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RECENT DEVELOPMENTS IN INVERSE SEMIGROUP THEORY

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Abstract. After reviewing aspects of the development of inverse semigroup theory, we describe an approach to studying them which views them as 'non-commutative meet semilattices'. This leads to non-commutative versions of Stone duality and connections with the contemporary theory of étale groupoids and infinite groups analogous to the Thompson groups.

1. Introduction

There is a constant tension in mathematics between the general and the particular; we would of course like to prove theorems in as great a generality as we can, but the more general we try to work the less we can say. We can of course work with individual structures but then we can lose ourselves in a morrass of details. The trick, or perhaps better said, the art, is to surf the boundary between the general and the specific. We need good, particular examples to motivate our theory, but our theory can then provide us with the right concepts to help us understand the specific. Finite semigroup theory is a good example of where this balancing act has been achieved. It is notoriously futile to attempt to study all finite semigroups but it is fruitful to study classes of semigroups such as varieties of finite semigroups. In addition, examples of finite semigroups can easily be constructed as the transition semigroups of finite-state automata and this has led to important connections between finite semigroups and the theory of regular languages. Thus in finite semigroup theory we have both a good source of examples and an approach that is general enough without being too general [66].

As is well-known, inverse semigroups arose from the theory of pseudogroups of transformations [30], but the subsequent development of the theory owed little to these origins. We shall return to pseudogroups of transformations later, but we begin with a brief retrospective on inverse semigroup theory.

As a separate field of semigroup theory, inverse semigroup theory was initiated in the 1950s in the papers of Ehresmann [11], Preston [58, 59, 60] and Wagner [73][1] though, as is almost always the case in mathematics, inverse semigroups had been sporadically studied in the decades leading up to their formal introduction.[2] The decade that began in 1970 with the first appearance of our journal, *Semigroup Forum*, would prove a fertile one for inverse semigroup theory; the theory of fundamental inverse semigroups (1970) [52], the theory of free inverse semigroups (1974) [67], and that of E-unitary inverse semigroups (1974) [45, 46] were motivated by the question of the extent to which inverse semigroups could be constructed from groups and semilattices whereas the description that Munn gave of the free inverse monoid (1974) [53] was ground-breaking in that the normal forms obtained were trees not strings. Petrich's book [57] is the best source for work on inverse semigroup theory up to the early 1980s. One particular theme that emerged was the

[1]This was the first definition of an inverse semigroup. See [20] for a discussion of the role of the Cold War in the development of mathematics at this time.

[2]"Each writer creates his precursors. His work modifies our conception of the past, as it will modify the future." Jorge Louis Borges.

role of groupoids[3] in studying inverse semigroups. It was the basis of Ehresmanns's work [11], who in fact used a class of ordered groupoids rather than a class of semigroups, but it only became more widely known through a paper of Boris Schein [68]. The apotheosis of this approach is undoubtedly Nambooripad's astonishing generalization of Ehresmann's work to arbitrary (von Neumann) regular semigroups [54]. The way in which ordered groupoid techniques may be used to prove theorems on inverse semigroups was the starting point for my own research and is summarized in my book [30].

It was clear that inverse semigroups were an important class of semigroups but was anyone listening in the wider mathematical world? In fact they were, but the fact they were had, at that time, no impact on the development of the field. In 1980, Jean Renault published as a monograph [62] his PhD thesis from 1978. This dealt with the construction of C^*-algebras from a class of topological groupoids but inverse semigroups jostle for attention throughout the book. Specifically, what we would call the polycyclic inverse monoids, introduced in 1970 [55], are used to construct a class of simple C^*-algebras, known as the *Cuntz C^*-algebras* [8]. Renault's fellow Berkeley student, Alexander Kumjian, wrote a thesis in 1980, which appeared as the paper [28], in which inverse semigroups take centre-stage in the construction of C^*-algebras. In this paper, he also clearly states the philosophy that underpins his use of inverse semigroups:

> *A localization may prifitably be viewed as a non-commutative analog of a countable basis; its affiliated inverse semigroup is to be viewed as the analog of a topology.*

Using inverse semigroups to construct C^*-algebras has become a major theme in the theory. It is best seen in the work of Ruy Exel — the paper [12] is merely representative of an extensive oeuvre — but the recent paper [1] makes an explicit connection with research on 0-bisimple inverse monoids. Paterson's book, which appeared in 1998 [56], updates Renault's but gives equal billing to inverse semigroups.

As a result of these developments, inverse semigroup theory in fact split: there was the 'mainstream' work on inverse semigroups carried out by algebraists and the 'clandestine' work that was carried out by C^*-algebra theorists. These two streams did not interact very much until the work of Johannes Kellendonk. Kellendonk is a theoretical physicist interested in the physical properties of aperiodic tilings. His main goal was to calculate the K_0-groups of some C^*-algebras associated with aperiodic tilings but, in trying to do this, he found the need to use certain algebraic structures that turned out to be inverse semigroups [22, 23, 24].[4] Remarkably, the idea Kellendonk used to construct an inverse semigroup from a tiling yielded the free inverse monoid when applied to the Cayley graph of a free group viewed as a tiling on a tree [47]. Aspects of Kellendonk's work are described in my book [30], but it was the paper by Daniel Lenz [41], which circulated from about 2002, that changed the mathematical landscape. Lenz began the process of transforming Kellendonk's work into purely algebraic terms with the finishing touches being supplied by the paper [37] in which the role of filters in Lenz's work is clarified. At around the same time Resende was studying a class of inverse semigroups using ideas from topos theory and quantales [65] in a way completely divorced from mainstream inverse semigroup theory.

