Structural Factorization of Latent Adjacency Matrix, with an application to Auto Industry Networks

Citation for published version:
Chakraborty, S, Bhattacharjee, A & Maiti, T 2021, 'Structural Factorization of Latent Adjacency Matrix, with an application to Auto Industry Networks', *Sankhya B*, vol. 83-B, no. Suppl. 2, pp. S185-S206.
https://doi.org/10.1007/s13571-019-00202-0

Digital Object Identifier (DOI):
10.1007/s13571-019-00202-0

Link:
Link to publication record in Heriot-Watt Research Portal

Document Version:
Peer reviewed version

Published In:
Sankhya B

Structural Factorization of Latent Adjacency Matrix, with an application to Auto Industry Networks

Sayan Chakraborty,[+,%] Arnab Bhattacharjee,[$] and Taps Maiti [+,#] [*]

Abstract

There is substantial interest in the current literature – spanning finance, economics, engineering and medical imaging – on the relationship structure between several nodes in a complex system. In this paper, we extend the literature by developing model and inference for complex networks in terms of latent factors, by extracting the hidden factors that plays significant role in the configuration of inter node relationships. Together, we extend inferences to applications where the underlying network structure is also latent; that is, the adjacency matrix is unobserved. Here, we consider a Bayesian variant of the matrix factorization technique to develop a structural model of the latent adjacency matrix. There are many potential applications. For illustration, we consider a latent network of firms in the in the US automotive sector, where the central object is to understand the impact of an economic shock on firms (or nodes of the network). An important question centers around the factors that affect the stability and resilience of inter node relationships in a network. In the automotive sector application, we would like to know whether these relationship structures are driven by collaborative or competitive environments? What are the effects of a collaborative or competitive role played by a specific firm on the configuration of the relationship network? Using accounting data on firm sales and costs, we use our proposed object oriented factorization methodology to provide explanation of the estimated network links between 3 major US auto manufacturers and their intermediate suppliers.

1 Introduction

Studying the patterns and relationships between the actors (nodes) of a complex system is of substantial interest in many research areas. Increasing complexity of economic relationships or of modern software systems have enhanced the challenges in isolating latent dynamic

[*]Correspondence: S. Chakraborty, Zillow Group, 1301 2nd Ave, Seattle, WA 98101. E-mail: sayan341@yahoo.in.
Affiliations: [+] Department of Statistics & Probability, Michigan State University, East Lansing, MI, U.S.A.; [$] Spatial Economics & Econometrics Centre (SEEC), Heriot-Watt University, U.K.; and [#] Eli Broad School of Business, Michigan State University, East Lansing, MI, U.S.A.; [%] Zillow Group, Seattle, WA, U.S.A.

patterns that represent the overall structure of the system. For example, it is very important to isolate a specific component that is critical for the functioning of a complex software system. Likewise, it is important to model flow of information across markets to understand a shock to one market or on a specific business sector would affect the stock market movements in the short and long run. Yet elsewhere, it is crucial to understand the impact of impulses in a specific location of the brain on other parts of the brain on elsewhere in the body. Answering these questions require not only a profound understanding of the relationship structure between different actors of the system but also subjective knowledge about the factors that cause these relationships.

Our central application is related to inter-firm networks. Recent studies on inter-firm relationships have revealed various characteristics of the firms, its managers and its ownership, and how those influence the neighboring firms. In this context, neighborhood definition is not unique. It can be modeled by linkages along the supply chain (Hertzel et al., 2008; Wang, 2012; Ahern and Harford, 2014; Itzkowitz, 2015), director networks (Renneboog and Zhao, 2011, 2014), or joint ventures and investment syndication (Wang and Wang, 2012). Questions have been asked about what are the known economic drivers (supply-side or demand-side) that drive interactions between firms (Ellis and McGuire, 1993; Venables 1996), and about the nature of the relationships themselves – whether cohesive networks of socially embedded ties or sparse networks rich in structural holes (Grandori and Soda 1995; Hite and Hesterly 2001). Recent research in finance has explored the functioning of the financial market placing emphasis on strategic interactions between firms and their industry rivals and suppliers. In this setting, economic or financial shocks can be transmitted through customer-supplier network and the entire industry can be affected by those shocks, since these can move through the links of firms in that industry, and beyond; see, for example, Ahern and Harford (2014).

A relationship between inter-firm linkages and business performance has been the focus of many studies. Vickery et al. (1999) found significant relationships between supply chain flexibility and different measures of performance in the US furniture industry. Likewise, Vonderembse and Tracy (1999) observed a link between supplier selection criteria and manufacturing performance. A key question that arise is: what roles do each of these firms play across different economic features that define the current status of relationship configuration of the given industry. Or indeed, what are the relevant economic features in this context? These economic features are usually unobserved within a given economic scenario under consideration and usually are evaluated based on subjective interpretations drawn from a given analysis for the scenario. Stronger inferences can be made through empirical evidences if the specific economic features under consideration can be structurally embedded within a model for the inter-firm network system.

From an engineering perspective, there is a substantial interest in analyzing the cause of system failure. There are several features that might cause a system to fail and understanding and isolating the problem by the importance of the features is of prime importance in this context. Brauckhoff et al. (2009) studied the pattern of an observed anomaly with respect to several system dimensions and isolated the most frequent (significant) dimensions using a randomized binning technique. Fernando and Diot (2010) studied the root cause of a system failure by re-scoring the anomaly detection engine after leaving out one dimension at a time. A very important aspect of this kind of analysis is the requirement of external features or deeper understanding of the data dimensions. The process of adequate causal detection of a problem requires extensive external engineering and hence it is quite important to have an unsupervised technique that analyzes the causal relationship within the data dimensions with respect to several latent factors to bring more automation to the failure detection process.

In this paper, we develop a novel approach that uses a pre-estimated relationship structure as labels and identifies the behaviors of the different nodes under consideration over various features of the system (for example, business collaborations or competition) and its statistical model. Here, we investigate a specific application in financial economics but the framework and proposed methods can be applied to other contexts in business, engineering, medical imaging, and so on. The problem under consideration is quite challanging. We observe structured data on different metrics across several dimensions of the nodes of a given system. Then, the goal is to draw inferences on the effect of the behaviour of those nodes on their peers over different features of the system. Hence, it is an unsupervised modeling problem and a strong indicator is required from the data itself for any classification of the nodes based on hidden factors. Hence, this paper extends the estimation approach of the spatial weight matrix developed in Bhattacharjee and Jensen-Butler (2013) and performs structural modeling on the estimated spatial weight matrix – our label for the second stage in our two stage modeling approach – using Bayesian classification modeling.

An important class of methods employed is the low rank matrix factorization of the estimated spatial weight matrix. A ready technique for matrix factorization in our problem could be Singular Value Decomposition (SVD) or some variant. However, since the goal of this paper is to draw interpretations from the hidden dimensions embedded within the adjacency matrix, we take an object oriented modeling approach where we explicitly extract the various relevant dimensions that might be interesting for a given problem. To this end, we follow an approach similar to the influential paper by Hoff et al. (2002) where a factorization technique through latent variable modeling was proposed for the relational matrix. The idea is that the probability of a relational edge between two nodes (or firms) are higher if these individuals are similar in the unobserved characteristic space.

