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ON THE STOCHASTIC NONLINEAR SCHRÖDINGER EQUATIONS WITH NON-SMOOTH ADDITIVE NOISE

TADAHIRO OH, OANA POCOVNICU, AND YUZHAO WANG

ABSTRACT. We study the stochastic nonlinear Schrödinger equations with additive stochastic forcing. By using the dispersive estimate, we present a simple argument, constructing a unique local-in-time solution with rougher stochastic forcing than those considered in the literature.

1. INTRODUCTION

1.1. Stochastic nonlinear Schrödinger equations. We consider the Cauchy problem of the following stochastic nonlinear Schrödinger equations (SNLS) with additive noise:¹

$$\begin{cases} i\partial_t u = \Delta u - |u|^{p-1}u + \phi\xi \\ u|_{t=0} = u_0, \end{cases} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad (1.1)$$

where $\xi(t, x)$ denotes a (Gaussian) space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$ and ϕ is a bounded operator on $L^2(\mathbb{R}^d)$. We say that u is a solution to (1.1) if it satisfies the following mild formulation (= Duhamel formulation):

$$u(t) = S(t)u_0 + i \int_0^t S(t-t')|u|^{p-1}u(t')dt' - i \int_0^t S(t-t')\phi\xi(dt'), \quad (1.2)$$

where $S(t) = e^{-it\Delta}$ denotes the linear Schrödinger propagator. The last term on the right-hand side represents the effect of the stochastic forcing and is called the stochastic convolution, which we denote by Ψ :

$$\Psi(t) := -i \int_0^t S(t-t')\phi\xi(dt'). \quad (1.3)$$

In the following, we assume that $\phi \in HS(L^2; H^s)$ for appropriate values of $s \geq 0$, namely, it is a Hilbert-Schmidt operator from $L^2(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$, guaranteeing that $\Psi \in C(\mathbb{R}_+; H^s(\mathbb{R}^d))$ almost surely [15]. See Section 2 for a further discussion on the stochastic convolution Ψ . Previously, de Bouard-Debussche [16] studied (1.1) in the energy-subcritical setting² and proved its well-posedness in $H^1(\mathbb{R}^d)$, assuming that $\phi \in HS(L^2; H^1)$. Our main goal in this paper is to present a simple construction of a unique local-in-time solution to (1.1) with a much rougher noise (and hence a rougher stochastic convolution) than those considered in [16].

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¹Since our interest is local in time, the defocusing/focusing nature of the equations does not play any role in this paper. Hence, we simply consider the defocusing equations.

²Namely, $1 < p < 1 + \frac{4}{d-2}$ when $d \geq 3$ and $1 < p < \infty$ when $d = 1, 2$. This guarantees that the scaling-critical regularity s_{crit} defined in (1.6) satisfies $s_{\text{crit}} < 1$.

Before discussing the well-posedness issue for SNLS (1.1), let us first go over the local well-posedness theory for the following deterministic nonlinear Schrödinger equations (NLS):

$$\begin{cases} i\partial_t u = \Delta u - |u|^{p-1}u \\ u|_{t=0} = u_0, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \quad (1.4)$$

The equation (1.4) enjoys the following dilation symmetry:

$$u(t, x) \longmapsto u^\lambda(t, x) = \lambda^{-\frac{2}{p-1}} u(\lambda^{-2}t, \lambda^{-1}x) \quad (1.5)$$

for $\lambda > 0$. Namely, if u is a solution to (1.4), then the scaled function u^λ is also a solution to (1.4) with the rescaled initial data. This dilation symmetry induces the following scaling-critical Sobolev regularity:

$$s_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1} \quad (1.6)$$

such that the homogeneous $\dot{H}^{s_{\text{crit}}}(\mathbb{R}^d)$ -norm is invariant under the dilation symmetry. This critical regularity s_{crit} provides a threshold regularity for well-posedness and ill-posedness of (1.4). Indeed, when $s \geq \max(s_{\text{crit}}, 0)$, the Cauchy problem (1.4) is known to be locally well-posed in $H^s(\mathbb{R}^d)$ [34, 9].³ On the other hand, it is known that NLS (1.4) is ill-posed in the scaling supercritical regime: $s < s_{\text{crit}}$. See [12, 24, 27].