In this article, I will argue that returning to the roots of inverse semigroup theory in the theory of pseudogroups of transformations is both natural and essential.

[3]Small categories in which every arrow is invertible.

[4]This is a recurring theme in the theory of C^*-algebras: namely, construct C^*-algebras using combinatorial data.

Needless to say, this is very much a personal viewpoint which does not claim to account for even a fraction of the developments in inverse semigroup theory over the last 50 years. In particular, one area I would have liked to explore in this article is the connection between inverse semigroup theory and topos theory [38, 14, 15, 16, 18, 25, 61] since this provides a modern setting for Ehresmann's work [11] on local structures.

Like the elephant in the story, there is no one correct way of thinking about inverse semigroups. Instead, there are a multiplicity of viewpoints each of which emphasizes an aspect of the theory. With this caveat in mind, it is the goal of this article to describe one way of thinking about inverse semigroups that connects them naturally with developments in the theory of C^*-algebras and, indeed, with the theory of certain kinds of infinite groups.

Acknowledgements I would like to thank all my colleagues with whom I have collaborated over the years and from whom I have learnt so much. My late colleague Douglas Munn was one of the prime movers in inverse semigroup theory and his work on inverse semigroup algebras shows that he was well ahead of the game [13].

2. From pseudogroups to inverse semigroups, and back

Inverse semigroups arose as algebraic versions of pseudogroups of transformations and they retain impressions of their origins encoded into their algebraic structure. In fact, we can regard the elements of an inverse semigroup as being arrows

$$a^{-1}a \xrightarrow{a} aa^{-1}$$

an approach formalized using the idea of an inductive groupoid and developed in my book [30]. We write $\mathbf{d}(a) = a^{-1}a$ and $\mathbf{r}(a) = aa^{-1}$ to remind us of abstract domains and ranges. We denote the set of idempotents of S by $\mathsf{E}(S)$ and recall that this is always a commutative subsemigroup. Inverse semigroups come equipped with an algebraically defined order $\leq$, called the *natural partial order*, which is the algebraic manifestation of the restriction ordering of partial bijections. This is defined by $a \leq b$ if and only if $a = be$ for some idempotent e. It is not true, however, that the union of any two partial bijections is again a partial bijection. For this to be true, they must satisfy an algebraic precondition: namely, that the two elements be compatible. Abstractly, this leads to the *compatibility relation* $\sim$ defined on any inverse semigroup by $a \sim b$ if and only if ab^{-1} and $a^{-1}b$ are both idempotents. It follows that the existence of joins in inverse semigroups will always require that the elements involved be pairwise compatible.

We can now define an (abstract) *pseudogroup* to be an inverse semigroup that has arbitrary compatible joins and where multiplication distributes over such joins. Pseudogroups of transformations are then pseudogroups of partial bijections on some set, usually pseudogroups of partial homeomorphisms on a topological space. Pseudogroups in the abstract sense were studied by Boris Schein [69] who proved that every inverse semigroup could be completed to such a pseudogroup. However, Schein was primarily interested in developing general algebraic constructions, as the title of his paper suggests, rather than re-introducing pseudogroups of transformations as objects of study. But in his paper, Pedro Resende [65] does exactly this: what we have called 'pseudogroups' he refers to as 'abstract complete pseudogroups' and he reminds us of something important: the semilattice of idempotents of such a pseudogroup is a *frame*; that is, a complete, infinitely distributive lattice.

Frames provide an alternative setting for the notion of 'space'. Specifically, one can either take the points of a topological space as the primary objects of study in which case the open subsets are a secondary class of objects, this is the approach adopted in the classical theory of topological spaces, or one can take the lattice of open subsets as the primary object and then points are secondary objects. However, it is important to understand that topological spaces and frames are not equivalent in general. The reason why resides in the notion of a 'point' and how to recapture that notion from the frame of open sets. To explain this, we need some definitions. Let F be any frame. A subset $A \subseteq F$ is called a *filter* if $a, b \in A$ implies that $a \wedge b \in A$, and $a \in A$ and $a \leq b$ implies that $b \in A$. If $0 \notin A$ we say that A is a *proper* filter. If A is a filter we say that it is *completely prime* if $\bigvee_{i \in I} e_i \in A$ implies that $e_i \in A$ for some i. Now let X be a topological space and denote its frame of open sets by $\Omega(X)$. If $x \in X$, then the set of all open sets that contain x, denoted by $\mathscr{U}_x$, is a proper completely prime filter. However, if a topological space has poor separation properties, it is quite possible for $\mathscr{U}_x = \mathscr{U}_y$ even though $x \neq y$ and, it is also quite possible for a proper completely prime filter in $\Omega(X)$ not to be of the form $\mathscr{U}_x$ for any point $x \in X$. A topological space in which neither of these possibilities occur is said to be *sober*. It is possible to endow the set of all proper completely prime filters $\mathrm{pt}(F)$ of a frame F with a topology. The space X is homeomorphic to $\mathrm{pt}(\Omega(x))$ precisely when X is sober. By the same token, if A is frame such that A is isomorphic to $\Omega(\mathrm{pt}(A))$ then we say it is *spatial*. There are topological spaces that are not sober and frames which are not spatial which accounts for the difference between spaces and frames. If $f \colon X \to Y$ is a continuous function then $f^{-1} \colon \Omega(Y) \to \Omega(X)$ is a morphism of frames. Since maps are reversed in passing from spaces to frames, it is usual to work with the opposite category to the category of frames, which is called the category of *locales*. Thus, this alternative approach to the study of spaces goes under the name of the *theory of locales* and the book by Peter Johnstone [21] is based on this point of view.

