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THE BOOLEANIZATION OF AN INVERSE SEMIGROUP

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Abstract. We prove that the forgetful functor from the category of Boolean inverse semigroups to the category of inverse semigroups with zero has a left adjoint. This left adjoint is what we term the 'Booleanization'. We establish the exact theoretical connection between the Booleanization of an inverse semigroup and Paterson's universal groupoid of the inverse semigroup and we explicitly compute the concrete Booleanization of the polycyclic inverse monoid P_n and demonstrate its affiliation with the Cuntz-Toeplitz algebra.

1. Introduction

The goal of this paper is to prove the following theorem and to study some of its consequences.

Theorem 1.1 (Booleanization)**.** *Let S be an inverse semigroup with zero. There is a Boolean inverse semigroup $\mathsf{B}(S)$ together with an embedding $\beta \colon S \to \mathsf{B}(S)$ such that if $\theta \colon S \to T$ is any homomorphism to a Boolean inverse semigroup T then there is a unique morphism $\gamma \colon \mathsf{B}(S) \to T$ such that $\theta = \beta\gamma$.*

Although we originally constructed $\mathsf{B}(S)$ in [16], we restricted the class of homomorphisms considered and so did not prove universality in full generality. This lacuna will be filled here.

We refer the reader to [10] for background on inverse semigroups, to [25] for background on étale groupoids and to [29] for the theory of Boolean inverse semigroups. All distributive lattices will be assumed to have a bottom but not necessarily a top; if they have a top we say they are *unital*. We use the term *Boolean algebra* to mean what is often termed a 'generalized Boolean algebra'. Thus a Boolean algebra is a distributive lattice in which each principal order ideal is a unital Boolean algebra. The inverse semigroups in this paper will usually have a zero and for those that do homomorphisms between them will be required to preserve zero. The order on inverse semigroups will be the *natural partial order*. The semilattice of idempotents of an inverse semigroup S is denoted by $\mathsf{E}(S)$. More generally, if X is any subset of S then $\mathsf{E}(X) = \mathsf{E}(S) \cap X$. In addition, define

$$X^{\uparrow} = \{s \in S \colon \exists x \in X, x \leq s\} \text{ and } X^{\downarrow} = \{s \in S \colon \exists x \in X, s \leq x\}.$$

If $X = \{x\}$ then we write simply $x^{\uparrow}$ and $x^{\downarrow}$, respectively. If s is an element of an inverse semigroup define $\mathbf{d}(s) = s^{-1}s$ and $\mathbf{r}(s) = ss^{-1}$. The *compatibility relation* $\sim$ in an inverse semigroup is defined by $s \sim t$ if and only if $s^{-1}t$ and st^{-1} are both idempotents. A set that consists of elements which are pairwise compatible is said to be *compatible*. The *orthogonality relation* $\perp$ in an inverse semigroup with zero is defined by $s \perp t$ if and only if $s^{-1}t = 0 = st^{-1}$. Observe that $s \perp t$ if and only if $\mathbf{d}(a) \perp \mathbf{d}(b)$ and $\mathbf{r}(a) \perp \mathbf{r}(b)$. A set that consists of elements which are pairwise orthogonal is said to be *orthogonal*. The proof of the following is straightforward.

Lemma 1.2. *Let S be an inverse semigroup with zero and let $s, t, a \in S$. If $s \perp t$ then both $sa \perp ta$ and $as \perp at$.*

Key words and phrases. Inverse semigroups, Stone duality, Boolean algebras.

Lemma 1.3. *In an arbitrary inverse semigroup, let $a \sim b$. Then the following are equivalent:*

(1) $a \perp b$.
(2) $\mathbf{d}(a) \perp \mathbf{d}(b)$.
(3) $\mathbf{r}(a) \perp \mathbf{r}(b)$.

Proof. By symmetry, it is enough to prove the equivalence of (1) and (2). Clearly, (1) implies (2). We prove that (2) implies (1). To do this, we need to prove that $\mathbf{r}(a) \perp \mathbf{r}(b)$. We are given that $\mathbf{d}(a) \perp \mathbf{d}(b)$, which is equivalent to $ab^{-1} = 0$, and and also given that $a^{-1}b$ is an idempotent. Therefore $aa^{-1}bb^{-1} = a(a^{-1}b)b^{-1} \leq ab^{-1} = 0$. We have proved that $\mathbf{r}(a) \perp \mathbf{r}(b)$ and so $a \perp b$. □

An inverse semigroup is said to be a $\wedge$-*semigroup* if and only if it has all binary meets. An inverse semigroup is said to be *distributive* if it has binary joins of compatible elements and multiplication distributes over such joins. A *pseudogroup* is an inverse semigroup with zero in which every compatible subset has a join and where multiplication distributes over all such joins. Observe that the semilattice of idempotents of a pseudogroup is a *frame* in the sense of [5]. A distributive inverse semigroup is *Boolean* if its semilattice of idempotents is a Boolean algebra. A *morphism* between distributive inverse semigroups is a homomorphism of inverse semigroups with zero that maps binary compatible joins to binary compatible joins. A $\wedge$-*morphism* between distributive $\wedge$-semigroups preserves binary meets.

Proofs of the following can be found in [10, Lemma 1.4.11, Lemma 1.4.12, Lemma 1.4.14].

Lemma 1.4. *Let S be an inverse semigroup.*

(1) *$s \sim t$ if and only if $s \wedge t$ exists and $\mathbf{d}(s \wedge t) = \mathbf{d}(s)\mathbf{d}(t)$ and $\mathbf{r}(s \wedge t) = \mathbf{r}(s)\mathbf{r}(t)$.*
(2) *If $s \sim t$ then $s \wedge t = st^{-1}t = ss^{-1}t$.*
(3) *$s^{\downarrow}$ is a compatible set.*

Definition. If $s \sim t$ then the meet $s \wedge t$ guaranteed by part (1) of Lemma 1.4 is called a *compatible meet*. Observe by part (2) of Lemma 1.4 that such meets can be constructed purely algebraically.

Remark 1.5. Morphisms between distributive inverse semigroups are required to preserve binary compatible joins but are not required to preserve any binary meets that might exist. However, compatible meets are preserved by any homomorphism. This simple observation will prove useful in establishing the universality of our construction.

We shall need a little more on the algebraic properties of Boolean inverse semigroups. Let B be a Boolean algebra. If $e, f \in B$ are such that $f \leq e$ then there is a unique element $e \setminus f$ such that $e = (e \setminus f) \vee f$ and $(e \setminus f) \perp f$. In fact, $e \setminus f$ is the largest element of B that is less than e and orthogonal to f. The following definition is crucial in what follows.

Definition. Let S be an inverse semigroup whose semilattice of idempotents forms a Boolean algebra. If $a, b \in S$ with $b \leq a$, define

$$a \setminus b = a(\mathbf{d}(a) \setminus \mathbf{d}(b)).$$

Observe that $b \perp (a \setminus b)$ by Lemma 1.3. If S is, in fact, a Boolean inverse semigroup then $a = b \vee (a \setminus b)$; in this case, $a \setminus b$ is the largest element less than or equal to a which is orthogonal to b. The proof of the following is straightforward.

Lemma 1.6. *Let S be a Boolean inverse monoid. Let $b \leq a$. Suppose that $x \leq a$, $x \perp b$ and $a = b \vee x$. Then $x = a \setminus b$.*

The following shows how to write a compatible join of two elements as a binary orthogonal join of two elements.

Lemma 1.7. *Let a and b be compatible elements in an inverse semigroup whose semilattice of idempotents forms a Boolean algebra under the natural partial order.*

1. *$a \setminus (a \wedge b)$ and b are orthogonal.*
2. *$a \vee b$ exists if and only if $(a \setminus (a \wedge b)) \vee b$ exists in which case they are equal.*

Proof. (1) The element $a \wedge b$ exists by Lemma 1.4. Thus $a \setminus (a \wedge b)$ exists by Lemma 1.6. Clearly, $\mathbf{d}(a \setminus (a \wedge b))\mathbf{d}(b) = 0$. By Lemma 1.3, it follows that $(a \setminus (a \wedge b)) \perp b$.

(2) It is enough to prove that $a, b \leq c$ if and only if $(a \setminus (a \wedge b)), b \leq c$. Only one direction needs an actual proof. Suppose that $(a \setminus (a \wedge b)), b \leq c$. We need to prove that $a \leq c$. But this follows from the fact that $a = (a \setminus (a \wedge b)) \vee (a \wedge b)$ by Lemma 1.6. □

The following result is frequently invoked in proofs and follows easily from Lemma 1.7

Proposition 1.8. *The following are equivalent.*

1. *S is a Boolean inverse semigroup.*
2. *S has all binary* orthogonal *joins, multiplication distributes over such joins, and its semilattice of idempotents forms a Boolean algebra with respect to the natural partial order.*

A useful first step in the proof of Theorem 1.1 is the observation that we may restrict to the case where the inverse semigroup S is actually a distributive inverse semigroup. This follows from the fact that the forgetful functor from the category of distributive inverse semigroups to the category of inverse semigroups has a left adjoint which is easy to construct. We recall this construction here. Let S be an inverse semigroup. The *Schein completion* of S [10, Theorem 1.4.23, Theorem 1.4.24] is the *pseudogroup* $\mathsf{C}(S)$ whose elements are the compatible order ideals of S. Multiplication is subset multiplication; the order is subset inclusion; the idempotents are order ideals of $\mathsf{E}(S)$. In fact, C is left adjoint to the forgetful functor from the category of pseudogroups to the category of inverse semigroups. We now describe the 'finite' elements of $\mathsf{C}(S)$. Let T be a pseudogroup. An element $a \in T$ is said to be *finite* if $a \leq \bigvee_{i \in I} b_i$ implies that there is a finite subset $\{1, \ldots, n\} \subseteq I$ such that $a \leq \bigvee_{i=1}^{n} b_i$.[1] The set of finite elements of T is denoted by $\mathsf{K}(T)$. The following is a slightly sharper statement of [16, Lemma 3.3].

Lemma 1.9. *Let S be a pseudogroup.*

1. *a is finite if and only if a^{-1} is finite.*
2. *If a is any element and e is a finite idempotent $e \leq a^{-1}a$ (respectively, $e \leq aa^{-1}$) then ae (respectively, ea) is finite.*
3. *a is finite if and only if $a^{-1}a$ is finite (respectively, aa^{-1} is finite).*
4. *If a and b are finite and $\mathbf{d}(a) = \mathbf{r}(b)$ then ab is finite.*

The following is [16, Lemma 3.4].

Lemma 1.10. *Let S be a pseudogroup.*

1. *The finite elements of S form an inverse subsemigroup if and only if the finite idempotents form a subsemilattice.*
2. *If the finite elements form an inverse subsemigroup they form a distributive inverse subsemigroup.*

[1] One could argue that the term 'compact' is more appropriate, but we follow the terminology in [5].

Define

$$\mathsf{D}(S) = \mathsf{K}(\mathsf{C}(S)). \tag{1}$$

An element A of $\mathsf{C}(S)$ is said to be *finitely generated* if $A = \{a_1, \ldots, a_m\}^{\downarrow}$ where $\{a_1, \ldots, a_m\}$ is a finite compatible subset of S.

Lemma 1.11. *Let S be an inverse semigroup.*

(1) The finite elements in $\mathsf{C}(S)$ are precisely the finitely generated ones.
(2) $\mathsf{D}(S)$ is a distributive inverse semigroup and there is an embedding $\delta \colon S \to \mathsf{D}(S)$ given by $s \mapsto s^{\downarrow}$.

Proof. (1) It is immediate from the definition that finitely generated elements of $\mathsf{C}(S)$ are finite. We prove the converse. Let A be a finite element of $\mathsf{C}(S)$. Then $A \leq \bigvee_{s \in A} s^{\downarrow}$. Thus $A \leq \bigvee_{i=1}^{n} s_i^{\downarrow}$ for some finite set of elements $\{s_i \colon 1 \leq i \leq n\} \subseteq A$. Clearly, $A = \bigcup_{i=1}^{n} s_i^{\downarrow}$ and so A is finitely generated.

(2) By Lemma 1.10, it is enough to prove that the product of two finite idempotents is finite. This is straightforward. □

Theorem 1.12 (Distributive completion)**.** *Let S be an inverse semigroup. There is a distributive inverse semigroup $\mathsf{D}(S)$ together with an embedding $\delta \colon S \to \mathsf{D}(S)$ such that if $\theta \colon S \to T$ is any homomorphism to a distributive inverse semigroup T there is a unique morphism $\gamma \colon \mathsf{D}(S) \to T$ such that $\theta = \delta\gamma$.*

Proof. Let $\theta \colon S \to T$ be a homomorphism to a distributive inverse semigroup T. We prove that there is a unique morphism $\gamma \colon \mathsf{D}(S) \to T$ such that $\theta = \phi\gamma$. Define

$$\gamma(\{a_1, \ldots, a_n\}^{\downarrow}) = \bigvee_{i=1}^{n} \theta(a_i).$$

Observe that the right hand side above is defined because $\{a_1, \ldots, a_n\}$ is a compatible subset of S and homomorphisms preserve compatibility. We prove that γ is well-defined. Suppose that $\{a_1, \ldots, a_n\}^{\downarrow} = \{b_1, \ldots, b_m\}^{\downarrow}$. Then for each $1 \leq i \leq n$, we can write $a_i = b_i'$ where $b_i' \leq b_j$ for some j. It follows that $\bigvee_{i=1}^{n} \theta(a_i) \leq \bigvee_{j=1}^{m} \theta(b_j)$. By symmetry, we get equality and so γ is well-defined. The proofs of the remaining parts of the theorem are now routine. □

Remark 1.13. Let S be an inverse semigroup and let $\theta \colon S \to T$ be a homomorphism to a Boolean inverse semigroup T. Then by Theorem 1.12, there is a unique morphism $\theta' \colon \mathsf{D}(S) \to T$ such that $\theta'\delta = \theta$. Thus we can construct our Booleanization first for distributive inverse semigroups.