In applications where the adjacency matrix is unobserved, we propose a two stage approach. The first stage is similar to Bhattacharjee and Jensen-Butler (2013), where we capture the diffusion of excess impact to the response from the causally linked nodes (firms in our case) through a spatial (or network) error model. We isolate any systematic effects from the response and assume the residuals carry all remaining information about the latent relationships between the nodes. At the second stage, we use the estimated binary spatial weight matrix ($\widehat{W}$) to structurally identify the latent positions (or factors) $Z = (z_1, z_2, ..., z_N)$ that not only provide an interpretation of the model in a lower dimentional latent space but also inferences on the positions of the different nodes in latent space and their mutual distances. We apply our model and inferences to estimate the network structure between the US auto industry firms, where our data constitute the costs and sales turnover for leading manufacturers Ford, GM and Chrysler, together with 21 of their leading intermediate suppliers considered in Ramcharran (2001). In effect, we go beyond the regression modeling approach of Ramcharran (2001) to infer on interactions between the firms using their latent positions in a two dimentional Euclidean space. The results provide exciting new evidences on the importance of the factors that drives inter-firm relationships and positions of firms in the estimated factor space.

The rest of the paper is organized as follows. Section 2 outlines our model, followed by a discussion of proposed estimation methods. Section 3 discusses our empirical findings. Section 4 concludes.

2 Model and Estimation

Our modeling methodology consists of two stages. In the first, we estimate the adjacency structure between the nodes of the system. Then, in the second stage, we perform structural factor modeling of the linkages with respect to several relevant factor dimensions. The first stage is optional in applications hwre the relationship structure between the nodes is known *a priori*.

In the first stage, we consider a linear model. In our application, this is the error correction model:

$$\Delta y_{it} = \alpha_i + \lambda_i \Delta x_{it} + (1 - \gamma_i)\left(y_{i,t-1} - \theta_i x_{i,t-1}\right) + \eta_{it}, \tag{1}$$

where y_{it} and x_{it} are respectively the response and the predictor for the node i at time t. $\alpha_i + \lambda_i \triangle x_{it}$ determines the short-run relationship with the response Δy_{it}. A key aspect of the above model is the term $(y_{i,t-1} - \theta_i x_{i,t-1})$. Under the assumption that y_{it} and x_{it} have a long run equilibrium relationship, this is the departure from equilibrium arising from an unanticipated shock (error) at time $t-1$. Then, $(1-\gamma_i)\left(y_{i,t-1} - \theta_i x_{i,t-1}\right)$ in a partial adjustment at time t to the unanticipated shock that occurred at time $t-1$. The assumtion here is $y's$ and $x's$ take some time (here a lag of 1 period) to react to past shocks, and it takes $1/(1-\gamma_i)$ periods to return to equilibrium. Then, the parameter $(1-\lambda_i)$ determines the speed at which the system corrects back to the equilibrium relationship $y_{i,t-1} - \theta_i x_{i,t-1} = c$ where c is a constant in equilibrium. We assume the errors η_{it} are iid across time.

Observe that η_{it} are the errors that are unexplained by the model covariates or cannot be captured through the formulated short-run or long-run relationships between the response and the predictors. The corresponding residuals are key ingredients for modeling the latent relationships through a systematic formulation via a relational structure among them. We consider the residual of a specific node as the spillovers of the residuals from the neighboring nodes. This is a very important assumption for our model that introduces a structure among the residuals analogous to the spatial error model as follows:

$$\eta_t = RW\eta_t + \varepsilon_t$$

Here W is typically a symmetric adjacency matrix with $w_{ij} = 1$ if node i and j has edge (or relationship) between them and 0 otherwise, with $w_{ii} = 0$ for $1 \leq i, j \leq N$ (assuming N total nodes). η_t is a $(N \times 1)$ vector of residuals at time t. Hence, the spillover of the residuals arises only through the nodes in the neighborhood; that is, whether they have an edge connecting them. The strength of the spillover can be captured by the autocorrelation parameter $R = diag(\rho_1, \rho_2, ..., \rho_N)$. A standard assumption is that $(I - W)$ is nonsingular. This ensures identification of the model in reduced form. The adjacency matrix W controls the propagation of the residuals in the unobserved latent space of the nodes. We follow an approach similar to Bhattacharjee and Jensen-Butler (2013), where they showed that the spatial weight (network adjacency) matrix is fully identified from observed (estimated) spatial autocovariances of the residuals.

Next, we use the estimated (as above) adjacency matrix to perform structural matrix factorization. As discussed before, it is not only important to obtained the significant factors that help configuring the adjacency structure, but also to interpret these from the application subject perspectives (for example, economic perspectives). Hence our structural factorization is performed by manually observing different features of the relations between the nodes. Since the main object in our application is to model the effects of nodes on economic collaborations and competitions, we need to estimate $\widehat{W}$ in a similarly interpretable form. Hence, the estimates of W obtained by the method of Bhattacharjee and Jensen-Butler (2013) are used to generate bootstrap samples for the spatial weight matrix and these bootstrap samples are used as inputs to the following multiple testing problem:

$$H_{0ij} :\mid w_{ij} \mid= 0 \quad vs \quad H_{1ij} :\mid w_{ij} \mid= 1 \quad \forall \quad 1 \leq i < j \leq N. \tag{2}$$

The estimated $0-1$ adjacency matrix $\widehat{W}$ through multiple testing works as pre-estimated labels of network ties between the nodes. A positive $(+1)$ link between two nodes signifies a collaborative relationship between the respective firms. Likewise, a negative (-1) link between two nodes signifies a inverse relationship reflecting competittition. A neutral (0) link between two nodes shows there is no relationship at all. This pre-estimation of the adjacency matrix helps providing a source of labelling to the unsupervised problem with some degree of sampling variation. This is a very important transition that supports modeling of such a challenging unsupervised problem (in our specific application, interpreting economic impacts of various firms on different economic factors such as collaborations or competitions) through supervised techniques.

In the second stage of the modeling, we need to factorize $\widehat{W}$ to obtain the factor scores of different nodes under considerations. A common approach for factorizing the estimated matrix $\widehat{W}$ would be to perform a singular value decomposition as follows:

$$\widehat{W} = UDU^{'} \tag{3}$$

where $Z = UD^{1/2}$ are the factors in the transformed space and the V is the loading matrix. Here Z is a low rank transformation of the pre-estimated matrix $\widehat{W}$ and we call that as the latent positions for the nodes. The above factorization is deterministic and does not involve any randomness. Since we consider the adjacency matrix W to be an instance from a family of matrices defined through some probability distribution, we divert to a probabilistic model for the matrix factorization as:

$$\begin{aligned} logodds(|\widehat{w}_{i,j}| = 1) \qquad &= \alpha + u_i^{'} D u_j + e_{ij} \\ &= \alpha + z_i^{'} z_j + e_{ij} \end{aligned}$$

The term $z_i^{'} z_j$ measures the projection of z_i over z_j which means the term measures the allignment between z_i and z_j. If z_i and z_j are similar (has a low cosine between them), then the odds that i^{th} and j^{th} node being connected is high. On the other hand, if z_i and z_j are very different (almost orthogonal with a high cosine), then the odds that i^{th} and j^{th} node being connected is lower. Hence, to draw interpretation of such modeling, we observe that log odds of any relationship (either positive or negative) is directly proportional to the distances between z_i and z_j. This means, the closer the nodes i and j to each other in the factor space, the more chance is of being related either in terms of collaborations or competitions.

A more generic formulation of the above modeling can be:

$$logodds(|\widehat{w}_{i,j}| = 1) = \alpha + \gamma f(z_i, z_j) + e_{ij}, \tag{4}$$

where $f(z_i, z_j)$ is any distance measure between z_i and z_j.

The interpretation of the above model is natural in specific application contexts. The relationship between node i and j depends on their location in the characteristic space defined through z. For example, in a software system, metric i might be highly related to metric j with respect to a specific software release but i might be related to some other metric k and totally unrelated to the metric j if the refernce factor involves common shared database between i and k. On the other hand, for an industry segment, a supplier might be related to a giant manufacturer with respect to a collaborative reference but might be totally unrelated if the reference factor is the geographical region of operations. Hence, the context is very important while defining the relationship between two nodes and the equation (4) provides the control over the factor space for the relationship $w_{i,j}$ between node i and j.