We now see that the semilattice of idempotents of a pseudogroup represents the algebraic trace of the original topological space. This lattice approach to the theory of topological spaces goes back to the work of Ehresmann [10] and it is no accident that it arises in his work on abstracting the notion of a pseudogroup of transformations. At this point, it is worth mentioning a curious fact: nowhere does Johnstone refer to pseudogroups despite the fact that he cites Ehresmann's paper where frames are defined within the context of pseudogroups. This is connected to the more general fact that pseudogroups of transformations have suffered a strange fate: they are constantly referred to in papers, but have yet to be fully admitted to the canon of acceptable algebraic structures. They exist in a sort of mathematical limbo.

Studying pseudogroups is all well and good, but the requirement to have all compatible joins seems too strong. It turns out that we can weaken this condition by generalizing an idea from frame theory. We first define a *distributive inverse semigroup* to be an inverse semigroup that has all binary compatible joins and where multiplication distributes over such joins. A *morphism* of distributive inverse semigroups is a homomorphism that preserves binary, compatible joins. In effect, we are replacing the arbitrary joins in pseudogroups by the finite joins in distributive inverse semigroups. Despite appearances, there is a very close connection between distributive inverse semigroups and pseudogroups via the notion of 'coherence'. We say that the compatible subset X of a pseudogroup S is a *covering* of the element a if $a \leq \bigvee X$. An element $a \in S$ in a pseudogroup S is said to be *finite* if for any compatible subset $X \subseteq S$ such that $a \leq \bigvee X$ there exists a finite subset Y of X such that $a \leq \bigvee Y$. In other words, every covering has a finite subcovering. In the case of the frames of open sets of a topological space the finite elements are just the compact ones. We denote the set of finite elements of a pseudogroup S by $\mathsf{K}(S)$. The following is proved as [36, Lemma 3.4].

Lemma 2.1. *Let S be a pseudogroup.*

(1) The finite elements of S form an inverse subsemigroup if and only if the finite idempotents form a subsemigroup.
(2) If the finite elements form an inverse subsemigroup they form a distributive inverse semigroup.
(3) Every element of S is a join of finite elements if and only if every idempotent is a join of finite idempotents.

A pseudogroup S is said to be *coherent* if the set of its finite elements forms a distributive inverse subsemigroup and if every element of S is a join of finite elements. A pseudogroup morphism between coherent pseudogroups is said to be *coherent* if it preserves finite elements. We can construct examples of coherent pseudogroups from distributive inverse semigroups by modifying Schein's construction described in [68]. Given a distributive inverse semigroup S, define $\mathrm{Idl}(S)$ to consist of the compatible order ideals of S which are closed under binary compatible joins. It is possible to endow $\mathrm{Idl}(S)$ with a multiplication that makes it a pseudogroup. The following is [36, Proposition 3.5] and [36, Lemma 3.6].

Theorem 2.2.

(1) A pseudogroup S is coherent if and only if there exists a distributive inverse semigroup T such that S is isomorphic to $\mathrm{Idl}(T)$. In fact, any coherent pseudogroup S is canonically isomorphic to $\mathrm{Idl}(\mathsf{K}(S))$.
(2) The category of distributive inverse semigroups and their morphisms is equivalent to the category of coherent pseudogroups and coherent pseudogroup morphisms.

To complete the picture, we define a distributive inverse semigroup to be *Boolean* if its semilattice of idempotents is a generalized Boolean algebra. We have been dealing with joins above and this immediately raises the question of how meets, as discussed by Leech [40], fit into the picture. The answer is: very easily, as the following result shows [33, Part 3 of Lemma 2.5].

Lemma 2.3. *Suppose that $\bigvee_{i=1}^m a_i$ and $b \wedge (\bigvee_{i=1}^m a_i)$ both exist. Then all the meets $b \wedge a_i$ exist, the join $\bigvee_{i=1}^m b \wedge a_i$ exists and we have that*

$$b \wedge \left(\bigvee_{i=1}^m a_i \right) = \bigvee_{i=1}^m b \wedge a_i.$$

These definitions lead to the central idea of this paper: that we regard pseudogroups as 'non-commutative frames'. The following table summarizes what we mean.

Commutative	**Non-commutative**
meet semilattice	inverse semigroup
frame	pseudogroup
(non-unital) distributive lattice	distributive inverse semigroup
generalized Boolean algebra	Boolean inverse semigroup

3. Boolean inverse semigroups

The previous section contained generalities. In this section, we shall describe a class of pseudogroups — the Boolean inverse semigroups — in more detail. These have turned out to be of wide interest and even have a book dedicated to them [75].