Remark 1.14. Suppose that S is an inverse monoid. Then $1^{\downarrow}$ is a finitely generated compatible order ideal. It follows that $\mathsf{D}(S)$ is a monoid and that δ is also a monoid homomorphism.

Example 1.15. It is tempting to believe that if S were already a distributive inverse semigroup then it should be isomorphic to $\mathsf{D}(S)$. We construct a simple counterexample to show that this is not true. Let $S = I_2$, the symmetric inverse monoid on a set with 2 elements. This has 7 elements consisting of the identity and the transposition $t = (12)$, the idempotents $e : 1 \mapsto 1$, $f : 2 \mapsto 2$ and 0, and the elements $a : 1 \mapsto 2$ and $b : 2 \mapsto 1$. The distributive completion consists of the principal order ideals $1^{\downarrow}, t^{\downarrow}, e^{\downarrow}, f^{\downarrow}, 0^{\downarrow}, a^{\downarrow}, b^{\downarrow}$ together with the following compatible order ideals: $\{e, f, 0\}$ and $\{a, b, 0\}$. Thus in this case $\mathsf{D}(S)$ has a different cardinality from S and so they cannot be isomorphic.

An inverse semigroup is said to be a *weak semilattice* [26] if the intersection of any two principal order ideals is finitely generated.

Lemma 1.16. *Let S be an inverse semigroup. Then S is a weak semilattice if and only if $\mathsf{D}(S)$ is a $\wedge$-semigroup.*

Proof. The natural partial order in $\mathsf{D}(S)$ is subset inclusion. Suppose first that S is a weak semilattice. Let $A, B \in \mathsf{D}(S)$ where $A = \{a_1, \ldots, a_m\}^{\downarrow}$ and $B = \{b_1, \ldots, b_n\}^{\downarrow}$. For each pair (i, j), where $1 \leq i \leq m$ and $1 \leq j \leq n$, choose a finite set of generators $C_{i,j}$ of the order ideal $a_i^{\downarrow} \cap b_j^{\downarrow}$. Then it is easy to see that $A \cap B = (\bigcup_{(i,j)} C_{i,j})^{\downarrow}$. It follows that $\mathsf{D}(S)$ is a $\wedge$-semigroup. Conversely, suppose that $\mathsf{D}(S)$ is a $\wedge$-semigroup. Let $a, b \in S$. Then $a^{\downarrow}, b^{\downarrow} \in \mathsf{D}(S)$. By assumption, $a^{\downarrow} \wedge b^{\downarrow}$ exists. Thus we can write $a^{\downarrow} \wedge b^{\downarrow} = \{c_1, \ldots, c_m\}^{\downarrow} \in \mathsf{D}(S)$. It follows that $\{c_1, \ldots, c_m\}^{\downarrow} \subseteq a^{\downarrow} \cap b^{\downarrow}$. Let $c \leq a, b$. Then $c^{\downarrow} \subseteq a^{\downarrow}, b^{\downarrow}$. Thus $c^{\downarrow} \subseteq \{c_1, \ldots, c_m\}^{\downarrow}$ and so $c \in \{c_1, \ldots, c_m\}^{\downarrow}$. It follows that $a^{\downarrow} \cap b^{\downarrow} = \{c_1, \ldots, c_m\}^{\downarrow}$, and therefore S is a weak semilattice. □

Let S be an inverse semigroup. A *filter* in S is a subset A such that $A = A^{\uparrow}$ and whenever $a, b \in A$ there exists $c \in A$ such that $c \leq a, b$. A filter is *proper* if it does not contain zero. In what follows, any results stated about filters are proved in [13, 14, 16]. Observe that A is a filter if and only if A^{-1} is a filter. If A and B are filters then $(AB)^{\uparrow}$ is a filter. Define $\mathbf{d}(A) = (A^{-1}A)^{\uparrow}$ and $\mathbf{r}(A) = (AA^{-1})^{\uparrow}$. Then both $\mathbf{d}(A)$ and $\mathbf{r}(A)$ are filters. It is easy to check that A is proper if and only if $\mathbf{d}(A)$ is proper (respectively, $\mathbf{r}(A)$ is proper). Observe that for each $a \in A$ we have that $A = (a\mathbf{d}(A))^{\uparrow} = (\mathbf{r}(A)a)^{\uparrow}$. We denote the set of proper filters on S by $\mathcal{L}(S)$.[2] If $A, B \in \mathcal{L}(S)$, define the partial operation $A \cdot B$ if and only if $\mathbf{d}(A) = \mathbf{r}(B)$ in which case $A \cdot B = (AB)^{\uparrow}$. In this way, $\mathcal{L}(S)$ becomes a groupoid in which the identities are the filters that contain idempotents; indeed, these are precisely the filters that are also inverse subsemigroups.

Let S be a distributive inverse semigroup. A *prime filter* in S is a proper filter $A \subseteq S$ such that if $a \vee b \in A$ then $a \in A$ or $b \in A$. Denote the set of all prime filters of S by $\mathsf{G}(S)$. It can be checked that A is a prime filter if and only if $\mathbf{d}(A)$ (respectively, $\mathbf{r}(A)$) is a prime filter. Define a partial multiplication $\cdot$ on $\mathsf{G}(S)$ by $A \cdot B$ exists if and only if $\mathbf{d}(A) = \mathbf{r}(B)$, in which case $A \cdot B = (AB)^{\uparrow}$. With respect to this partial multiplication, $\mathsf{G}(S)$ is a groupoid where the identities are the prime filters that contain idempotents. For this reason, it is convenient to define a prime filter to be an *identity* if it contains an idempotent. Proofs of all of the above claims can be found in [16].

Let G be any discrete groupoid with set of identities G_o.[3] A subset $X \subseteq G$ is said to be a *local bisection* if $x, y \in X$ and $\mathbf{d}(x) = \mathbf{d}(y)$ then $x = y$, and if $x, y \in X$ and $\mathbf{r}(x) = \mathbf{r}(y)$ then $x = y$. It is easy to check that a subset $X \subseteq G$ is a local bisection precisely when $X^{-1}X, XX^{-1} \subseteq G_o$. A local bisection is called simply a *bisection* if in fact $X^{-1}X = G_o = XX^{-1}$. Denote the set of all local bisections of G by $\mathsf{L}(G)$. Endow it with the binary operation of subset multiplication. The following is well-known [25, page 12].

Proposition 1.17. *Let G be a discrete groupoid. Then $\mathsf{L}(G)$ is a pseudogroup in which the natural partial order is subset inclusion.*

Let S be a distributive inverse semigroup. For each $a \in S$ define V_a to be the set of all prime filters that contains a. The following is proved in [16].

Proposition 1.18. *Let S be a distributive inverse semigroup.*

(1) If $a \not\leq b$ then there is a prime filter that contains a and omits b.
(2) $V_a \subseteq V_b$ if and only if $a \leq b$.

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[2]Observe that this notation refers only to proper filters.

[3]This set is also termed the 'set of units' in the literature and denoted by $G^{(0)}$.

(3) *V_a consists entirely of identities if and only if a is an idempotent.*
(4) *The sets V_a form a basis for a topology with respect to which they are compact-open local bisections. In this way, $\mathsf{G}(S)$ becomes an étale groupoid who space of identities is a* locally compact *spectral space — this means that it is sober and has a basis of compact-open sets closed under binary intersections.*
(5) *If $a \sim b$ then $V_a \cup V_b = V_{a \vee b}$.*

Lemma 1.19. *Let S be a Boolean inverse semigroup and let P be a prime filter and let $b \leq a$. Then the following are equivalent.*
(1) *$a \in P$ and $b \notin P$.*
(2) *$a \setminus b \in P$.*

Proof. (1)⇒(2). Observe that $a = b \vee (a \setminus b)$. Either $b \in P$ or $a \setminus b \in P$, since P is a prime filter. It follows that $a \setminus b \in P$.

(2)⇒(1). Since $a = b \vee (a \setminus b)$ we know that $a \in P$. But if $b \in P$ then we would have $b \wedge (a \setminus b) = 0 \in P$, which is impossible. Thus $b \notin P$. □

The philosophy that underlies this paper can be usefully summarized by the following table:

Commutative	**Non-commutative**
Meet semilattice	Inverse semigroup
Frame	Pseudogroup
Distributive lattice	Distributive inverse semigroup
Boolean algebra	Boolean inverse semigroup

2. Proof of Theorem 1.1

This splits into two parts. First, we construct the Boolean inverse semigroup $\mathsf{B}(S)$ and the map $\beta \colon S \to \mathsf{B}(S)$; this reiterates what we showed in [16]. Second, which is the new part, we prove that β has the requisite universal properties.

If S is an inverse semigroup then we can replace it by $\mathsf{D}(S)$ when calculating its Booleanization by Remark 1.13. Thus in what follows we shall assume that S is distributive.

We introduce some important notation. Let S be a distributive inverse semigroup. If $b \leq a$ where $a, b \in S$ define $V_{a;b} = V_a \setminus V_b$. Since this is a subset of a local bisection it is itself a local bisection. The following is immediate by Proposition 1.18.

Lemma 2.1. *Let S be a distributive inverse semigroup. Then $V_{a;b} = \varnothing$ if and only if $a = b$.*

If $e \in \mathsf{E}(S)$ then we denote by $\mathcal{V}_e$ the set of prime filters in $\mathsf{E}(S)$ containing e. If $f \leq e$ define $\mathcal{V}_{e;f} = \mathcal{V}_e \setminus \mathcal{V}_f$.

Notation. Keep clear the typographical distinction between V_a, which is a set of prime filters *in* S, and $\mathcal{V}_e$, which is the set of prime filters *in* $\mathsf{E}(S)$.

The properties of the operation $(a, b) \mapsto a \setminus b$ in a Boolean inverse semigroup motivate this paper. They are summarized below. Observe that whenever we write $s \setminus t$ we assume that $t \leq s$.

Lemma 2.2. *Let S be a Boolean inverse semigroup.*
(1) *$(s \setminus t)^{-1} = s^{-1} \setminus t^{-1}$.*
(2) *If a is any element then $(s \setminus t)a = sa \setminus ta$.*
(3) *$s \setminus (u \vee v) = (s \setminus u)s^{-1}(s \setminus v)$.*

(4) $(s \setminus t)(u \setminus v) = su \setminus (sv \vee tu)$.
(5) $V_{a;b} = V_{c;d}$ *if and only if* $a \setminus b = c \setminus d$.
(6) *If* $a \leq b \leq c$ *then* $(c \setminus b) \leq (c \setminus a)$.

Proof. (1) Straightforward.

(2) Observe that from $t \leq s$ we obtain $ta \leq sa$. From $s = t \vee (s \setminus t)$ we get that $sa = ta \vee (s \setminus t)a$ and from $t \perp (s \setminus t)$ we get that $ta \perp (s \setminus t)a$. It follows by Lemma 1.6 that $(s \setminus t)a = sa \setminus ta$.

(3) This follows from the fact that in a Boolean algebra $e \setminus (i \vee j) = (e \setminus i)(e \setminus j)$.

(4) Observe that $sv, tu \leq su$ and so $sv \sim tu$ meaning that the join $sv \vee tu$ exists in S. We have the following argument, where we use parts (3) and (2).

$$\begin{aligned} su \setminus (sv \vee tu) &= (su \setminus tu)u^{-1}s^{-1}(su \setminus sv) \\ &= (s \setminus t)uu^{-1}s^{-1}s(u \setminus v) \\ &= (s \setminus t)s^{-1}suu^{-1}(u \setminus v) \\ &= (s \setminus t)(u \setminus v). \end{aligned}$$

(5) We prove that in a Boolean inverse semigroup $V_{a;b} = V_{a \setminus b}$. The result then follows by part (2) of Proposition 1.18. Let $A \in V_{a;b}$. Then $a \in A$ and $b \notin A$. But $a = b \vee (a \setminus b)$ and A is a prime filter. Thus $a \setminus b \in A$. In the other direction, let $A \in V_{a \setminus b}$. Then $a \in A$ since $a \setminus b \leq a$. But A is a prime filter and $a = b \vee a \setminus b$ so that $b \in A$ or $a \setminus b \in A$. However $b \in A$ and $a \setminus b \in A$ implies that $0 = b \wedge (a \setminus b) \in A$, which is impossible. Thus $a \setminus b \in A$, as required.

(6) This follows from the fact that $\mathbf{d}(a) \leq \mathbf{d}(b) \leq \mathbf{d}(c)$ and that $(\mathbf{d}(c) \setminus \mathbf{d}(b)) \leq (\mathbf{d}(c) \setminus \mathbf{d}(a))$. □

2.1. **Construction.** Let S be a distributive inverse semigroup. We begin by constructing the discrete groupoid $\mathsf{G}(S)$ of prime filters on S and then the pseudogroup $\mathsf{L}(\mathsf{G}(S))$ by Proposition 1.17. By Proposition 1.18 there is an embedding $\iota \colon S \to \mathsf{L}(\mathsf{G}(S))$ given by $\iota(a) = V_a$. We shall construct $\mathsf{B}(S)$ as an inverse subsemigroup of $\mathsf{L}(\mathsf{G}(S))$ that contains the image of ι.

