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Thirty-Five Moment Theory for Dilute Smooth Hard Sphere Gases

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Abstract. This paper derives a higher-order transport equations for rarefied gases and are developed within the framework of kinetic theory. In particular, a thirty-five moment theory by considering all zeroth-order to full fourth-order velocity moments of the particle distribution function for a dilute smooth hard-sphere gas is derived by employing the well-known moment closure approach, namely, Grad's method of moments. The linear stability analysis is performed on the the derived Grad type 35-moment equations to examine the spatial and temporal stability for small perturbations in space and time, respectively. Based on the plane wave solution analysis, we confirm that the thirty-five moment system is spatially and temporally stable.

INTRODUCTION

Modeling transport phenomena in a gas under rarefied conditions is one of the hardest problems in computational fluid dynamics. One needs to develop accurate hydrodynamic models that can be used for the modeling of flows in the rarefied regime - the well-known standard hydrodynamic equations are inappropriate since continuum hypotheses are violated. Rarefied gas flows are well described by the Boltzmann equation and the solution to the same via Discrete Simulation Monte Carlo (DSMC) technique will be more useful, but this method is restrictive with regards to computational requirements [1, 2]. Due to the complexities associated in solving Boltzmann equation, there have been significant attempts made to formulate alternative solution methods that can provide an accurate description of a gas in the rarefied regime. Further, modeling rarefied or non-equilibrium gas flows requires a 'beyond Navier-Stokes-order' description in terms of an extended set of hydrodynamic fields [3, 4].

Moment closure methods play an important role in handling the behaviour of non-equilibrium gases by assuming the distribution function with more degrees of freedom. In general it is assumed that the addition of more moments in a closure approximation gives rise to a system of moment equations, which approximate non-equilibrium flows accurately [4]. Moment closure approximations render a protracted range of physical credibility over the standard continuum approaches and they can model non-equilibrium gas flows with minify expense as compared to particle-based approaches [5]. A best known classical technique for obtaining a closed moment system is the Grad's moment method [6], in which the distribution function is expanded in terms of Hermite polynomials in the components of the fluctuating velocity with the Maxwellian as weight function. The method introduces new sets of unknowns such as stress tensor, heat fluxes and higher-order moments of the distribution function. The closure can be obtained by truncating the Hermite polynomial expansion. By using the truncated distribution function, one can obtain a closed moment system up to the desired moments of interest. In this work, we employ the Grad's method of moments to develop higher-order moment equations - up to first 35-moments, for dilute smooth hard-sphere molecular gas. The current work is initially motivated from the previous work of S. L. Brown [7], who derived 35-moment model in which the original Boltzmann collision term is approximated by a well-known BGK approximation and based on the expansion similar to that of Grad procedure [6], but about an anisotropic Gaussian instead of the Maxwellian.

This paper is organized as follows. In the next section we briefly discuss about the kinetic theory of hard-sphere gases and derive an extended hydrodynamic equations, namely, 35-moment model based on Grad's method of moments. The following section is devoted to check the spatial and temporal stability of 35-moment equations. Finally, conclusions are drawn at the end.

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KINETIC THEORY AND EXTENDED HYDRODYNAMIC EQUATIONS

The fundamental equation of kinetic theory of gases is the Boltzmann equation that governs the dynamics of a gas. We consider a dilute gas of smooth hard spheres of mass m and diameter d that interact via binary collisions. The state of the gas is described by the single-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ which characterizes the spatial and velocity distribution of individual hard sphere particles. The evolution of $f(\mathbf{x}, \mathbf{v}, t)$ is governed by the Boltzmann equation. In the absence of external forces the Boltzmann equation, reads

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} = d^2 \int_{\mathbf{g}\cdot\mathbf{k}>0} \left[f(\mathbf{x}, \mathbf{v}_1'', t)\, f(\mathbf{x}, \mathbf{v}'', t) - f(\mathbf{x}, \mathbf{v}_1, t)\, f(\mathbf{x}, \mathbf{v}, t) \right] (\mathbf{g}\cdot\mathbf{k})\, \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{v}_1, \tag{1}$$

where we have made use of the assumption of molecular chaos in the right hand side of (1), which is the Boltzmann collision integral term and it accounts for the rate of change of particle distribution function f due to binary collisions. In Eq. (1), $(\mathbf{v}, \mathbf{v}_1)$ and $(\mathbf{v}'', \mathbf{v}_1'')$ represents the pre-collisional velocities of colliding pair of particles for direct and inverse collisions, respectively; $\mathbf{g} = (\mathbf{v}_1 - \mathbf{v})$ denote the relative velocity of colliding particles; $\mathbf{k}$ is the unit contact-vector pointing from the center of the particle denoted by index 1 to the center of the other particle without index. In the following we provide a brief sketch of the derivation of extended hydrodynamic equations in terms of the first 35-moments, starting from the Boltzmann equation (1).