To work efficiently with a Boolean inverse semigroup S, it is useful to define some further operations. If $a \in S$, define $\mathbf{e}(a) = \mathbf{d}(a) \vee \mathbf{r}(a)$, which we call the *extent* of a. If $ab^{-1} = 0 = a^{-1}b$ then we say that a and b are *orthogonal*, denoted by $a \perp b$. If $y \leq x$ define $(x \setminus y) = x(\mathbf{d}(x) \setminus \mathbf{d}(y))$. If the Boolean inverse semigroup is actually a monoid then each idempotent e has a complement $\bar{e}$.

Boolean inverse semigroups arise naturally as soon as you are interested in inverse semigroups as subsemigroups of the multiplicative semigroups of rings. The following result is implicit in [56], explicit in [75] and also proved in [35].

Theorem 3.1. *Let S be an inverse subsemigroup with zero of the multiplicative monoid of a ring R. Then there is a Boolean inverse semigroup S'' such that $S \subseteq S'' \subseteq R$.*

The above theorem is a very natural extension of the work carried out by Stone back in the 1930s that led to the classical theory of Stone duality [21].

Studying the finite semigroups in a class is usually futile but in the case of Boolean inverse semigroups, it is very easy since they can be explicitly described. Let G^0 be a (finite) group with zero adjoined. The set of $n \times n$ *rook matrices* over G^0, denoted by $R_n(G^0)$, consists of all $n \times n$ matrices which have at most one non-zero entry from G in each row and each column. It is not hard to prove that $R_n(G^0)$ is a Boolean inverse monoid. We call it a *rook matrix semigroup over the group G*.

Example 3.2. The $n \times n$ rook matrices over the trivial group form an inverse monoid isomorphic to I_n, the symmetric inverse monoid on n letters [70].

The following theorem synthesizes results from [32, Theorem 4.18] and [44].

Theorem 3.3. *Every finite Boolean inverse monoid is isomorphic to a finite direct product of rook matrix semigroups over finite groups. Furthermore, the Boolean inverse semigroup is fundamental if and only if it is isomorphic to a finite direct product of finite symmetric inverse monoids.*

The above theorem is important because it tells us that fundamental Boolean inverse semigroups should be regarded as generalizations of finite direct products of symmetric inverse monoids.

We now describe the form taken by some important classes of inverse semigroups when restricted to the Boolean case.

Let S be a Boolean inverse monoid with group of units $\mathsf{U}(S)$. An important feature of such semigroups is that units can be constructed from certain non-unital elements. A non-zero element $s \in S$ is called an *infinitesimal* if $s^2 = 0$. An *involution* is a non-identity unit g such that $g^2 = 1$.

Lemma 3.4. *Let S be a Boolean inverse monoid. If a is an infinitesimal then*

$$g = a^{-1} \vee a \vee \overline{\mathbf{e}(a)}$$

is an involution above a.

Proof. Let a be an infinitesimal. Then $a \perp a^{-1}$. It follows that $a \vee a^{-1}$ is defined. This element has the property that $\mathbf{d}(a \vee a^{-1}) = \mathbf{r}(a \vee a^{-1})$. Thus $g = a^{-1} \vee a \vee \overline{\mathbf{e}(a)}$ is a unit and a simple calculation shows that $g^2 = 1$. □

Example 3.5. The above lemma is in fact the basis for building transpositions in finite symmetric groups. For example, the partial bijection $1 \mapsto 2$ is an infinitesimal which gives rise to the transposition (12).

This raises the question of whether infinitesimals exist. This is dealt with by the following result. Recall that an inverse semigroup is *Clifford* if its idempotents are central.

Proposition 3.6. *Let S be a Boolean inverse semigroup. The following are equivalent:*

(1) S is Clifford.
(2) S contains no infinitesimals.

Proof. (1)⇒(2). Let a be an infinitesimal. Then $a^2 = 0$ and so $a^{-1}aaa^{-1} = 0$. By assumption, $a^{-1}a = aa^{-1}$. Thus $a^{-1}a = 0$ from which it follows that $a = 0$, a contradiction.

(2)⇒(1). Suppose that there were an element $a \in S$ such that $a^{-1}a \neq aa^{-1}$. If $a^{-1}aaa^{-1} = 0$ then a would be an infinitesimal. Thus $e = \mathbf{d}(a)\mathbf{r}(b) \neq 0$. Since $e \leq \mathbf{d}(a)$ there is an idempotent f such that $f \leq \mathbf{d}(a)$ and $fe = 0$ using the fact that the semilattice of idempotents is a generalized Boolean algebra. Observe that $af \neq 0$ but then $(af)^2 = afaf = a\mathbf{d}(a)f\mathbf{r}(a)f = aefa = 0$ and af is an infinitesmial, which is a contradiction. It follows that no such element a can exist and so S is a Clifford semigroup. □

Thus if a Boolean inverse semigroup is not Clifford, it will contain infinitesimals.

Recall that an inverse monoid is *factorizable* if every element is beneath a unit. Let S be a Boolean inverse monoid. We say that $\mathscr{D}$ *preserves complementation* if $e \,\mathscr{D}\, f$ implies that $\overline{e} \,\mathscr{D}\, \overline{f}$ for all idempotents e and f.

Proposition 3.7. *Let S be a Boolean inverse monoid. Then the following are equivalent:*

(1) $\mathscr{D}$ preserves complementation
(2) S is factorizable.

