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# Solutions Sets to Systems of Equations in Hyperbolic Groups Are EDT0L in PSPACE

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## Abstract

We show that the full set of solutions to systems of equations and inequations in a hyperbolic group, with or without torsion, as shortlex geodesic words, is an EDT0L language whose specification can be computed in  $\text{NSPACE}(n^2 \log n)$  for the torsion-free case and  $\text{NSPACE}(n^4 \log n)$  for the torsion case. Our work combines deep geometric results by Rips, Sela, Dahmani and Guirardel on decidability of existential theories of hyperbolic groups, work of computer scientists including Plandowski, Jeż, Diekert and others on PSPACE algorithms to solve equations in free monoids and groups using compression, and an intricate language-theoretic analysis.

The present work gives an essentially optimal formal language description for all solutions in all hyperbolic groups, and an explicit and surprising low space complexity to compute them.

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## 1 Introduction

Hyperbolic groups were introduced by Gromov in 1987 [25], and play a significant role in group theory and geometry [12, 33, 40]. Virtually free groups, small cancellation groups, and the fundamental groups of extensive classes of negative curvature manifolds are important examples (see [1] for background). In a certain probabilistic sense made precise in [26, 37, 41], almost all finitely generated groups are hyperbolic. They admit very efficient solutions to the word and conjugacy problems [21, 27, 28], and extremely nice language-theoretic properties, for example the set of all geodesics over any generating set is regular (see Lemma 16), and forms a biautomatic structure [22]. They are exactly the groups which admit context-free multiplication tables [23], and have a particularly simple characterisation in terms of rewriting systems [6, 35] (see Lemma 13).



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In this paper we consider systems of equations and inequations in hyperbolic groups, building on and generalising work done in the area of solving equations over various groups and monoids in PSPACE. Starting with work of Plandowski [38], many prominent researchers have given PSPACE algorithms [7, 14, 16, 17, 18, 30, 31] to find (all) solutions to systems of equations over free monoids, free groups, partially commutative monoids and groups, and virtually free groups (that is, groups which have a free subgroup of finite index).

The satisfiability of equations over torsion-free hyperbolic groups is decidable by the work of Rips and Sela [39], who reduced the problem in hyperbolic groups to solving equations in free groups, and then calling on Makanin’s algorithm [36]. Kufleitner proved PSPACE for decidability in the torsion-free case [34], without an explicit complexity bound, by following Rips-Sela and then using Plandowski’s result [38]. Dahmani and Guirardel radically extended Rips and Sela’s work to all hyperbolic groups (with torsion), by reducing systems of equations to systems over virtually free groups, which they then reduced to systems of *twisted* equations over free groups [11]. In terms of describing solution sets, Grigorchuck and Lysionok gave efficient algorithms for the special case of quadratic equations [24].

Here we combine Rips, Sela, Dahmani and Guirardel’s approach with recent work of the authors with Diekert [7, 14, 15] to obtain the following results.

► **Theorem 1 (Torsion-free).** *Let  $G$  be a torsion-free hyperbolic group with finite symmetric generating set  $S$ . Let  $\Phi$  be a system of equations and inequations of size  $n$  (see Section 2 for a precise definition of input size). Then the set of all solutions, as tuples of shortlex geodesic words over  $S$ , is EDT0L. Moreover there is an NSPACE( $n^2 \log n$ ) algorithm which on input  $\Phi$  prints a description for the EDT0L grammar.*

► **Theorem 2 (Torsion).** *Let  $G$  be a hyperbolic group with torsion with finite symmetric generating set  $S$ . Let  $\Phi$  be a system of equations and inequations of size  $n$  (see Section 2 for a precise definition of input size). Then the set of all solutions, as tuples of shortlex geodesic words over  $S$ , is EDT0L. Moreover there is an NSPACE( $n^4 \log n$ ) algorithm which on input  $\Phi$  prints a description for the EDT0L grammar.*

A corollary of Theorems 1 and 2 is that the existential theory for hyperbolic groups can be decided in NSPACE( $n^2 \log n$ ) for torsion-free and NSPACE( $n^4 \log n$ ) for groups with torsion. Another consequence of our work is that we can decide in the same space complexity as above whether or not the solution set is empty, finite or infinite (see [8]).

EDT0L is a surprisingly low language complexity for this problem. EDT0L languages are playing an increasingly useful role in group theory, not only in describing solution sets to equations in groups [7, 14, 17], but more generally [4, 5, 9].

The paper is organised as follows. We briefly set up some notation for solution sets and input size in Section 2. We then give an informal description of the entire argument for the torsion-free case in Section 3. This overview uses various concepts which are defined more carefully afterwards, but we hope that having the entire argument in one place is useful for the reader to understand the “big picture” before descending into the details. Section 4 develops necessary material on EDT0L and space complexity. Section 5 covers the necessary background on hyperbolic groups, including the key step to obtain a full solution set (as tuples of shortlex geodesics) from a *covering solution set* (see Definition 3(iii)). In Section 6 we use Rips and Sela’s *canonical representatives* (see [8, Appendix A]) in torsion-free hyperbolic groups, to reduce the problem of finding solutions in a torsion-free hyperbolic group to finding solutions in the free group on the same generators as the hyperbolic one. We show that if the input system has size  $n$  then the resulting system in the free group has size  $O(n^2)$ . Applying [7] produces a covering solution set in  $O(n^2 \log n)$  nondeterministic space, from

which we obtain the full set of solutions as shortlex geodesics in the original group, as an EDTOL language, in the same space complexity. In Section 7 we prove the general case for hyperbolic groups with torsion, following Dahmani and Guirardel who construct canonical representatives in a graph containing the Cayley graph of the hyperbolic group, and working in an associated virtually-free group.

## 2 Notations for equations and solution sets

Let  $G$  be a fixed group with finite symmetric generating set  $S$ . Let  $\pi: S^* \rightarrow G$  be the natural projection map. Let  $\{X_1, \dots, X_m\}$ ,  $m \geq 1$ , be a set of variables to which we adjoin their formal inverses  $X_i^{-1}$  and denote by  $\mathcal{X}$  the union  $\{X_i, X_i^{-1} \mid 1 \leq i \leq m\}$ . Let  $\mathcal{C} = \{a_1, \dots, a_k\} \subseteq G$  be a set of constants and

$$\Phi = \{\varphi_j(\mathcal{X}, \mathcal{C}) = 1\}_{j=1}^h \cup \{\varphi_j(\mathcal{X}, \mathcal{C}) \neq 1\}_{j=h+1}^s \quad (1)$$

be a set of  $s$  equations and inequations in  $G$ , where the length of each (in)equation is  $l_i$ . Then the total length of the equations is  $n = \sum_{i=1}^s l_i$ , and we take  $|\Phi| = n$  as the input size in the remainder of the paper.