3 Application to Automobile Industry Networks

We use the Compustat data for 16 years, from 1997 to 2013, on 'Cost of Goods Sold' and 'Sales Turnover' for three major US based auto manufacturers: (1) GM, (2) Ford, (3) Chrysler and their 21 major suppliers. As we might expect dynamicity in the relationships between the big manufacturers and the suppliers over a larger time window, we divide the data into two segments and run separate independent analysis for the data between $1997 - 2005$ and $2005 - 2013$. We first reduce the two variables to real terms using the corresponding consumer price index for each year, and take logarithms of our measures.

At the base of our analysis is an error correction model relating the logarithm of costs of a firm to the logarithm of its sales turnover. Let us assume y_{it} and x_{it} denote the costs and sales respectively, in algorithms, for firm i at time t. The key idea of the error correction model is to capture not only the short run relationships among $y's$ and $x's$ but also to model their long run equilibrium relationship. We consider potential nonstationarity of the response and predictor and hence model the first differences of the response and the covariates to remove any effect due to spurious regression. Thus, following Bhattacharjee et al. (2014), we test and then assume that log-costs and log-sales are in long run equilibrium, that is, they are cointegrated. This cointegrating relationship determines a distinct equilibrium profit margin for each firm. Then, log costs and log sales also partially adjust in each year to the disequilibrium in the previous year; this is called the error correction mechanism. The key assumption here is the inter-firm linkages that induce in the errors of the model correlation across the firms.

Hence we fit the error correction model (1) by least squares and extract the residuals. Next, we use the residuals to compute the inter-firm covariance matrix, and use the covariance matrix to calculate a bootstrap sample from W. For this purpose, we apply the methodology in Bhattacharjee and Jensen-Butler (2013) based on symmetric interactions, but not binary dyads which is assumed in our case. Then, we obtain the bootstrap samples to test for each element of the W matrix as described in equation (2). Hence, we finally obtain an estimate of the binary adjacency matrix as $\widehat{W}$.

In the second stage of the modeling, we define equation (4) specifically through a nested logistic model with $f(\cdot)$ being the Euclidean distance measure between two points. Here, we take a conditional independence approach by assuming the presence or absence of a connection between two firms is independent of all other connections, given the unobserved positions in the latent space of the two firms.

$$\mathbb{P}(\widehat{W} \mid Z, \theta) = \prod_{i,j} \mathbb{P}(\widehat{w}_{i,j} \mid z_i, z_j, \theta)$$

Here θ and Z are respectively model parameters and the unknown latent positions.

We also assume that the alignment of the error structure η's has a significant impact on deciding the connection between the i^{th} and the j^{th} firm. Hence we can write our logistic model as:

$$logodds(\mid \widehat{w}_{i,j} \mid = 1 \mid z_{1i}, z_{1j}, \alpha_1, \beta_1, \gamma_1) = \alpha_1 - \beta_1 \mid\mid \eta_i - \eta_j \mid\mid - \gamma_1 \mid z_{1i} - z_{1j} \mid + e_{1ij} \quad (5)$$

Here α_1 is the additive constant. β_1 and γ_1 are the scaling factors for the corresponding distance measures and z_1's are the elements from the first dimension of the latent factor matrix Z. Here the distance measures are directly related to the proximity between the error structure for a pair of firms and their latent positions. Further, as the networks can generate either positive or negative externalities, we set up a nested model where we assume having positive and negative connection between two firms is independent of all other ties given the latent positions of the two firms and the significance of the coennection between two firms. Hence, we can write:

$$logodds(\widehat{w}_{i,j} = 1 \mid z_{2i}, z_{2j}, \alpha_2, \beta_2, \gamma_2, \mid \widehat{w}_{i,j} \mid = 1) = \alpha_2 - \beta_2 \mid\mid \eta_i - \eta_j \mid\mid - \gamma_2 \mid z_{2i} - z_{2j} \mid + e_{2ij} \quad (6)$$

Equation (7) above signifies modeling of another economic aspect for our example application which directly classifies collaborations and competitions which was treated as similar in equation (6). Hence, the closer the firms i and j to each other in the second factor space, the more chance is of being collaborative than being competitive.

In our implementation, we consider a rank 2 approximation of the adjacency matrix W as Z = (z_1, z_2) and we model each separate dimension of Z in a nested fashion. It is important to note that the $\gamma's$ and Z are not identifiable together since they both work as a factor in the term $\gamma_1 \mid z_{1i} - z_{1j} \mid$ or $\gamma_2 \mid z_{2i} - z_{2j} \mid$. Moreover, $logP(\widehat{W} \mid Z, \theta) = logP(\widehat{W} \mid Z^*, \theta)$

for any Z^* that is equivalent to Z under the operations of reflection, rotation or transition (Hoff, 2003). Hence, we can set a reference scale of the distance between any three z_i's by fixing them to a specific point in the Euclidean space. Since our latent space is a two dimensional Euclidean space, this makes all the model components identifiable. Specifically, we keep the latent position of GM and Ford fixed (we are fixing two firm positions instead of three) at some predefined values as we are only identifying a single dimension at every step in the nested modeling defined in equation (6) and (7). This creates a reference scale for the distance measure as well as the reference points in the underlying Euclidean space.

For the estimation of the model, we follow a Bayesian methodology. We first consider the z_i's to be independent of each other, that is,

$$z_{i1}, \ldots, z_{in} \overset{iid}{\sim} N\left(0, \sigma_{z_i}\right), i = 1, 2$$

We formulate mutually independent priors for α_1, α_2, β_1, β_2, γ_1, γ_2, ${\sigma_{z_1}}^2$, ${\sigma_{z_2}}^2$ as:

$$\pi(\alpha_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{{\alpha_i}^2}{\sigma^2_{\alpha_i}}} \qquad -\infty < \alpha_i < \infty \qquad \forall i = 1, 2$$

$$\pi(\beta_i) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{{\beta_i}^2}{\sigma^2_{\beta_i}}} \qquad 0 \leq \beta_i < \infty \qquad \forall i = 1, 2$$

$$\pi(\gamma_i) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{{\gamma_i}^2}{\sigma^2_{\gamma_i}}} \qquad 0 \leq \gamma_i < \infty \qquad \forall i = 1, 2$$

$$\pi({\sigma_{z_i}}^2) = \frac{2^{-\frac{\nu_1}{2}}}{\Gamma(\frac{\nu_1}{2})} e^{-\frac{1}{2{\sigma_{z_i}}^2}} {\sigma_{z_i}}^{-\nu_1 - 2} \qquad {\sigma_{z_i}}^2 > 0 \qquad \forall i = 1, 2$$

Given a network data $\widehat{W} = \widehat{w}_{i,j}$, our goal is to estimate the unknown parameters of the model, denoted as $\theta = (Z, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \sigma^2_{z_1}, \sigma^2_{z_2})$. The conditional distribution of the parameters given the information in the data is $p(\theta \mid \widehat{W}) = p(\widehat{W} \mid \theta) \times p(\theta)/p(\widehat{W})$.

The MCMC based inference constructs a dependent sequence of θ values as follows:

- Sample a parameter θ^* from a proposal distribution $J(\theta|\theta_k)$;
- Compute the acceptance probability

$$r = max\left(1, \frac{p(\widehat{W} \mid \theta^*)p(\theta^*)J(\theta_k \mid \theta^*)}{p(\widehat{W} \mid \theta)p(\theta)J(\theta^* \mid \theta_k)}\right);$$

- Set $\theta_{k+1} = \theta^*$ with probability r and $\theta_{k+1} = \theta_k$ with probability $1 - r$.