Lemma 2.3. *Let S be a distributive inverse semigroup.*

(1) $V_{s;t}^{-1} = V_{s^{-1};t^{-1}}$.
(2) $V_{s;t}V_{u;v} = V_{su;sv \vee tu}$.
(3) *Let* $a \sim b$, $c \sim d$ *and* $c \vee d \leq a \vee b$. *Then* $V_{(a \vee b);(c \vee d)} = V_{a;(c \vee d)a^{-1}a} \cup V_{b;(c \vee d)b^{-1}b}$.

Proof. (1) The proof follows by two simple observations. First, A is a prime filter if and only if A^{-1} is a prime filter. Second, A is a prime filter that contains s and omits t if and only if A^{-1} is a prime filter that contains s^{-1} and omits t^{-1}.

(2) This is proved as part (1) of [16, Proposition 5.5] though in this paper we have taken the opportunity to make the obvious simplification in the statement of the result.

(3) Let $P \in V_{(a \vee b);(c \vee d)}$. Then $a \vee b \in P$ and $c \vee d \notin P$. Suppose that $a \in P$. If $(c \vee d)a^{-1}a \in P$ then $c \vee d \in P$ which is a contradiction. By a similar argument we deduce that $V_{(a \vee b);(c \vee d)} \subseteq V_{a;(c \vee d)a^{-1}a} \cup V_{b;(c \vee d)b^{-1}b}$. Now let $P \in V_{a;(c \vee d)a^{-1}a}$. Then $a \in P$ and $(c \vee d)a^{-1}a \notin P$. It follows that $a \vee b \in P$. Suppose that $c \vee d \in P$. Then $(c \vee d)a^{-1}a \in P$ which is a contradiction. It follows that $P \in V_{(a \vee b);(c \vee d)}$. By a similar argument we have proved the claim. □

By Lemma 2.3, the set $\mathsf{V}(S) = \{V_{s;t} \colon t < s\}$ forms an inverse subsemigroup of $\mathsf{L}(\mathsf{G}(S))$. More generally, let V be an inverse subsemigroup of a distributive inverse semigroup L. Define $V^{\vee}$ to be the set of all elements of L which are compatible joins of finite compatible subsets of V. The proof of the following is straightforward.

Lemma 2.4. *Let V be an inverse subsemigroup of a distributive inverse semigroup L. Then $V^{\vee}$ is a distributive inverse semigroup.*

Definition. In the light of Lemma 2.4, define $\mathsf{B}(S)$ to be the set of all unions within $\mathsf{L}(\mathsf{G}(S))$ of finite compatible subsets of $\mathsf{V}(S)$ and define $\beta\colon S \to \mathsf{B}(S)$ by $\beta(s) = V_s$.

Then $\mathsf{B}(S)$ is a distributive inverse semigroup and β is a morphism of distributive inverse semigroups.

Lemma 2.5. *Let S be a distributive inverse semigroup.*

(1) *Let $b < a$ and $d < c$. If $V_{a;b} \subseteq V_{c;d}$ then $V_{\mathbf{d}(a);\mathbf{d}(b)} \subseteq V_{\mathbf{d}(c);\mathbf{d}(d)}$.*
(2) *Let $f < e$ and $j < i$ be idempotents. Then $V_{e;f} \subseteq V_{i;j}$ if and only if $\mathcal{V}_{e;f} \subseteq \mathcal{V}_{i;j}$.*

Proof. (1) Let $F \in V_{\mathbf{d}(a);\mathbf{d}(b)}$. Then $A = (aF)^{\uparrow} \in V_{a;b}$. Thus $A \in V_{c;d}$. But then $F = \mathbf{d}(A) \in V_{\mathbf{d}(c);\mathbf{d}(d)}$.

(2) The crux of the proof is that if A is a prime filter that contains an idempotent then $A = \mathsf{E}(A)^{\uparrow}$ where $\mathsf{E}(A)$ is a prime filter in the distributive lattice of idempotents. Suppose that $V_{e;f} \subseteq V_{i;j}$. Let $F \in \mathcal{V}_{e;f}$. Then $F^{\uparrow} \in V_{e,f}$. Thus $F^{\uparrow} \in V_{i,j}$. It follows that $F \in \mathcal{V}_{i;j}$. Conversely, suppose that $\mathcal{V}_{e;f} \subseteq \mathcal{V}_{i;j}$. Let $A \in V_{e;f}$. Then $\mathsf{E}(A) \in \mathcal{V}_{e;f}$ and so $\mathsf{E}(A) \in \mathcal{V}_{i;j}$. Thus $A \in V_{i;j}$. ☐

Lemma 2.6. *Let S be a distributive inverse semigroup. If every element of $V_{a;b}$ is an identity prime filter then $V_{a;b} = V_{\mathbf{d}(a);\mathbf{d}(b)}$.*

Proof. If $a = b$ the result is trivial. Let $A \in V_{a,b}$. Then since A is an identity prime filter, $\mathbf{d}(A) = A$. It follows that $\mathbf{d}(a) \in A$. Suppose that $\mathbf{d}(b) \in A$. Then $b = a\mathbf{d}(b) \in A$, since A is an identity. This is a contradiction. Thus $\mathbf{d}(b) \notin A$. Hence $A \in V_{\mathbf{d}(a);\mathbf{d}(b)}$. It follows that $V_{a;b} \subseteq V_{\mathbf{d}(a);\mathbf{d}(b)}$. Let $F \in V_{\mathbf{d}(a);\mathbf{d}(b)}$. Then $A = (aF)^{\uparrow}$ is a prime filter. Clearly, $A \in V_a$. If $b \in A$ then $\mathbf{d}(b) \in F$, which is a contradiction. Thus $A \in V_{a;b}$. But by what we proved above $\mathbf{d}(a) \in A$. Thus there is an idempotent $e \in A$ such that $e \leq \mathbf{d}(a), a$. It follows that $e \in F$ and so $a \in F$. We have proved that $A = F$ and so $F \in V_{a;b}$. ☐

Proposition 2.7. *Let S be a distributive inverse semigroup. Then $\mathsf{B}(S)$ is a Boolean inverse semigroup and β is a morphism.*

Proof. It only remains to prove that the idempotents of $\mathsf{B}(S)$ form a Boolean algebra. The idempotents in $\mathsf{B}(S)$ are those local bisections which are subsets of the identity space of the groupoid $\mathsf{G}(S)$. They are therefore finite joins of elements of the form $V_{a;b}$ where the only elements of $V_{a;b}$ are identity prime filters. By Lemma 2.6, we have that $V_{a;b} = V_{\mathbf{d}(a);\mathbf{d}(b)}$. It now follows by Lemma 2.5 that there is an order isomorphism between $\mathsf{E}(\mathsf{B}(S))$ and $\mathsf{B}(\mathsf{E}(S))$ induced by the map $V_{e;f} \mapsto \mathcal{V}_{e;f}$. That $\mathsf{B}(\mathsf{E}(S))$ is a Boolean algebra can be easily verified, indeed, it is the Booleanization of the distributive lattice $\mathsf{E}(S)$ [5, Proposition II.4.5]. This gives us the result. ☐

Remark 2.8. Let S be a distributive inverse monoid. Then V_1 is the identity of $\mathsf{B}(S)$ and the map β is a monoid homomorphism.

2.2. **Universality.** We have described the semigroup $\mathsf{B}(S)$ in terms of prime filters. To prove the universality of this construction, we need to convert certain results about prime filters into purely algebraic and order-theoretic results. The motivation for doing this came from [28, Section 2.3]. Our first lemma exemplifies our approach.

Lemma 2.9. *Let D be a distributive lattice and let $e, f, i, j \in D$ such that $f < e$ and $j < i$. Then $\mathcal{V}_{e;f} \subseteq \mathcal{V}_{i;j}$ if and only if $e = f \vee (e \wedge i)$ and $f \wedge j = e \wedge j$.*

Proof. Suppose first that $e = f \vee (e \wedge i)$ and $f \wedge j = e \wedge j$. Let $F \in \mathcal{V}_{e;f}$ Then $e \in F$ and $f \notin F$. It follows that $e \wedge i \in F$ and so $i \in F$. Suppose that $j \in F$. Then $e \wedge j \in F$ and so $f \wedge j \in F$ meaning that $f \in F$, which is a contradiction. It follows that $j \notin F$.

To prove the converse, let $\mathcal{V}_{e;f} \subseteq \mathcal{V}_{i;j}$. Clearly, $f \vee (e \wedge i) \leq e$. Suppose that $e \not\leq f \vee (e \wedge i)$. Then there is a prime filter F such that $e \in F$ and $f \vee (e \wedge i) \notin F$. Clearly, $f \notin F$. Thus, by assumption, $i \in F$ and $j \notin F$. But then $e, i \in F$ and so $e \wedge i \in F$ implying that $f \vee (e \wedge i) \in F$, which is a contradiction. It follows that $e = f \vee (e \wedge i)$. Clearly, $f \wedge j \leq e \wedge j$. Suppose that $e \wedge j \not\leq f \wedge j$. Then there is a prime filter F such that $e \wedge j \in F$ and $f \wedge j \notin F$. Clearly, $f \notin F$ and so $F \in \mathcal{V}_{e;f}$. Thus $i \in F$ and $j \notin F$ but this contradicts the fact that $j \in F$. It follows that $e \wedge j = f \wedge j$. □

Lemma 2.10. *Let S be a distributive inverse semigroup and let $a, b, c, d \in S$ be such that $b < a$ and $d < c$.*

1. *$V_{a;b} \subseteq V_{c;d}$ if and only if $\mathcal{V}_{\mathbf{d}(a);\mathbf{d}(b)} \subseteq \mathcal{V}_{\mathbf{d}(c);\mathbf{d}(d)}$ and there exists $x \leq c$ such that $a = b \vee x$.*
2. *$V_{a;b} \subseteq V_{c;d}$ if and only if there exist $d \leq b' < a' \leq c$ such that $V_{a;b} = V_{a';b'}$.*

Proof. (1) We use Proposition 1.18 throughout. Suppose that $V_{a;b} \subseteq V_{c;d}$. Then $V_a \subseteq V_c \cup V_b$. Thus $V_a = (V_a \cap V_c) \cup V_b$. Since $V_a \cap V_c$ is open (and non-empty) we may cover it with sets of the form V_{x_i}, where $i \in I$, and $V_{x_i} \subseteq V_a \cap V_c$. If we adjoin V_b we thereby have a cover of V_a. Since V_a is compact we may write $(V_a \cap V_c) \cup V_b = (\bigcup_{i=1}^{m} V_{x_i}) \cup V_b$ for some finite subset $\{1, \ldots, m\} \subseteq I$. We have that $\bigcup_{i=1}^{m} V_{x_i} \subseteq V_a \cap V_c$. Thus the elements x_i are pairwise compatible and so have a join x, say, where $x \leq a, c$. It follows that $V_a = V_x \cup V_b$. Since $x, b \leq a$ the elements x and b are compatible and so $a = x \vee b$. The fact that $\mathcal{V}_{\mathbf{d}(a);\mathbf{d}(b)} \subseteq \mathcal{V}_{\mathbf{d}(c);\mathbf{d}(d)}$ follows by Lemma 2.5.

To prove the converse, suppose that $V_{\mathbf{d}(a);\mathbf{d}(b)} \subseteq V_{\mathbf{d}(c);\mathbf{d}(d)}$ and there exists a non-zero $x \leq c$ such that $a = b \vee x$. Let $A \in V_{a;b}$. Then by Lemma 2.5, we have that $\mathbf{d}(A) \in V_{\mathbf{d}(a);\mathbf{d}(b)}$ and so by $\mathbf{d}(A) \in V_{\mathbf{d}(c);\mathbf{d}(d)}$. By assumption, $a \in A$ and so since $b \notin A$ we have that $x \in A$ and so $c \in A$. Suppose that $d \in A$. Then $\mathbf{d}(d) \in \mathbf{d}(A)$ which is a contradiction. It follows that $A \in V_{c;d}$.

(2) By part (1) there exists $x \leq c$ such that $a = b \vee x$. The meet $b \wedge x$ exists since $b, x \leq a$ by Lemma 1.4. Likewise, the meet $d \wedge x$ exists since $d, x \leq c$. But $b \wedge x, d \wedge x \leq x$ and so $(b \wedge x) \vee (d \wedge x)$ exists. It is routine to check that

$$V_{a;b} = V_{x;(x \wedge b) \vee (x \wedge d)}.$$

Observe that $x \leq c$. In what follows, we may therefore assume, without loss of generality, that in fact $a \leq c$. It is immediate that $b' = b \vee d \leq a \vee d = a'$ with both joins existing. It is routine to check that

$$V_{a;b} = V_{a';b'}.$$

But

$$d \leq b' = b \vee d < a' = a \vee d \leq c.$$

The proof of the converse is immediate. □

The following result is the first step in proving universality.