A tuple  $(g_1, \dots, g_m) \in G^m$  solves an equation [resp. inequation]  $\varphi_j$  in  $\Phi$  if replacing each variable  $X_i$  by  $g_i$  (and  $X_i^{-1}$  by  $g_i^{-1}$ ) produces an identity [resp. inequality] in the group as follows:

$$\varphi_j(g_1, \dots, g_m, a_1, \dots, a_k) = 1 \text{ [resp. } \varphi_j(g_1, \dots, g_m, a_1, \dots, a_k) \neq 1].$$

A tuple  $(g_1, \dots, g_m) \in G^m$  solves  $\Phi$  if it simultaneously solves  $\varphi_j$  for all  $1 \leq j \leq s$ .

### ► Definition 3.

(i) The group element solution set to  $\Phi$  is the set

$$\text{Sol}_G(\Phi) = \{(g_1, \dots, g_m) \in G^m \mid (g_1, \dots, g_m) \text{ solves } \Phi\}.$$

(ii) Let  $T \subseteq S^*$  and  $\#$  a symbol not in  $S$ . The full set of  $T$ -solutions is the set

$$\text{Sol}_{T,G}(\Phi) = \{w_1\# \dots \# w_m \mid w_i \in T, (\pi(w_1), \dots, \pi(w_m)) \text{ solves } \Phi\}.$$

(iii) A set  $L \subseteq \{w_1\# \dots \# w_m \mid w_i \in S^*, 1 \leq i \leq m\}$  is a covering solution set to  $\Phi$  if

$$\{(\pi(w_1), \dots, \pi(w_k)) \mid w_1\# \dots \# w_m \in L\} = \text{Sol}_G(\Phi).$$

## 3 Overview of the proof

In a free group, the equation  $xy = z$  has a solution in reduced words (that is, words which do not contain factors  $aa^{-1}$  for any  $a \in S$ ) if and only if there exist words  $P, Q, R$  with  $x = PQ, y = Q^{-1}R, z = PR$  in the free monoid with involution over  $S$  ([7, Lemma 4.1]). In a hyperbolic group this direct reduction to cancellation-free equations is no longer true: a triangle  $xy = z$  where  $x, y, z$  are replaced by geodesics looks as in Figure 1a.

Rips and Sela [39] proved that in a torsion-free hyperbolic group one can define certain special words called *canonical representatives* so that a system of equations of the form  $X_j Y_j = Z_j, 1 \leq j \leq O(n)$  has solutions which are canonical representatives with the properties that their prefixes and suffixes coincide, as shown in Figure 1b, and the inner circle is the concatenation of three words with lengths in  $O(n)$ . Moreover, these canonical representatives are  $(\lambda, \mu)$ -quasigeodesics (Definition 15) where the constants  $\lambda, \mu$  depend only on the group.



(a) Using geodesics.

(b) Using canonical representatives.

■ **Figure 1** Solutions to  $xy = z$  in the Cayley graph of a hyperbolic group.

We use these facts to devise the following algorithm, presented here for the torsion-free case. We treat the hyperbolic group  $G$  with finite generating set  $S$  as a constant. On input a system of equations and inequations as in (1) of size  $n$ :

1. Replace inequations by equations (by using a new variable and requiring that this variable is not trivial in the group, as explained in Section 6.3).
2. Triangulate the system, so that all equations have the form  $X_j Y_j = Z_j$ . The size of the resulting system is still in  $O(n)$ . Suppose there are  $q \in O(n)$  such equations.
3. Enumerate, one at a time, all possible tuples  $\mathbf{c} = (c_{11}, c_{12}, c_{13}, \dots, c_{q1}, c_{q2}, c_{q3})$  of words (say, in lex order) so that the length  $\ell(c_{ji})$  with respect to  $S$  is bounded by a constant in  $O(n)$ . Note that the size of each tuple (the sum of the lengths of the  $c_{ij}$ ) is in  $O(n^2)$ .
4. For each tuple  $\mathbf{c}$ , run Dehn's algorithm to check  $c_{j1} c_{j2} c_{j3} =_G 1$  for  $1 \leq j \leq q$ . If this holds for all  $j$ , write down a system of  $3q$  equations

$$X_j = P_j c_{j1} Q_j, Y_j = Q_j^{-1} c_{j2} R_j, Z_j = P_j c_{j3} R_j.$$

Note that the resulting system,  $\Phi_{\mathbf{c}}$ , has size in  $O(n^2)$ .

5. We now call the algorithm of the authors and Diekert [7] to find all solutions to  $\Phi_{\mathbf{c}}$  in the free group generated by  $S$ . This algorithm, on input of size  $O(n^2)$ , runs in  $\text{NSPACE}(n^2 \log n)$ , and prints a description of the EDT0L grammar which generates all tuples of solutions as reduced words in  $S^*$ . Specifically it prints nodes and edges of a trim NFA which is the rational control for the EDT0L grammar (see Definition 4 below). Modify the algorithm so that the nodes printed include the label  $\mathbf{c}$  which has length  $O(n^2)$  (so does not affect the complexity).
6. Delete the current system stored, and move to the next tuple  $\mathbf{c}$ .
7. At the end, print out a new start node and  $\epsilon$  edges to the start node of the NFA for the system  $\Phi_{\mathbf{c}}$  for all  $\mathbf{c}$  already printed.

The NFA that is printed gives an EDT0L grammar that generates a language of tuples which is a covering solution to the original system in the hyperbolic group. To obtain the full set of solutions as shortlex geodesic words we need to perform further steps. Using the facts that canonical representatives are  $(\lambda, \mu)$ -quasigeodesics, and

- the full set of  $(\lambda, \mu)$ -quasigeodesics,  $Q_{S, \lambda, \mu}$
- the set of all pairs  $\{(u, v) \in Q_{S, \lambda, \mu} \mid u =_G v\}$
- the set of all shortlex geodesics in  $G$

are all regular, we can obtain from the covering solution an ET0L language, in the same space complexity (by Proposition 9 below), which represents the full set of solutions in shortlex geodesic words. Then finally, because of the special form of our solutions, we can apply a version of the *Copying Lemma* of Ehrenfeucht and Rozenberg [19] to show that in fact the resulting language of shortlex representatives is EDT0L in  $\text{NSPACE}(n^2 \log n)$ .

Details for handling the case of hyperbolic groups with torsion also follows this general scheme, however finding the analogue of canonical representatives is harder in this case, so further work is required, and we describe this in Section 7.

## 4 E(D)T0L in PSPACE

### 4.1 ET0L and EDT0L languages

Let  $C$  be an alphabet. A *table* for  $C$  is a finite subset of  $C \times C^*$ . If  $(c, v)$  is in some table  $t$ , we say that  $(c, v)$  is a *rule* for  $c$ . A table  $t$  is *deterministic* if for each  $c \in C$  there is exactly one  $v \in C^*$  with  $(c, v) \in t$ .

