\mathfrak{r}(u) \otimes \mathrm{id}\big)\big(\mathrm{id} \otimes \mathfrak{r}(u+v)\big)\big(\mathfrak{r}(v) \otimes \mathrm{id}\big)\\ &= \big(\mathrm{id} \otimes \mathfrak{r}(v)\big)\big(\mathfrak{r}(u+v) \otimes \mathrm{id})(\mathrm{id} \otimes \mathfrak{r}(u)\big),\end{aligned} \tag{5.1}$$

$$\begin{aligned}&\big(\mathrm{id} \times \kappa(v)\big)\mathfrak{r}(u+v)\big(\mathrm{id} \times \kappa(u)\big)\mathfrak{r}(u-v)\\ &= \mathfrak{r}(u-v)\big(\mathrm{id} \times \kappa(u)\big)\mathfrak{r}(u+v)\big(\mathrm{id} \times \kappa(v)\big),\end{aligned} \tag{5.2}$$

where u and v are in $\mathbb{C}$. Here $\mathfrak{r}(u) : V \otimes V \to V \otimes V$ and $\kappa(u) : V \to V$, where $V = \mathbb{C}X$ is the vector space spanned by elements of X. These parameter dependent solutions $\mathfrak{r}(u)$ and $\kappa(u)$ are used to define *transfer matrices* which are certain maps $T(u) : V^{\otimes N} \to V^{\otimes N}$. The details may be found in the paper [28], but the key point to note is that the properties (5.1) and (5.2) lead to the result that $T(u)T(v) = T(v)T(u)$ for all $(u, v) \in \mathbb{C} \times \mathbb{C}$. This commutativity is the key defining property of a quantum integrable system.

In this section, we describe a very simple way in which rational parameter dependence may be introduced into our r and k matrices.

Let (X, r) be an involutive non-degenerate set-theoretic solution and write

$$r(x, y) = (\sigma_x(y), \tau_y(x)), \quad x, y \in X.$$

Let $V = \mathbb{C}X$ be the linear space over the field of complex numbers spanned by elements from X. For $u \in \mathbb{C}$ let

$$R(u) : V \otimes V \to V \otimes V, \quad R(u) = \mathrm{id} + ur,$$

where $r : V \otimes V \to V \otimes V$ is defined by $r(x \otimes y) = \sigma_x(y) \otimes \tau_y(x)$ for $x, y \in X$.

Notice that $R(u)$ is a rational solution of the (parameter dependent) Yang–Baxter equation

$$(R(u) \otimes \mathrm{id})(\mathrm{id} \otimes R(u+v))(R(v) \otimes \mathrm{id}) = (\mathrm{id} \otimes R(v))(R(u+v) \otimes \mathrm{id})(\mathrm{id} \otimes R(v)).$$

Let $k : X \to X$ be a function (for example a reflection on X). We can extend this function linearly to a linear map $k\colon V \to V$. For a map $k : V \to V$ and fixed $u \in \mathbb{C}$ define map $K(u) : V \to V$ by $K(u)(x) = uk(x)$ for $x \in X$.

Theorem 5.1. *Let (X, r) be an involutive non-degenerate solution to the YBE, $k : X \to X$ be an involutive reflection of (X, r), $K_1(u) = K(u) \otimes id$ and $K_2(u) = id \otimes K(u)$. Then*

$$K_2(v)R(u + v)K_2(u)R(u - v) = R(u - v)K_2(u)R(u + v)K_2(v).$$

Proof. To prove the claim it is sufficient to show that

$$k_2R(u + v)k_2R(u - v) = R(u - v)k_2R(u + v)k_2.$$

By using the fact that $R(u) = \mathrm{id} + ur$ we get that for each $x, y \in X$ we have

$$\begin{aligned} k_2R(u + v)k_2R(u - v) &= k_2(\mathrm{id} + (u + v)r)k_2(\mathrm{id} + (u - v)r) \\ &= k_2k_2 + (u - v)k_2k_2r + (u + v)k_2rk_2 + (u + v)(u - v)k_2rk_2r \end{aligned}$$

and

$$\begin{aligned} R(u - v)k_2R(u + v)k_2 &= (\mathrm{id} + (u - v)r)k_2(\mathrm{id} + (u + v)r)k_2 \\ &= k_2k_2 + (u - v)rk_2k_2 + (u + v)k_2rk_2 + (u + v)(u - v)rk_2rk_2. \end{aligned}$$