Proposition 2.11. *Let S be a distributive inverse semigroup and let $\alpha \colon S \to T$ be a morphism to a distributive inverse semigroup T.*

1. *If $V_{a;b} = V_{c;d}$ then $V_{\alpha(a);\alpha(b)} = V_{\alpha(c);\alpha(d)}$.*
2. *If $V_{a;b} \subseteq V_{c;d}$ then $V_{\alpha(a);\alpha(b)} \subseteq V_{\alpha(c);\alpha(d)}$.*

Proof. (1) Let $V_{a;b} = V_{c;d}$. Then by Lemma 2.10. $V_{\mathbf{d}(a);\mathbf{d}(b)} = V_{\mathbf{d}(c);\mathbf{d}(d)}$ and there exists $x \leq c$ such that $a = b \vee x$ and $y \leq a$ such that $c = d \vee y$ By Lemma 2.5 we have that $\mathcal{V}_{\mathbf{d}(a);\mathbf{d}(b)} = \mathcal{V}_{\mathbf{d}(c);\mathbf{d}(d)}$. Lemma 2.9 tells us that $\mathcal{V}_{\mathbf{d}(a);\mathbf{d}(b)} = \mathcal{V}_{\mathbf{d}(c);\mathbf{d}(d)}$ is equivalent to a purely lattice-theoretic conditions. Thus, since α is a morphism of distributive lattices, we have that $\mathcal{V}_{\alpha(\mathbf{d}(a));\alpha(\mathbf{d}(b))} = \mathcal{V}_{\alpha(\mathbf{d}(c));\alpha(\mathbf{d}(d))}$. By Lemma 2.5, we therefore have that $V_{\alpha(\mathbf{d}(a));\alpha(\mathbf{d}(b))} = V_{\alpha(\mathbf{d}(c));\alpha(\mathbf{d}(d))}$. From $a = b \vee x$ where $x \leq a, c$ and $c = d \vee y$ where $y \leq a, c$ we get that $\alpha(a) = \alpha(b) \vee \alpha(x)$ where $\alpha(x) \leq \alpha(a), \alpha(c)$ and $\alpha(c) = \alpha(d) \vee \alpha(y)$ where $\alpha(y) \leq \alpha(a), \alpha(c)$. By Lemma 2.10 again, it follows that $V_{\alpha(a);\alpha(b)} = V_{\alpha(c);\alpha(d)}$.

(2) Let $V_{a;b} \subseteq V_{c;d}$. By Lemma 2.10, there exist $d \leq b' < a' \leq c$ such that $V_{a;b} = V_{a';b'}$. But by part (1), $V_{\alpha(a);\alpha(b)} = V_{\alpha(a');\alpha(b')}$. Now $\alpha(d) \leq \alpha(b') < \alpha(a') \leq \alpha(c)$. It is routine to check that $V_{\alpha(a');\alpha(b')} \subseteq V_{\alpha(c);\alpha(d)}$. Thus $V_{\alpha(a);\alpha(b)} \subseteq V_{\alpha(c);\alpha(d)}$, as required. □

Lemma 2.12. *Let S be a distributive inverse semigroup and let $\alpha \colon S \to T$ be a morphism to a Boolean inverse semigroup T. Then the map $V_{a;b} \mapsto \alpha(a) \setminus \alpha(b)$ is well-defined.*

Proof. Suppose that $V_{a;b} = V_{c;d}$. Then by Proposition 2.11, we have that $V_{\alpha(a);\alpha(b)} = V_{\alpha(c);\alpha(d)}$. Thus by part (5) of Lemma 2.2 together with the fact that T is a Boolean inverse semigroup, we have that $\alpha(a) \setminus \alpha(b) = \alpha(c) \setminus \alpha(d)$. □

Remark 2.13. Observe in the above lemma that if $V_{a;b} = \varnothing$ then $a \setminus b = 0$ by Lemma 2.1.

Lemma 2.14. *Let S be a distributive inverse semigroup and let $\alpha \colon S \to T$ be a morphism to a Boolean inverse semigroup T. Suppose that $V_{a;b} = V_{a;b}^2 = V_{a^2;ba \vee ab}$ Then $\alpha(a) \setminus \alpha(b)$ is an idempotent in T.*

Proof. By Lemma 2.12 we have that $\alpha(a) \setminus \alpha(b) = \alpha(a^2) \setminus \alpha(ba \vee ab)$. But α is a morphism and so $\alpha(a^2) \setminus \alpha(ba \vee ab) = \alpha(a)^2 \setminus (\alpha(b)\alpha(a) \vee \alpha(a)\alpha(b)) = (\alpha(a) \setminus \alpha(b))^2$. It follows that $\alpha(a) \setminus \alpha(b)$ is an idempotent. □

The following is an important result.

Lemma 2.15. *Let S be a distributive inverse semigroup and let $\alpha \colon S \to T$ be a morphism to a Boolean inverse semigroup T. If $V_{a;b} \sim V_{c;d}$ in $\mathsf{B}(S)$ then $(a \setminus b) \sim (c \setminus d)$.*

Proof. By assumption, $V_{a;b}^{-1} V_{c;d}$ and $V_{a;b} V_{c;d}^{-1}$ are both idempotents in $\mathsf{B}(S)$. We concentrate on $V_{a;b}^{-1} V_{c;d}$ since the proofs for $V_{a;b} V_{c;d}^{-1}$ will be similar. By Lemma 2.3, we have that $V_{a;b}^{-1} V_{c;d} = V_{a^{-1}c; a^{-1}d \vee b^{-1}c}$. By assumption, this is an idempotent and so $\alpha(a^{-1}c) \setminus \alpha(a^{-1}d \vee b^{-1}c)$ is also an idempotent by Lemma 2.14. Using the fact that α is a morphism, this element becomes $\alpha(a)^{-1}\alpha(c) \setminus (\alpha(a)^{-1}\alpha(d) \vee \alpha(b)^{-1}\alpha(c))$ which is equal to $(\alpha(a) \setminus \alpha(b))^{-1}(\alpha(c) \setminus \alpha(d))$ by Lemma 2.2. The result now follows. □

Lemma 2.16. *Let S be a distributive inverse semigroup. Let $a, b, a_i, b_i \in S$, where $1 \leq i \leq m$, be such that $b \leq b_i \leq a_i \leq a$, and where $\{V_{a_i;b_i} \colon 1 \leq i \leq m\}$ is a compatible subset of $\mathsf{B}(S)$. Then*

$$V_{a;b} = \bigcup_{i=1}^{m} V_{a_i;b_i}$$

if and only if the following three conditions hold:

(1) $a = \bigvee_{i=1}^{m} a_i$.
(2) $b = \bigwedge_{i=1}^{m} b_i$, *a compatible meet.*

(3) *For all* $X \subseteq \{1, \ldots, m\} = Y$*, where* $X \neq \emptyset$ *and* $X \neq Y$*, we have that*

$$\bigwedge_{i \in X} b_i \leq \bigvee_{j \in Y \setminus X} a_j,$$

where $\bigwedge_{i \in X} b_i$ *is a compatible meet.*

Proof. Observe that since $a_i, b_i \leq a$ for all $1 \leq i \leq m$ the set $\{a_i, b_i \colon 1 \leq i \leq m\}$ is compatible. Thus all joins and meets which are needed actually exist.

Suppose first that $V_{a;b} = \bigcup_{i=1}^m V_{a_i;b_i}$. We prove that (1), (2) and (3) all hold. (1) holds. Clearly $a \geq \bigvee_{i=1}^m a_i$. To prove the reverse inequality, let P be a prime filter that contains a but omits $\bigvee_{i=1}^m a_i$. Then P must omit b. Thus $P \in V_{a;b}$. It follows that $a_i \in P$ for some i but this is a contradiction.

The proof that (2) holds is similar.

We now show that (3) holds. Suppose that $X \subseteq Y$ where $X \neq \emptyset$ and $X \neq Y$. Let P be a prime filter that contains $\bigwedge_{i \in X} b_i$ but omits $\bigvee_{j \in Y \setminus X} a_j$. Let $i \in X$. Then $b_i \in P$ and so $a \in P$. But if $b \in P$ then all a_i would be P. Thus P contains a and omits b. Thus $P \in V_{a;b}$. It follows that P contains a_k but omits b_k, for some k. Now $k \notin Y \setminus X$ and so $k \in X$. But then by assumption $b_k \in P$ which is a contradiction.

We now assume that (1), (2) and (3) all hold. We prove that $V_{a;b} = \bigcup_{i=1}^m V_{a_i;b_i}$. Let $P \in V_{a;b}$. Then P contains a and omits b. By (1), P must contain at least one a_i and by (2), P must omit at least one b_j. Let $X \subseteq \{1, \ldots, m\} = Y$ be the proper, non-empty subset of *all* $i \in Y$ such that $b_i \in P$. Then $\bigwedge_{i \in X} b_i \in P$ and so by (3), we must have that $\bigvee_{j \in Y \setminus X} a_j \in P$. Thus for some j we have that $a_j \in P$. But, by assumption, $b_j \notin P$. It follows that $P \in V_{a_j;b_j}$. That the reverse inclusion holds is immediate. ☐

Proposition 2.17. *Let* $\alpha \colon S \to T$ *be a morphism between distributive inverse semigroups. Define* $\gamma \colon \mathsf{B}(S) \to \mathsf{B}(T)$ *by*

$$\beta\left(\bigcup_{i=1}^m V_{a_i;b_i}\right) = \bigcup_{i=1}^m V_{\alpha(a_i);\alpha(b_i)}$$

where $\{V_{a_i;b_i} \colon 1 \leq i \leq m\}$ *is a compatible subset of* $\mathsf{B}(S)$*. Then* β *is well-defined.*

Proof. The proof in the case $i = 1$ follows by Proposition 2.11.

Next consider the case where

$$V_{a;b} = \bigcup_{i=1}^m V_{a_i;b_i}.$$

By Lemma 2.10, we may assume without loss of generality that $b \leq b_i \leq a_i \leq a$. We now use Lemma 2.16 and Remark 1.5 to deduce that

$$V_{\alpha(a);\alpha(b)} = \bigcup_{i=1}^m V_{\alpha(a_i);\alpha(b_i)}.$$

Finally, suppose that

$$\bigcup_{i=1}^m V_{a_i;b_i} = \bigcup_{j=1}^n V_{c_j;d_j}$$

where $\{V_{a_i;b_i} \colon 1 \leq i \leq m\}$ and $\{V_{c_j;d_j} \colon 1 \leq j \leq n\}$ are compatible subsets of $\mathsf{B}(S)$. Then

$$V_{a_i;b_i} \subseteq \bigcup_{j=1}^n V_{c_j;d_j}$$

for each i. Thus

$$V_{a_i;b_i} = \bigcup_{j=1}^{n} V_{c_j;d_j} V_{a_i;b_i}^{-1} V_{a_i;b_i} = \bigcup_{j=1}^{n} V_{c_j a_i^{-1} a_i; c_j a_i^{-1} b_i \vee c_j b_i^{-1} a_i \vee d_j a_i^{-1} a_i}$$

by Lemma 2.3 making essential use of the fact that we are working in an inverse semigroup. With a small calculation, it is easy to see that

$$V_{c_j a_i^{-1} a_i; c_j a_i^{-1} b_i \vee c_j b_i^{-1} a_i \vee d_j a_i^{-1} a_i} \subseteq V_{c_j;d_j}.$$

Thus by the previous result and part (2) of Proposition 2.11, we deduce that

$$V_{\alpha(a_i);\alpha(b_i)} \subseteq \bigcup_{j=1}^{n} V_{\alpha(c_j);\alpha(d_j)}.$$

Consequently,

$$\bigcup_{i=1}^{m} V_{\alpha(a_i);\alpha(b_i)} \subseteq \bigcup_{j=1}^{n} V_{\alpha(c_j);\alpha(d_j)}.$$

By symmetry we get equality. □

Lemma 2.18. *Let S be a Boolean inverse semigroup. Suppose that*

$$V_{a;b} \subseteq \bigcup_{i=1}^{m} V_{c_i;d_i}$$

where $\{V_{(c_i;d_i)} : 1 \leq i \leq n\}$ is a compatble subset of $\mathsf{B}(S)$. Then

$$a \setminus b \leq \bigvee_{i=1}^{m} (c_i \setminus d_i).$$

Proof. We use Lemma 1.19. We know that $V_{a;b} = \varnothing$ if and only if $a = b$. So, in what follows we may assume that $b < a$. Let P be a prime ideal that contains $a \setminus b$. Then $a \in P$ and $b \notin P$. Thus $P \in V_{a;b}$. By assumption, $P \in V_{c_i;d_i}$ for some i. Thus $c_i \in P$ and $d_i \notin P$. It follows that $c_i \setminus d_i \in P$ and so $\bigvee_{i=1}^{m}(c_i \setminus d_i) \in P$. By part (1) of Proposition 1.18, we have that

$$a \setminus b \leq \bigvee_{i=1}^{m} (c_i \setminus d_i).$$

□

We can now complete the proof of Theorem 1.1.