If  $t$  is a table and  $u \in C^*$  then we write  $u \xrightarrow{t} v$  to mean that  $v$  is obtained by applying rules from  $t$  to each letter of  $u$ . That is,  $u = a_1 \dots a_n$ ,  $a_i \in C$ ,  $v = v_1 \dots v_n$ ,  $v_i \in C^*$ , and  $(a_i, v_i) \in t$  for  $1 \leq i \leq n$ . If  $H$  is a set of tables and  $r \in H^*$  then we write  $u \xrightarrow{r} v$  to mean that there is a sequence of words  $u = v_0, v_1, \dots, v_n = v \in C^*$  such that  $v_{i-1} \xrightarrow{t_i} v_i$  for  $1 \leq i \leq n$  where  $r = t_1 \dots t_n$ . If  $R \subseteq H^*$  we write  $u \xrightarrow{R} v$  if  $u \xrightarrow{r} v$  for some  $r \in R$ .

► **Definition 4** ([2]). *Let  $\Sigma$  be an alphabet. We say that  $L \subseteq \Sigma^*$  is an ET0L language if there is an alphabet  $C$  with  $\Sigma \subseteq C$ , a finite set  $H \subset \mathcal{P}(C \times C^*)$  of tables, a regular language  $R \subseteq H^*$  and a letter  $c_0 \in C$  such that*

$$L = \{w \in \Sigma^* \mid c_0 \xrightarrow{R} w\}.$$

*In the case when every table  $h \in R$  is deterministic, i.e. each  $h \in R$  is in fact a homomorphism, we write  $L = \{r(c_0) \in \Sigma^* \mid r \in R\}$  and say that  $L$  is EDT0L. The set  $R$  is called the rational control, the symbol  $c_0$  the start symbol and  $C$  the extended alphabet.*

### 4.2 Space complexity for E(D)T0L

Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be a function. Recall an algorithm is said to run in  $\text{NSPACE}(f(n))$  if it can be performed by a non-deterministic Turing machine with a read-only input tape, a write-only output tape, and a read-write work tape, with the work tape restricted to using  $O(f(n))$  squares on input of size  $n$ . The following definition formalises the idea of producing some E(D)T0L language (such as the solution set of some system of equations) in  $\text{NSPACE}(f(n))$ , where the language is the output of a computation with input (such as a system of equations) of size  $n$ . We use the notation  $L(\mathcal{A})$  to denote the language accepted by an automaton  $\mathcal{A}$ .

► **Definition 5.** *Let  $\Sigma$  be a (fixed) alphabet and  $f: \mathbb{N} \rightarrow \mathbb{N}$  a function. If there is an  $\text{NSPACE}(f(n))$  algorithm that on input  $\Omega$  of size  $n$  outputs the specification of an ET0L language  $L_\Omega \subseteq \Sigma^*$ , then we say that  $L_\Omega$  is ET0L in  $\text{NSPACE}(f(n))$ .*

*Here the specification of  $L_\Omega$  consists of:*

- *an extended alphabet  $C \supseteq \Sigma$ ,*
- *a start symbol  $c_0 \in C$ ,*
- *a finite list of nodes of a (trim) NFA  $\mathcal{A}$ , labeled by some data, some possibly marked as initial and/or final,*
- *a finite list  $\{(u, v, h)\}$  of edges of  $\mathcal{A}$  where  $u, v$  are nodes and  $h \in \mathcal{P}(C \times C^*)$  is a table such that  $L_\Omega = \{w \in \Sigma^* \mid c_0 \xrightarrow{L(\mathcal{A})} w\}$ .*

*A language  $L_\Omega$  is EDT0L in  $\text{NSPACE}(f(n))$  if, in addition, every table  $h$  labelling an edge of  $\mathcal{A}$  is deterministic.*

Note that the entire print-out is not required to be in  $O(f(n))$  space. Previous results of the authors with Diekert can now be restated as follows.

► **Theorem 6** ([7, Theorem 2.1]). *The set of all solutions to a system of size  $n$  of equations (with rational constraints), as reduced words, in a free group is EDTOL in  $\text{NSPACE}(n \log n)$ .*

► **Theorem 7** ([15, Theorem 45]). *The set of all solutions to a system of size  $n$  of equations (with rational constraints), as words in a particular quasigeodesic normal form over a certain finite generating set, in a virtually free group is EDTOL in  $\text{NSPACE}(n^2 \log n)$ .*

► **Remark 8.** In our applications below we have  $\Omega$  representing some system of equations and inequations, with  $|\Omega| = n$ , and we construct algorithms where the extended alphabet  $C$  has size  $|C| \in O(n)$  in the torsion-free case and  $|C| \in O(n^2)$  in the torsion case. This means we can write down the entire alphabet  $C$  as binary strings within our space bounds. Moreover, each element  $(c, v)$  of any table we construct has  $v$  of (fixed) bounded length, so we can write down entire tables within our space bounds.

### 4.3 Closure properties

It is well known [32, Theorem 2.8] that ETOL is a full AFL (closed under homomorphism, inverse homomorphism, finite union, intersection with regular languages). Here we show the space complexity of an ETOL language is not affected by these operations.

► **Proposition 9.** *Let  $\Sigma, \Gamma$  be finite alphabets of fixed size,  $M$  an NFA of constant size with  $L(M) \subseteq \Sigma^*$ , and  $\varphi: \Gamma^* \rightarrow \Sigma^*$ ,  $\psi: \Sigma^* \rightarrow \Gamma^*$  homomorphisms. If  $L_{\Omega_1}, L_{\Omega_2} \subseteq \Sigma^*$  are  $E(D)TOL$  in  $\text{NSPACE}(f(n))$  (on inputs  $\Omega_1, \Omega_2$ , respectively, with  $|\Omega_1|, |\Omega_2| \in O(n)$ ) then*

- (homomorphism)  $\psi(L_{\Omega_1})$  is  $E(D)TOL$  in  $\text{NSPACE}(f(n))$ ,
- (intersection with regular)  $L_{\Omega_1} \cap L(M)$  is  $E(D)TOL$  in  $\text{NSPACE}(f(n))$ ,
- (union)  $L_{\Omega_1} \cup L_{\Omega_2}$  is  $E(D)TOL$  in  $\text{NSPACE}(f(n))$ ,
- (inverse homomorphism)  $\varphi^{-1}(L_{\Omega_1})$  is  $ETOL$  in  $\text{NSPACE}(f(n))$ .

The proof is straightforward keeping track of complexity in the standard proofs [3, 10]. Note EDTOL is not closed under inverse homomorphism [20].