The macroscopic field variables are obtained via a coarse-graining procedure over the single-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$. For any polynomial of the particle velocity $\psi(\mathbf{v})$, its coarse-grained average value which is referred as it's moment with the particle distribution function is defined as

$$\langle \psi(\mathbf{v}) \rangle = \int \psi(\mathbf{v}) f(\mathbf{x}, \mathbf{v}, t) \mathrm{d}\mathbf{v}. \tag{2}$$

Recall that to obtain the well-known 13-moment theory of Grad [6], one has to take $\psi = m\{1, v_i, \frac{1}{3}C^2, C_{\langle i}C_{j\rangle}, \frac{1}{2}C^2 C_i\}$ that yields the relevant field variables for the 13-moment system: density (ρ), momentum $(\rho\, u_i)$, temperature (θ), stress tensor (σ_{ij}), heat-flux vector (q_i), where $\mathbf{C} = \mathbf{v} - \mathbf{u}$ is the peculiar velocity. In the realm of obtaining higher-order extended hydrodynamics, namely the 35-moment system, one has to consider ψ to be

$$\psi = m\{1, v_i, \frac{1}{3}C^2, C_{\langle i}C_{j\rangle}, \frac{1}{2}C^2\, C_i, C^4, C_{\langle i}C_jC_{k\rangle}, C^2 C_{\langle i}C_{j\rangle}, C_{\langle i}C_jC_kC_{l\rangle}\}, \tag{3}$$

in which the following additional higher moments of the velocity distribution function are added to the existed 13-moment system of Grad.

$$\left.\begin{aligned} Q_{ijk} &= \int m\, C_{\langle i}C_jC_{k\rangle} f(\mathbf{x}, \mathbf{v}, t)\mathrm{d}\mathbf{v}, \\ \mathcal{R} &= \int m\, C^4 f(\mathbf{x}, \mathbf{v}, t)\mathrm{d}\mathbf{v}, \\ \mathcal{R}_{ij} &= \int m\, C^2 C_{\langle i}C_{j\rangle} f(\mathbf{x}, \mathbf{v}, t)\mathrm{d}\mathbf{v}, \\ \mathcal{R}_{ijkl} &= \int m\, C_{\langle i}C_jC_kC_{l\rangle} f(\mathbf{x}, \mathbf{v}, t)\mathrm{d}\mathbf{v}. \end{aligned}\right\} \tag{4}$$

The transport equations for the macroscopic field variables can be derived from the Boltzmann equation (1). The transport equation for any moment $\psi(\mathbf{v})$ is now obtained by multiplying the Boltzmann equation (1) by $\psi(\mathbf{v})$ and integrating the resulting equation over the velocity space $\mathbf{v}$. This leads to the following moment transfer (coarse-grained) equation

$$\frac{\partial}{\partial t}\int \psi\, f\, \mathrm{d}\mathbf{v} + \underline{\frac{\partial}{\partial x_i}\int \psi\, \mathbf{v}_i\, f\, \mathrm{d}\mathbf{v}} - \int \left[\frac{\partial \psi}{\partial t} + \mathbf{v}_i \frac{\partial \psi}{\partial x_i} \right] f \mathrm{d}\mathbf{v} = \frac{d^2}{2} \int \mathrm{d}\mathbf{v} \int \mathrm{d}\mathbf{v}_1 \int_{(\mathbf{g}\cdot\mathbf{k})>0} \mathrm{d}\mathbf{k}\, (g\, \cdot\, k)\, f_1\, f\, \mathbf{\Xi}[\psi(\mathbf{v})], \tag{5}$$

where

$$\mathbf{\Xi}\,[\psi(\mathbf{v})] = \left[\psi(\mathbf{v}') + \psi(\mathbf{v}_1') - \psi(\mathbf{v}) - \psi(\mathbf{v}_1) \right], \tag{6}$$

with $(\mathbf{v}, \mathbf{v}_1)$ and $(\mathbf{v}', \mathbf{v}_1')$ represents the pre and post collisional velocities of collision pair particles. Note that in the above moment transfer equation (5), we have made use of the symmetric properties of the collision integral.

It is noteworthy to mention here that the moment transfer equation (5) is not closed due to the presence of underlined term in it. In other words, if one obtains an n^{th} order system of moment transport equations from the moment transfer equation (5) then there is an $(n+1)^{th}$ order velocity moment contained in the moment system. Hence, an constitutive equation or relation is required which governs the transport of that $(n+1)^{th}$ higher-order moment. This problem of closure is remedied if the form of the non-equilibrium distribution function f is known in terms of n^{th} order moments so that the highest-order velocity moment contained in the moment system can be related to the lower-order moment quantities. At the end of this section, based on the Hermite polynomial expansion we derive the non-equilibrium distribution function up to fourth-order in velocity moments. In the next subsection, we write down the macroscopic transport equations for first 35-moments.

Macroscopic transport equations for 35-moments

The extended hydrodynamic equations for 35-moments $(\rho, \rho u_i, \theta, \sigma_{ij}, q_i, \Delta, \mathcal{Q}_{ijk}, \mathcal{R}_{ij}, \mathcal{R}_{ijkl})$ are obtained with the choice of $\psi(\mathbf{v})$ given in (3). One can derive the macroscopic transport equations for the 35-moment system by making use of the moment transfer equation (5) as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0, \tag{7}$$