Since k is an involution, $k_2k_2r = r = rk_2k_2$. Moreover, $rk_2rk_2 = rk_2rk_2$. This concludes the proof. □

Remark 5.2. *In Theorem 5.1 we can alternatively take*

$$K(u)(x) = f(u)k(x),$$

where $f : \mathbb{C} \to \mathbb{C}$ is an arbitrary function.

Some solutions to the reflection equation constructed earlier in this paper are involutive. We now present three examples of involutive reflections:

Example 5.3. *Let A be a nilpotent ring, $X \subseteq A$ be a subset such that the restriction r of r_A to $X \times X$ is a solution to the Yang–Baxter equation. Let $c \in A$ be such that $c * c = 0$, define $k(x) = x + x * c$, then*

$$rk_2rk_2 = k_2rk_2r.$$

Moreover, if $a + a = 0$ for every $a \in A$, then k is an involution. This means that k and r give solutions to the parameter depended equation as in Theorem 5.1.

Example 5.4. *Let A be a nilpotent ring and $X \subseteq A$ be a subset such that the restriction r of r_A to $X \times X$ is a solution to the Yang–Baxter equation. Define $k(x) = x + x^{n-1}$, then $rk_2rk_2 = k_2rk_2r$. Moreover, if $a + a = 0$ for every $a \in A$, then*

$$k^2(x) = k(x + x^{n-1}) = x + x^{n-1} + x^{n-1} = x.$$

Hence k is an involution. So k and r give solutions to the parameter dependent equation as in Theorem 5.1.

Example 5.5. *Let A be a nilpotent ring such that $a + a = 0$ for every $a \in A$. Let I be an ideal of A and $g\colon A \to A$ be such that $g(A) \subseteq I$ and $g(a + x) = g(a)$ for all $a \in A$ and $x \in I$. Assume that $g(x)^2 = 0$ for every $x \in A$. Then $k(x) = x + x * g(x)$ is an involution and a reflection of (A, r_A). So k and r give solutions to the parameter dependent equation as in Theorem 5.1.*

Notice that the linear mapping K from Theorem 5.1, when translated to a matrix form will have exactly one non-zero entry in each column, because it comes from a set-theoretic solution to YBE. Since the setting of Theorem 5.1 is no longer set-theoretic, is natural to ask to find K-matrices $k : V \to V$ which have many nonzero entries in their rows and columns. We notice that a small modification of Theorem 3.1 gives large classes of such K-matrices.

Theorem 5.6. *Let A be a two-sided brace, $X \subseteq A$ be a subset such that the restriction $r = r_A|_{X\times X}$ is a solution (X, r) to the YBE. Let $g_1, \dots, g_n \in \mathbb{C}$ and $f_1, \dots, f_n\colon X \to X$ be $\mathcal{G}(X, r)$-equivariant maps. Let $V = \mathbb{C}X$ be the vector space with basis in the elements of X and denote by $\oplus$ the addition of V. Let $k : V \to V$ defined as*

$$k(x) = f_1(x) \cdot g_1 \oplus f_2(x) \cdot g_2 \oplus \dots \oplus f_n(x) \cdot g_n$$

for $x \in X$ and then extended by linearity to V. If $k(x)$ is involutive, then $K(u) = uk$ satisfies the reflection equation

$$K_2(v)R(u + v)K_2(u)R(u - v) = R(u - v)K_2(u)R(u + v)K_2(v),$$

where $R(u) = \mathrm{id} + ur$ and $K_2(u) = id \otimes K(u)$.