► **Proposition 10** (Projection onto a factor). *If  $L_{\Omega} \subseteq \Sigma^*$  is  $E(D)TOL$  in  $\text{NSPACE}(f(n))$  on an input  $\Omega$  of size  $n$ , and for some fixed integer  $s$  all words in  $L_{\Omega}$  have the form  $u_1\# \dots \#u_s$  with  $u_i \in (\Sigma \setminus \{\#\})^*$ , and  $1 \leq i \leq j \leq s$ , then*

$$L = \{u_i\# \dots \#u_j \mid u_1\# \dots \#u_i\# \dots \#u_j\# \dots \#u_s \in L_{\Omega}\}$$

is  $E(D)TOL$  in  $\text{NSPACE}(f(n))$ .

### 4.4 From ETOL to EDTOL

In computing the full solution set to equations as shortlex geodesic words, we will need to take inverse homomorphism. Even though in general the image under an inverse homomorphism of an EDTOL language is just ETOL, because of the special structure of solution sets we can apply the *Copying Lemma* of Ehrenfeucht and Rozenberg [19] to show the following.

► **Proposition 11.** *Let  $S$  be an alphabet and  $h: S \rightarrow S'$  be a homomorphism of from  $S$  to a disjoint alphabet  $S' = \{s' \mid s \in S\}$  defined by  $h(s) = s'$ . Let  $\wr$  be a symbol not in  $S \cup S'$  and define  $h(\wr) = \wr$ . Let  $L_1$  be a set of words of the form  $w \wr h(w)$  where  $w \in S^*$ . If  $L_1$  is  $ETOL$  in  $\text{NSPACE}(f(n))$ , then  $L_2 = \{w \mid w \wr h(w) \in L_1\}$  is  $EDTOL$  in  $\text{NSPACE}(f(n))$ .*

**Proof.** By [19], any nondeterministic table in the grammar for  $L_1$  can be replaced by a finite number of deterministic tables (essentially, if nondeterminism allowed some letter  $c \in C$  to produce two different results, then some word in  $L_1$  would not have the form  $w \wr h(w)$ ). So without loss of generality we can replace a table  $f$  containing  $(c, v_1), \dots, (c, v_k)$  by  $k$  tables  $f_i$  containing  $(c, v_i)$  only). This modification is clearly in the same space bound. Project onto the prefix using Proposition 10.  $\blacktriangleleft$

## 5 Hyperbolic groups

### 5.1 Definitions

Recall the *Cayley graph* for a group  $G$  with respect to a finite symmetric generating set  $S$  is a directed graph  $\Gamma(G, S)$  with vertices labeled by  $g \in G$  and a directed edge  $(g, h)$  labeled by  $s \in S$  whenever  $h =_G gs$ . Let  $\ell(p)$ ,  $i(p)$  and  $f(p)$  resp. be the length, initial and terminal vertices of a path  $p$  in the Cayley graph. A path  $p$  is *geodesic* if  $\ell(p)$  is minimal among the lengths of all paths  $q$  with the same endpoints. If  $x, y$  are two points in  $\Gamma(G, S)$ , we define  $d(x, y)$  to be the length of a shortest path from  $x$  to  $y$  in  $\Gamma(G, S)$ .

► **Definition 12** ( $\delta$ -hyperbolic group (Gromov)). *Let  $G$  be a group with finite symmetric generating set  $S$ , and let  $\delta \geq 0$  be a fixed real number. If  $p, q, r$  are geodesic paths in  $\Gamma(G, S)$  with  $f(p) = i(q)$ ,  $f(q) = i(r)$ ,  $f(r) = i(p)$ , we call  $[p, q, r]$  a geodesic triangle. A geodesic triangle is  $\delta$ -slim if  $p$  is contained in a  $\delta$ -neighbourhood of  $q \cup r$ , that is, every point on one side of the triangle is within  $\delta$  of some point on one of the other sides. (See for example Figure 1a.) We say  $(G, S)$  is  $\delta$ -hyperbolic if every geodesic triangle in  $\Gamma(G, S)$  is  $\delta$ -slim. We say  $(G, S)$  is hyperbolic if it is  $\delta$ -hyperbolic for some  $\delta \geq 0$ .*

It is a straightforward to show that being hyperbolic is independent of choice of finite generating set. Thus we say  $G$  is hyperbolic if  $(G, S)$  is for some finite generating set  $S$ .

► **Lemma 13** (Dehn presentation).  *$G$  is hyperbolic if and only if there is a finite list of pairs of words  $(u_i, v_i) \in S^* \times S^*$  with  $|u_i| > |v_i|$  and  $u_i =_G v_i$  such that the following holds: if  $w \in S^*$  is equal to the identity of  $G$  then it contains some  $u_i$  as a factor.*

This gives an algorithm to decide whether or not a word  $w \in S^*$  is equal to the identity: while  $\ell(w) > 0$ , look for some  $u_i$  factor. If there is none, then  $w \neq_G 1$ . Else replace  $u_i$  by  $v_i$  (which is shorter). This procedure is called *Dehn's algorithm*.

► **Lemma 14.** *Dehn's algorithm runs in (linear time and) linear space.*

► **Definition 15** (Quasigeodesic). *For  $\lambda \geq 1, \mu \geq 0$  real numbers, a path  $p$  in  $\Gamma(G, S)$  is a  $(\lambda, \mu)$ -quasigeodesic if for any subpath  $q$  of  $p$  we have  $\ell(q) \leq \lambda d(i(q), f(q)) + \mu$ .*

Throughout this article, we assume  $G$  is a fixed hyperbolic group which we treat as a constant for complexity purposes. We assume we are given  $(G, S)$ , the constant  $\delta$ , the finite list of pairs for Dehn's algorithm, and any other constants depending only on the group.

### 5.2 Languages in hyperbolic groups

► **Proposition 16.** *Let  $G$  be a fixed hyperbolic group with finite generating set  $S$ ,  $\lambda \geq 1, \mu \geq 0$  constants with  $\lambda \in \mathbb{Q}$  and  $\mu$  sufficiently large. Then the following sets are regular languages.*

1. *The set of all geodesics over  $S$ .*
2. *The set of all shortlex geodesics over  $S$ .*
3. *The set of all  $(\lambda, \mu)$ -quasigeodesics,  $Q_{S, \lambda, \mu} \subseteq S^*$ .*

*Furthermore, the set of all pairs of words  $(u, v) \in Q_{S, \lambda, \mu}^2$  such that  $u =_G v$  is accepted by an asynchronous 2-tape automaton.*

See [22, 29].



### 5.3 Main reduction result

Here is our key technical result.