This algorithm produces a sequence of θ values having a distribution which is approximately equal to the target distribution $p(\theta \mid \widehat{W})$. A point estimate of θ is often taken to be the posterior mean, which is approximated by the average of the sampled θ values. The details of the Gibbs steps for the algorithm is discussed in the Appendix. We run the Gibbs update 3000 times with a burn-in sample of 2000 iterations. However, in all cases, our computations are well committed and we obtain satisfactory convergence at around 1000 iterations.

Table 1: Probabilities of Linkages for Chrysler, Ford and GM with their Suppliers between 1997-2005

	GM			Ford			Chrysler		
Suppliers	No Connection	Negative Connection	Positive Connection	No Connection	Negative Connection	Positive Connection	No Connection	Negative Connection	Positive Connection
ALCAN	0.2851423	0.4212563281	0.2936013277	0.5368694	0.319358191	0.1437724284	0.9997248	1.896068e-04	8.560192e-05
HONEYWELL	0.5687251	0.2557556601	0.1755192354	0.3122095	0.477586950	0.2102035825	0.9973084	1.870344e-03	8.212645e-04
ARVIN	0.9982684	0.0010407504	0.0006908436	0.9950255	0.003474412	0.0015001273	0.4529265	3.803860e-01	1.666875e-01
COOPER	0.2887126	0.4298005355	0.2814868886	0.5409240	0.314984555	0.1440914086	0.9997255	1.884596e-04	8.600483e-05
DANA	0.2889388	0.4538339871	0.2572271670	0.5410138	0.306010725	0.1529754491	0.9997288	1.798665e-04	9.130000e-05
DEERE	0.3152356	0.3989132929	0.2858510740	0.5738726	0.290381672	0.1357457211	0.9997618	1.636616e-04	7.450338e-05
EATON	0.4995229	0.2833634919	0.2171135951	0.2564970	0.497508642	0.2459943900	0.9979078	1.411069e-03	6.810888e-04
GE	0.2516759	0.4077351123	0.3405890076	0.4944437	0.328268674	0.1772876734	0.9996671	2.198818e-04	1.130339e-04
SPX	0.6865350	0.1796781734	0.1337868428	0.4295075	0.386368064	0.1841244332	0.9955147	3.048868e-03	1.436475e-03
GOODRICH	0.6889803	0.1736551031	0.1373645748	0.4321334	0.377132278	0.1907343169	0.9954223	3.058647e-03	1.519041e-03
GOODYEAR	0.4852940	0.3008771241	0.2138289080	0.2458011	0.517047682	0.2371512436	0.9980196	1.369471e-03	6.109401e-04
JC	0.7129252	0.1672898719	0.1197849658	0.4609147	0.369404590	0.1696807328	0.9948257	3.563083e-03	1.611239e-03
KUHLMAN	0.9991770	0.0004439544	0.0003790823	0.9976352	0.001520325	0.0008445159	0.3258842	4.592333e-01	2.148825e-01
AEROQUIP	0.9848760	0.0088014849	0.0063225053	0.9947610	0.003572874	0.0016660950	0.9999983	1.146825e-06	5.278083e-07
OWENS	0.4905606	0.3215552082	0.1878841818	0.2505248	0.500685453	0.2487897941	0.9980134	1.317327e-03	6.692472e-04
PPG	0.2515288	0.4344366981	0.3140344637	0.4927363	0.346701757	0.1605619886	0.9996717	2.256162e-04	1.026421e-04
ROCKWELL	0.3016617	0.4027294128	0.2956089334	0.5547486	0.303151041	0.1421003805	0.9997420	1.773659e-04	8.067256e-05
SMITH (A O)	0.2393531	0.4677349914	0.2929118793	0.4749066	0.359266717	0.1658267121	0.9996466	2.406005e-04	1.127677e-04
TRW	0.2388713	0.4686293796	0.2924993260	0.4773120	0.360030438	0.1626575571	0.9996484	2.415440e-04	1.100456e-04
UTD TECH	0.7113147	0.1720077995	0.1166775354	0.4599903	0.375562592	0.1644470984	0.9949342	3.523163e-03	1.542603e-03
WALBRO	0.9971139	0.0017863703	0.0010997665	0.9916734	0.005658263	0.0026682961	0.5834277	2.810266e-01	1.355457e-01

Table 2: Probabilities of Linkages for Chrysler, Ford and GM with their Suppliers between 2005-2013

	GM			Ford			Chrysler		
Suppliers	No Connection	Negative Connection	Positive Connection	No Connection	Negative Connection	Positive Connection	No Connection	Negative Connection	Positive Connection
ALCAN	0.2073361	0.4571592756	0.3355046377	0.4151757	0.364114351	0.2207099769	0.9993869	3.755307e-04	2.376122e-04
HONEYWELL	0.7051529	0.1750840698	0.1197630682	0.3806125	0.397608618	0.2217789171	0.9941996	3.627449e-03	2.172995e-03
ARVIN	0.9884343	0.0069467795	0.0046188716	0.9970171	0.001919591	0.0010633122	0.9999987	8.299413e-07	4.770497e-07
COOPER	0.1909933	0.4838831541	0.3251235185	0.4785898	0.329888240	0.1915219838	0.9995168	3.104197e-04	1.727876e-04
DANA	0.2025631	0.4752485930	0.3221883545	0.4966277	0.323273717	0.1800985964	0.9995576	2.802519e-04	1.621507e-04
DEERE	0.2066247	0.4866114973	0.3067637889	0.5057617	0.313295490	0.1809427880	0.9995714	2.773110e-04	1.512399e-04
EATON	0.5841546	0.2500255871	0.1658197638	0.2663108	0.474307907	0.2593813091	0.9964614	2.260859e-03	1.277721e-03
GE	0.1980482	0.4764770483	0.3254747498	0.4213538	0.369369174	0.2092770380	0.9993853	3.945296e-04	2.202112e-04
SPX	0.8055858	0.1163472287	0.0780669747	0.5151947	0.312045831	0.1727594617	0.9899225	6.392943e-03	3.684580e-03
GOODRICH	0.7606404	0.1447726566	0.0945869864	0.4488009	0.352980311	0.1982188065	0.9921570	5.048923e-03	2.794040e-03
GOODYEAR	0.6353432	0.2189089476	0.1457478824	0.3104557	0.445133515	0.2444107948	0.9956033	2.811686e-03	1.585049e-03
JC	0.7793299	0.1321682717	0.0885018654	0.4757907	0.339014642	0.1851946159	0.9911863	5.602484e-03	3.211241e-03
KUHLMAN	0.9981634	0.0009261417	0.0009104513	0.9929806	0.003816409	0.0032029424	0.4922226	3.073474e-01	2.004301e-01
AEROQUIP	0.9985989	0.0008446579	0.0005564747	0.9946280	0.003352341	0.0020196607	0.3709223	3.975621e-01	2.315156e-01
OWENS	0.6304952	0.2242875088	0.1452173390	0.3072405	0.441406017	0.2513535214	0.9957979	2.680050e-03	1.522072e-03
PPG	0.1955058	0.4852461563	0.3192480785	0.4170490	0.376034345	0.2069166274	0.9993920	3.898464e-04	2.181914e-04
ROCKWELL	0.2580737	0.4524882283	0.2894380387	0.5727451	0.271373518	0.1558813885	0.9996718	2.133453e-04	1.148950e-04
SMITH (A O)	0.2019653	0.4748030434	0.3232316739	0.4084756	0.382696242	0.2088281112	0.9993681	4.033383e-04	2.285925e-04
TRW	0.9920751	0.0047410531	0.0031838409	0.9979562	0.001313083	0.0007307346	0.9999991	5.728848e-07	3.270294e-07
UTD TECH	0.7967928	0.1201261441	0.0830810288	0.5033381	0.316229414	0.1804324404	0.9904007	6.041806e-03	3.557540e-03
WALBRO	0.9764152	0.0142820904	0.0093026912	0.9938077	0.003954282	0.0022380554	0.9999973	1.731482e-06	9.714756e-07

Table 1 and Table 2 reports the estimated probabilities of having no connection (shown in blue), a positive connection (shown in green) and a negative connection (shown in red) by the three major auto manufacturers with their key suppliers for the years $1997 - 2005$ and $2005 - 2013$ respectively. Here a link between company i and j signifies a connection between firm i and j, either positive or negative. Positive $(+1)$ connections are collaborative, and negative (-1) connections are strongly competitive. That is to say, if $\widehat{w}_{ij} = \widehat{w}_{ji} = +1$, a positive shock to firm i would induce a collaborative positive change in firm j, and vice versa. On the other hand, if firms i and j are competing with each other, that is $\widehat{w}_{ij} = \widehat{w}_{ji} = -1$, then a positive shock to firm i would imply a negative effect on firm j, and vice versa. It is evident that the networks that the three big manufacturers have with their suppliers are moderately dense with some high probabilities of being connected with their suppliers.