Let us now introduce two important critical regularities. When $s_{\text{crit}} = 0$, we say that the Cauchy problem (1.4) is *mass-critical*. This corresponds to the case $p = 1 + \frac{4}{d}$. When $s_{\text{crit}} < 0$, i.e. $p < 1 + \frac{4}{d}$ (and $s_{\text{crit}} > 0$, i.e. $p > 1 + \frac{4}{d}$, respectively), we say that (1.4) is mass-subcritical (and mass-supercritical, respectively). When $s_{\text{crit}} = 1$, we say that the Cauchy problem (1.4) is *energy-critical*. This corresponds to the case $p = 1 + \frac{4}{d-2}$. When $s_{\text{crit}} < 1$, i.e. $p < 1 + \frac{4}{d-2}$ (and $s_{\text{crit}} > 1$, i.e. $p > 1 + \frac{4}{d-2}$, respectively), we say that (1.4) is energy-subcritical (and energy-supercritical, respectively). In the following, we use the same terminology for SNLS (1.1).

One of the main ingredients in establishing local well-posedness of (1.4) is the following Strichartz estimates [33, 36, 19, 23]:

$$\|S(t)u_0\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \leq C_{d,q,r} \|u_0\|_{L_x^2(\mathbb{R}^d)}, \quad (1.7)$$

which holds true for any Schrödinger admissible pair (q, r) , satisfying

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad (1.8)$$

with $2 \leq q, r \leq \infty$ and $(q, r, d) \neq (2, \infty, 2)$. In [16], de Bouard-Debussche used the Strichartz estimates to show that the stochastic convolution Ψ almost surely belongs to a right Strichartz space. As a result, under the assumption that $\phi \in HS(L^2; H^1)$, they proved that SNLS (1.1) is locally well-posed in $H^1(\mathbb{R}^d)$ in the energy-subcritical case: $1 < p < 1 + \frac{4}{d-2}$ when $d \geq 3$ and $1 < p < \infty$ when $d = 1, 2$.⁴ Now, let $s \geq \max(s_{\text{crit}}, 0)$. Then, by slightly modifying the argument in [16] with Lemma 2.1 below, it is easy to see

³When p is not an odd integer, we need to impose an extra assumption such as $p \geq [s] + 1$ due to the non-smoothness of the nonlinearity. See also Remark 1.4. Note that this condition can be relaxed or eliminated in some situations. See, for example, [22].

⁴In [16], they also proved global well-posedness of SNLS (1.1). The well-posedness issue for SNLS with multiplicative noise was also considered in the same paper. See also Cheung-Mosincat [10] for analogous well-posedness results of SNLS with additive and multiplicative noises in the periodic setting.

that SNLS (1.1) is locally well-posed⁵ in $H^s(\mathbb{R}^d)$, provided that $\phi \in HS(L^2; H^s)$. In particular, (1.1) is locally well-posed in $L^2(\mathbb{R}^d)$ in the mass-(sub)critical case, provided that $\phi \in HS(L^2; L^2)$. Therefore, we focus our attention on the mass-supercritical case in the following.

We point out that so far we assumed that the noise had the same spatial regularity as that of initial data. On the one hand, the aforementioned ill-posedness results tell us that we can not take (deterministic) initial data below the scaling-critical regularity s_{crit} . On the other hand, we are allowed to take different regularities for initial data and the noise. Indeed, in the following, we treat rough stochastic noises that have regularities below the scaling critical regularity s_{crit} , while keeping (deterministic) initial data above the scaling critical regularity.