► **Proposition 17** (Covering to full solution sets). *Let  $G$  be a hyperbolic group with finite symmetric generating set  $S$ . Let  $h : S \rightarrow S'$  and  $Q_{S,\lambda,\mu}$  be as defined above,  $\#, \wr$  symbols not in  $S \cup S'$ ,  $h(\#) = \#, h(\wr) = \wr$ , and  $\mathcal{T} \subseteq Q_{S,\lambda,\mu}$  a regular set of quasigeodesic words in bijection with  $G$ . Suppose  $L_1 \subseteq (S \cup S' \cup \{\#, \wr\})^*$  consists of words of the form*

$$u_1\# \dots \#u_r \wr h(v_1)\# \dots \#h(v_r), \quad u_i, v_i \in Q_{S,\lambda,\mu}, u_i =_G v_i, 1 \leq i \leq r.$$

If  $L_1$  is ETOL in NSPACE( $f(n)$ ), then

1.  $L_Q = \{w_1\# \dots \#w_r \wr h(z_1)\# \dots \#h(z_r) \mid \exists u_1\# \dots \#u_r \wr h(v_1)\# \dots \#h(v_r) \in L_1, w_i =_G z_i =_G u_i, w_i, z_i \in Q_{S,\lambda,\mu}\}$  is ETOL in NSPACE( $f(n)$ ).
2.  $L_{\mathcal{T}} = \{w_1\# \dots \#w_r \mid \exists u_1\# \dots \#u_r \wr h(v_1)\# \dots \#h(v_r) \in L_1, w_i =_G u_i, w_i \in \mathcal{T}\}$  is EDTOL in NSPACE( $f(n)$ ).

The proof involves a series of operations as in Proposition 9–11, see [8] for further details. Note that the set of all shortlex geodesics is a suitable choice for  $\mathcal{T}$  in the proposition.

## 6 Reduction from torsion-free hyperbolic to free groups

Section 3 contains an overview of the general algorithm for solving equations in torsion-free hyperbolic groups. Here we provide further details, and give a proof of the soundness and completeness of our algorithm. The algorithm relies on the existence and special properties of canonical representatives, whose construction is very technical (details are provided in [8]). Their existence guarantees that the solutions of a system in a torsion-free hyperbolic group generated by  $S$  can be found by solving an associated system in the free group on  $S$ , while the fact that they are quasigeodesics (see [8, Prop. 30]) allows us to apply the results of the previous sections to obtain the EDTOL characterisation of solutions in shortlex normal form.

► **Proposition 18.** *Let  $G$  be a torsion-free hyperbolic group, with finite symmetric generating set  $S$ . Let  $\Phi$  be a system of equations and inequations of input size  $n$  as in Section 2. Let  $h : S \rightarrow S', \#, \wr$  be as in Proposition 17. Then there exist  $\lambda \geq 1, \mu \geq 0$  and*

$$L = \{w_1\# \dots \#w_m \wr h(w_1)\# \dots \#h(w_m) \mid w_i \in Q_{S,\lambda,\mu}, 1 \leq i \leq m\}$$

such that  $\{w \mid w \wr h(w) \in L\}$  is a covering solution for  $\Phi$ , and  $L$  is EDTOL in NSPACE( $n^2 \log n$ ).

Applying Proposition 17 immediately gives Theorem 1.

**Proof.** We produce a language  $L$  of quasigeodesic words over  $S$  such that the projection of any tuple in  $L$  is in the group element solution set  $\text{Sol}_G(\Phi)$  (soundness). We then prove (using [39, Corollary 4.4]) that any solution in  $\text{Sol}_G(\Phi)$  is the projection of some tuple in  $L$  (completeness). Our proof follows the outline presented in Section 3.

### 1. Preprocessing.

- (Remove inequations) We first transform  $\Phi$  into a system consisting entirely of equations by adding a variable  $x_D$  to  $\mathcal{X}$  and replacing any inequation  $\varphi_j(\mathcal{X}, \mathcal{A}) \neq 1$  by  $\varphi_j(\mathcal{X}, \mathcal{A}) = x_D$ , with the constraint  $x_D \neq_G 1$ .

- (Triangulation) We transform each equation into several equations of length 3, by introducing new variables. This can always be done (see the discussion in [7, Section 4]), and it produces approximately  $\sum_{i=1}^s l_i \in O(n)$  triangular equations with set of variables  $\mathcal{Z}$  where  $m \leq |\mathcal{Z}| \in O(n)$  and  $\mathcal{X} \subset \mathcal{Z}$ . From now on assume that the system  $\Phi$  consists of  $q \in O(n)$  equations of the form  $X_j Y_j = Z_j$  where  $1 \leq j \leq q$ .
2. **Lifting  $\Phi$  to the free group on  $S$ .** In [39, Theorem 4.2] Rips and Sela define a constant, which they call “ $bp$ ”, that roughly bounds the circumference of the “centres” of the triangles whose edges are canonical representatives. We denote here  $bp$  by  $\rho$ , and note that  $\rho \in O(q) = O(n)$  depends on  $\delta$  and linearly on  $q$ . As described in Section 3 we run in lex order through all possible tuples of words  $\mathbf{c} = (c_{11}, c_{12}, c_{13}, \dots, c_{q1}, c_{q2}, c_{q3})$  with  $c_{ji} \in S^*$ ,  $\ell(c_{ji}) \leq \rho \in O(n)$ . For each tuple  $\mathbf{c}$  we use Dehn’s algorithm to check  $c_{j1}c_{j2}c_{j3} =_G 1$ , and if this holds for all  $1 \leq j \leq q$  we then construct a system  $\Phi_{\mathbf{c}}$  of equations of the form

$$X_j = P_j c_{j1} Q_j, Y_j = Q_j^{-1} c_{j2} R_j, Z_j = P_j c_{j3} R_j, \quad 1 \leq j \leq q, \quad (2)$$

which has size  $O(n^2)$ . In order to avoid an exponential size complexity we write down each system  $\Phi_{\mathbf{c}}$  one at a time, so the space required for this step is  $O(n^2)$ . Let  $\mathcal{Y} \supset \mathcal{Z} \supset \mathcal{X}$  be the new set of variables.

3. **Some observations.** We pause to make the following observations. Any solution to  $\Phi_{\mathbf{c}}$  in the free group  $F(S)$  is guaranteed to be a solution to  $\Phi$  in the original hyperbolic group  $G$ . Thus if  $S_1 \subseteq F(S)^m$  is a group element solution to  $\Phi_{\mathbf{c}}$  then  $\pi(S_1)$  is a group element solution to  $\Phi$  in  $G$ . This will show soundness below.

Secondly, if  $(g_1, \dots, g_m) \in G^m$  is a solution to  $\Phi$  in the original hyperbolic group, [39, Theorem 4.2 and Corollary 4.4] (see Theorem 31 in [8]) guarantees that there exist canonical representatives  $w_i \in Q_{S,\lambda,\mu}$  with  $w_i =_G g_i$  for  $1 \leq i \leq m$ , which have reduced forms  $u_i =_G w_i$  for  $1 \leq i \leq m$ , and our construction is guaranteed to capture any such collection of words. This will show completeness below.