Proof. Let S be a distributive inverse semigroup and let $\theta \colon S \to T$ be any morphism to a Boolean inverse semigroup T. Define $\gamma \colon \mathsf{B}(S) \to T$ by

$$\gamma\left(\bigcup_{i=1}^{m} V_{a_i;b_i}\right) = \bigvee_{i=1}^{m} \theta(a_i) \setminus \theta(b_i)$$

where $\{V_{a_i;b_i} : 1 \leq i \leq m\}$ is a compatible subset of $\mathsf{B}(S)$. We prove that this function is well-defined. Observe first that the definition of γ at least makes mathematical sense by Lemma 2.15. Now, suppose that

$$\bigcup_{i=1}^{m} V_{a_i;b_i} = \bigcup_{j=1}^{n} V_{c_j;d_j}$$

where $\{V_{c_j;d_j} : 1 \leq j \leq n\}$ is also a compatible subset of $\mathsf{B}(S)$. Then by Proposition 2.17

$$\bigcup_{i=1}^{m} V_{\theta(a_i);\theta(b_i)} = \bigcup_{j=1}^{n} V_{\theta(c_j);\theta(d_j)}.$$

Thus for each i, we have that

$$V_{\theta(a_i);\theta(b_i)} \subseteq \bigcup_{j=1}^{n} V_{\theta(c_j);\theta(d_j)}.$$

By Lemma 2.18, we therefore have that $\theta(a_i) \setminus \theta(b_i) \leq \bigvee_{j=1}^{n}(\theta(c_j); \theta(d_j))$. The result now follows. This is a homomorphism by Lemma 2.3 and a morphism by construction. Clearly, $\theta = \beta\gamma$ and it is immediate that γ is the unique morphism making the diagram of maps commute. □

As a reminder, $\mathsf{B}(S)$, the *Booleanization* of the distributive inverse semigroup S, is defined after Lemma 2.4. The map $\beta \colon S \to \mathsf{B}(S)$ takes the element s to the set V_s of prime filters containing s.

Proposition 2.19. *Let S be an inverse semigroup. Then S is a weak semilattice if and only if $\mathsf{B}(S)$ is a $\wedge$-semigroup.*

Proof. By Lemma 1.16, we may assume that S is distributive. Thus we need to prove that S is a $\wedge$-semigroup if and only if $\mathsf{B}(S)$ is a $\wedge$-semigroup. Observe that to prove $\mathsf{B}(S)$ is a $\wedge$-semigroup, we need only consider intersections of the form $V_{a;b} \cap V_{c;d}$. Suppose that S is a $\wedge$-semigroup. Then it is routine to check that

$$V_{a;b} \cap V_{c;d} = V_{(a\wedge c);((b\wedge c)\vee(a\wedge d))}$$

and makes sense. Conversely, suppose that $V_{a;b} \cap V_{c;d}$ is an element of $\mathsf{B}(S)$. Then $V_{a;b} \cap V_{c;d} = \bigcup_{i=1}^{m} V_{x_i;y_i}$. By Lemma 2.10, we may assume that $b \leq y_i \leq x_i \leq a$ and $d \leq y_i \leq x_i \leq c$ for $1 \leq i \leq m$. Let $a, b \in S$. Then by the above $V_a \cap V_b = \bigcup_{i=1}^{m} V_{x_i;y_i}$. Put $c = \bigvee_{i=1}^{m} x_i$, well-defined since $x_i \leq a, b$ and so $\{x_1, \ldots, x_n\}$ is a compatible subset. We claim that $V_a \cap V_b = V_c$. To prove this, let $a, b \in P$, a prime filter. Then $x_i \in P$ for some i. Thus $c \in P$. Conversely, suppose that $c \in P$, a prime filter. Then $x_i \in P$ for some i. Thus $a, b \in P$. We finish off by proving that $a \wedge b$ exists and equals c. Clearly, $c \leq a, b$. Let $d \leq a, b$. Then $V_d \subseteq V_a, V_b$. Thus $V_d \subseteq V_c$. It follows by part (2) of Proposition 1.18 that $d \leq c$. We have therefore proved that all binary meets exist in S. □

2.3. **Paterson's universal groupoid.** In this section, we shall complete the description of the connection between the Booleanization of an inverse semigroup and Paterson's universal groupoid first discussed in [16, Section 5.1]. Paterson described his universal groupoid in [20, Sections 4.3, 4,4], the significant part of this being [20, Proposition 4.4.2] where he proves that the inverse semigroup he denotes by S'' is isomorphic to the Boolean inverse semigroup associated with his universal groupoid. There are two steps in constructing S''. The first is the construction of what he calls S' [20, Page 176] which is isomorphic to the inverse semigroup $\mathsf{V}(S)$ defined prior to Lemma 2.4; our semigroup appears simpler than the one described by Paterson because we are able to assume that our source inverse semigroup is distributive. The second is the construction of the semigroup S'', defined in [20, page 190], which is the semigroup $\mathsf{V}(S)^{\vee} = \mathsf{B}(S)$. Observe that the construction of Paterson's groupoid has been increasingly viewed from a more algebraic perspective [7, 8, 19, 15].

Let S be an inverse semigroup with zero. We know that $\mathsf{B}(S) \cong \mathsf{B}(\mathsf{D}(S))$. The first step, therefore, is to relate the prime filters in $\mathsf{D}(S)$ to, what will turn out to be, the filters in S.

Lemma 2.20. *Let S be an inverse semigroup with zero. Then there is an order isomorphism between the proper filters of S and the prime filters of $\mathsf{D}(S)$.*

Proof. Let F be a proper filter in S. Define

$$F^u = \{A \in \mathsf{D}(S) \colon A \cap F \neq \varnothing\}.$$

We prove that F^u is a prime filter in $\mathsf{D}(S)$. Let $A, B \in F^u$. Then $a \in A \cap F$ and $b \in B \cap F$ for some a and some b. In particular, $a, b \in F$ and so, by assumption, there is a non-zero element $c \in F$ such that $c \leq a, b$. Now $c^\downarrow \in \mathsf{D}(S)$ and $c^\downarrow \subseteq A, B$, since both A and B are order ideals. It is clear that F^u is closed upwards. Thus F^u is a proper filter (proper since it cannot contain $0^\downarrow$). It remains to show that it is prime. Let $A \cup B \in F^u$, where A and B are compatible elements of $\mathsf{D}(S)$. Then $F \cap (A \cup B) \neq \varnothing$. It follows that $F \cap A \neq \varnothing$ or $B \cap B \neq \varnothing$ and so $A \in F^u$ or $B \in F^u$, as required.

Let F_1 and F_2 be proper filters of S such that $F_1 \subseteq F_2$. Then it is immediate that $F_1^u \subseteq F_2^u$.

Let P be a prime filter in $\mathsf{D}(S)$. Define

$$P^d = \{s \in S \colon s^\downarrow \in P\}.$$

We prove that P^d is a proper filter in S. Observe first that since P is a prime filter $0^\downarrow \notin P$. Thus $0 \notin P^d$. Let $s, t \in P^d$. Then $s^\downarrow, t^\downarrow \in P$. But P is a prime filter and so there is an element $A \in P$ such that $A \subseteq s^\downarrow, t^\downarrow$. Now if $A = \{a_1, \ldots, a_m\}^\downarrow$ then $A = \bigvee_{i=1}^m a_i^\downarrow$. But P is a prime filter and so $a_i^\downarrow \in P$ for some i. But then $a_i \in P^d$ and $a_i \leq s, t$. Let $s \in P^d$ and $s \leq t$. Then $s^\downarrow \in P$ but $s^\downarrow \subseteq t^\downarrow$ and so $t^\downarrow \in P$ giving $t \in P^d$. We have therefore proved that P^d is a proper filter.

Let P_1 and P_2 be prime filters in $\mathsf{D}(S)$ such that $P_1 \subseteq P_2$. Then it is immediate that $P_1^d \subseteq P_2^d$.

It remains to iterate these two constructions. Let F be a proper filter in S. We prove that $(F^u)^d = F$. Observe that

$$s \in (F^u)^d \Leftrightarrow s^\downarrow \in F^u \Leftrightarrow s^\downarrow \cap F \neq \varnothing \Leftrightarrow s \in F,$$

where we use the fact that F is a filter. Let P be a prime filter in $\mathsf{D}(S)$. We prove that $(P^d)^u = P$. Observe that

$$A \in (P^d)^u \Leftrightarrow A \cap P^d \neq \varnothing \Leftrightarrow s^\downarrow \subseteq A \text{ for some } s^\downarrow \in P \Leftrightarrow A \in P.$$

□

Let S be an inverse semigroup with zero. Denote by $\mathcal{L}(S)$ the set of all proper filters of S. This set becomes a groupoid in the following way. If A is a proper filter define $\mathbf{d}(A) = (A^{-1}A)^\uparrow$ and $\mathbf{r}(A) = (AA^{-1})^\uparrow$; both $\mathbf{d}(A)$ and $\mathbf{r}(A)$ are proper filters. Define $A \cdot B = (AB)^\uparrow$ if and only if $\mathbf{d}(A) = \mathbf{r}(A)$. Then with this partial binary operation, the set $\mathcal{L}(S)$ is a groupoid. The identities are precisely the proper filters that contain idempotents. The order isomorphism of Lemma 2.20 can be extended to an isomorphism of groupoids.

Lemma 2.21. *Let S be an inverse semigroup with zero. Then $F \mapsto F^u$ defines a functor from $\mathcal{L}(S)$ to $\mathsf{G}(\mathsf{D}(S))$, $P \mapsto P^d$ defines a functor from $\mathsf{G}(\mathsf{D}(S))$ to $\mathcal{L}(S)$ and these functors are mutually inverse.*

Proof. We prove that $F \mapsto F^u$ is a functor. If F is an identity filter then every element of F lies above an idempotent of F. We prove that F^u contains an identity which is enough to show that F^u is an identity. Let $A \in F^u$. Then $A \cap F \neq \varnothing$. Then there exists $a \in A$ and $a \in F$. But A is an order ideal and a is above an idempotent, e say, in F. Thus $e \in A$ and $e \in F$. It follows that $A \in F^u$ contains an idempotent and so is an identity. □

Lemma 2.22. *Let S be an inverse semigroup with zero. For each $a \in S$ define U_a to be the set of all proper filters that contain a. Put $\sigma = \{U_a \colon a \in S\}$.*

(1) $U_0 = \varnothing$.
(2) $U_a = U_b$ *if and only if* $a = b$.
(3) $U_a^{-1} = U_{a^{-1}}$.
(4) $U_a U_b = U_{ab}$.
(5) U_a *is a local bisection.*
(6) $U_a \cap U_b = \bigcup_{x \leq a,b} U_x$.

Proof. (1) Immediate. (2) This follows by the fact that $a^{\uparrow} \in U_a$. (3) Straightfoward. (4) Let $a \in A$ and $b \in B$, where A and B are filters. Then from the definition of $A \cdot B$ we have that $ab \in A \cdot B$. It follows that $U_a U_b \subseteq U_{ab}$. To prove the reverse inclusion, let $ab \in C$, a filter. Since $\mathbf{d}(ab) \leq \mathbf{d}(b)$ and so $B = (b\mathbf{d}(C))^{\uparrow}$ is a well-defined filter. Since $b\mathbf{d}(ab) \in B$ we have that $\mathbf{d}(a)\mathbf{r}(b) \in \mathbf{r}(B)$. It follows that $\mathbf{d}(a) \in \mathbf{r}(B)$ and so $A = (a\mathbf{r}(B))^{\uparrow}$ is a well-defined filter. Clearly, $A \in U_a$, $B \in U_b$ and $C = A \cdot B$. (5) This is immediate from the standard properties of filters. (6) Straightfoward. □

We therefore have an injective homomorphism $\upsilon \colon S \to \mathsf{L}(\mathcal{L}(S))$. The identities of $\mathcal{L}(S)$ are the filters that contain idempotents (which is equivalent to saying that the filter is an inverse subsemigroup). Let $a \in S$ and $a_1, \ldots, a_m$. Define

$$U_{a;a_1,\ldots,a_m} = U_a \cap U_{a_1}^c \cap \ldots \cap U_{a_m}^c.$$

Clearly, $U_{a;a_1,\ldots,a_m}$ is a local bisection and so an element of $\mathsf{L}(\mathcal{L}(S))$. The proof of the following is immediate.

Lemma 2.23. *Let S be an inverse semigroup. Then*

$$U_{a;a_1,\ldots,a_m} = U_{a;a_1} \cap \ldots \cap U_{a;a_m}$$

where the intersection is a compatible meet in the inverse semigroup since $U_{a;a_i} \subseteq U_a$.

Define Ω to be the set of all sets of the form $U_{a;a_1,\ldots,a_m}$. It is easy to check that σ is a basis for a topology on $\mathcal{L}(S)$. In fact, using arguments similar to those in [16, Lemma 2.7, Proposition 2.8], it follows that $\mathcal{L}(S)$ is an étale topological groupoid.

If S is now a distributive inverse semigroup then the set $\pi = \{V_a \colon a \in S\}$ forms the basis of a topology for $\mathsf{G}(S)$ which makes it an étale topological groupoid. The functor of Lemma 2.21 is actually continuous and so links the filter topology constructed from S with the prime filter topology constructed from $\mathsf{D}(S)$.

Proposition 2.24. *Let S be an inverse semigroup with zero. Then the étale topological groupoids $\mathcal{L}(S)$ and $\mathsf{G}(\mathsf{D}(S))$ are isomorphic under the maps defined in Lemma 2.21.*

Proof. The basic open set V_s is mapped to the basic open set $V_{s^{\downarrow}}$ and the inverse image of $V_{s^{\downarrow}}$ is V_s. □

Recall that a *(locally compact) Boolean space* is a 0-dimensional, locally compact Hausdorff space. An étale topological groupoid is said to be *Boolean* if its identity space is a Boolean space.[4] The topologies on the groupoids $\mathcal{L}(S)$ and $\mathsf{G}(\mathsf{D}(S))$ now have to be refined in order that both groupoids become Boolean groupoids.