Proof. To prove the claim it is sufficient to show that

$$K_2R(u + v)K_2R(u - v) = R(u - v)K_2R(u + v)K_2.$$

By using the fact that $R(u) = id + ur$ we get that for each $x, y \in X$ we have

$$\begin{aligned} K_2R(u+v)K_2R(u-v) &= K_2(id+(u+v)r)K_2(id+(u-v)r) \\ &= K_2K_2 + (u-v)K_2K_2r + (u+v)K_2rK_2 + (u+v)(u-v)K_2rK_2r. \end{aligned}$$

Notice also that

$$\begin{aligned} R(u-v)K_2R(u+v)K_2 &= (id+(u-v)r)K_2(id+(u+v)r)K_2 \\ &= K_2K_2 + (u-v)rK_2K_2 + (u+v)K_2rK_2 + (u+v)(u-v)rK_2rK_2. \end{aligned}$$

Since K is an involution, $K_2K_2r = r = rK_2K_2$. It remains to show that

$$rK_2rK_2 = K_2rK_2r.$$

We calculate the left hand side for $x, y \in X$:

$$\begin{aligned} rK_2rK_2(x,y) &= rK_2r(x, g_1f_1(y) \oplus g_2f_2(x) \oplus \ldots \oplus g_nf_n(y)) \\ &= \oplus_{i=1}^n g_i rK_2r(x, f_i(y)) \\ &= \oplus_{i=1}^n g_i rK_2(\sigma_x(f_i(y)) \otimes \tau_{f_i(y)}(x)) \\ &= \oplus_{j=1}^n \oplus_{i=1}^n g_jg_i(c_{i,j} \otimes d_{i,j}), \end{aligned}$$

where

$$c_{i,j} = \sigma_a(f_j(b)), \quad d_{i,j} = \tau_{f_j(b)}(a), \quad a = \sigma_x(f_i(y)), \quad b = \tau_{f_i(y)}(x).$$

Let A^1 be natural extension of the ring A by an identity element, then

$$d_{i,j} = (1 + c_{i,j})^{-1} * a = (1 + c_{i,j})^{-1} * \sigma_x(f_i(y))$$

where $(1+c_{i,j})^{-1} * (1+c_{i,j}) = 1$. Observe that because each $f_i(x)$ is $\mathcal{G}(X,r)$-equivariant, we get

$$c_{i,j} = \sigma_a(f_j(b)) = f_j(\sigma_a(b)) = f_j(x)$$

since r is involutive. Therefore

$$d_{i,j} = (1 + c_{i,j})^{-1} * a = (1 + c_{i,j})^{-1} * \sigma_x(f_i(y)) = (1 + c_{i,j})^{-1} * f_i(\sigma_x(y)).$$

We now calculate the left hand side of our equation for $x, y \in X$:

$$\begin{aligned} K_2rK_2r(x,y) &= K_2rK_2(\sigma_x(y), \tau_y(x)) \\ &= \oplus_{j=1}^n g_jK_2r(\sigma_x(y), f_j(\tau_y(x))) \\ &= \oplus_{i=1}^n \oplus_{j=1}^n g_ig_j(e_{i,j} \otimes q_{i,j}), \end{aligned}$$

where

$$e_{i,j} = \sigma_p(f_j(q)), \quad q_{i,j} = f_i(\tau_{f_j(q)}(p)), \quad p = \sigma_x(y), \quad q = \tau_y(x).$$

Notice that because f_j is $\mathcal{G}(X,r)$-equivariant we get

$$e_{i,j} = \sigma_p(f_j(q)) = f_j(\sigma_p(q)) = f_j(x).$$

It follows that $e_{i,j} = c_{i,j}$.

We will now calculate $q_{i,j}$: We know that $\tau_{f_j(q)}(p) = (e_{i,j}+1)^{-1} * p$ where $(e_{i,j}+1)^{-1} * (e_{i,j}+1) = 1$. We now calculate

$$q_{i,j} = f_i(\tau_{g_j(q)}(p)) = f_i((e_{i,j}+1)^{-1} * p) = (c_{i,j}+1)^{-1} * f_i(p).$$

Since $p = \sigma_x(y)$, we get $q_{i,j} = d_{i,j}$ and the result follows. $\square$

Remark 5.7. *We can guarantee that the above map k is involutive by assuming the appropriate relations on the underlying ring A and on elements f_i. Notice that the matrix associated to the map $K : V \to V$ will usually have around n nonzero entries in each column.*

Remark 5.8. *Let (X, r) and $k : V \to V$ be constructed as in Theorem 5.6, then k is a solution to the set theoretic reflection equation*

$$K_2 r K_2 r = r K_2 r K_2$$

where $K_2 = id \otimes k$.