Proof. (1)⇒(2). Suppose that $\mathscr{D}$ preserves complementation. We prove that S is factorizable. Let $a \in S$. Put $e = a^{-1}a$ and $f = aa^{-1}$. Then $e \,\mathscr{D}\, f$. By assumption, $\overline{e} \,\mathscr{D}\, \overline{f}$. Thus there is an element b such that $\overline{e} \xrightarrow{b} \overline{f}$. The elements a and b are orthogonal and so their join $g = a \vee b$ exists. But then $g^{-1}g = 1 = gg^{-1}$ and so g is an invertible element and, by construction, $a \leq g$. Thus S is factorizable.

(2)⇒(1). Suppose that $e \,\mathscr{D}\, f$. Then there is an element a such that $e \xrightarrow{a} f$. By factorizability, there is an invertible element g such that $a \leq g$. Thus $a = ge$ and so $f = geg^{-1}$. Put $b = g\bar{e}$. Then $\mathbf{d}(b) = \bar{e}$. We now calculate $\mathbf{r}(b)$. Observe that

$$f\mathbf{r}(b) = fg\bar{e}g^{-1} = geg^{-1}g\bar{e}g^{-1} = 0$$

and

$$bb^{-1} \vee aa^{-1} = g\bar{e}g^{-1} \vee geg^{-1} = g(\bar{e} \vee e)g^{-1} = 1.$$

It follows that $bb^{-1} = \bar{f}$ and so $\overline{e} \,\mathscr{D}\, \overline{f}$, as required. □

Let S be a Boolean inverse monoid. Then the group of units of S acts on the set of idempotents of S by $e \mapsto geg^{-1}$, where $e \in \mathsf{E}(S)$ and $g \in \mathsf{U}(S)$. We call this the *natural action* on the Boolean algebra of idempotents.

Proposition 3.8. *Let S be a Boolean inverse monoid. Then S is fundamental if and only if the natural action of $\mathsf{U}(S)$ on $\mathsf{E}(S)$ is faithful.*

Proof. Suppose first that S is fundamental. Let g be a unit such that $geg^{-1} = e$ for all idempotents e. Then $ge = eg$ for all idempotents e. By assumption, g is an idempotent and so the group identity. Thus the natural action is faithful. Conversely, suppose that the natural action is faithful. Let $a \in S$ be such that $ae = ea$ for all idempotents e. In particular, $a = a(a^{-1}a) = (a^{-1}a)a$. Thus $aa^{-1} = (a^{-1}a)aa^{-1}$ giving $aa^{-1} \leq a^{-1}a$ which by symmetry yields $a^{-1}a = aa^{-1}$. Put $g = a \vee \overline{(a^{-1}a)}$, a unit. Observe that $ge = eg$ for all idempotents e. But the natural action is faithful. Thus g is the identity and so a is an idempotent from which it follows that S is fundamental. □

The operation of taking joins is only partially defined so it would seem that Boolean inverse semigroups cannot be defined using only algebraic equations. In fact, by changing the signature used to describe them, Boolean inverse semigroups do form a variety. This was discovered by Wehrung [75] and we describe that result now. It requires the introduction of two new operations. Let $a, b \in S$, a Boolean inverse semigroup. Put $e = \mathbf{d}(a) \setminus \mathbf{d}(a)\mathbf{d}(b)$ and $f = \mathbf{r}(b) \setminus \mathbf{r}(a)\mathbf{r}(b)$. Define

$$a \ominus b = fae.$$

We call this operation *left skew difference*. It is easy to check that $a \ominus b$ is the largest element less than or equal to a and orthogonal to b. Define

$$a \mathbin{\triangledown} b = (a \ominus b) \vee b.$$

We call this operation *left skew join of a and b*. If $a \sim b$ then $a \mathbin{\triangledown} b = a \vee b$. It follows that left skew difference is an everywhere defined proxy for the partially defined join. The following is proved as [75, Theorem 3.2.4, Theorem 3.4.11]. See [2] for background from the theory of universal algebras.

Theorem 3.9. *Boolean inverse semigroups are algebraic structures with respect to a signature that includes left skew difference and left skew join in addition to multiplication and inversion. Furthermore, with respect to this signature there is a Mal'cev term.*