1.2. Main results. In the following, we use s_0 and s to denote the regularities of initial data u_0 and the noise (i.e. $\phi \in HS(L^2; H^s)$), respectively. Our main goal is to lower the value of s , while keeping $s_0 \geq s_{\text{crit}}$. In order to achieve this goal, we work within the L_x^r -framework, $r > 2$, by exploiting the following dispersive estimate:

$$\|S(t)u_0\|_{L_x^r(\mathbb{R}^d)} \leq \frac{C_r}{|t|^{\frac{d}{2}-\frac{d}{r}}} \|u_0\|_{L_x^{r'}(\mathbb{R}^d)} \quad (1.9)$$

for any $2 \leq r \leq \infty$ with $\frac{1}{r} + \frac{1}{r'} = 1$. Another key ingredient is the space-time integrability of the stochastic convolution. By a small modification of the argument in [16], we show that, the stochastic convolution Ψ almost surely belongs to

$$L^q([0, T]; W^{s,r}(\mathbb{R}^d)) \quad (1.10)$$

for any $1 \leq q < \infty$ and finite $r \geq 2$ such that $r \leq \frac{2d}{d-2}$ when $d \geq 3$, provided that $\phi \in HS(L^2; H^s)$. See Lemma 2.1 below. Note that the pair (q, r) is no longer restricted to be Schrödinger admissible. In particular, while keeping $r = \frac{2d}{d-2}$ and sending q to ∞ , we basically gain almost one spatial derivative.⁶ This allows us to prove the following improved local well-posedness result.⁷

Theorem 1.1. (i) Energy-subcritical case:⁸ *Let $d \geq 1$ and $1 + \frac{4}{d} < p < \infty$. When $d \geq 3$, assume that $p < 1 + \frac{4}{d-2}$ in addition.*

(i.a) *Let $s_0 \geq \frac{d}{2} - \frac{d}{p+1}$. Then, given $u_0 \in H^{s_0}(\mathbb{R}^d)$, there exists a unique local-in-time solution u to SNLS (1.1), provided that $\phi \in HS(L^2; L^2)$. Moreover, the solution u lies in the class:*

$$\begin{aligned} & \Psi + C([0, T]; L^{p+1}(\mathbb{R}^d)) \cap C([0, T]; L^2(\mathbb{R}^d)) \\ & \subset C([0, T]; L^2(\mathbb{R}^d)), \end{aligned}$$

where $T = T_\omega$ is almost surely positive.

⁵Once again, an extra assumption such as $p \geq [s] + 1$ is needed when p is not an odd integer.

⁶Recall that $H^{s+1}(\mathbb{R}^d) \subset W^{s, \frac{2d}{d-2}}(\mathbb{R}^d)$.

⁷Here, we use the term “well-posedness” in a loose sense as in [6]. See also Remark 1.2 (ii).

⁸As we mentioned before, we assume that (d, p) satisfies the mass-supercritical condition.

(i.b) Let $s_0 > s_{\text{crit}}$. Then, given $u_0 \in H^{s_0}(\mathbb{R}^d)$, there exists a unique local-in-time solution u to SNLS (1.1), provided that $\phi \in HS(L^2; L^2)$. Moreover, the solution u lies in the class:

$$\begin{aligned} & \Psi + L^q([0, T]; L^{p+1}(\mathbb{R}^d)) \cap C([0, T]; L^2(\mathbb{R}^d)) \\ & \subset C([0, T]; L^2(\mathbb{R}^d)), \end{aligned}$$

where $q = q(d, p) > 2$ is finite and $T = T_\omega$ is almost surely positive.