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial \left(\rho u_i u_j\right)}{\partial x_j} + \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \tag{8}$$

$$\rho \left(\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i}\right) + \frac{2}{3}\left(\rho \theta \frac{\partial u_i}{\partial x_i} + \sigma_{ij} \frac{\partial u_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i}\right) = 0, \tag{9}$$

$$\frac{\partial \sigma_{ij}}{\partial t} + \frac{\partial (\sigma_{ij} u_k)}{\partial x_k} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + 2p\frac{\partial u_{\langle i}}{\partial x_{j\rangle}} + 2\sigma_{k\langle i}\frac{\partial u_{j\rangle}}{\partial x_k} + \frac{\partial \mathcal{Q}_{ijk}}{\partial x_k} = \sigma^s_{ij}, \tag{10}$$

$$\begin{aligned}\frac{\partial q_i}{\partial t} + \frac{\partial (q_i u_j)}{\partial x_j} - \frac{5}{2}\theta\left(\rho\frac{\partial \theta}{\partial x_i} + \theta\frac{\partial \rho}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j}\right) - \frac{\sigma_{ij}}{\rho}\left(\rho\frac{\partial \theta}{\partial x_j} + \theta\frac{\partial \rho}{\partial x_j} + \frac{\partial \sigma_{jk}}{\partial x_k}\right)\\ + \frac{7}{5}q_j\frac{\partial u_i}{\partial x_j} + \frac{2}{5}q_i\frac{\partial u_j}{\partial x_j} + \frac{2}{5}q_k\frac{\partial u_k}{\partial x_i} + \frac{1}{2}\frac{\partial \mathcal{R}_{ij}}{\partial x_j} + \frac{1}{6}\frac{\partial \mathcal{R}}{\partial x_i} + \mathcal{Q}_{ijk}\frac{\partial u_j}{\partial x_k} = q^s_i,\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial \mathcal{Q}_{ijk}}{\partial t} + \frac{\partial \left(\mathcal{Q}_{ijk} u_l\right)}{\partial x_l} + \frac{\partial \mathcal{R}_{ijkl}}{\partial x_l} + \frac{3}{7}\frac{\partial \mathcal{R}_{\langle ij}}{\partial x_{k\rangle}} - \frac{3}{\rho}\sigma_{\langle ij}\frac{\partial p}{\partial x_{k\rangle}} - \frac{3}{\rho}\sigma_{\langle ij}\frac{\partial \sigma_{k\rangle l}}{\partial x_l}\\ + 3\mathcal{Q}_{l\langle ij}\frac{\partial u_{k\rangle}}{\partial x_l} + \frac{12}{5}q_{\langle i}\frac{\partial u_{k\rangle}}{\partial x_j} = \mathcal{Q}^s_{ijk},\end{aligned} \tag{12}$$

$$\frac{\partial \mathcal{R}}{\partial t} + \frac{\partial (\mathcal{R} u_i)}{\partial x_i} + \underline{\frac{\partial \mathcal{S}_i}{\partial x_i}} - \frac{8}{\rho}q_i\frac{\partial (p\delta_{ij} + \sigma_{ij})}{\partial x_j} + 4\mathcal{R}_{ij}\frac{\partial u_j}{\partial x_j} + \frac{4}{3}\mathcal{R}\frac{\partial u_i}{\partial x_i} = \mathcal{R}^S, \tag{13}$$

$$\begin{aligned}\frac{\partial \mathcal{R}_{ij}}{\partial t} + \frac{\partial \left(\mathcal{R}_{ij} u_k\right)}{\partial x_k} + \underline{\frac{\partial \mathcal{N}_{ijk}}{\partial x_k} + \frac{2}{5}\frac{\partial \mathcal{S}_{\langle i}}{\partial x_{j\rangle}}} + 2\mathcal{R}_{ijkl}\frac{\partial u_l}{\partial x_k} + \frac{14}{15}\mathcal{R}\frac{\partial u_{\langle i}}{\partial x_{j\rangle}} + 2\mathcal{R}_{k\langle i}\frac{\partial u_{j\rangle}}{\partial x_k}\\ + \frac{4}{5}\mathcal{R}_{k\langle i}\frac{\partial u_k}{\partial x_{j\rangle}} + \frac{6}{7}\mathcal{R}_{\langle ij}\frac{\partial u_{k\rangle}}{\partial x_k} - \frac{2}{\rho}\mathcal{Q}_{ijk}\frac{\partial p_{kl}}{\partial x_l} - \frac{28}{5\rho}q_{\langle i}\frac{\partial p_{j\rangle k}}{\partial x_k} = \mathcal{R}^s_{ij},\end{aligned} \tag{14}$$

$$\frac{\partial \mathcal{R}_{ijkl}}{\partial t} + \frac{\partial \left(\mathcal{R}_{ijkl} u_k\right)}{\partial x_k} + \underline{\frac{\partial \mathcal{N}_{ijklm}}{\partial x_k} + \frac{4}{9}\frac{\partial \mathcal{N}_{\langle ijk}}{\partial x_{l\rangle}}} - 4\frac{\mathcal{Q}_{\langle ijk}}{\rho}\frac{\partial p_{l\rangle m}}{\partial x_m} + 4\mathcal{R}_{m\langle ijk}\frac{\partial u_{l\rangle}}{\partial x_m} + \frac{12}{7}\mathcal{R}_{\langle ij}\frac{\partial u_k}{\partial x_{l\rangle}} = \mathcal{R}^s_{ijkl}, \tag{15}$$

where the angular brackets over subscripts denote traceless part of respective tensors; for example, traceless part of any second order tensor A_{ij} is given by

$$A_{\langle ij\rangle} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) - \frac{1}{3}A_{kk}\,\delta_{ij}, \tag{16}$$