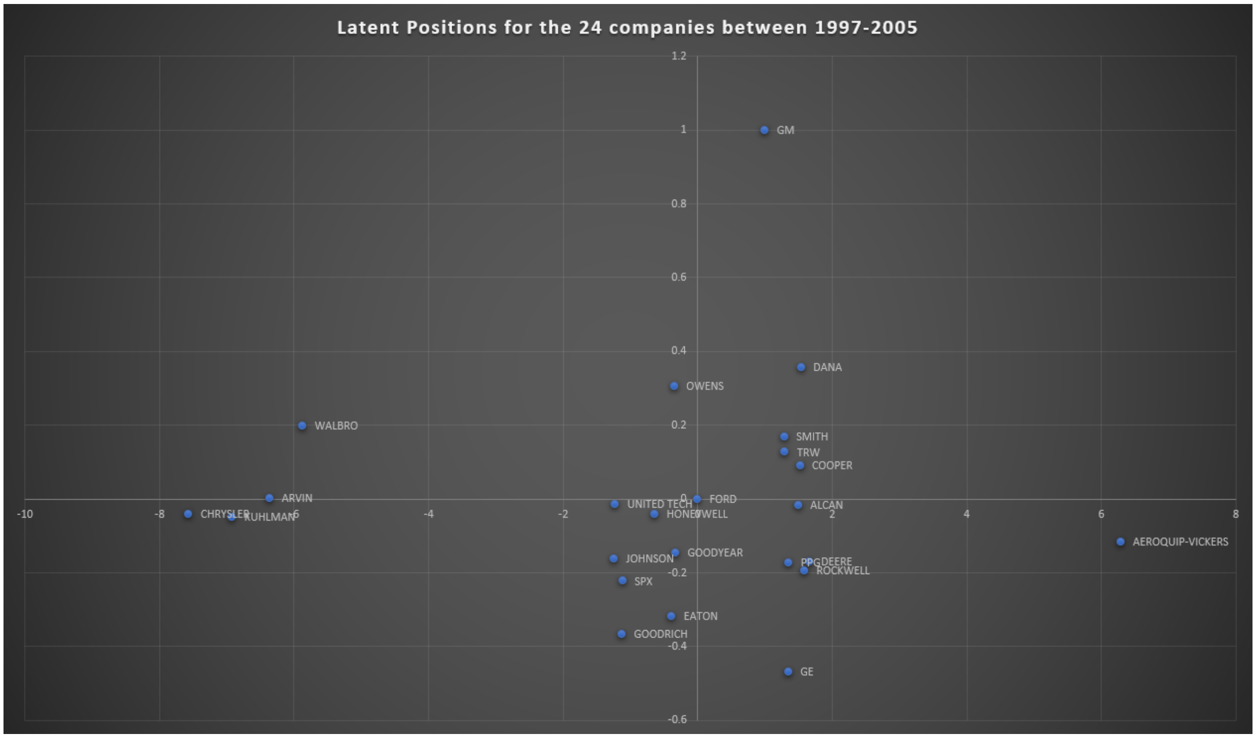

Figure 1: Positions of the 24 firms in the 2-dimensional latent space between 1997-2005

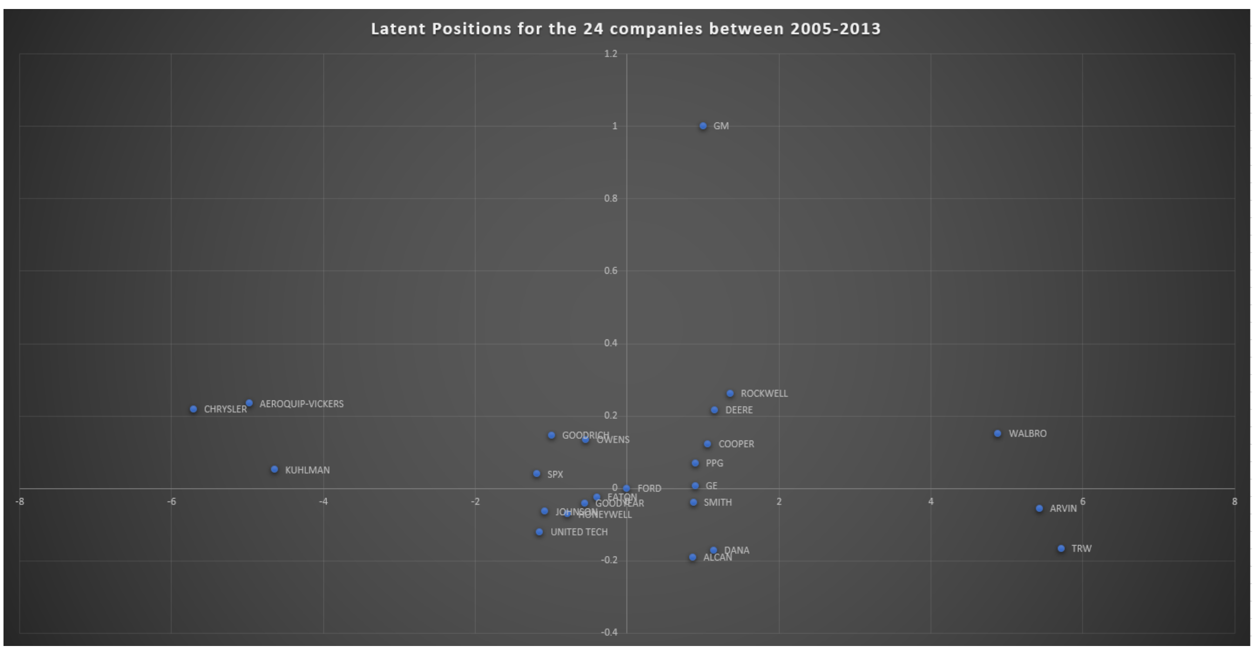

Figure 2: Positions of the 24 firms in the 2-dimensional latent space between 2005-2013

A very interesting fact can be observed from Figure 1 and Figure 3 is that the difference of the variabilities over the two different axis. The plot for the firms (different auto industry firms) in the 2-dimensional Euclidean space shows a very high variability over the x (z_1) axis and a low variability over the y (z_2) axis. Now, as the z_1 dimension induces the significance of the ties, it can be concluded that the firms are very diversely aligned in the relational space. This statement is also evident from the probabilities of ties shown in Table 1 and Table 2. We can see that the probabilities of having a tie ranges from 0.001 to 0.999. This finding heuristically makes sense as we expect the firms from the same industry domain would have some diversities in terms of the business ties with different other firms. An interesting observation comes out from the z_2 dimension where we observe very limited variability (except GM). As, z_2 induces the characteristics of a significant tie, we can conclude that the data depicts much less diversity in terms of collaborations or competitiveness in the market. We also observe the dynamic pattern of the connection over time. Although the overall structure of the latent positions are nearly very identical, we still se some movent in the graph. For example, the social position of $ARVIN$ has made a significant shift over the two time period. Moreover, Chrysler seem to have much weaker relationships with the given set of suppliers compared to Ford and GM.