(ii) Energy-(super)critical case: Let $d \geq 3$ and $p \geq 1 + \frac{4}{d-2}$ be an odd integer. Fix $s_0 > s_{\text{crit}}$ and $s > s_{\text{crit}} - 1$. Then, given $u_0 \in H^{s_0}(\mathbb{R}^d)$, there exists a unique local-in-time solution u to SNLS (1.1), provided that $\phi \in HS(L^2; H^s)$. Moreover, the solution u lies in the class:

$$\begin{aligned} & \Psi + C([0, T]; W^{s_1, \frac{2d}{d-2}-\delta}(\mathbb{R}^d)) \cap C([0, T]; H^{s_1}(\mathbb{R}^d)) \\ & \subset C([0, T]; H^{s_1}(\mathbb{R}^d)), \end{aligned}$$

where $s_1 = \min(s_0 - 1, s)$, $\delta = \delta(s_1) > 0$ is sufficiently small, and $T = T_\omega$ is almost surely positive.

The structure of the mild formulation (1.2) states that any solution u can be written as⁹

$$u = v + \Psi. \quad (1.11)$$

We then study the following fixed point problem for the residual term $v := u - \Psi$:

$$v(t) = S(t)u_0 + \int_0^t S(t-t')\mathcal{N}(v + \Psi)(t')dt', \quad (1.12)$$

where $\mathcal{N}(u) = i|u|^{p-1}u$. In [16], de Bouard-Debussche studied the fixed point problem (1.12) for v in terms of the standard L_x^2 -based theory for NLS (1.4). In particular, the solution v to (1.12) was constructed in $C([0, T]; H^1(\mathbb{R}^d))$ intersected with an appropriate Strichartz space. In the following, we instead work in the L_x^r -framework with $r > 2$ and directly solve the fixed point problem in $C([0, T]; W^{s_1, r}(\mathbb{R}^d))$ ¹⁰ by applying the dispersive estimate (1.9).

On the one hand, the spatial regularity s_1 of v in Theorem 1.1, i.e. $s_1 = 0$ in (i) and $s_1 = s_{\text{crit}} - 1 + \varepsilon$ in (ii) is below the scaling critical regularity s_{crit} defined in (1.6) (when $\varepsilon < 1$). On the other hand, given any $1 \leq r \leq \infty$, we can also consider the scaling-critical Sobolev regularity adapted to the L^r -based Sobolev spaces:

$$s_{\text{crit}}(r) = \frac{d}{r} - \frac{2}{p-1}$$

such that the homogeneous $\dot{W}^{s_{\text{crit}}(r), r}$ -norm is invariant under the dilation symmetry (1.5). Note that we have $s_{\text{crit}}(r) < s_{\text{crit}} = s_{\text{crit}}(2)$ for $r > 2$. For example, in the energy-(super)critical case, the gain of spatial integrability (1.10) of the stochastic convolution Ψ allows us to work in the L_x^r -based Sobolev space¹¹ with $r = \frac{2d}{d-2}$, thus lowering the critical regularity from $s_{\text{crit}} = s_{\text{crit}}(2)$ to $s_{\text{crit}}(r) = s_{\text{crit}} - 1$ with $r = \frac{2d}{d-2}$. This heuristically explains how the regularity threshold $s_{\text{crit}} - 1$ appears in Theorem 1.1 (ii). Moreover, note that, by working only within the L^r -based Sobolev space with $s_1 > s_{\text{crit}}(r)$, we have made

⁹The decomposition (1.11) is often referred to as the Da Prato-Debussche trick [14] in the field of stochastic parabolic PDEs. Such an idea also appears in McKean [26] and Bourgain [6] in the context of (deterministic) dispersive PDEs with random initial data, preceding [14]. See also de Bouard-Debussche [16] and Burq-Tzvetkov [8].

¹⁰For Theorem 1.1 (i.b), we need to work in $L^q([0, T]; L^{p+1}(\mathbb{R}^d))$.

¹¹For a technical reason, we need to take $r = \frac{2d}{d-2} - \delta$ for some small $\delta > 0$ in the proof of Theorem 1.1.

the problem *subcritical*. Indeed, all the spatial function spaces such as $L^{p+1}(\mathbb{R}^d)$ appearing in Theorem 1.1 are subcritical in the sense described above.