Let $\mathbf{s}, \mathbf{t} \in \mathsf{D}(S)$ where $\mathbf{t} \leq \mathbf{s}$ in $\mathsf{D}(S)$. We shall describe the set $V_{\mathbf{s};\mathbf{t}}$ explicitly. Suppose that $\mathbf{s} = \{s_1, \ldots, s_m\}^{\downarrow}$ and $\mathbf{t} = \{t_1, \ldots, t_n\}^{\downarrow}$. In $\mathsf{D}(S)$, we have that $\mathbf{s} = \bigvee_{i=1}^m s_i^{\downarrow}$ and $\mathbf{t} = \bigvee_{j=1}^n t_j^{\downarrow}$. By repeated application of part (3) of Lemma 2.3, we may restrict out attention to the sets of the form $V_{s^{\downarrow};t_1^{\downarrow} \vee \ldots \vee t_n^{\downarrow}}$.

[4]These are frequently referred to in the literature as *ample groupoids*. We prefer our term because it makes clear that we are generalizing classical Stone duality.

We now describe Paterson's universal groupoid[5] $\mathsf{G}_u(S)$ of the inverse semigroup S. The underlying groupoid is still $\mathcal{L}(S)$ but a different topology is defined using Ω as a basis. With respect to the topology with basis Ω, the groupoid is called the *universal groupoid* and is denoted by means of $\mathsf{G}_u(S)$ where 'u' stands for 'universal'.

Define the topological groupoid $\mathsf{G}(\mathsf{D}(S))^{\dagger}$ to have the same underlying groupoid as $\mathsf{G}(\mathsf{D}(S))$ but with the topology having the basis the sets $V_{s;t}$. We call this the *patch topology.*[6]

Proposition 2.25. *Let S be an inverse semigroup with zero. The universal groupoid $\mathsf{G}_u(S)$ is isomorphic to $\mathsf{G}(\mathsf{D}(S))^{\dagger}$, meaning that the groupoids are algebraically isomorphic and topologically homeomorphic.*

Proof. We shall use the same groupoid isomorphism as in Proposition 2.24 derived from Lemma 2.21. Observe that the basic open set $U_{s;s_1,\ldots,s_m}$ is mapped to the basic open set $V_{s\downarrow;\{s_1,\ldots,s_m\}\downarrow}$ and that the inverse image of $V_{s\downarrow;\{s_1,\ldots,s_m\}\downarrow}$ is $U_{s;s_1,\ldots,s_m}$. □

At this point, we apply non-commutative Stone duality [16]. If G is a Boolean groupoid then the set of all compact-open local bisections of G, denoted by $\mathsf{KB}(G)$, is a Boolean inverse semigroup. The following theorem establishes the exact connection between the Booleanization of an inverse semigroup and its associated universal groupoid, the bridge between the two being provided by Proposition 2.25.

Theorem 2.26. *Let S be an inverse semigroup with zero. Then $\mathsf{B}(S) \cong \mathsf{KB}(\mathsf{G}_u(S))$.*

2.4. **Computations.** In this section, we describe how to compute the Booleanization of a distributive inverse semigroup in a practical way. Let B be a Boolean inverse semigroup. Let D be an inverse subsemigroup of B where D is distributive in its own right. We are interested in the way that D sits inside B. For clarity, denote the join in D by $\vee$ and the join in B by $\cup$. Let $a, b \in D$ be compatible. Thus both $a \vee b \in D$ and $a \cup b \in B$ exist. By definition, $a \cup b \leq a \vee b$ but there is no reason for them to be equal. For this to be the case, $(\mathsf{E}(D), \vee)$ must be a subalgebra of $(\mathsf{E}(B), \vee)$. This leads us to the following definition. Let B be a Boolean inverse semigroup and let D be an inverse subsemigroup of B which is distributive. We say that D is a *distributive subalgebra* of B if the distributive lattice $\mathsf{E}(D)$ is a subalgebra, with respect to meets, joins and bottom, of the Boolean algebra $\mathsf{E}(B)$. In this case, joins in D are identical to joins in B. With meets, we have to be more careful but the only meets we shall be interested in are the compatible ones which are constructed purely algebraically by Lemma 1.4. By Lemma 2.2 and Boolean algebra, we have the following.

Lemma 2.27. *Let S be a Boolean inverse semigroup and let D be an inverse subsemigroup distributive in its own right.*

1. *Put D' equal to all elements of S of the form $a \setminus b$ where $b \leq a$ and $a, b \in D$. Then D' is an inverse subsemigroup of S.*
2. *Put D'' equal to all joins of compatible, finite subsets of D'. Then D'' is a Boolean inverse semigroup.*

Our goal now is to determine what conditions need to be imposed to ensure that D'' is in fact equal to $\mathsf{B}(D)$, the Booleanization of D. Let D be a distributive subalgebra of the Boolean inverse semigroup S. Put $\mathsf{B}_S(D) = D''$ using the above notation. We call this the *Boolean hull* of D in S.

Theorem 2.28. *Let D be a distributive subalgebra of the Boolean inverse semigroup S. Then the Boolean hull of D in S is isomorphic to the Booleanization of D.*

[5]We have modified Paterson's construction in the obvious way to deal with the case where the inverse semigroup has a zero.

[6]Compare with [5].

Proof. By Theorem 1.1 and its proof, there is a morphism $\gamma\colon \mathsf{B}(S) \to \mathsf{B}_S(D)$ given by

$$\gamma\left(\bigcup_{i=1}^{m} V_{a_i;b_i}\right) = \bigvee_{i=1}^{m} a_i \setminus b_i$$

where $\{V_{a_i;b_i} \colon 1 \leq i \leq m\}$ is a compatible subset of $\mathsf{B}(S)$. From the construction of $\mathsf{B}_S(D)$ this morphism is surjective and so it just remains to prove that it is injective to prove the theorem. The crux of the proof is to show that if

$$a \setminus b \leq \bigvee_{i=1}^{m} a_i \setminus b_i$$

then

$$V_{a;b} \subseteq \bigcup_{i=1}^{m} V_{a_i;b_i}.$$

The important point to remember is that the first inequality holds in S whereas the second in D. Observe that $\mathbf{d}(a \setminus b) = \mathbf{d}(a) \setminus \mathbf{d}(b)$. Now

$$(a_i \setminus b_i)(\mathbf{d}(a) \setminus \mathbf{d}(b)) = (a_i\mathbf{d}(a)) \setminus (a_i\mathbf{d}(b) \vee b_i\mathbf{d}(a)).$$

Put $a_i' = a_i\mathbf{d}(a)$ and $b_i' = \mathbf{d}(b) \vee b_i\mathbf{d}(a)$. Observe that by our assumption $a_i', b_i' \in D$. Thus

$$a \setminus b = \bigvee_{i=1}^{m} a_i' \setminus b_i'.$$

By Lemma 2.2 and Lemma 2.10, we may assume that $b \leq b_i' \leq a_i' \leq a$. We need to be careful about notation in what follows. Let $s \in S$. We shall write V_s^S for the set of all prime filters in S that contain s. By Lemma 2.2, we have that

$$V_{a;b}^S = \bigcup_{i=1}^{m} V_{a_i';b_i'}^S.$$

By Lemma 2.16 this translates into results about joins and compatible meets for elements of B and so are equal to joins and compatible meets in D since D is a distributive subalgebra of S. Thus applying Lemma 2.16 in the opposite direction gives us $V_{a;b} = \bigcup_{i=1}^{m} V_{a_i';b_i'}$. But $V_{a_i';b_i'} \subseteq V_{a_i;b_i}$ and from this our result follows. ☐

3. Applications and examples

3.1. **Representations of inverse semigroups in rings.** Observe that in this section, we deal with monoids; the extension to semigroups is straightforward.

Marshall H. Stone, a functional analyst, became interested in Boolean algebras through his work on the spectral theory of symmetric operators which in turn led to an interest in algebras of commuting projections. Such algebras are naturally Boolean algebras: in fact, Stone proved that Boolean algebras and Boolean rings[7] were two different ways of viewing the same class of structures [27]. Slightly more generally, Foster [3] proved that the set of idempotents of any commutative ring was a Boolean algebra when the following definitions were made: $e \vee f = e + f - ef$, $e \wedge f = e \cdot f$ and $e' = 1 - e$. In this section, we shall be interested in inverse semigroups as subsemigroups of the multiplicative monoids of rings; in particular, inverse semigroups as subsemigroups of the multiplicative monoids-with-involution of C^*-algebras. We begin with a simple lemma.

Lemma 3.1. *Let S be an inverse submonoid (with zero) of the multiplicative monoid of a ring R. Suppose that the following two conditions hold:*

[7] A Boolean ring is a ring in which every element is an idempotent. A simple exercise shows that such rings are always commutative.

(1) If $a, b \in S$ are orthogonal then $a + b \in S$.
(2) If $e \in S$ is an idempotent then $1 - e \in S$.

Then S is a Boolean inverse monoid.

Proof. Let e and f be orthogonal idempotents in S. We prove first that $e \vee f$ exists in S and equals $e + f$. Clearly, $e + f$ is an idempotent and belongs to S by assumption. Observe that $e(e+f) = e$ and $f(e+f) = f$. Thus $e, f \leq e+f$. Suppose that $e, f \leq i$, where i is an idempotent in S. Then $i(e + f) = ie + if = e + f$. Thus $e + f \leq i$. We have therefore proved that $e \vee f = e + f$. Now let a and b be orthogonal elements of S. We prove that $a \vee b$ exists in S and is equal to $a + b$. Put $c = a + b$. Then $ca^{-1}a = a$ and $cb^{-1}b = b$. Thus $a, b \leq c$. But $\mathbf{d}(c) = \mathbf{d}(a) + \mathbf{d}(b) = \mathbf{d}(a) \vee \mathbf{d}(b)$. It follows that $a \vee b = a + b$. We have therefore shown that S has all binary orthogonal joins and multiplication distributes over such joins.

We now prove that $\mathsf{E}(S)$ is a Boolean algebra. Let $e, f \in \mathsf{E}(S)$. Define $e \circ f = e + f - ef$ but $e + f - ef = e(1 - f) + f$ and $e(1 - f)$ and f are orthogonal. It follows that $\mathsf{E}(S)$ is closed under the binary operation $\circ$. Let e and f be arbitrary idempotents. We prove that $e \vee f$ exists and equals $e \circ f$. Observe that $e(e \circ f) = e$ and $f(e \circ f) = f$ so that $e, f \leq e \circ f$. Now let $e, f \leq i$. It is easy to check that $i(e \circ f) = e \circ f$. Thus $e \circ f \leq i$. We have therefore proved that $e \vee f = e \circ f$. It is clear that $e(i \vee j) = ei \vee ej$ and it is easy to show that $e \vee (ij) = (e \vee i)(e \vee j)$. It is now routine to prove that $\mathsf{E}(S)$ is a Boolean algebra.

The lemma now follows by an application of Proposition 1.8. □

Our first main result is a slight generalization of a construction to be found in [20, pp 175–176, pp 190–193] although our proof is completely algebraic and there is no appeal to [30]. In the proof below, the construction of S' deals with part (2) of Lemma 3.1 and that of S'' deals with part (1) of Lemma 3.1.

Proposition 3.2. *Let S be an inverse submonoid (with zero) of the multiplicative monoid of a ring R. Then there is a Boolean inverse submonoid S'' such that $S \subseteq S'' \subseteq R$.*

Proof. Observe first that if e is an idempotent then $1 - e$ is an idempotent and if $ef = fe$ then $e(1 - f) = (1 - f)e$. Define

$$E' = \{e(1 - e_1) \dots (1 - e_n) \colon e, e_1, \dots, e_n \in \mathsf{E}(S)\} \cup \mathsf{E}(S).$$

It is easy to see that E' is a commutative idempotent subsemigroup of R containing $\mathsf{E}(S)$. In addition, E' is closed under conjugation by elements of S; to prove this, let

$$\mathbf{e} = e(1 - e_1) \dots (1 - e_m)$$

and $s \in S$. Then

$$s^{-1}\mathbf{e}s = s^{-1}e(1 - e_1) \dots (1 - e_m)s.$$

But

$$s^{-1}e(1 - e_1) \dots (1 - e_m)s = s^{-1}es \cdot s^{-1}(1 - e_1)s \cdot \dots \cdot s^{-1}(1 - e_m)s$$

whereas $s^{-1}(1 - i)s = s^{-1}s - s^{-1}is = s^{-1}s(1 - s^{-1}is)$. The claim now follows.

Put $S' = SE'$. Let $\mathbf{s} = se(1 - e_1) \dots (1 - e_m)$. Then $\mathbf{s} = s(s^{-1}se)(1 - e_1) \dots (1 - e_m)$. Thus we may assume, whenever convenient, that $e \leq s^{-1}s$. Next, $e(1 - e_1) = e - ee_1 = e - e(ee_1) = e(1 - ee_1)$. It follows that we may also assume, whenever convenient, that $e_1, \dots, e_m \leq e$. We prove that S' is an inverse semigroup with semilattice of idempotents E'.

First, we prove closure under multiplication. Let $\mathbf{s} = se(1 - e_1) \dots (1 - e_m)$ and $\mathbf{t} = tf(1 - f_1) \dots (1 - f_n)$. Then $\mathbf{st} = se(1 - e_1) \dots (1 - e_m)tf(1 - f_1) \dots (1 - f_n)$. Write $t = tt^{-1}t$. Then $\mathbf{st} = st[t^{-1}e(1 - e_1) \dots (1 - e_m)t]f(1 - f_1) \dots (1 - f_n)$. But we proved above that E' is closed under conjugation by elements of S. It follows that S' is closed under multiplication.