which is traceless and δ_{ij} is the identity tensor. In above equations we also introduced pressure tensor p_{ij} which can be decomposed as it's trace part pressure (p) and deviatoric part (σ_{ij}) as $p_{ij} = p\,\delta_{ij} + \sigma_{ij}$. The underlined terms on the left-hand-side in Eq. (13) - Eq. (15) are of "higher-order" moments and are given by

$$\left.\begin{array}{rcl}\mathcal{S}_i &=& \int m\, C^2 C^2 C_i f(\mathbf{x},\mathbf{v},t)\mathrm{d}\mathbf{v},\\ \mathcal{N}_{ijk} &=& \int m\, C^2 C_{\langle i}C_j C_{k\rangle} f(\mathbf{x},\mathbf{v},t)\mathrm{d}\mathbf{v},\\ \mathcal{N}_{ijklm} &=& \int m\, C_{\langle i}C_j C_k C_l C_{m\rangle} f(\mathbf{x},\mathbf{v},t)\mathrm{d}\mathbf{v}.\end{array}\right\} \tag{17}$$

The source terms on the right-hand-side of Eq. (10) - Eq. (15) are given by

$$\sigma^s_{ij} = \frac{md^2}{2}\int_{\mathbf{g}\cdot\mathbf{k}>0} \Xi\,[\,C_{\langle i}\,C_{j\rangle}]\,f_1\,f\,(\mathbf{g}\cdot\mathbf{k})\,\mathrm{d}\mathbf{k}\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}, \tag{18}$$

$$q^s_i = \frac{md^2}{4}\int_{\mathbf{g}\cdot\mathbf{k}>0} \Xi\,[\,C^2\,C_i\,]\,f_1\,f\,(\mathbf{g}\cdot\mathbf{k})\,\mathrm{d}\mathbf{k}\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}, \tag{19}$$

$$Q^s_{ijk} = \frac{md^2}{2}\int_{\mathbf{g}\cdot\mathbf{k}>0} \Xi\,[\,C_{\langle i}C_jC_{k\rangle}\,]\;f_1\,f\,(\mathbf{g}\cdot\mathbf{k})\,\mathrm{d}\mathbf{k}\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}, \tag{20}$$

$$\mathcal{R}^s = \frac{md^2}{2}\int_{\mathbf{g}\cdot\mathbf{k}>0} \Xi\,[\,C^2\,C^2\,]\,f_1\,f\,(\mathbf{g}\cdot\mathbf{k})\,\mathrm{d}\mathbf{k}\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}, \tag{21}$$

$$\mathcal{R}^s_{ij} = \frac{md^2}{2}\int_{\mathbf{g}\cdot\mathbf{k}>0} \Xi\,[\,C^2 C_{\langle i}C_{j\rangle}\,]\,f_1\,f\,(\mathbf{g}\cdot\mathbf{k})\,\mathrm{d}\mathbf{k}\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}, \tag{22}$$

$$\mathcal{R}^s_{ijkl} = \frac{md^2}{2}\int_{\mathbf{g}\cdot\mathbf{k}>0} \Xi\,[\,C_{\langle i}C_jC_kC_{l\rangle}\,]\,f_1\,f\,(\mathbf{g}\cdot\mathbf{k})\,\mathrm{d}\mathbf{k}\mathrm{d}\mathbf{v}_1\mathrm{d}\mathbf{v}. \tag{23}$$

Equations (7), (8) and (9) represent the conservation laws for mass, momentum and energy, respectively, while equations (10), (11), (12), (13), (14) and (15) represent the balance equations for the deviatoric part of the pressure tensor, the heat-flux vector, the deviatoric part of third-order moment, the fourth-order fully contracted moment, the single contracted fourth-order moment and the deviatoric part of fourth-order moment, respectively. Note that equations (13 – 15) do not form a closed set since they contain additional higher moments, $\mathcal{S}_i$, $\mathcal{N}_{ijk}$, and $\mathcal{N}_{ijklm}$, as defined in (17), which in turn require knowledge about the non-equilibrium single-particle distribution function.

Higher-order distribution function: constitutive relations and source terms

In order to close the 35-moment system [Eq. (7) - Eq. (15)], the non-equilibrium distribution function up to fourth-order in velocity moments is obtained from a Hermite expansion [6] around the equilibrium distribution function as

$$f(\mathbf{x},\mathbf{v},t) = f^{\mathrm{eq}}\sum_{i=0}^{N}\frac{1}{i!}a^{(i)}\,\mathcal{H}^{(i)} \;\approx\; f^{\mathrm{eq}}\left[a^{(0)}\,\mathcal{H}^{(0)} + a^{(1)}_i\,\mathcal{H}^{(1)}_i + \frac{1}{2!}a^{(2)}_{ij}\,\mathcal{H}^{(2)}_{ij} + \frac{1}{3!}a^{(3)}_{ijk}\,\mathcal{H}^{(3)}_{ijk} + \frac{1}{4!}a^{(4)}_{ijkl}\,\mathcal{H}^{(4)}_{ijkl}\right] \tag{24}$$