Classical algebra makes much use of abelian invariants. This has not been true of inverse semigroup theory but we now describe the construction of an abelian invariant whose definition was motivated by that of the K_0-group for C^*-algebras [19]. In fact, there are strong analogies between C^*-algebras and Boolean inverse semigroups. If X is an infinite set, we denote by $I_f(X)$ the set of all partial bijections of X with finite domain. This forms a Boolean inverse *semigroup*, not a Boolean inverse *monoid*, since the complement of a finite set will not be finite. In [75, Section 6.8], Wehrung introduces the *tensor product*, $S \otimes T$, of Boolean inverse semigroups S and T. We shall not make the formal definition here; we simply note that it derives from a suitable notion of a 'bi-additive' map. If S is any Boolean inverse semigroup, it is natural to call the Boolean inverse semigroup $S \otimes I_f(X)$ the *stabilization* of S to purloin terminology from C^*-algebras. Define $M_\omega(S)$ to be the set of all $\mathbb{N} \times \mathbb{N}$-matrices with a finite number of non-zero entries from S that satisfy the *generalized rook conditions*: if a and b are in distinct columns but lie in the same row then $a^{-1}b = 0$; if a and b are in distinct rows but lie in the same column then $ab^{-1} = 0$. It is not hard to show that $M_\omega(S)$ is a Boolean inverse semigroup under 'matrix multiplication' where the role of addition is taken by compatible joins. Furthermore, $S \otimes I_f(\mathbb{N}) \cong M_\omega(S)$. This Boolean inverse semigroup has the property that if e and f are any two idempotents then there exist orthogonal idempotents e' and f' such that $e \mathscr{D} e'$ and $f \mathscr{D} f'$. Denote the $\mathscr{D}$-class containing the idempotent e by $[e]$. Then on the set $\mathsf{E}(M_\omega(S))/\mathscr{D}$ we may define a binary operation by $[e] \oplus [f] = [e' \vee f']$ where e' and f' are such that $e' \perp f'$, $e \mathscr{D} e'$ and $f \mathscr{D} f'$. In fact, with respect to this operation $\mathsf{E}(M_\omega(S))/\mathscr{D}$ is a commutative monoid. If S is any Boolean inverse semigroup, we define its *type monoid* $\mathsf{T}(S)$ to be the monoid $(\mathsf{E}(M_\omega(S))/\mathscr{D}, \oplus)$. The type monoid was originally defined, in a different way, in [74] and [27] motivated by C^*-algebras and Tarski's work on cardinal algebras [72] but by making use of tensor products, it is easy to show that it is equivalent to the way the type monoid is defined in [75, Chapter 4]. Recall that a commutative monoid $(M, +)$ is called a *refinement monoid* if it satisfies the following condition: if $a_1 + a_2 = b_1 + b_2$ then there are elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$.

Theorem 3.10. *The type monoid is a commutative refinement monoid with trivial group of units. Moreover, the construction $S \mapsto \mathsf{T}(S)$ is functorial.*

In addition to exploring connections between Boolean inverse semigroups and the Banach-Tarski paradox [4, 27], the main application of the type monoid so far has been in establishing the exact connection between Boolean inverse monoids and MV algebras — another generalization of Boolean algebras. An *MV algebra* is a structure $(A, \oplus, \neg, 0)$ where $(A, \oplus, 0)$ is an abelian monoid with identity 0 and zero $\neg 0$ such that $\neg\neg x = x$ and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. The theory of MV algebras grew out of multi-valued logic, whence the 'MV' [5]. See [50, 51]. In order that the type monoid $\mathsf{T}(S)$ of a Boolean inverse monoid S be an MV algebra we shall want to define $\neg[e] = [\bar{e}]$. This implies, by Proposition 3.7, that S should be factorizable. In fact, we also need that the poset $S/\mathscr{J}$ be a lattice. Accordingly, we define a *Foulis monoid* to be any factorizable Boolean inverse monoid in which $S/\mathscr{J}$ is a lattice. The following was proved in [39] by emulating the theory of AF C^*-algebras [3].

Theorem 3.11. *Let S be a countable Foulis monoid. Then its type monoid $\mathsf{T}(S)$ is an MV algebra and every countable MV algebra is isomorphic to one of this type.*

The above theorem was generalized to arbitrary MV algebras using quite different techniques in [75, Theorem 5.2.10]. See also [43] for further examples.

We conclude this section with a theoretical result which suggests that it might be profitable to study the structure of Boolean inverse semigroups in more detail. An *additive ideal* in a Boolean inverse semigroup S is an ideal (in the usual sense of semigroup theory) that is closed under binary compatible joins. Clearly, both $\{0\}$ and S are additive ideals. If these are the only ones we say that the Boolean inverse semigroup is 0-*simplifying*. The following relation was introduced in [41] and is important in handling 0-simplifying semigroups. Let e and f be two non-zero idempotents in S. Define $e \preceq f$ if and only if there exists a set of elements $X = \{x_1, \ldots, x_m\}$ such that $e = \bigvee_{i=1}^{m} \mathbf{d}(x_i)$ and $\mathbf{r}(x_i) \leq f$ for $1 \leq i \leq m$. We can write this formally as $e = \bigvee \mathbf{d}(X)$ and $\bigvee \mathbf{r}(X) \leq f$. We say that X is a *pencil* from e to f. The relation $\preceq$ is a preorder on the set of idempotents. Observe that if I is a non-zero $\vee$-closed ideal then $f \in I$ and $e \preceq f$ implies that $e \in I$. Define the equivalence relation $e \equiv f$ if and only if $e \preceq f$ and $f \preceq e$. The following was proved as part of [41, Lemma 7.8].

Lemma 3.12. *Let S be a Boolean inverse semigroup. Then $\equiv$ is the universal relation on the set of non-zero idempotents if and only if S is* 0*-simplifying.*

The importance of simplicity conditions in algebra is not that they produce 'building blocks' but that they guarantee a certain homogeneity of structure. This is illustrated by our next theorem. An element a in a Boolean inverse semigroup is said to be *finite* if the number of elements below a, under the natural partial order, is finite. A Boolean inverse semigroup is said to be *semisimple* if every element is finite. A non-zero element a is called an *atom* if $b \leq a$ implies that $b = a$ or $b = 0$. A Boolean inverse semigroup is said to be *atomless* if it has no atoms.