Thirdly, note that the constraint that a word  $w \in S^*$  must be a  $(\lambda, \mu)$ -quasigeodesic and satisfy  $w =_G 1$  implies that  $\ell(w) \leq \mu$ . Therefore we can construct a DFA  $\mathcal{D}$  which accepts all words in  $S^*$  equal to 1 in the hyperbolic group  $G$  of length at most  $\mu$  in constant space (using for example Dehn’s algorithm). In our next step, we will use this rational constraint to handle the variable  $x_D$  added in the first step above (to remove inequalities).

Now let us complete the construction by finding the covering solution required.

4. **Covering solution set.** We now run the algorithm from [7] (which we will refer to as the CDE algorithm) which takes input  $\Phi_{\mathbf{c}}$ , which has size in  $O(n^2)$ , plus the rational constraint  $x_D \notin L(\mathcal{D})$ , plus for each  $y \in \mathcal{Y}$  the rational constraint that the solution for  $y$  is a word in  $Q_{S,\lambda,\mu}$ . Since these constraints have constant size (depending only on the group  $G$ , not the system  $\Phi$ ), they do not contribute to the  $O(n^2)$  size of the input to the CDE algorithm.

We make two modifications to the details of the CDE algorithm. First, every node printed by the algorithm should include the additional label  $\mathbf{c}$ . (This ensures the NFA we print for each system  $\Phi_{\mathbf{c}}$  is distinct.) This does not affect the complexity since  $\mathbf{c}$  has size in  $O(n^2)$ .

Second, so that we can apply Proposition 11 later, we modify the form of “extended equations” in [7] by inserting the factor  $\mathcal{h}(W)$  in the appropriate position(s). This simply increases the size of the nodes by a factor (of two).

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We run the CDE algorithm to print an NFA (possibly empty) for each  $\Phi_{\mathbf{c}}$ , which is the rational control for an EDTOL grammar that produces all solutions as freely reduced words for elements of  $F(S)$  which correspond to solutions as  $(\lambda, \mu)$ -quasigeodesics to the same system  $\Phi_{\mathbf{c}}$  in the hyperbolic group. If  $(w_1, \dots, w_m)$  is a solution in canonical representatives to  $\Phi$  then  $(u_1, \dots, u_m, \dots, u_{|Y|})$  will be included in the solution to  $\Phi_{\mathbf{c}}$  output by the CDE algorithm, with  $u_i$  the reduced forms of  $w_i$  for  $1 \leq i \leq m$ . This shows completeness once we union the grammars from all systems  $\Phi_{\mathbf{c}}$  together.

Adding a new start node with edges to each of the start nodes of the NFA's with label  $\mathbf{c}$ , we obtain a rational control for the EDTOL grammar generating  $L$  as required. The space required is exactly that required by the CDE algorithm on input  $O(n^2)$ , which is  $\text{NSPACE}(n^2 \log n)$ . ◀

### 7 Reduction from hyperbolic with torsion to virtually free groups

In the case of a hyperbolic group  $G$  with torsion, the general approach of Rips and Sela can still be applied, but the existence of canonical representatives is not always guaranteed (see Delzant [13, Rem.III.1]). To get around this, Dahmani and Guirardel “fatten” the Cayley graph  $\Gamma(G, S)$  of  $G$  to a larger graph  $\mathcal{K}$  which contains  $\Gamma(G, S)$  (in fact  $\Gamma(G, S)$  with midpoints of edges included), and solve equations in  $G$  by considering equalities of paths in  $\mathcal{K}$ . More precisely,  $\mathcal{K}$  is the 1-skeleton of the barycentric subdivision of a Rips complex of  $G$  (see [8] for definitions).

► **Definition 19.** *Let  $\gamma, \gamma'$  be paths in  $\mathcal{K}$ .*

- (i) *We denote by  $i(\gamma)$  the initial vertex of  $\gamma$ , by  $f(\gamma)$  the final vertex of  $\gamma$ , and by  $\bar{\gamma}$  the reverse of  $\gamma$  starting at  $f(\gamma)$  and ending at  $i(\gamma)$ .*
- (ii) *We say that  $\gamma$  is reduced if it contains no backtracking, that is, no subpath of length 2 of the form  $e\bar{e}$ .*
- (iii) *We write  $\gamma\gamma'$  for the concatenation of  $\gamma, \gamma'$  if  $i(\gamma') = f(\gamma)$ .*
- (iv) *Two paths in  $\mathcal{K}$  are homotopic if one can obtain a path from the other by adding or deleting backtracking subpaths. Each homotopy class has a unique reduced representative.*

Let  $V$  be the set of all homotopy classes  $[\gamma]$  of paths  $\gamma$  in  $\mathcal{K}$  with  $i(\gamma) = 1_G$ , and  $f(\gamma) \in G$ . For  $[\gamma], [\gamma'] \in V$  define their product  $[\gamma][\gamma'] = [\gamma^v\gamma']$ , where  $\gamma^v\gamma'$  denotes the concatenation of  $\gamma$  and the translate  ${}^v\gamma'$  of  $\gamma'$  by  $v = f(\gamma)$ , and let  $[\gamma]^{-1}$  be the homotopy class of  $v^{-1}\bar{\gamma}$ . Then  $V$  is a group that projects onto  $G$  by the final vertex map  $f$ , that is,  $f : V \twoheadrightarrow G$  is a surjective homomorphism. Moreover, since  $G$  has an action on  $\mathcal{K}$  induced by the natural action on its Rips complex,  $V$  will act on  $\mathcal{K}$  as well. This gives rise to an action of  $V$  onto the universal cover  $T$  (which is a tree) of  $\mathcal{K}$ , and [11, Lemma 9.9] shows that the quotient  $T/V$  is a finite graph (isomorphic to  $\mathcal{K}/G$ ) of finite groups, and so  $V$  is virtually free.

We assume that the algorithmic construction (see [11, Lemma 9.9]) of a presentation for  $V$  is part of the preprocessing of the algorithm, will be treated as a constant, and will not be included in the complexity discussion.

The first step in solving a system  $\Phi$  of equations in  $G$  is to translate  $\Phi$  into identities between quasigeodesic paths (with start and end point in  $G$ ) in  $\mathcal{K}$ , defined as  $\mathcal{QG}_{\lambda_1, \mu_1}(V)$  in (5) in [8], paths which can be seen as the analogues of the canonical representatives from the torsion-free case. This can be done by Proposition 9.8 [11]. The second step in solving  $\Phi$  is to express the equalities of quasigeodesic paths in  $\mathcal{K}$  in terms of equations in the virtually free group  $V$  based on  $\mathcal{K}$ . Finally, Proposition 9.10 [11] shows it is sufficient to solve the systems of equations in  $V$  in order to obtain the solutions of the system  $\Phi$  in  $G$ .