with $f^{\mathrm{eq}} = \left(n/(2\,\pi\,\theta)^{\frac{3}{2}}\right)e^{-C^2/(2\,\theta)}$ being the equilibrium distribution function; $n = \rho/m$ is the particle number density; $\mathcal{H}^{(i)}$ denotes i^{th}-order Hermite polynomials in components of particle velocity and $a^{(i)}$ are the corresponding expansion coefficients which are related to the moments of the distribution function. Note that the number of terms N retained in the above expansion (24) is dictated by the physical considerations and it is well-known that in the framework of extended hydrodynamics [3, 6] the macroscopic state of a gas can be characterized by the ten, thirteen, fourteen, twenty one, twenty six and thirty five basic field variables for 10-moment, 13-moment, 14-moment, 21-moment, 26-moment and 35-moment closures, respectively. The resulting distribution function for 35-moment theory reads

$$\begin{aligned} f|_{35} \;=\; f^{\mathrm{eq}}\Bigg[1 &+ \frac{\sigma_{ij}}{2\,\rho\,\theta^2}\,C_i\,C_j + \frac{q_i}{5\,\rho\,\theta^3}\left(C^2 - 5\,\theta\right)C_i + \underline{\left(\frac{C^4 - 10\,C^2\,\theta + 15\,\theta^2}{8\,\theta^2}\right)\Delta} \\ &+ \underline{\frac{Q_{ijk}}{6\,\rho\,\theta^3}\,C_i\,C_j\,C_k} + \underline{\frac{(\mathcal{R}_{ij} - 7\,\theta\,\sigma_{ij})}{28\,\rho\,\theta^4}(\,C^2 - 7\,\theta\,)\,C_i\,C_j} + \underline{\frac{\mathcal{R}_{ijkl}}{24\,\rho\,\theta^4}\,C_i\,C_j\,C_k\,C_l}\Bigg]. \end{aligned} \tag{25}$$

In the above Eq. (25), we have introduced the dimensionless non-equilibrium part of the fully contracted fourth moment $\mathcal{R}$ denoted by Δ for convenient and it is defined via [8]

$$\Delta = \frac{1}{15\,\rho\,\theta^2}(\mathcal{R} - \mathcal{R}^{eq}) = \frac{1}{15\,\rho\,\theta^2}\int m\,C^4\,(f - f^{\rm eq})\,{\rm d}\mathbf{v}, \tag{26}$$

and it is straightforward to verify that $\mathcal{R}^{eq} = 15\rho\theta^2$ and at equilibrium Δ vanishes. Omitting all the underlined terms in (25) results in the well-known distribution function for 13-moment theory of Grad [6]. Deleting last three underlined terms in (25) gives the distribution function for 14-moment theory [8] and keeping the terms up to (i) the first two underlined terms and (ii) the first three underlined terms in (25) results in the distribution function for 21-moment and 26-moment theories [9], respectively.

Now using knowledge of the non-equilibrium distribution (25), we can determine the constitutive relations of higher-order moments (17) in terms of lower-order moments. The higher-order moments (17) which are required to close the 35-moment system are evaluated as

$$\mathcal{S}_{i|35} = 28\,\theta\,q_i, \qquad \mathcal{N}_{ijk|35} = 9\,\theta Q_{ijk}, \qquad \mathcal{N}_{ijklm|35} = 0, \tag{27}$$

with the subscript 35 denoting that (27) holds for the 35-moment theory. The collisional source terms (18) - (23) are evaluated as:

$$\sigma^s_{ij} = -\frac{16}{5\,\tau_r}\,\sigma_{ij}, \qquad q^s_i = -\frac{32}{15\,\tau_r}\,q_i, \qquad Q^s_{ijk} = -\frac{24}{5\,\tau_r}\,Q_{ijk}, \tag{28}$$

$$\mathcal{R}^s = -\frac{32}{\tau_r}\,\rho\,\theta^2\,\Delta, \qquad \mathcal{R}^s_{ij} = -\frac{2}{105\,\tau_r}\left(247\,\mathcal{R}_{ij} - 469\,\theta\,\sigma_{ij}\right), \qquad \mathcal{R}^s_{ijkl} = -\frac{1832}{315\,\tau_r}\,\mathcal{R}_{ijkl}. \tag{29}$$

Note that we have considered only linear terms of respective moments while evaluating the source terms and the integrals are calculated by using the computer algebra software *MATHEMATICA* and following the work of Gupta & Torrilhon [10]. Note that the linear source terms for beyond 26-moments for dilute hard-sphere gas and also for dilute granular gas has been derived in the PhD thesis of M. H. L. Reddy [11] and the full nonlinear source terms for 26-moment theory are evaluated for both dilute molecular gas as well as for dilute granular gas in [9, 10]. In the above equations (28) – (29), τ_r is a relaxation time given by

$$\tau_r = \frac{m}{\rho\,d^2\,\sqrt{\pi\,\theta}}. \tag{30}$$

Finally, insertion of Eq. (27) – Eq. (29) into the balance equations (7) – (15) yields a closed set of 35-moment equations for a smooth hard-sphere dilute molecular gas.

LINEAR STABILITY ANALYSIS

In this section, we examine the stability of the 35-moment system along with two other lower-order moment theories, namely, Grad 13-moment system and Grad type 21-moment system to small perturbations. We consider 35-moment model derived in the previous section in a one dimensional flow configuration. For this case, all field variables depend only on single spatial co-ordinate x and time t. The one-dimensional field equations for 35-moment system, involves nine field variables namely, the density $\rho(x,t)$, the velocity $u(x,t)$, the scalar temperature $\theta(x,t)$, the longitudinal stress $\sigma(x,t)$, the heat flux $q(x,t)$, the $Q(x,t)$ [i.e, $Q_{111}(x,t)$], the dimensionless fourth-order fully contracted moment $\Delta(x,t)$, the single contracted fourth order moment $\mathcal{R}_1(x,t)$ [i.e,$\mathcal{R}_{11}(x,t)$] and the full traceless part $\mathcal{R}_2(x,t)$ [i.e,$\mathcal{R}_{1111}(x,t)$].