A question naturally arises: to what extent is the estimated network structure related to the supply chain network of these firms? While a full analysis of this question is beyond the scope of this paper, we conduct some preliminary analysis. We collected data on relationship values of each supplier with Ford (Bloomberg SPLC database, August 2015) and verify the correlation of a positive connection with logarithm of relationship value. This correlation coefficient is statistically significant and evaluates to 0.41. This demonstrates that inter-firm network are quite complex, and while the supply chain provides a partial explanation for the network structure, more elaborate analysis of demand and supply side influences are required to understand these inter-firm networks better. Our modeling approach estimates positions in latent space that can provide very useful insights in such analyses.

Extension to higher rank models

In order to extend the idea to a higher dimensional model, we consider another hierarchy where we want to capture the social distance of two nodes (or firms) via a third node. That is, we consider modeling for,

$$\begin{aligned}&logodds(|\widehat{w}_{i,j}\widehat{w}_{i,k}| = 1 \mid z_{3i}, z_{3j}, \alpha_3, \beta_3, \gamma_3, \mid \widehat{w}_{i,j} \mid = 1, |\widehat{w}_{i,k}| = 1)\\ &= \alpha_3 - \beta_3 \mid\mid \eta_i - \eta_j \mid\mid -\gamma_3(\mid z_{3i} - z_{3j} \mid + \mid z_{3i} - z_{3k} \mid) + e_{3ij}\end{aligned}$$

Moreover, we also consider modeling the competitiveness between two firms via a third firm as,

$$\begin{aligned}&logodds(\widehat{w}_{i,j}\widehat{w}_{i,k} = 1 \mid z_{4i}, z_{4j}, \alpha_4, \beta_4, \gamma_4, \mid \widehat{w}_{i,j} \mid = 1, |\widehat{w}_{i,k}| = 1)\\ &= \alpha_4 - \beta_4 \mid\mid \eta_i - \eta_j \mid\mid -\gamma_4(\mid \mid z_{4i} - z_{4j} \mid - \mid z_{4i} - z_{4k} \mid \mid) + e_{4ij}\end{aligned}$$

The intution behind the above formulation is the relationship type will be opposite between the two nodes if the difference of distances become high.

Although the interpretation of the factors obtained through the traditional SVD based matrix factorization is not possible in this context, we still wish to check the performance in terms of accuracy of the nested latent space model compared to the traditional SVD method. Hence, we performed factorization via SVD on the estimated adjacency matrix $\widehat{W}$. We also compare our results with the logistic SVD method described by de Leeuw (2006) by taking absolute values for the edges ignoring the relationship charasteristics (signs).

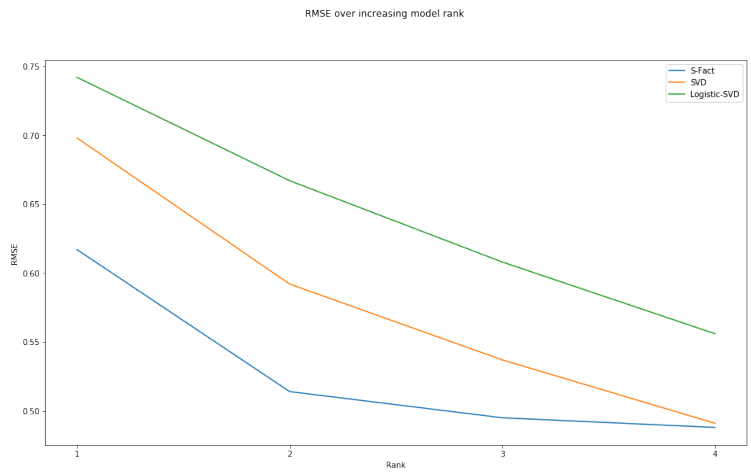

Figure 3: RMSE comparison over increasing model ranks

The above plot shows the performance of the the proposed structured matrix factorization model with the traditional SVD and the logistic SVD model. We can see that the proposed structural factorization model performs quite well with interpretable factors compared to the other matrix factorization methods. Although, the effect of the higher ranks seem to be diminishing quite rapidly compared to the other matrix factorization method but the main advantage of this method stands out in terms of the factor interpretations compared to the traditional SVD and the logistic SVD.

4 Discussion

The main contribution of the paper comes from the hierarchical objective modeling of the hidden factors of a complex system. It is important to note that we consider an economic application and hence assume that a labeled response is available for the pre-construction of the adjacency structure. But the availability of response variable is not guaranteed in many other application such as fault detection in a software system. But the two stages of the modeling process are quite independent and the first stage of pre-estimation of the adjacency matrix can potentially be substituted with any observed system graph or lineage.

The methodology described in the modeling section provides a tool to transform the subjective understanding about the problem domain to an objective modeling scenario. Hence, the method requires domain expertise to gain interpretable insights from the estimated model. For example, the problem of understanding the impact of a specific economic shock or an investment decision in the stock market is akin to understanding the impact over several individual stocks contributing to the index. It is expected that the behavior of two different stocks might be similar with respect to exposure on one risk in the market (say, market risk) but might be totally different along another risk (say, size). Hence the formulation of the model needs to be aligned to the specific application context. For example, interesting analyses could constitute a conditional latent position model on the probability of stock k having a relationship with i and j, given an edge between i and j.

The alignment of the findings from the data analysis with the Bloomberg SPLC database asserts usefulness of the model and the idea that it can be extended to extract relationship information about the nodes with respect to several latent features.

References

[1] Ahern KR, Harford J (2014). The importance of industry links in merger waves. *Journal of Finance* **69**(2), 527-576.

[2] Bailey N, Holly S, Pesaran MH (2016). A two stage approach to spatio-temporal analysis with strong and weak cross-sectional dependence. *Journal of Applied Econometrics* **31**(1), 249-280.

[3] Besag J (2001). *Markov Chain Monte Carlo for Statistical Inference.* Center for Statistics and the Social Sciences.

[4] Bhattacharjee A, Castro E, Maiti T, Marques J (2016). Endogenous Spatial Regression and Delineation of Submarkets: A New Framework with Application to Housing Markets. *Journal of Applied Econometrics* **31**(1), 32-57.

[5] Bhattacharjee A, Higson C, Holly S (2014). Asymmetric Price Adjustment, Sticky Costs and Operating Leverage over the Business Cycle. Working Paper No. **1402**, *Spatial Economics and Econometrics Centre* (Heriot-Watt University, UK).

[6] Bhattacharjee A, Holly S (2013). Understanding interactions in social networks and committees. *Spatial Economic Analysis* **8**(1), 23-53.

[7] Bhattacharjee A, Jensen-Butler C (2013). Estimation of the spatial weights matrix under structural constraints. *Regional Science and Urban Economics* **43**, 617-634.

[8] Borgatti SP, Mehra A, Brass DJ, Labianca G (2009). Network analysis in the social sciences. *Science* **323**(5916), 892-895.

[9] Brauckhoff D, Dimitropoulos X, Wagner A, Salamatian K (2012). Anomaly extraction in backbone networks using association rules. *IEEE/ACM Transactions on Networking (TON)* **20**(6), 1788-1799.

[10] Chakraborty, S (2016). Estimation of statistical network and region-wise variable selection. *Michigan State University.*

[11] De Leeuw (2006). Principal component analysis of binary data by iterated singular value decomposition. *Computational statistics data analysis* **50**(1), 21-39.