Remark 1.2. (i) Our argument for proving Theorem 1.1 is of subcritical nature in the sense that the local existence time T depends on the H^{s_0} -norm of initial data (and a space-time norm of the stochastic convolution). It is possible to improve Theorem 1.1 (i.b) so that it also holds when $s_0 = s_{\text{crit}}$ by relying on the critical local well-posedness theory (in terms of initial data). See Remark 3.3.

(ii) Theorem 1.1 establishes existence of unique solutions to (1.1). Note that the (spatial) regularity of the noise is rougher than that of the initial data in Theorem 1.1 (i.a) and (i.b). As such, the solution inherits the rougher regularity of the noise and it only lies in $C([0, T]; L^2(\mathbb{R}^d))$. The situation is slightly more subtle in Theorem 1.1 (ii). Also, note that, in view of the aforementioned ill-posedness results, the map: $(u_0, \phi\xi) \mapsto u$ is not continuous, when the noise has spatial regularity $s < s_{\text{crit}}$. By the use of the Da Prato-Debussche trick, however, the map sending an enhanced data set (u_0, Ψ) to a solution u is continuous, where the stochastic convolution Ψ is measured in an appropriate space-time function norm.

Our next goal is to study the Cauchy problem (1.1) with random initial data and prove almost sure local well-posedness for (random) initial data of lower regularities. More precisely, given a function u_0 on \mathbb{R}^d , we consider a randomization of u_0 adapted to the so-called Wiener decomposition [35] of the frequency space: $\mathbb{R}^d = \bigcup_{n \in \mathbb{Z}^d} Q_n$, where Q_n is the unit cube centered at $n \in \mathbb{Z}^d$.

Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\text{supp } \psi \subset [-1, 1]^d \quad \text{and} \quad \sum_{n \in \mathbb{Z}^d} \psi(\xi - n) \equiv 1 \quad \text{for any } \xi \in \mathbb{R}^d.$$

Then, given a function u_0 on \mathbb{R}^d , we have

$$u_0 = \sum_{n \in \mathbb{Z}^d} \psi(D - n)u_0,$$

where $\psi(D - n)$ is defined by $\psi(D - n)u_0(x) = \int_{\mathbb{R}^d} \psi(\xi - n)\widehat{u}_0(\xi)e^{2\pi i x \cdot \xi} d\xi$, namely, the Fourier multiplier operator with symbol $\mathbf{1}_{Q_n}$ conveniently smoothed. This decomposition leads to the following randomization of u_0 adapted to the Wiener decomposition. Let $\{g_n\}_{n \in \mathbb{Z}^d}$ be a sequence of independent mean-zero complex-valued random variables (with independent real and imaginary parts), satisfying the following exponential moment bound:

$$\mathbb{E}[e^{\gamma_1 \text{Re } g_n + \gamma_2 \text{Im } g_n}] \leq e^{c(\gamma_1^2 + \gamma_2^2)} \quad (1.13)$$

for all $\gamma_1, \gamma_2 \in \mathbb{R}$ and $n \in \mathbb{Z}^d$. Note that (1.13) is satisfied by standard complex-valued Gaussian random variables, Bernoulli random variables, and any random variables with compactly supported distributions. We then define the Wiener randomization¹² of u_0 by

$$u_0^\omega := \sum_{n \in \mathbb{Z}^d} g_n(\omega)\psi(D - n)u_0. \quad (1.14)$$

Given $u_0 \in H^s(\mathbb{R}^d)$, it is easy to see that its Wiener randomization u_0^ω in (1.14) lies in $H^s(\mathbb{R}^d)$ almost surely. One can also show that, under some non-degeneracy condition, there is no smoothing upon randomization in terms of differentiability; see, for example,

¹²It is also called the unit-scale randomization in [17].