In the virtually free group  $V$  we will use the results from [15]. Let  $Y$  be the generating set of  $V$  and  $T \subseteq Y^*$  the set of normal forms for  $V$  over  $Y$  as in [15, Remark 44, page 50], and let

$$\text{Sol}_{T,V}(\Psi) = \{w_1 \# \dots \# w_m \in T^n \mid (\pi(w_1), \dots, \pi(w_m)) \text{ solves } \Psi \text{ in } V\}$$

be the language of  $T$ -solutions in  $V$  of a system  $\Psi$  of size  $|\Psi| = O(k)$ ; by [15] the language  $\text{Sol}_{T,V}(\Psi)$  consists of  $(\lambda_Y, \mu_Y)$ -quasigeodesics and is EDTOL in  $\text{NSPACE}(k^2 \log k)$  over  $Y$ .

► **Proposition 20.** *Let  $G$  be a hyperbolic group with torsion, with finite symmetric generating set  $S$ . Let  $\Phi$  be a system of equations and inequations with  $|\Phi| = n$  as in Section 2. Let  $h : S \rightarrow S', \#, \wr$  be as in Proposition 17. Then there exist  $\lambda \geq 1, \mu \geq 0$  and*

$$L = \{w_1 \# \dots \# w_m \wr h(v_1) \# \dots \# h(v_m) \mid w_i, v_i \in Q_{S,\lambda,\mu}, w_i =_G v_i, 1 \leq i \leq m\}$$

such that  $\{w \mid w \wr h(v) \in L\}$  is a covering solution for  $\Phi$ , and  $L$  is ETOL in  $\text{NSPACE}(n^4 \log n)$ .

Again applying Proposition 17 immediately gives Theorem 2.

Before proving Proposition 20 we need to show how one can translate between elements and words in  $V$  over the generating set  $Y$ , and elements and words in  $G$  over  $S$  via the graph  $\mathcal{K}$ , so that the EDTOL characterisation of languages is preserved.

► **Notation.** *Let  $Z$  be some generating set of  $V$  and let  $\pi : Z^* \rightarrow V$  be the standard projection map from words to group elements in  $V$ .*

- (i) *For each  $z_i \in Z$  there exists a unique reduced path  $p_i$  in  $\mathcal{K}$  with  $i(p_i) = 1_G$  and  $f(p_i) \in G$ ; by concatenation for each word  $w = z_{i_1} \dots z_{i_k}$  over  $Z$  there is then a unique path denoted*

$$p_w = p_{i_1} \dots p_{i_k} \tag{3}$$

with  $i(p_w) = 1_{\mathcal{K}} = 1_G$  and  $f(p_w) \in G$ .

- (ii) *For each  $z_i \in Z$ , assign a geodesic path  $\gamma_i$  in the Cayley graph  $\Gamma(G, S)$  such that  $i(\gamma_i) = 1_G$  and  $f(\gamma_i) = f(p_i) \in G$ , where  $p_i$  as in (i). Let  $\sigma : Z^* \rightarrow S^*$  be the map/substitution given by  $\sigma(z_i) = \gamma_i$ ; by concatenation one can associate to each word  $w = z_{i_1} \dots z_{i_k}$  over  $Z$  a path in  $\Gamma(G, S)$  denoted*

$$\gamma_w = \gamma_{i_1} \dots \gamma_{i_k} = \sigma(w) \tag{4}$$

with  $i(p_w) = 1_G$  and  $f(\gamma_w) = f(w) \in G$ .

- (iii) *There exists a unique reduced path, denoted  $p_{\pi(w)}$ , which is homotopic to  $p_w$ .*

**Proof of Proposition 20.** The algorithm to produce the language of solutions for  $\Phi$  is similar to that outlined in Section 3 and detailed in the proof of Proposition 18, but it applies to different groups. The triangulation of  $\Phi$  and introduction of a variable with rational constraint to deal with the inequations proceeds in the same manner. Again, we suppose after preprocessing we have  $q \in O(n)$  triangular equations.

Then for  $\kappa \in O(n)$  as in [8, Prop 36] define  $V_{\leq \kappa} = \{[\gamma] \in V \mid \gamma \text{ reduced and } \ell_{\mathcal{K}}(\gamma) \leq \kappa\}$ . One lifts the system  $\Phi$  in  $G$  to a finite set of systems  $\Psi_{\mathbf{c}}$  in the virtually free group  $V$ , one system for each  $q$ -tuple  $\mathbf{c}$  of triples  $(c_1, c_2, c_3)$  with  $c_i \in V_{\leq \kappa}$  and such that  $f(c_1 c_2 c_3) = 1_G$ , as in [8, Prop 36]. We enumerate these tuples by enumerating triples of words  $(v_1, v_2, v_3)$  over the generating set  $Y$  of  $V$  with  $\ell_Y(v_i) \leq \kappa_Y$ , where  $\kappa_Y \in O(q)$  is a constant depending on  $\kappa$ , as in Lemma 21(ii). By Lemma 21(ii) the tuples of path triples  $(p_{v_1}, p_{v_2}, p_{v_3})$  (see (3))

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in  $\mathcal{K}$  contain all  $q$ -tuples of triples  $(c_1, c_2, c_3)$  with  $c_i \in V_{\leq \kappa_Y}$ , up to homotopy. Then for each triple  $(v_1, v_2, v_3)$  we check whether  $f(v_1 v_2 v_3) = 1_G$ , and this is done by checking whether  $\sigma(v_1)\sigma(v_2)\sigma(v_3) =_G 1$  using the Dehn algorithm in  $G$ .

Then each system  $\Psi_{\mathbf{c}}$  is obtained as in (2) in the proof of Proposition 18 and has input size  $O(q^2) \in O(n^2)$  since it has  $O(q)$  equations, each of length in  $O(q)$ , and the factors  $c_i$  inserted also have length in  $O(q)$ . For each system  $\Psi_{\mathbf{c}}$  over  $V$  we apply the algorithm in [15] and obtain the set of solutions  $\text{Sol}_{T,V}(\Psi_{\mathbf{c}})$  as an EDTOL in  $\text{NSPACE}((q^2)^2 \log(q^2)) = \text{NSPACE}(n^4 \log n)$  of  $(\lambda_Y, \mu_Y)$ -quasigeodesics over  $Y$ .

Now let  $\mathcal{Q}\text{Sol}_{T,V}(\Psi_{\mathbf{c}})$  be the set of all  $(\lambda'_1, \mu'_1)$ -quasigeodesics which represent solutions of  $\Psi_{\mathbf{c}}$  in  $V$  over  $Y$ . By Proposition 17 this language is ETOL and by Corollary 22 it contains at least one word over  $Y$  for each solution in  $\mathcal{Q}\mathcal{G}_{\lambda_1, \mu_1}(V)$ .