Theorem 3.13 (The Dichotomy theorem)**.** *Each* 0*-simplifying Boolean inverse semigroup is either atomless or semisimple.*

Proof. Let S be a 0-simplifying Boolean inverse semigroup. Suppose that $a \in S$ is finite. We shall prove that every element in S is finite. In fact, we shall prove that every idempotent is finite since, x is finite if and only if $\mathbf{d}(x)$ is finite. In particular, $e = \mathbf{d}(a)$ is a finite idempotent. Let f be any non-zero idempotent. Then by Lemma 3.12, we have that $e \equiv f$. In particular, $f \preceq e$. There is therefore a pencil $X = \{x_1, \ldots, x_m\}$ from f to e. Without loss of generality, we may assume that all elements of X are non-zero. By definition, $f = \bigvee_{i=1}^{m} \mathbf{d}(x_i)$ and $\mathbf{r}(x_i) \leq e$ for all $1 \leq i \leq m$. But e is an atom and so $\mathbf{r}(x_i) = e$ for all $1 \leq i \leq m$. It follows that each of the idempotents $f_i = \mathbf{d}(x_i)$ is an atom and so f is a join of the atoms $F = \{f_1, \ldots, f_m\}$. We now prove that f is finite. Let $g \leq f$ be any element. Then $g = \bigvee_{i=1}^{m} g \wedge f_i$. But f_i is an atom, and so $g \wedge f_i$ is either equal to f_i or zero. It follows that g is a join of a subset of F. But F is finite and so has only a finite number of subsets. It follows that f is finite. □

Applications of Boolean inverse semigroups can be found in [9] and [7].

4. Non-commutative Stone duality

In this section, we shall describe the most important theorems so far that deal with pseudogroups: these are the non-commutative analogues of classical Stone duality. The origins of these theorems lie in the work of Johannes Kellendonk [22, 23, 24] as mediated through the papers [41, 37], inspired by [65, 71] and then developed in a sequence of papers [31, 32, 33, 36, 26]. The first two chapters of [21] are a good introduction to the classical theory though restricted to the unital case.

The one remaining question is why pseudogroups of transformations are not more visible than they are. The answer is: étale groupoids of germs. From each pseudogroup of transformations, one can construct an étale groupoid and it is this étale groupoid which is often studied rather than the original pseudogroup. I think there are two reasons for this: a minor and a major one. The minor reason is that groupoids are well established structures and so mathematicians feel comfortable using them. The major reason is that the standard methods of algebraic topology can be applied to étale groupoids to yield abelian group invariants. The construction of the groupoid of germs of a pseudogroups is described in [30, page 64] and [63]. See [64], for a good account of the theory of topological groupoids. This construction would appear to pose a problem for (abstract) pseudogroups but this is exactly the same problem that arises when dealing with frames rather than topological spaces, and can be dealt with in a similar way. First of all, we replace topological spaces by étale topological groupoids. A groupoid is just a small category in which every arrow is invertible and a topological groupoid is a groupoid equipped with a topology in which all the structure maps are required to be continuous. A topological groupoid is *étale* if the domain and range maps in the groupoid are local homeomorphisms. Before going any further, we should explain the significance of the class of étale groupoids. This is what Resende proved in the following theorem [64, 65].

Theorem 4.1. *Let G be an étale groupoid. Then the frame of open sets, $\Omega(G)$, is a monoid under multiplication of subsets.*

To paraphrase the above theorem: étale groupoids have a strong algebraic character. We shall now describe the relationship that exists between étale groupoids and pseudogroups. One direction is easy and requires the following notion. Let G be a groupoid with set of identities G_o. A subset $X \subseteq G$ is called a *partial bisection* if $X^{-1}X, XX^{-1} \subseteq G_o$. Denote the set of open partial bisections of an étale groupoid G by $\mathsf{B}(G)$.

Proposition 4.2. *Let G be an étale groupoid. Then $\mathsf{B}(G)$ is a pseudogroup under subset multiplication.*

To go in the other direct requires us to generalize completely prime filters from frames to pseudogroups. Let S be an inverse semigroup with zero. A subset $A \subseteq S$ is called a *filter* if $a, b \in A$ implies that there is a $c \in A$ such that $c \leq a, b$ and if

$a \in A$ and $a \leq b$ implies that $b \in A$. A filter is said to be *proper* if $0 \notin A$. If Y is any subset of S we define $Y^{\uparrow}$ to be the set of all elements greater than or equal to an element of Y. If A is a filter define $\mathbf{d}(A) = (A^{-1}A)^{\uparrow}$ and $\mathbf{r}(A) = (AA^{-1})^{\uparrow}$. It is easy to check that both of these sets are filters and are proper if and only if A is proper. Now let S be a pseudogroup. A proper filter $A \subseteq S$ is said to be *completely prime* if $\bigvee_{i \in I} a_i \in A$ implies that $a_i \in A$ for some i. Denote the set of all completely prime filters on S by $\mathsf{G}_{cp}(S)$. It is easy to check that if A is a completely prime filter so too are $\mathbf{d}(A)$ and $\mathbf{r}(A)$ [36, Lemma 2.2]. If A and B are completely prime filters such that $\mathbf{d}(A) = \mathbf{r}(B)$ then $A \cdot B = (AB)^{\uparrow}$ is also a completely prime filter [36, Lemma 2.4]. Let $s \in S$. Define X_s to be the set of all completely prime filters in S that contain s. Put $\tau = \{X_s \colon s \in S\}$. Then we have the following.