Dimensionless linearized 35-moment system

For linear stability analysis, the 35-moment model is linearized by introducing small perturbation from equilibrium ground state. An equilibrium ground state is defined by the flow variables ρ_0, $u_0 = 0$, θ_0, $p^0 = \rho_0\,\theta_0$, $u^0 = 0$ and all remaining field variables vanish at the equilibrium. A perturbation to the equilibrium ground state is introduced as follows:

$$\rho = \rho^0\,(1 + \rho^*), \quad \theta = \theta_0\,(1 + \theta^*), \quad u = \sqrt{\theta_0}\,u^*, \quad p = p^0\,(1 + p^*), \quad \sigma = \rho_0\,\theta_0\,\sigma^*,$$
$$q = \rho_0\,\theta_0\,\sqrt{\theta_0}\,q^*, \quad Q = \rho_0\,\theta_0\,\sqrt{\theta_0}\,Q^*, \quad \Delta = \Delta^*, \quad \mathcal{R}_1 = \rho_0\,\theta_0^2\,\mathcal{R}_1^*, \quad \mathcal{R}_2 = \rho_0\,\theta_0^2\,\mathcal{R}_2^* \tag{31}$$

where the asterisked variables represents dimensionless perturbations of corresponding field variables with $p^* = \rho^* + \theta^*$, and the subscript 0 denotes an equilibrium ground state flow parameters. The dimensionless space and time variables are specified using a characteristic length L and a characteristic time τ by the expressions

$$x = Lx^*, \quad t = \frac{L}{\sqrt{\theta_0}}\, t^*. \tag{32}$$

The dimensionless linearized version of 35-moment system can be obtained by substituting the values of reduced field variables from Eq. (31) and Eq. (32) in the 35-moment transport equations, and holding only the linear terms in the perturbation filed variables. The final set of linearized 35-moment system is as follows:

$$\frac{\partial \rho^*}{\partial t} + \frac{\partial u^*}{\partial x^*} = 0, \qquad \frac{\partial u^*}{\partial t^*} + \frac{\partial \rho^*}{\partial x^*} + \frac{\partial \theta^*}{\partial x^*} + \frac{\partial \sigma^*}{\partial x^*} = 0, \tag{33}$$

$$\frac{\partial \theta^*}{\partial t^*} + \frac{2}{3}\frac{\partial u^*}{\partial x^*} + \frac{2}{3}\frac{\partial q^*}{\partial x^*} = 0, \tag{34}$$

$$\frac{\partial \sigma^*}{\partial t^*} + \frac{4}{3}\frac{\partial u^*}{\partial x^*} + \frac{8}{15}\frac{\partial q}{\partial x} + \frac{\partial Q^*}{\partial x^*} + \frac{1}{\mathrm{Kn}}\sigma^* = 0, \tag{35}$$

$$\frac{\partial q^*}{\partial t^*} + \frac{5}{2}\frac{\partial \theta^*}{\partial x^*} - \frac{5}{2}\frac{\partial \sigma^*}{\partial x^*} + \frac{5}{2}\frac{\partial \Delta^*}{\partial x^*} + \frac{1}{2}\frac{\partial \mathcal{R}_1^*}{\partial x^* v} + \frac{2}{3\mathrm{Kn}} q = 0, \tag{36}$$

$$\frac{\partial Q^*}{\partial t^*} + \frac{9}{35}\frac{\partial \mathcal{R}_1^*}{\partial x^*} + \frac{\partial \mathcal{R}_2^*}{\partial x^*} + \frac{3}{2\mathrm{Kn}} Q^* = 0, \tag{37}$$

$$\frac{\partial \Delta^*}{\partial t^*} + \frac{8}{15}\frac{\partial q^*}{\partial x^*} + \frac{2}{3\mathrm{Kn}}\Delta^* = 0, \tag{38}$$

$$\frac{\partial \mathcal{R}_1^*}{\partial t^*} + \frac{28}{3}\frac{\partial u^*}{\partial x^*} + \frac{112}{15}\frac{\partial q^*}{\partial x^*} + 9\frac{\partial Q^*}{\partial x^*} - \frac{67}{24\,\mathrm{Kn}}\sigma^* + \frac{247}{168\,\mathrm{Kn}}\mathcal{R}_1^* = 0, \tag{39}$$

$$\frac{\partial \mathcal{R}_2^*}{\partial t^*} + \frac{16}{7}\frac{\partial Q^*}{\partial x^*} + \frac{229}{126\,\mathrm{Kn}}\mathcal{R}_2^* = 0. \tag{40}$$

In the above linearized system, Kn is the Knudsen number and is given by

$$\mathrm{Kn} = \frac{\mu}{\mathrm{L}\rho_0\sqrt{\theta_0}} = \frac{\mu}{\mu_0}, \tag{41}$$

with μ_0 being the reference viscosity coefficient chosen such that the Knudsen number, Kn, is set equal to unity.