Then  $\sigma(\mathcal{Q}\text{Sol}_{T,V}(\Psi_{\mathbf{c}}))$  is ETOL since ETOL languages are preserved by substitutions, and by [8, Prop 36]  $\mathcal{S} = \cup_{\mathbf{c}} \sigma(\mathcal{Q}\text{Sol}_{T,V}(\Psi_{\mathbf{c}}))$  contains  $\text{Sol}_G(\Phi)$ , so it is a covering solution set of  $\Phi$ . By Lemma 23 the set  $\mathcal{S}$  consists of at least one  $(\lambda_G, \mu_G)$ -quasigeodesic over  $S$  for each solution, and then by intersection with the regular set  $\mathcal{Q}_{S, \lambda_G, \mu_G}$  of quasigeodesics in  $G$  over  $S$  we obtain a set of solutions for  $\Phi$  consisting of  $(\lambda_G, \mu_G)$ -quasigeodesics.

Finally, we run the modified DE algorithm (inserting the additional  $\mathcal{H}(W)$  and label  $\mathbf{c}$  for each node printed) to print an NFA for each  $\Phi_{\mathbf{c}}$  for the EDTOL grammar for  $\text{Sol}_{T,V}(\Psi_{\mathbf{c}})$ , which we union using an extra start node as before. From the above work this grammar generates the language  $L$  as required.  $\blacktriangleleft$

### ► Lemma 21.

- (i) If  $c \in V$  and the reduced path representing  $c$  in  $\mathcal{K}$  is an  $(a, b)$ -quasigeodesic, then there exists a word  $w$  on  $Y$  representing  $c$  such that  $w$  is an  $(a', b')$ -quasigeodesic, where  $a', b'$  depend on  $a, b$  and  $Y$ .
- (ii) If  $c \in V$  and the length of the reduced path representing  $c$  in  $\mathcal{K}$  is  $\leq L$ , then there exists a word  $w$  on  $Y$  representing  $c$  such that  $\ell_Y(w) \leq L_Y$ , where  $L_Y$  depends on  $L$  and  $Y$ .

► **Corollary 22.** For any element  $v \in \mathcal{Q}\mathcal{G}_{\lambda_1, \mu_1}(V)$  there is a  $(\lambda'_1, \mu'_1)$ -quasigeodesic word over  $Y$  representing  $v$ , where  $\lambda'_1, \mu'_1$  depend on  $\lambda_1, \mu_1$  and  $Y$ .

► **Lemma 23.** Let  $w$  be a  $(\lambda'_1, \mu'_1)$ -quasigeodesic word over  $Y$ . Then if the reduced path  $p_{\pi(w)}$  is  $(a, b)$ -quasigeodesic in  $\mathcal{K}$  the (unreduced) path  $p_w$  is  $(a_{\mathcal{K}}, b_{\mathcal{K}})$ -quasigeodesic in  $\mathcal{K}$ , where  $(a_{\mathcal{K}}, b_{\mathcal{K}})$  depend on  $a, b, \lambda'_1, \mu'_1$  and  $Y$ .

Moreover,  $\sigma(w)$  is a  $(\lambda_G, \mu_G)$ -quasigeodesic over  $S$  in the hyperbolic group  $G$ , where  $\lambda_G, \mu_G$  depend on  $\lambda_1, \mu_1$  and  $Y$ .

**Proof.** Consider the generating set  $Z = Y \cup V_{\leq 3}$  for  $V$  and let  $\lambda_Z, \mu_Z$  be such that any  $(\lambda'_1, \mu'_1)$ -quasigeodesic over  $Y$  is  $(\lambda_Z, \mu_Z)$ -quasigeodesic over  $Z$ . Let  $M = \max\{\ell_{\mathcal{K}}(p_y) \mid y \in Y\}$ . That is,  $M$  is the maximal length of a generator in  $Y$  with respect to the associated reduced path length in  $\mathcal{K}$ . We will show the statement in the lemma holds for  $(a_{\mathcal{K}}, b_{\mathcal{K}}) = (a, b + M\mu_Z)$ .

We say that a subpath  $s_w$  of  $p_w$  is a maximal backtrack if  $p_w = ps_w p'$ ,  $s_w$  is homotopic to an empty path (via the elimination of backtrackings), and  $s_w$  is not contained in a longer subpath of  $p_w$  with the same property. This implies there is a point  $A$  on  $p_w$  such that  $s_w$  starts and ends at  $A$ , and such a maximal backtrack traces a tree in  $\mathcal{K}$ . We can then write  $p_w = a_1 s_1 a_2 \dots s_{n-1} a_n$ , where  $a_i$  are (possibly empty) subpaths of  $p_w$  and  $s_i$  are maximal backtracks; thus  $p_{\pi(w)} = a_1 a_2 \dots a_n$ . If  $\ell_{\mathcal{K}}(s_i) \leq M\mu_Z$  for all  $i$ , then the result follows immediately. Otherwise there exists an  $s_i$  with  $\ell_{\mathcal{K}}(s_i) > M\mu_Z$ , and we claim that we can write  $s_i$  in terms of a word over  $Z$  that is not a quasigeodesic, which contradicts the assumption that  $w$  is quasigeodesic.

To prove the claim, suppose  $i(s_i) = f(s_i) = A$ . We have two cases: in the first case  $A \in G$  then  $\pi(s_i) =_V 1$  and  $s_i$  corresponds to a subword  $v$  of  $w$  for which  $l_{\mathcal{K}}(p_v) \geq M\mu_Z$ . But  $v$  represents a word over  $Z$ , so  $l_{\mathcal{K}}(p_v) \leq l_Z(v)M$ , and altogether  $M\mu_Z \leq l_{\mathcal{K}}(p_v) \leq l_Z(v)M$ . Since  $|v|_Z = 0$  and  $v$  is a  $(\lambda_Z, \mu_Z)$ -quasigeodesic word over  $Z^*$ ,  $l_Z(v) \leq \mu_Z$ , which contradicts  $l_Z(v) \geq \mu_Z$  from above.

In the second case  $A \notin G$ , so take a point  $B \in G$  at distance 1 from  $A$  in  $\mathcal{K}$  (this can always be done), and modify the word  $w$  to get  $w'$  over  $Z$  so that  $p_{w'}$  in  $\mathcal{K}$  includes the backtrack  $[AB, BA]$  off the path  $p_w$ . Also modify  $s_i$  to obtain a new backtrack  $s'_i$ . Clearly  $\pi(p_w) = \pi(p_{w'})$  and  $\pi(s_i) = \pi(s'_i)$ , and  $s'_i$  becomes a maximal backtrack of  $p_{w'}$  which can be written as a word over the generators  $Z$  that represents the trivial element in  $V$ . We can then apply the argument from the first case.

The fact that  $\sigma(w)$  is a  $(\lambda_G, \mu_G)$ -quasigeodesic over  $S$  in the hyperbolic group  $G$  follows from the fact that  $\mathcal{K}$  and  $\Gamma(G, S)$  are quasi-isometric.  $\blacktriangleleft$

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