[12] Ellis RP, McGuire TG (1993). Supply-side and demand-side cost sharing in health care. *Journal of Economic Perspectives* **7**(4), 135-151.

[13] Frank O, Strauss D (1986). Markov graphs. *Journal of the American Statistical Association* **81**(395), 832-842.

[14] Geletkanycz MA, Boyd BK (2011). CEO outside directorships and firm performance: A reconciliation of agency and embeddedness views. *Academy of Management Journal* **54**, 335-352.

[15] Grandori A, Soda G (1995). Inter-firm networks: antecedents, mechanisms and forms. *Organization Studies* **16**(2), 183-214.

[16] Handcock MS (2000). *Progress in Statistical Modeling of Drug User and Sexual Networks*. Unpublished manuscript, University of Washington, Center for Statistics and the Social Sciences.

[17] Handcock MS, Raftery AE, Tantrum JM (2007). Model-Based Clustering for Social Networks (with discussion). *Journal of the Royal Statistical Society Series A* **170**(2), 301-354.

[18] Hertzel MG, Li Z, Officer MS, Rodgers KJ (2008). Inter-firm linkages and the wealth effects of financial distress along the supply chain. *Journal of Financial Economics* **87**(20), 374-387.

[19] Hite JM, Hesterly WS (2001). Research notes and commentaries. The evolution of firm networks: From emergence to early growth of the firm. *Strategic Management Journal* **22**(3), 275-286.

[20] Hoff PD (2003). Random effects models for network data. In: Breiger, K.C.R. and Pattison, P. (eds.), *Dynamic Social Network Modeling and Analysis: Workshop Summary and Papers*, 303-312, Washington DC: National Academy Press.

[21] Hoff PD, Raftery AE and Handcock MS (2002). Latent Space Approaches to Social Network Analysis. *Journal of the American Statistical Association* **97**(460), 1090-1098.

[22] Holland PW, Leinhardt S (1981). An Exponential Family of Probability Distributions for Directed Graphs. *Journal of the American Statistical Association* **76**(373), 33-50.

[23] Horton J, Millo Y, Serafeim G (2012). Resources or power? Implications of social networks on compensation and firm performance. *Journal of Business Finance and Accounting* **39**(3-4), 399-426.

[24] Itzkowitz J (2015). Buyers as stakeholders: How relationships affect suppliers' financial constraints. *Journal of Corporate Finance* **31**, 54-66.

[25] Larcker DF, Sob EC, Wang CCY (2013). Boardroom centrality and firm performance. *Journal of Accounting and Economics* **55**(2-3), 225-250.

[26] Larcker DF, Tayan B (2010). Director Networks: Good for the Director, Good for Shareholders. *Rock Center for Corporate Governance at Stanford University Closer Look Series: Topics, Issues and Controversies in Corporate Governance* No. **CGRP-08**.

[27] Narasimhan R, Jayaram J (1998). Causal linkages in supply chain management: an exploratory study of North American manufacturing firms. *Decision Sciences* **29**(3), 579-605.

[28] Nowicki K, Snijders TAB (2001). Estimation and Prediction for Stochastic Blockstructures. *Journal of the American Statistical Association* **96**(455), 1077-1087.

[29] Omer TC, Shelley M, Tice F (2012). Do Well-Connected Directors Improve Firm Performance?. Working paper, *University of Nebraska-Lincoln*.

[30] Ramcharran H (2001). Inter-Firm Linkages and Profitability in the Automobile Industry: The Implications for Supply Chain Management. *Journal of Supply Chain Management* **37**(4), 11-17.

[31] Renneboog L, Zhao Y (2011). Us knows us in the UK: On director networks and CEO compensation. *Journal of Corporate Finance* **17**(4), 1132-1157.

[32] Renneboog L, Zhao Y (2014). Director networks and takeovers. *Journal of Corporate Finance* **28**, 218-234.

[33] Silveira F, Diot C (2010). URCA: Pulling out anomalies by their root causes. In: *INFOCOM 2010 Proceedings IEEE*, IEEE, 1-9.

[34] Snijders TAB (2002). Markov Chain Monte Carlo estimation of exponential random graph models. *Journal of Social Structure* **3**(2), 1-40.

[35] Strauss D, Ikeda M (1990). Pseudolikelihood estimation for social networks. *Journal of the American Statistical Association* **85**(409), 204-212.

[36] Swaminathan JM, Smith SF, Sadeh NM (1998). Modeling supply chain dynamics: A multiagent approach. *Decision Sciences* **29**(3), 607-632.

[37] Venables AJ (1996). Equilibrium locations of vertically linked industries. *International Economic Review* **37**(2), 341-359.

[38] Vickery S, Calantone R, Dröge C (1999). Supply chain flexibility: an empirical study. *Journal of Supply Chain Management* **35**(2), 16-24.

[39] Vonderembse MA, Tracey M (1999). The impact of supplier selection criteria and supplier involvement on manufacturing performance. *Journal of Supply Chain Management* **35**(3), 33-39.

[40] Wang J (2012). Do firms' relationships with principal customers/suppliers affect shareholders' income?. *Journal of Corporate Finance* **18**(4), 860-878.

[41] Wang L, Wang S (2012). Endogenous networks in investment syndication. *Journal of Corporate Finance* **18**(3), 640-663.

[42] Wang YJ, Wong GY (1987). Stochastic blockmodels for directed graphs. *Journal of the American Statistical Association* **82**(397), 8-19.

[43] Wasserman S, Pattison P (1996). Logit models and logistic regressions for social networks: I. An introduction to Markov graphs and p^*. *Psychometrika* **61**(3), 401-425.

Appendix: Gibbs Steps for Latent Space Model

For notational convenience in the Appendix, we use W in place of $\widehat{W}$ from the context of modeling section. We set initial values for the parameters as $\alpha_{01}, \alpha_{02}, \beta_{01}, \beta_{02}, \gamma_{01}, \gamma_{02}, {\sigma_{0z_1}}^2, {\sigma_{0z_2}}^2$ and we update the parameters according to the following Gibbs steps.

.1 $(s+1)$th Gibbs Step for updating $(\alpha_1, \beta_1, \gamma_1)$:

The full conditional distribution of $\alpha_1, \beta_1, \gamma_1$ is given by:

$$
\begin{aligned}
& P(\alpha_1, \beta_1, \gamma_1 | \sigma_{z_1}^2, Z_1, W, \eta) \\
\propto\ & \left(\prod_{i>j} \frac{e^{\left(\alpha_1 - \beta_1 \left\| \widetilde{\eta}_i - \widetilde{\eta}_j \right\| - \gamma_1 |z_{1i} - z_{1j}|\right)|w_{ij}|}}{1 + e^{\alpha_1 - \beta_1 \left\| \widetilde{\eta}_i - \widetilde{\eta}_j \right\| - \gamma_1 |z_{1i} - z_{1j}|}} \right) \times e^{-\left(\frac{\alpha_1^2}{2\sigma_{\alpha_1}^2} + \frac{\beta_1^2}{2\sigma_{\beta_1}^2} + \frac{\gamma_1^2}{2\sigma_{\gamma_1}^2} \right)} \\
=\ & K_{\alpha_1, \beta_1, \gamma_1 | \sigma_{z_1}^2, Z_1, W, \eta}
\end{aligned}
$$

which is not a closed form expression of any distribution. Hence we need to perform Metropolis-Hastings with a symmetric proposal distribution for $\alpha_1, \beta_1, \gamma_1$. The r-th step for the Metropolis-Hastings algorithm is given by,

1. Generate ${\alpha_{1s}}^{r*}$ from $N({\alpha_{1s}}^{r-1}, \sigma_{met})$, ${\beta_{1s}}^{r*}$ from $N({\beta_{1s}}^{r-1}, \sigma_{met})$ and ${\gamma_{1s}}^{r*}$ from $N({\gamma_{1s}}^{r-1}, \sigma_{met})$.
2. Calculate:
$$
u = \frac{K_{\alpha_{1s}^{r*}, \beta_{1s}^{r*}, \gamma_{1s}^{r*} | {\sigma^2}_{z_1 s}^{r-1}, Z_{1s}, W, \eta}}{K_{\alpha_{1s}^{r-1}, \beta_{1s}^{r-1}, \gamma_{1s}^{r-1} | {\sigma^2}_{z_1 s}^{r-1}, Z_{1s}, W, \eta}}
$$
3. Set $\alpha_{1s}^r = \alpha_{1s}^{r*}$, $\beta_{1s}^r = \beta_{1s}^{r*}$, $\gamma_{1s}^r = \gamma_{1s}^{r*}$ with probability u otherwise continue with the value of step $r-1$ with probability $1-u$.
4. Repeat the above steps n_{met} times.