Lemma B.1 in [8]. The main point of the randomization (1.14) is its improved integrability. For example, under the assumption (1.13), u_0^ω almost surely belongs to $W^{s,r}(\mathbb{R}^d)$ for any finite $r \geq 2$. Moreover, by restricting our attention to local-in-time intervals, the random linear solution $S(t)u_0^\omega$ satisfies the Strichartz estimate (1.7) for *any* finite $q, r \geq 2$ almost surely. See Lemma 3.5 below. This gain of space-time integrability allows us to take random initial data at the same low regularity as the stochastic forcing.

Theorem 1.3. (i) Energy-subcritical case: *Let d and p be as in Theorem 1.1 (i). Then, given $u_0 \in L^2(\mathbb{R}^d)$, SNLS (1.1) is almost surely locally well-posed with respect to the Wiener randomization u_0^ω defined in (1.14), provided that $\phi \in HS(L^2; L^2)$. More precisely, there exists a unique local-in-time solution $u = u^\omega$ to SNLS (1.1) with $u|_{t=0} = u_0^\omega$ in the class:*

$$\begin{aligned} S(t)u_0^\omega + \Psi + C([0, T]; L^{p+1}(\mathbb{R}^d)) \cap C([0, T]; L^2(\mathbb{R}^d)) \\ \subset C([0, T]; L^2(\mathbb{R}^d)), \end{aligned}$$

where $T = T_\omega$ is almost surely positive.

(ii) Energy-(super)critical case: *Let d and p be as in Theorem 1.1 (ii). Moreover, let $s = s_{\text{crit}} - 1 + \varepsilon$ for some small $\varepsilon > 0$. Then, given $u_0 \in H^s(\mathbb{R}^d)$, SNLS (1.1) is almost surely locally well-posed with respect to the Wiener randomization u_0^ω defined in (1.14), provided that $\phi \in HS(L^2; H^s)$. More precisely, there exists a unique local-in-time solution $u = u^\omega$ to SNLS (1.1) with $u|_{t=0} = u_0^\omega$ in the class:*

$$\begin{aligned} S(t)u_0^\omega + \Psi + C([0, T]; W^{s, \frac{2d}{d-2}-\delta}(\mathbb{R}^d)) \cap C([0, T]; H^s(\mathbb{R}^d)) \\ \subset C([0, T]; H^s(\mathbb{R}^d)), \end{aligned}$$

where $\delta = \delta(s) > 0$ is sufficiently small and $T = T_\omega$ is almost surely positive.

In view of the probabilistic Strichartz estimates (Lemma 3.5), we see that $\tilde{\Psi} := S(t)u_0^\omega + \Psi$ solving

$$\begin{cases} i\partial_t \tilde{\Psi} = \Delta \tilde{\Psi} + \phi \xi \\ \tilde{\Psi}|_{t=0} = u_0^\omega, \end{cases} \quad (1.15)$$

satisfies the same regularity properties, both in terms of differentiability and integrability, as the stochastic convolution Ψ in (1.3). Then, by decomposing u as

$$u = v + \tilde{\Psi},$$

Theorem 1.3 follows from repeating the argument in the proof of Theorem 1.1.

We conclude this introduction with several remarks.

Remark 1.4. In Theorems 1.1 and 1.3, we assumed that p is an odd integer in the energy-(super)critical case. One may apply the fractional chain rule [13] and remove this restriction in certain situations. For conciseness of the presentation, however, we do not pursue this direction in this paper.