Let $\mathbf{s} = se(1-e_1)\dots(1-e_m)$ and define $\mathbf{s}^{-1} = e(1-e_1)\dots(1-e_m)s^{-1}$. Then $\mathbf{s}^{-1} = e(1-e_1)\dots(1-e_m)s^{-1} = s^{-1}[se(1-e_1)\dots(1-e_m)s^{-1}]$ and we now use the fact that E' is closed under conjugation by elements of S. It follows that if $\mathbf{s} \in S'$ then $\mathbf{s}^{-1} \in S'$. It is easy to check that $\mathbf{s} = \mathbf{s}\mathbf{s}^{-1}\mathbf{s}$ and $\mathbf{s}^{-1} = \mathbf{s}^{-1}\mathbf{s}\mathbf{s}^{-1}$.

Thus S' is a regular semigroup.

To prove that S' is inverse it is enough to prove that $\mathsf{E}(S') = E'$ since a regular semigroup is inverse if its idempotents commute. Let $\mathbf{s} = se(1-e_1)\dots(1-e_m)$ and suppose that $\mathbf{s}^2 = \mathbf{s}$. As we indicated above, we may assume that $e \leq s^{-1}s$ and that $e_1,\dots,e_m \leq e$. We prove that $\mathbf{s} \in E'$. By assumption,

$$se(1-e_1)\dots(1-e_m) = se(1-e_1)\dots(1-e_m)se(1-e_1)\dots(1-e_m).$$

Thus multiplying this equation on the left by s^{-1} we obtain

$$e(1-e_1)\dots(1-e_m) = e(1-e_1)\dots(1-e_m)se(1-e_1)\dots(1-e_m).$$

Now write $s = (ss^{-1})s$ and move the ss^{-1} to the front to get

$$e(1-e_1)\dots(1-e_m) = s[s^{-1}e(1-e_1)\dots(1-e_m)s]e(1-e_1)\dots(1-e_m).$$

It follows from this equation that

$$e(1-e_1)\dots(1-e_m) = [s^{-1}e(1-e_1)\dots(1-e_m)s]e(1-e_1)\dots(1-e_m).$$

Thus

$$e(1-e_1)\dots(1-e_m) = se(1-e_1)\dots(1-e_m) = \mathbf{s}.$$

We have therefore proved that S' is an inverse semigroup in its own right.

Now define $S'' \subseteq R$ to be the set of all finite sums of orthogonal elements of S'. If $\{a_1,\dots,a_m\}$ and $\{b_1,\dots,b_n\}$ are orthogonal subsets of an inverse semigroup so too is $\{a_1b_1,\dots a_ib_j,\dots,a_mb_n\}$. It follows that S'' is closed under multiplication. If $\{a_1,\dots,a_m\}$ is an orthogonal subset of an inverse semigroup so too is $\{a_1^{-1},\dots,a_m^{-1}\}$. Thus if $a_1+\dots+a_m \in S''$ then $a_1^{-1}+\dots+a_m^{-1} \in S''$. Observe that

$$(a_1+\dots+a_m)(a_1^{-1}+\dots+a_m^{-1}) = a_1a_1^{-1}+\dots+a_ma_m^{-1}$$

and

$$(a_1^{-1}+\dots+a_m^{-1})(a_1+\dots+a_m) = a_1^{-1}a_1+\dots+a_m^{-1}a_m$$

and so $(a_1+\dots+a_m)(a_1^{-1}+\dots+a_m^{-1})(a_1+\dots+a_m) = a_1+\dots+a_m$. We have therefore shown that S'' is a regular semigroup. We use again the fact that an inverse semigroup is a regular semigroup whose idempotents commute. Thus to show that S'' is inverse, it is enough to prove that the idempotents in S'' are precisely the elements of the form $e_1+\dots+e_m$, where $e_1,\dots,e_m$ are idempotents in S' and form an orthogonal subset. Suppose that $\sum_{i=1}^m a_i$ is an idempotent in S'' where $\{a_1,\dots,a_m\}$ is an orthogonal subset of S'. Then

$$\left(\sum_{i=1}^m a_i\right)^2 = \sum_{i=1}^m a_i.$$

Multiply both sides of this equation on the left by $a_1a_1^{-1}$. Then

$$a_1 = a_1^2 + a_1a_2 + \dots + a_1a_m.$$

Now multiply both sides of this equation on the right by $a_1^{-1}a_1$. It follows that $a_1 = a_1^2$. By symmetry, it follows that each of the elements $a_1,\dots,a_m$ is an idempotent. Thus $a_1+\dots+a_m$ is an idempotent. We have therefore proved that S'' is an inverse monoid with zero.

By Lemma 3.1, to prove that S'' is a Boolean inverse monoid it is enough to prove that $\mathsf{E}(S'')$ is closed under the operation $\mathbf{e} \mapsto 1-\mathbf{e}$. Referring back to the proof of

Lemma 3.1, we see that if $e, f \in \mathsf{E}(S)$ then $e \circ f = e(1-f) + f$ which is an element of $\mathsf{E}(S'')$. Let $e_1, \ldots, e_m \in \mathsf{E}(S)$. Define $[e_1, \ldots, e_m] = (\ldots((e_1 \circ e_2) \circ e_3) \ldots \circ e_m)$; in other words, associate to the left. Then $[e_1, \ldots, e_m] \in \mathsf{E}(S'')$. We now have the following two identities. Let $e_1, \ldots, e_m \in \mathsf{E}(S)$. Then

$$1 - \left(e \prod_{i=1}^{m} (1 - e_i) \right) = (1 - e) + e[e_1, \ldots, e_m].$$

This can be proved from the following identity

$$\prod_{i=1}^{m} (1 - e_i) = 1 - [e_1, \ldots, e_m],$$

which can be proved by induction. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be an orthogonal set of elements in $\mathsf{E}(S')$. Then

$$1 - \left(\sum_{i=1}^{m} \mathbf{e}_i \right) = \prod_{i=1}^{m} (1 - \mathbf{e}_i),$$

where the proof is straightforward. □

If $S \subseteq R$ define $\mathsf{B}_R(S) = S''$ above which we call the *Booleanization of S in R*.

Wehrung [29, Theorem 6-1.7] gives an alternative construction in the case of rings with involution which can be traced back to the work of Renault [23, 24] on Cartan subalgebras of C^*-algebras as well as Kumjian [9]. This is more adapted to representations of inverse semigroups by partial isometries in C^*-algebras which we shall return to at the end of this section.

Let R be a unital ring and let S be an inverse monoid with zero. A *representation* of S in R is a homomorphism of monoids $\theta \colon S \to R$ which maps zero to zero. The image of θ is an inverse semigroup and so there is a Boolean inverse monoid $\mathsf{B}_R(\mathrm{im}(\theta)) \subseteq R$. Thus, by restricting the codomain of θ and by a mild abuse of notation, $\theta \colon S \to \mathsf{B}_R(\mathrm{im}(\theta))$. By Theorem 1.1, there is a morphism $\theta^* \colon \mathsf{B}(S) \to \mathsf{B}_R(\mathrm{im}(\theta))$. Thus, by extending the codomain of θ^* and by a mild abuse of notation, $\theta^* \colon \mathsf{B}(S) \to R$. However, θ^* has the additional property that if $a, b \in \mathsf{B}(S)$ are orthogonal then $\theta^*(a \vee b) = \theta^*(a) + \theta^*(b)$. Let S be a Boolean inverse monoid and R a ring. Then a representation $\phi \colon S \to R$ is called an *additive representation* if $a \perp b$ in S implies that $\phi(a \vee b) = \phi(a) + \phi(b)$. We have therefore proved the following.

Proposition 3.3. *Let $\theta \colon S \to R$ be a representation of the inverse monoid S in the unital ring R. Then there is a unique additive representation $\theta^* \colon \mathsf{B}(S) \to R$ such that $\theta^* \beta = \theta$.*

The case where the ring is actually a C^*-algebra is of particular interest. There are then some minor modifications to the definitions. Let S be an inverse semigroup and C a C^*-algebra. A representation $\theta \colon S \to C$ is a *$*$-representation* if $\theta(s^{-1}) = \theta(a)^*$. This implies that S is being represented by partial isometries in the C^*-algebra. The following is almost immediate from the above calculations with obvious ammendments.

Proposition 3.4. *Let $\theta \colon S \to C$ be a $*$-representation of the inverse semigroup S in the C^*-algebra C. Then there is a unique additive $*$-representation $\Theta \colon \mathsf{B}(S) \to C$ such that $\theta = \Theta\beta$.*

Let S be an inverse semigroup with zero. We can construct the *contracted semigroup algebra* $\mathbb{C}_0 S$ — this is the usual semigroup algebra $\mathbb{C}S$ factored out by the ideal $\mathbb{C}z$ where z is the zero of S. The following is a re-interpretation of what Paterson proves. It is proved by combining [20, Proposition 4.4.3] with Theorem 2.26.

Theorem 3.5. *Let S be an inverse semigroup with zero. Then $\mathsf{B}(S) \cong \mathsf{B}_{\mathbb{C}_0 S}(S)$.*

3.2. **The Booleanization of the polycyclic monoids P_n.** In this section, we shall carry out an explicit computation of the Booleanization of an important family of inverse semigroups. Our computations will rely on the perspective provided by Section 2.4. Our inverse semigroups will actually be monoids, but Remarks 1.14 and 2.8 tell us that this will be taken care of in our construction.[8]

Let $A = \{a_1, \ldots, a_n\}$, where $n \geq 2$ and finite, and denote by A^* the free monoid generated by A with concatenation as its multiplication. Any subset of A^* is called a *language (over A)*. If $x, y \in A^*$ we write $x \leq_p y$ if and only if $x = yz$ for some string z. We say that y is a *prefix* of x. We call $\leq_p$ the *prefix ordering.* A *prefix code* is a set of finite strings which are pairwise $\leq_p$-incomparable. A maximal prefix code is a prefix code that cannot be a proper subset of another prefix code. A subset $R \subseteq A^*$ is called a *right ideal* if $RA^* \subseteq R$. For each right ideal R, there is a unique prefix code P such that $R = PA^*$ [1, Lemma A.1]. We say that R is *finitely generated* if P is finite. The intersection of two finitely generated right ideals of A^* is also finitely generated [1, Lemma 3.3]. A right ideal R is said to be *essential* if R has a non-empty intersection with every right ideal. The essential right ideals are precisely those right ideals PA^* where P is a maximal prefix code [1, Lemma A.1]. The following result is important.

Lemma 3.6. *Let $R = PA^*$ be a finitely generated right ideal of A^*. Then R is essential if and only if $A^* \setminus R$ is finite.*

Proof. Suppose first that R is essential. Then P is a finite maximal prefix code. By [1, Lemma A1(4)], it is immediate that $A^* \setminus PA^*$ is a finite set. Conversely, suppose that $A^* \setminus R$ is finite. Let x be any finite string. The set xA^* is infinite so cannot be disjoint from the set R since the complement of R is finite. It follows that $xA^* \cap R \neq \varnothing$. ☐

Let R_1 and R_2 be right ideals of A^*. A *morphism* $\theta \colon R_1 \to R_2$ is a function such that $\theta(xu) = \theta(x)u$ for all $x \in R_1$ and $u \in A^*$. Let $\theta \colon P_1A^* \to P_2A^*$ be an ismorphism where P_1 and P_2 are prefix codes. Then the restriction of θ to P_1 induces a bijection $\mathsf{T}(\theta) \colon P_1 \to P_2$ called the *table* of θ. More generally, any bijection from P_1 to P_2 is called a *table*. There is a bijection between isomorphisms $P_1A^* \to P_2A^*$ and tables $P_1 \to P_2$.

It is now easy to show that the subset $R(A^*)$ of $I(A^*)$ consisting of all isomorphisms between the finitely generated right ideals of A^* is a distributive inverse monoid.

The *polycyclic monoid* P_n, where $n \geq 2$, is the monoid with zero given by the following monoid presentation

$$P_n = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} \colon a_i^{-1}a_i = 1 \text{ and } a_i^{-1}a_j = 0 \text{ if } i \neq j \rangle.$$

We refer the reader to [11] for all the details and [10, Section 9.3] (though beware that the notation is slightly different in the latter). The goal of this section is to compute $\mathsf{B}(P_n)$. The first step is to compute the distributive completion of P_n. This was, in fact accomplished in [11], where we proved that $\mathsf{D}(P_n)$ is isomorphic to the inverse monoid of right ideal isomorphisms between the finitely generated right ideals of A^*. In fact, we proved a slightly different theorem there so we shall first explain why in the case of the polycyclic inverse monoids it accomplishes what we claim. There, we in fact constructed *orthogonal completions.* However, polycyclic inverse monoids have a special property that ensures orthogonal completions and

[8]A key part of our computation is the description of the Stone space of the set of all reverse definite languages over a fixed alphabet. This space is actually described in [22] though for completely different reasons from ours.

distributive completions are the same thing; an inverse semigroup S is said to be *ramified*[9] if for $a, b, c \in S$ $a \leq b, c$ implies that $b \leq c$ or $c \leq b$.

Lemma 3.7. *The polycyclic inverse monoids are ramified.*

Proof. Suppose that $yx^{-1} \leq vu^{-1}, zw^{-1}$. Then $(y, x) = (v, u)p = (z, w)q$ for some $p, q \in A^*$. Then $y = vp = zq$ and $x = up = wq$. The strings v and z are prefix comparable. Without loss of generality suppose that $v = zr$ for some string r. Then $q = rp$ and $u = wr$. Thus $(v, u) = (z, w)r$ and so $v^{-1}u \leq zw^{-1}$. □

The significance of being ramified is explained by the following lemma.