Example 5.9. *Let A be a nilpotent ring such that the sum of three copies of any element from this ring is zero. Let $c \in A$ be such that $c * c = 0$ and*

$$c_1 = c, \qquad c_2 = 2c, \qquad c_3 = 3c = 0.$$

For each $i \in \{1, 2, 3\}$ let $f_i(x) = x * c_i + x$ and

$$L(x) = (2/3) f_1 \oplus (2/3) f_2 + (-1/3) f_3.$$

A straightforward calculation shows that $L(L(x)) = x$. Then L is an involutive map which satisfies the assumptions of Theorem 5.6, so it gives a solution to a parameter dependent reflection equation.

Remark 5.10. *In Theorem 5.6 we can take functions $f_i : V \to V$ defined as follows: for a given i let either $f_i(x) = x * c_i$ or $f_i(x) = x + x * c_i$ for $x \in X$ and then extend them by linearity to V.*

Theorem 5.11. *Let (X, r) be an involutive non-degenerate solution to the YBE, let $V = \mathbb{C}X$ and denote by r the linear extension of r to $V \otimes V$. Let $k \colon X \to X$ be a reflection of (V, r) so $rK_2rK_2 = K_2rK_2r$, where $K_2 = \mathrm{id} \otimes k$. Assume that either $k^2 = \mathrm{id}$ or $k^2 = 0$. Denote $K_2(u) = \mathrm{id} \otimes K(u)$ and $R(u) = \mathrm{id} + ur$, where*

$$K(u) = \mathrm{id} + uk.$$

Then $K(u)$ satisfies the reflection equation

$$K_2(v)R(u + v)K_2(u)R(u - v) = R(u - v)K_2(u)R(u + v)K_2(v).$$

Proof. Denote $C(u, v) = K_2(v)R(u + v)K_2(u)$. We have to show that

$$C(u, v)R(u - v) = R(u - v)C(v, u).$$

This is equivalent to show that

$$C(u, v) - C(v, u) + (u - v)(C(v, u)r - rC(v, u)) = 0,$$

since $R(u - v) = \mathrm{id} + (u - v)r$.

Because all maps are $\mathbb{C}$-linear we have $K_2(u) = \mathrm{id} + uK_2$. Notice that

$$\begin{aligned} C(u, v) = {} & v(u + v)K_2 r + u(u + v)rK_2 \\ & + \mathrm{id} + (u + v)(K_2 + r) + uv(u + v)K_2 r K_2 + uvK_2^2. \end{aligned}$$

Therefore

$$C(u,v)-C(v,u)=(v^2-u^2)(K_2r-rK_2).$$

A straightforward calculation shows that

$$C(u,v)r-rC(v,u)=(u+v)(K_2r-rK_2)$$

since by assumption $rK_2rK_2=K_2rK_2r$ and $K_2^2r=rK_2^2$. Therefore

$$C(u,v)-C(v,u)+(u-v)(C(v,u)r-rC(v,u))=0,$$

as required. □

Remark 5.12. *Notice that conditions from Theorem 5.11 include conditions* $K(\lambda)K(-\lambda)=\mathrm{id}$ *for each* $\lambda\in C$*, and* $K(0)=\mathrm{id}$*, from* [12] *provided that* $k^2=0$*. On the other hand if* $k^2=\mathrm{id}$ *then we can define* $K(u)=\frac{\mathrm{id}+ku}{\sqrt{1-u^2}}$ *for* $u\neq 1,-1$ *and the conditions from* [12] *are satisfied.*

Notice that Theorem 5.6 also holds if we assume that $k^2=0$ instead of the assumption $k^2=\mathrm{id}$ and it gives a rich source of maps k which could be used in Theorem 5.11.

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