Plane wave solutions

We assume that the disturbances ϕ^* to follow the plane wave solutions of the form

$$\phi^* = \phi_a^* \exp[\, i\,(\omega\, t^* - \mathrm{k}\, x^*)], \tag{42}$$

where ω is the complex wave frequency, k is the complex wave number, and ϕ_a^* is the complex amplitude, so we have

$$\frac{\partial \phi^*}{\partial t^*} = i\,\omega\,\phi^*, \quad \frac{\partial \phi^*}{\partial x^*} = -i\,\mathrm{k}\,\phi^*. \tag{43}$$

Substitution of the plane wave solution (42) into the linearized 35-moment system (33) - (40) yields the homogeneous system $\mathbf{A}(\omega, \mathrm{k})\,\phi^* = 0$, where

$$\mathbf{A} = \begin{bmatrix} i\omega & 0 & -i\mathrm{k} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\omega & -\frac{2}{3}i\mathrm{k} & 0 & -\frac{2}{3}i\mathrm{k} & 0 & 0 & 0 & 0 \\ -i\mathrm{k} & -i\mathrm{k} & i\omega & -i\mathrm{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{4}{3}i\mathrm{k} & i\omega + \frac{1}{\mathrm{Kn}} & -\frac{8}{15}i\mathrm{k} & -i\mathrm{k} & 0 & 0 & 0 \\ 0 & -\frac{5}{2}i\mathrm{k} & 0 & \frac{5}{2}i\mathrm{k} & i\omega + \frac{2}{3}\frac{1}{\mathrm{Kn}} & 0 & -\frac{5}{2}i\mathrm{k} & -\frac{1}{2}i\mathrm{k} & 0 \\ 0 & 0 & 0 & 0 & 0 & i\omega + \frac{3}{2}\frac{1}{\mathrm{Kn}} & 0 & -\frac{9}{15}i\mathrm{k} & -i\mathrm{k} \\ 0 & 0 & 0 & 0 & -\frac{8}{15}i\mathrm{k} & 0 & i\omega + \frac{2}{3}\frac{1}{\mathrm{Kn}} & 0 & 0 \\ 0 & 0 & -\frac{28}{3}i\mathrm{k} & -\frac{67}{24}\frac{1}{\mathrm{Kn}} & -\frac{112}{15}i\mathrm{k} & -9i\mathrm{k} & 0 & i\omega + \frac{247}{168}\frac{1}{\mathrm{Kn}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{16}{7}i\mathrm{k} & 0 & 0 & i\omega + \frac{229}{126}\frac{1}{\mathrm{Kn}} \end{bmatrix} \tag{44}$$

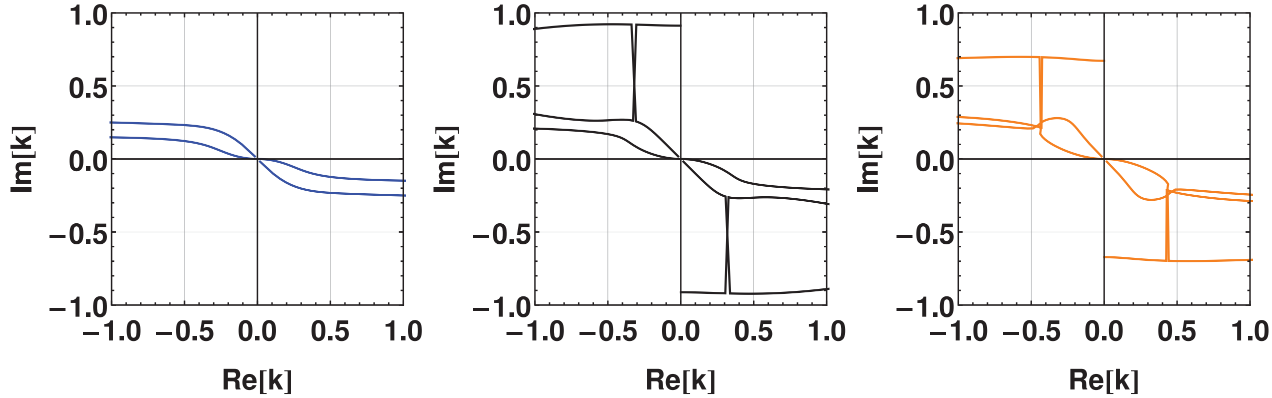

FIGURE 1. Spatial stability of moment systems: left, middle and right panels shows results for Grad 13-moment, 21-moment, and 35-moment system, respectively. The roots k(ω) of the dispersion relation are plotted in the complex plane with wave frequency ω as parameter and $\omega \in [0.001, 100]$. First and third quadrants represent the unstable region while second and fourth quadrants represent the stable region.

and

$$\phi^* = \left[\ \rho^*, \theta^*, u^*, \sigma^*, q^*, \mathcal{Q}^*, \Delta^*, \mathcal{R}_1^*, \mathcal{R}_2^*\ \right]^T. \tag{45}$$