Proposition 4.3. *Let S be a pseudogroup. Then the set $\mathsf{G}_{cp}(S)$ equipped with the partial product $\cdot$ and the topology with basis τ is an étale topological groupoid.*

There are natural maps $\varepsilon \colon S \to \mathsf{B}(\mathsf{G}_{cp}(S))$, given by $s \mapsto X_s$, and $\eta \colon G \to \mathsf{G}_{cp}(\mathsf{B}(S))$, given by $g \mapsto F_g$, where F_g is the set of all open partial bisections in $\mathsf{B}(G)$ that contain the element g. In neither case is the map necessarily an isomorphism. It is for this reason that we actually obtain an adjunction between a suitable category of pseudogroups and a suitable category of étale groupoids (where by 'suitable', I simply mean for appropriate choices of morphisms in each case). See [36, Theorem 2.22] for a full statement of this theorem. A much sharper result can be obtained if étale *topological* groupoids are replaced by étale *localic* groupoids. This is the basis of Resende's paper [65].

The above results can be cut down to the case of distributive inverse semigroups by using the notion of coherence — one replaces open partial bisections by compact-open partial bisections. See [36, Section 3]. But the case of most interest is where we replace pseudogroups by Boolean inverse semigroups. We define a *Boolean space* to be a locally compact Hausdorff space with a basis of clopen subsets. An étale topological groupoid is said to be *Boolean* if its identity space is a Boolean space. An *ultrafilter* in a Boolean inverse semigroup is a maximal proper filter. It is possible to endow the set of ultrafilters on a Boolean inverse semigroup with a groupoid multiplication in much the same way as we did that of completely prime filters. If S is a Boolean inverse semigroup, we denote its groupoid of ultrafilters by $\mathsf{G}(S)$. This is endowed with a topology supplied by the basis $\beta = \{U_s \colon s \in S\}$ where U_s is the set of ultrafilters that contain s. If G is a Boolean groupoid then $\mathsf{KB}(G)$ denotes the set of all compact-open partial bisections. We now have the following, much sharper theorem.

Theorem 4.4.

1. *Let S be a Boolean inverse semigroup. Then $\mathsf{G}(S)$ is a Boolean groupoid.*
2. *Let G be a Boolean groupoid. Then $\mathsf{KB}(G)$ is a Boolean inverse semigroup.*
3. *If S is a Boolean inverse semigroup then $S \cong \mathsf{KB}(\mathsf{G}(S))$.*
4. *If G is a Boolean groupoid then $G \cong \mathsf{G}(\mathsf{KB}(G))$.*
5. *S is a Boolean inverse* monoid *if and only if the identity space of $\mathsf{G}(S)$ is* compact.

Example 4.5. Let S be a finite Boolean inverse monoid. Then $\mathsf{G}(S)$ is nothing other than a finite, discrete groupoid. It follows that $\mathsf{KB}(\mathsf{G}(S)) = \mathsf{L}(\mathsf{G}(S))$ the set of all partial bisections on $\mathsf{G}(S)$. Now, every groupoid is a disjoint union of connected components. If $G = G_1 \amalg G_2$ then $\mathsf{L}(G) = \mathsf{L}(G_1) \times \mathsf{L}(G_2)$. It follows that we need only describe the Boolean inverse semigroup set of all partial bisections on a finite connected discrete groupoid. Each finite, connected groupoid is isomorphic to one of the form $X \times G \times X$ where X is a finite set with n elements and G is a finite group. It is an easy exercise to check that $\mathsf{L}(X \times G \times X) \cong R_n(G^0)$. We are therefore able to derive easily the structure of finite Boolean inverse semigroups described in Theorem 3.3.

The following table acts as a dictionary between the theory of Boolean inverse monoids and the theory of Boolean groupoids (with compact identity space). I will not define the terms on the right-hand side, but on the left-hand side let me remark that an *inverse* $\wedge$*-semigroup* is an inverse semigroup in which each pair of elements has a meet (first studied in [40]) and the *Tarski algebra* is the unique countable atomless Boolean algebra.

Boolean inverse monoid	**Boolean groupoid**
$\wedge$-monoid	Hausdorff
fundamental	effective
countable	second-countable
Tarski algebra of idempotents	Cantor space of identities
0-simplifying	minimal
0-simple	purely infinite and minimal
group of units	topological full group

Define a *Tarski monoid* to be a countable, atomless Boolean inverse $\wedge$-monoid. We say that a Boolean inverse semigroup is *simple* if it is both 0-simplifying and fundamental. The classification of simple Tarski monoids is intimately connected, via the results of this section, with the study of certain étale groupoids [48, 49] which arise from both symbolic dynamics and the theory of infinite (Thompson-eque) groups. Some preliminary work has been carried out on this question [34] but we would anticipate the importation of the homology theory of étale groupoids into the theory of Boolean inverse semigroups. Loganathan [42] pioneered the use of such homological ideas when he provided a general setting for the work of Lausch [29] but it is an avenue that has remained largely unexplored. Finally, we can legitimately cite [6] and say that aspects of inverse semigroup theory are a contribution to noncommutative geometry.

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