Remark 1.5. In recent years, the well-posedness issue of the deterministic NLS (1.4) with respect to the random initial data u_0^ω in (1.14) has been studied intensively [1, 2, 7, 28, 3, 18]. The main idea is to study the fixed point problem for the residual term $v = u - S(t)u_0^\omega$, utilizing (a variant of) the Fourier restriction norm method [5, 20, 21] and carrying out rather tedious case-by-case analysis. In a recent paper [31], the second and third authors studied the deterministic NLS (1.4) with the random initial data u_0^ω in (1.14) by exploiting the dispersive estimate (1.9). In particular, they proved Theorem 1.3 above

when $\phi = 0$, i.e. when there is no stochastic noise. This argument with the dispersive estimate bypasses case-by-case analysis, which is closer in spirit to the almost sure local well-posedness argument for the nonlinear wave equations [8, 25, 30, 29]. See a survey paper [4] for a further discussion on the subject.

Remark 1.6. In [11], the second author with Cheung studied SNLS (1.1) on \mathbb{R}^d , $d \geq 3$, with the cubic nonlinearity ($p = 3$). By adapting the argument in [2] for the deterministic NLS with random initial data, they proved local well-posedness of (1.1) with stochastic forcing below the scaling-critical regularity, i.e. $\phi \in HS(L^2; H^s)$ with $s < s_{\text{crit}}$. Moreover, their work shows that the residual part $v = u - \Psi$ lies in $C([0, T]; H^{s_{\text{crit}}}(\mathbb{R}^d))$.

Notations: Given $T > 0$, we set $L_T^q B_x = L^q([0, T]; B(\mathbb{R}^d))$ and $C_T B_x = C([0, T]; B(\mathbb{R}^d))$, where $B(\mathbb{R}^d)$ denotes a Banach space of functions on \mathbb{R}^d .

2. ON THE STOCHASTIC CONVOLUTION

In this section, we study the regularity properties of the stochastic convolution Ψ in (1.3). Let us first recall the definition of a cylindrical Wiener process W on $L^2(\mathbb{R}^d)$. Fix an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $L^2(\mathbb{R}^d)$. Then, a cylindrical Wiener process W on $L^2(\mathbb{R}^d)$ is defined by the following random Fourier series:

$$W(t) = \sum_{n \in \mathbb{N}} \beta_n(t) e_n,$$

where $\{\beta_n\}_{n \in \mathbb{N}}$ is a family of mutually independent complex-valued Brownian motions. In terms of the cylindrical Wiener process W , we can express the stochastic convolution Ψ in (1.3) as

$$\Psi(t) = -i \int_0^t S(t-t') \phi dW(t') = -i \sum_{n \in \mathbb{N}} \int_0^t S(t-t') \phi(e_n) d\beta_n(t'). \quad (2.1)$$

By slightly modifying the argument in [16], we have the following lemma.

Lemma 2.1. *Suppose that $\phi \in HS(L^2; H^s)$ for some $s \in \mathbb{R}$. Then, the following statements hold almost surely:*

- (i) $\Psi \in C(\mathbb{R}_+; H^s(\mathbb{R}^d))$,
- (ii) *Given any $1 \leq q < \infty$ and finite $r \geq 2$ such that $r \leq \frac{2d}{d-2}$ when $d \geq 3$, we have $\Psi \in L^q([0, T]; W^{s,r}(\mathbb{R}^d))$ for any $T > 0$.*

Compare Part (ii) of Lemma 2.1 with [16], where (q, r) was restricted to be Schrödinger admissible. It is this gain of integrability in time which allows us to prove Theorems 1.1 and 1.3.

Proof. For (i), see [15]. Set $\langle \nabla \rangle = \sqrt{1 - \Delta}$. Given $\phi \in HS(L^2; H^s)$, let $\{\phi_k\}_{k \in \mathbb{N}} \subset HS(L^2; H^{s+\sigma})$, $\sigma > \frac{d}{2}$, such that ϕ_k converges to ϕ in $HS(L^2; H^s)$. Then, letting Ψ_k denote the stochastic convolution in (2.1) with ϕ replaced by ϕ_k , we see that Ψ_k converges to Ψ in $C(\mathbb{R}_+; H^s(\mathbb{R}^d))$ and that $\langle \nabla \rangle^s \Psi_k \in C(\mathbb{R}_+; H^\sigma(\mathbb{R}^d)) \subset C(\mathbb{R}_+; C(\mathbb{R}^d))$, where the inclusion follows from Sobolev's embedding theorem.