Recall that nontrivial solutions for a homogeneous system exist when the determinant of the coefficient matrix vanishes, i.e., $|\mathbf{A}| = 0$. The corresponding dispersion relation is then obtained as

$$\begin{aligned} &3\,\omega^3\left(\omega - i\right)\left(2\,\omega - 3\,i\right)\left(168\,\omega - 247\,i\right)\left(229 + 126\,i\,\omega\right)\left(-3\,\omega + 2\,i\right)^2 \\ &\quad + 2\,\mathrm{k}^2\,\omega\Big[\omega\Big(-13909487 + 3\,\omega\Big(3\,\omega\Big(9895613 + 3\,\omega\Big(126\Big(-11693 - 2184\,i\,\omega\Big)\omega + 3113335\,i\Big)\Big) \\ &\qquad\qquad - 16524844\,i\Big)\Big) + 1696890\,i\Big] \\ &\quad + 3\,\mathrm{k}^4\Big[1696890 + \omega\Big(21\,\omega\Big(-2637307 + 108\,\omega\Big(3\Big(9221 + 2520\,i\,\omega\Big)\omega - 37891\,i\Big)\Big) + 16870877\,i\Big)\Big] \\ &\quad + 54\,\mathrm{k}^6\Big[208654 - 147\,i\,\omega\Big(-9239 + 90\,\omega\Big(72\,\omega - 161\,i\Big)\Big)\Big] = 0. \end{aligned} \tag{46}$$

Firstly, we test the spatial stability by following the procedure given in [3, 12]. For spatial stability, a local disturbances of frequency ω is considered and it demands different signs of real and imaginary part of wave number k(ω). In other words, stability requirement is that the curves should not enter the first and third quadrant when $k(\omega)$ is plotted in the complex plane for different values of the frequency ω. Figure 1 illustrates the spatial stability diagrams of Grad 13-moment (left panel), 21-moment (middle panel) and 35-moment (right panel) equations for a purely real wave frequency $\omega \in [0.001, 100]$. The figure shows $\mathrm{k}(\omega) = \mathrm{Re(k)} + i\,\mathrm{Im(k)}$, which is obtained from the dispersion relation of respective moment equations. It is evident from Fig. 1. that the Grad 13-moment equations gives two modes whereas 21-moment and 35-moment equations gives three modes each, and all of them obey the stability condition. Finally, from Fig. 1 we conclude that all moment models discussed here are spatially stable.

Next, we consider the temporal stability against a disturbance of given wave number k. It is known that from plane wave solution (42) that the temporal stability questions about the imaginary part of the frequency ω and stability demands that $\mathrm{Im}(\omega)$ to be always positive for all wave numbers. The temporal stability diagrams of Grad 13-moment, 21-moment and 35-moment equations are depicted in left, middle and right panels of Fig. 2, respectively, for a purely real wave number $\mathrm{k} \in [0, 5]$. From Fig. 2, one can observe that none of the solutions enters the half plane $\mathrm{Im}(\omega) < 0$, which means that none of the solutions violates the stability criterion. Overall, Fig. 2 clarifies that 35-moment system is temporally stable.

Finally, we conclude that the thirty-five moment extended hydrodynamic model is developed for a dilute smooth hard-sphere gas by employing the Grad's moment method and the linear stability analysis confirmed that it is stable for small disturbances in time as well as in space. In a future work, we will perform the numerical simulations of

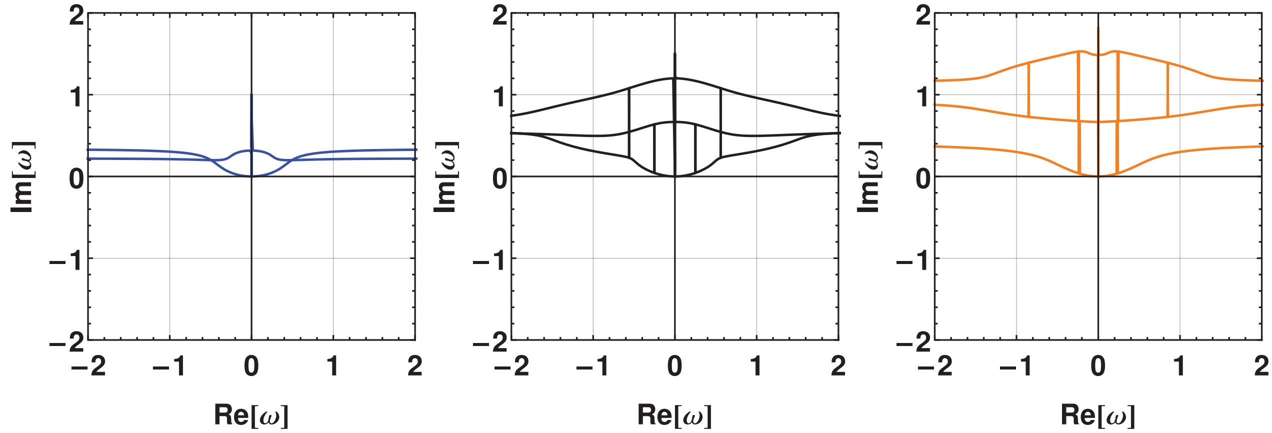

FIGURE 2. Temporal stability of moment systems: left, middle and right panels shows results for Grad 13-moment, 21-moment, and 35-moment system, respectively. The roots ω of the dispersion relation are plotted in the complex plane with wave number k as parameter and $k \in [0, 5]$. Upper half plane and lower half plane denotes the stable and unstable regions, respectively.

planar shock waves using 35-moment extended hydrodynamic model by adopting the numerical scheme discussed in [13, 14, 15].

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