Hence, we get n_{met} simulated samples from the distribution of $\alpha_1, \beta_1, \gamma_1 \mid \sigma_{z_1 s}^2, Z_{1s}, W, \eta$. We can use this simulated distribution update $\alpha_{1s}, \beta_{1s}, \gamma_{1s}$ to $\alpha_{1(s+1)}, \beta_{1(s+1)}, \gamma_{1(s+1)}$.

.2 $(s+1)$th Gibbs Step for updating $(\alpha_2, \beta_2, \gamma_2)$:

The full conditional distribution of $\alpha_2, \beta_2, \gamma_2$ is given by:

$$
\begin{aligned}
& P(\alpha_2, \beta_2, \gamma_2 \mid \sigma_{z_2}^2, Z_2, \{w_{ij} :\mid w_{ij} \mid= 1\}, \eta) \\
& \propto \left(\prod_{i>j} \frac{e^{(\alpha_2 - \beta_2 ||\widetilde{\eta}_i - \widetilde{\eta}_j|| - \gamma_2 |z_{2i} - z_{2j}|)(1 + w_{ij})/2}}{1 + e^{\alpha_2 - \beta_2 ||\widetilde{\eta}_i - \widetilde{\eta}_j|| - \gamma_2 |z_{2i} - z_{2j}|}} \right) \times e^{-(\frac{\alpha_2^2}{2\sigma_{\alpha_2}^2} + \frac{\beta_2^2}{2\sigma_{\beta_2}^2} + \frac{\gamma_2^2}{2\sigma_{\gamma_2}^2})} \\
& = K_{\alpha_2, \beta_2, \gamma_2 | \sigma_{z_2}^2, Z_2, W, \eta}
\end{aligned}
$$

which is not a closed form expression of any distribution. Hence we need to perform Metropolis-Hastings with a symmetric proposal distribution for $\alpha_2, \beta_2, \gamma_2$. The r-th step for the Metropolis-Hastings algorithm is given by,

1. Generate α_{2s}^{r*} from $N(\alpha_{2s}^{r-1}, \sigma_{met})$, β_{s2}^{r*} from $N(\beta_{2s}^{r-1}, \sigma_{met})$ and γ_{2s}^{r*} from $N(\gamma_{2s}^{r-1}, \sigma_{met})$.
2. Calculate:

$$u = \frac{K_{\alpha_{2s}^{k*}, \beta_{2s}^{r*}, \gamma_{2s}^{r*} | {\sigma^2}_{z_2 s}^{r-1}, Z_2, W, \eta}}{K_{\alpha_{2s}^{r-1}, \beta_{2s}^{r-1}, \gamma_{2s}^{r-1} | {\sigma^2}_{z_2 s}^{r-1}, Z_{2s}, W, \eta}}$$

3. Set $\alpha_{2s}^{r*} = \alpha_{2s}^{r*}$, $\beta_{2s}^{r} = \beta_{2s}^{r*}$, $\gamma_{2s}^{r*} = \gamma_{2s}^{r*}$ with probability u otherwise continue with the value of step $r-1$ with probability $1-u$.
4. Repeat the above steps n_{met} times.

Hence, we get n_{met} simulated samples from the distribution of $\alpha_2, \beta_2, \gamma_2 \mid \sigma^2_{z_2 s}, Z_{2s}, W, \eta$. We can use this simulated distribution to update $\alpha_{2s}, \beta_{2s}, \gamma_{2s}$ to $\alpha_{2(s+1)}, \beta_{2(s+1)}, \gamma_{2(s+1)}$.

.3 $(s+1)$th Gibbs Step for updating $\sigma^2_{z_1}$:

The posterior distribution of $\sigma^2_{z_1}$ is given by,

$$\sigma^2_{z_1} \mid Z_1 \sim (\sum_{i=1}^{N} {z_{1i}}^2 + 1) inv{\chi^2}_{\nu_1 + 4}$$

$\sigma^2_{z_1(s+1)}$ can be generated from the conditional $\sigma^2_{z_1} \mid Z_{1s}$

.4 $(s+1)$th Gibbs Step for updating $\sigma^2_{z_2}$:

The posterior distribution of $\sigma^2_{z_2}$ is given by,

$$\sigma^2_{z_2} \mid Z_2 \sim (\sum_{i=1}^{N} {z_{2i}}^2 + 1) inv{\chi^2}_{\nu_1 + 4}$$

$\sigma^2_{z_2(s+1)}$ can be generated from the conditional $\sigma^2_{z_2} \mid Z_{2s}$

.5 $(s+1)$th Gibbs Step for updating Z_1, Z_2:

The full conditional distribution of z_{1i} is given by:

$$
\begin{aligned}
& P(z_{1i}|\alpha_1, \beta_1, \gamma_1, \sigma^2_{z_1}, W, \eta) \\
\propto \quad & \left(\prod_{i \neq j} \frac{e^{\left(\alpha_1 - \beta_1 \left\| \widetilde{\eta}_i - \widetilde{\eta}_j \right\| - \gamma_1 |z_{1i} - z_{1j}|\right)|w_{ij}|}}{1 + e^{\alpha_1 - \beta_1 \left\| \widetilde{\eta}_i - \widetilde{\eta}_j \right\| - \gamma_1 |z_{1i} - z_{1j}|}} \right) \times e^{\frac{z_i^2}{\sigma^2_{z_1}}} \\
= \quad & K_{z_{1i}|\alpha_1, \beta_1, \gamma_1, \sigma^2_{z_1}, W, \eta}
\end{aligned}
$$

which is not a closed form expression of any distribution. Hence we need to perform Metropolis-Hastings with a symmetric proposal distribution for z_{1i}. The r-th step for the Metropolis-Hastings algorithm is given by,

1. Generate z^{r*}_{1is} from $N(z^{r-1}_{1is}, \sigma^2_{met})$.

2. Calculate:

$$
u = \frac{K_{z^{r*}_{1is}|\alpha_{1(s+1)}, \beta_{1(s+1)}, \gamma_{1(s+1)}, \sigma^2_{z_1(s+1)}, W, \eta)}}{K_{z^{r-1}_{1is}|\alpha_{1(s+1)}, \beta_{1(s+1)}, \gamma_{1(s+1)}, \sigma^2_{z_1(s+1)}, W, \eta)}}
$$

3. Set $z^r_{1is} = z^{r*}_{1is}$ with probability u otherwise continue with the value of step $r - 1$ with probability $1 - u$.

4. Repeat the above steps n_{met} times.

Hence, we get n_{met} simulated samples from the distribution of $z_{1i} \mid \alpha_{1(s+1)}, \beta_{1(s+1)}, \gamma_{1(s+1)}, \sigma^2_{z_1(s+1)}, W, \eta$. We can use this simulated distribution update z_{1is} to $z_{1i(s+1)}$.

We can update z_{2is} similarly.