Lemma 3.8. *Let S be a ramified inverse semigroup. Let $A = \{a_1, \ldots, a_n\}^{\downarrow}$ be any finitely generated compatible order ideal. Then $A = \{b_1, \ldots, b_m\}^{\downarrow}$ where $\{b_1, \ldots, b_m\}$ is an orthogonal set.*

Proof. Consider the element a_1 and any a_i where $2 \leq i \leq n$. Since $a_1 \sim a_i$ their meet exists. Suppose that $a_1 \wedge a_i = 0$. Then by Lemma 1.4, we deduce that a_1 and a_i are orthogonal. Suppose that $a_1 \wedge a_i \neq 0$. Then since S is ramified either $a_1 \leq a_i$ or $a_i \leq a_1$. Without loss of generality, suppose the former. Then a_1 may be discarded. This process can be repeated and we obtain in this way a subset of $\{a_1, \ldots, a_n\}$ which is orthogonal and still generates A. □

It now follows, as claimed, that [11] establishes $\mathsf{D}(P_n)$ as precisely the set of all isomorphisms between finitely generated right ideals of A^*. The idempotents of $\mathsf{D}(P_n)$ form a distributive lattice isomorphic to the distributive lattice of finitely generated right ideals of A^* under subset inclusion. The intersection of two finitely generated right ideals is a finitely generated right ideal and the union of two finitely generated right ideals is a finitely generated right ideal. It follows that $\mathsf{D}(P_n)$ is a distributive subalgebra of $I(A^*)$. Thus by Section 2.4, to compute the Booleanization of $\mathsf{D}(P_n)$ it will be enough to compute the Boolean hull of $\mathsf{D}(P_n)$ in $I(A^*)$. To do this, it is convenient to use terminology and notation from language theory [21].

We will use regular expressions to describe languages so $+$ means $\cup$ and singleton sets are denoted by their elements. A language L over A is said to be *definite*[10] if $L = X + YA^*$ where both X and Y are finite languages. It is well-known from the theory of regular languages [21] that the set of definite languages in A^* forms a Boolean algebra with respect to set intersection, union and complementation.

Lemma 3.9. *The set of definite languages is generated as a Boolean algebra by the finitely generated right ideals of A^*.*

Proof. Denote by $\mathscr{B}$ the Boolean subalgebra of $\mathsf{P}(A^*)$, the power set of A^*, generated by the finitely generated right ideals of A^*. Observe that $\{x\} = xA^* \setminus xAA^*$. Thus $\mathscr{B}$ contains all finite languages and so all unions of finite languages and finitely generated right ideals. Thus $\mathscr{B}$ contains all definite languages. But the set of definite languages is a Boolean algebra. □

It follows that the set of definite languages is the Booleanization of the distributive lattice of finitely generated right ideals. We shall construct a Boolean inverse submonoid of $I(A^*)$ whose Boolean algebra of idempotents is isomorphic to the set of definite languages over A. Before we do that, it is useful to make some simple observations about definite languages.

We now introduce some terms which are non-standard but useful. An element $x \in L$ of a definite language is said to be *unbounded* if $xA^* \subseteq L$ otherwise it is said to be *bounded*. Every definite language L can be written as a disjoint union

[9] We have borrowed this terminology from [4].

[10] Strictly speaking, this should be *reverse definite*.

$L = X_1 + X_2$ where X_1 are the bounded elements of L and X_2 are the unbounded elements. The set X_1 is finite and the set X_2 is a finitely generated right ideal. The set X_2 has a minimum generating set which is a prefix code [1]. We say that a definite language L is in *normal form* if it is written $L = X + YA^*$ where YA^* are all the unbounded elements, Y is a prefix code, and X are all the bounded elements.

Example 3.10. The following is adapted from [2]. Let $L = 0 + 201 + 212 + (00 + 20 + 01 + 02)(0 + 1 + 2)^*$. This is a definite language. We now convert it into normal form. We show first that 0 is unbounded. Observe that $0(0 + 1 + 2)^* = 0 + 00 + 01 + 02 + (00 + 01 + 02)(0 + 1 + 2)^*$. It follows that $L = 201 + 212 + (0 + 00 + 20 + 01 + 02)(0 + 1 + 2)^*$. But 202 is unbounded because $20(0 + 1 + 2)^* \subseteq L$. It follows that $L = 212 + (0 + 00 + 20 + 01 + 02 + 201)(0 + 1 + 2)^*$. Now we observe that $(0 + 00 + 20 + 01 + 02 + 201)(0 + 1 + 2)^* = (0 + 20)(0 + 1 + 2)^*$. Thus $L = 212 + (0 + 20)(0 + 1 + 2)^*$, which is in normal form.

The proof of the following is now routine.

Lemma 3.11. *Two definite languages are equal if and only if their normal forms are the same.*

Let L_1 and L_2 be definite languages. A bijection $\alpha \colon L_1 \to L_2$ is said to be *permissible* if it satisfies the following two conditions:

1. α maps bounded elements to bounded elements and unbounded elements to unbounded elements.
2. If $x \in L_1$ is an unbounded element and $y \in A^*$ is arbitrary then $\alpha(xy) = \alpha(x)y$.

For convenience, we list the notation we shall be using:

- $I(A^*)$ is the symmetric inverse monoids of all partial bijections on the set A^*.
- $I_f(A^*)$ is the inverse semigroup of all partial bijections between the finite subsets of A^*.
- $R(A^*)$ is the inverse semigroup of all isomorphisms between finitely generated right ideals of A^*. We assume this includes the empty function which is the zero of this inverse monoid.
- $CT(A^*)$ is the set of all permissible maps between definite languages. Clearly, the idempotent elements here are the identity functions on the definite languages.

It follows that each permissible map is a disjoint union of an element of $I_f(A^*)$ and an element of $R(A^*)$.

Let S be a distributive inverse semigroup. An *additive ideal* I of S is a semigroup ideal which is also closed under binary compatible joins. The proof of the following is immediate.

Lemma 3.12. *$I_f(A^*)$ is an additive ideal of $I(A^*)$.*

The proof of the following is also straightforward.

Lemma 3.13. *Let U be a distributive inverse semigroup. Let S and T be distributive inverse subsemigroups where both are closed under binary compatible joins and where S is an (additive) ideal. Put $V = \{s \vee t \colon s \in S, t \in T, s \perp t\}$. Then V is a distributive inverse subsemigroup of U.*

Proposition 3.14. *$CT(A^*)$ is a Boolean inverse $\wedge$-monoid.*

Proof. By Lemma 2.15, $CT(A^*)$ is a distributive inverse monoid with a Boolean algebra of idempotents and so is a Boolean inverse monoid. It remains to show that it has all binary meets. This is equivalent [18] to proving the following. Let $\alpha\colon L \to M$ be a permissible map between definite languages. Define $\mathrm{Fix}(\alpha) = \{x \in L\colon \alpha(x) = x\}$. Then F is a definite language. If $F = \varnothing$ then we are done, so in what follows we can assume that $F \neq \varnothing$. Let $L = L_1 + L_2A^*$ be the normal form of L. We prove that

$$\mathrm{Fix}(\alpha) = \mathrm{Fix}(\alpha|L_1) + \mathrm{Fix}(\alpha|L_2)A^*.$$

It is clear that the right hand side is contained in the left hand side. Observe that if x is an unbounded element of L and is fixed by α then α also fixes all elements of xA^*. We prove that the left hand side is contained in the right hand side. Let $x \in \mathrm{Fix}(\alpha)$ be unbounded. Then $x \in L_2A^*$. We can therefore write $x = py$ where $p \in L_2$. We have that $\alpha(x) = \alpha(py) = \alpha(p)y$. But by assumption $\alpha(x) = x$. Thus $\alpha(p) = p$. It follows that $p \in \mathrm{Fix}(\alpha|L_2)$ and so $x \in \mathrm{Fix}(\alpha|L_2)A^*$. If x is bounded then it is immediate that $x \in \mathrm{Fix}(\alpha|L_1)$. □

We now come to our main theorem.

Theorem 3.15. *The Boolean inverse monoid $CT(A^*)$ is the Booleanization of the polycyclic inverse monoid P_n.*

Proof. This is almost immediate by the results of Section 2.4. It devolves to checking that the Boolean hull of P_n in $I(A^*)$ is in fact $CT(A^*)$. Clearly, the idempotents of $\mathsf{B}_{I(A^*)}(P_n)$ are the same as the idempotents of $CT(A^*)$ and $\mathsf{B}_{I(A^*)}(P_n) \subseteq CT(A^*)$ since $\mathsf{D}(P_n) \subseteq CT(A^*)$. An element of $CT(A^*)$ is an orthogonal join of an isomorphism between two finitely generated right ideals of A^* and a bijection between two finite subsets of A^*. This latter map is itself an orthogonal join of the maps that take one element sets to one element sets. Let u, v be any two strings. Once we have shown that the partial bijection $u \mapsto v$ belongs to $\mathsf{B}_{I(A^*)}(P_n)$, our proof will be complete. Define $f\colon uA^* \to vA^*$ given by $f(ux) = vx$. Define $g\colon uAA^* \to vAA^*$ given by $g(uax) = vax$ where $a \in A$. It is clear that $g \leq f$ in the natural partial order. Observe that $f \setminus g$ is precisely the map $u \mapsto v$. □

We call the Boolean inverse monoid $CT(A^*)$ the *Cuntz-Toeplitz monoid (of degree n)* [6]. The rationale for this terminology will now be explained. We denote by C_n the *Cuntz inverse monoid* [12] and use the description given in [17, Section 5.2]. Denote by A^ω the set of all right-infinite strings over the alphabet A. The monoid C_n consists of all bijections $f\colon XA^\omega \to YA^\omega$, where X and Y are prefix codes, for which there exists an associated bijection $f_1\colon X \to Y$ such that $f(xw) = f_1(x)w$ where $x \in X$ and $w \in A^\omega$.

The behaviour of $\wedge$-morphisms between Boolean inverse $\wedge$-semigroups is analogous to that of the behaviour of homomorphisms between rings. Let $\theta\colon S \to T$ be a $\wedge$-morphism between two Boolean inverse $\wedge$-semigroups. Define the *kernel* of θ, denoted by $\ker(\theta)$, to be the set of all $s \in S$ such that $\theta(s) = 0$. It is easy to check that $\ker(\theta)$ is an additive ideal of S.

Lemma 3.16. *Let θ and ϕ be two surjective $\wedge$-morphisms between the Boolean inverse $\wedge$-semigroups S and T. Then $\theta = \phi$ if and only if* $\ker(\theta) = \ker(\phi)$.

Proof. Suppose that $\theta(a) = \theta(b)$. Then $\theta(a \setminus (a \wedge b)) = 0 = \theta(b \setminus (a \wedge b))$. By assumption $\phi(a \setminus (a \wedge b)) = 0 = \phi(b \setminus (a \wedge b))$. Thus

$$\phi(a) = \phi((a \setminus (a \wedge b) \vee (a \wedge b)) = \phi(a \wedge b).$$

By symmetry, we get that $\phi(a) = \phi(b)$ and symmetry again delivers the result. □

In the light of the above lemma, we may extend the usual notation from ring theory. Let S be a Boolean inverse $\wedge$-semigroup and let I be an additive ideal of S.

Denote by S/I the Boolean inverse $\wedge$-semigroup S/ε_I where ε_I is the congruence defined by $(a,b) \in \varepsilon_I$ if and only if $a \setminus (a \wedge b), b \setminus (a \wedge b) \in I$.

Proposition 3.17. $CT(A^*)/I_f(A^*) \cong C_n$.

Proof. Denote by $\equiv$ the congruence relation induced on $CT(A^*)$ by the additive ideal $I_f(A^*)$ We prove, first, the following. Suppose that R and R' are two finitely generated right ideals of A^* and that $1_R \cong 1_{R'}$. Then R is essential if and only if R' is essential. Suppose that R is essential. Then $A^* \setminus R$ is a finite set by Lemma 3.6. It follows that $1_{A^*} \cong 1_R$. Thus $1_{A^*} \cong 1_{R'}$. From the definition, $A^* \setminus R'$ is a finite set and so by Lemma 3.6, R' is also essential. It is immediate from the above that $1_{A^*} \cong 1_R$ precisely when R is essential. This is already enough to tell us that $CT(A^*)/I_f(A^*)$ is a homomorphic image of C_n by [12]. But C_n is congruence-free. Thus as long as the quotient is not trivial it will be isomorphic to C_n. But this is clear.

More concretely, we may also prove the result as follows. Let $f\colon (X_1 + Y_1A^*) \to (X_2 + Y_2A^*)$ be a permissible map where Y_1 and Y_2 are prefix codes. Then f induces a bijection $f_1\colon Y_1 \to Y_2$. Define $\Theta(f)\colon Y_1A^\omega \to Y_2A^\omega$ by $\Theta(f)(yw) = f_1(y)w$. It is clear that Θ is a surjective $\wedge$-morphism (and a monoid homomorphism) and that the kernel of Θ is $I_f(A^*)$. □

By Proposition 3.17, it follows that $CT(A^*)$ is the Boolean inverse monoid analogue of the *Cuntz-Toeplitz algebra*. See [6], for example.

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