Fix $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Then, from (2.1), we see that, as a linear combination of independent Wiener integrals, $\langle \nabla \rangle^s \Psi_k(t, x)$ is a mean-zero complex-valued Gaussian random

variable with variance $\sigma_k(t, x) = \|\langle \nabla \rangle^s \Psi_k(t, x)\|_{L^2(\Omega)}^2$. Recall that, for a mean-zero complex-valued Gaussian random variable g with variance σ , we have

$$\mathbb{E}[|g|^{2j}] = j! \cdot \sigma^j. \quad (2.2)$$

Given $\rho \geq 2$, let $\tilde{\rho}$ denote the smallest even integer such that $\tilde{\rho} \geq \rho$. Then, by Hölder's inequality and (2.2), we have¹³

$$\begin{aligned} \|\langle \nabla \rangle^s \Psi_k(t, x)\|_{L^\rho(\Omega)} &\leq \|\langle \nabla \rangle^s \Psi_k(t, x)\|_{L^{\tilde{\rho}}(\Omega)} \\ &= C_\rho \|\langle \nabla \rangle^s \Psi_k(t, x)\|_{L^2(\Omega)} \\ &\sim \left\| \left(\int_0^t |S(t-t') \langle \nabla \rangle^s \phi_k(e_n)(x)|^2 dt' \right)^{\frac{1}{2}} \right\|_{\ell_n^2} \\ &= \|S(\tau) \langle \nabla \rangle^s \phi_k(e_n)(x)\|_{\ell_n^2 L_\tau^2([0, t])} \end{aligned} \quad (2.3)$$

for any $\rho \geq 2$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

Now, fix $1 \leq q < \infty$ and finite $r \geq 2$ such that $r \leq \frac{2d}{d-2}$ when $d \geq 3$. Let $\tilde{q} = \tilde{q}(d, r) \geq 2$ be the unique index such that (\tilde{q}, r) is Schrödinger admissible, satisfying (1.8). Then, for $\rho \geq \max(q, r)$, it follows from Minkowski's integral inequality and (2.3) that

$$\begin{aligned} \|\|\Psi_k\|_{L_T^q W_x^{s, r}}\|_{L^\rho(\Omega)} &\leq \|\|\langle \nabla \rangle^s \Psi_k(t, x)\|_{L^\rho(\Omega)}\|_{L_T^q L_x^r} \\ &\leq C_\rho \|\|S(\tau) \langle \nabla \rangle^s \phi_k(e_n)(x)\|_{\ell_n^2 L_\tau^2([0, t])}\|_{L_T^q L_x^r} \end{aligned}$$

By Minkowski's integral inequality (with $r \geq 2$), Hölder's inequality in time, and then applying the Strichartz estimate (1.7),

$$\begin{aligned} &\leq T^{\frac{1}{q}} \|\|S(\tau) \langle \nabla \rangle^s \phi_k(e_n)\|_{L_\tau^2([0, T]; L_x^r)}\|_{\ell_n^2} \\ &\leq T^\theta \|\|S(\tau) \langle \nabla \rangle^s \phi_k(e_n)\|_{L_\tau^{\tilde{q}}([0, T]; L_x^r)}\|_{\ell_n^2} \\ &\leq T^\theta \|\|\phi_k(e_n)\|_{H_x^s}\|_{\ell_n^2} \\ &= T^\theta \|\phi_k\|_{HS(L^2; H^s)} < \infty \end{aligned}$$

for some $\theta = \theta(q, \tilde{q}) > 0$. Similarly, we have

$$\|\|\Psi_k - \Psi_j\|_{L_T^q W_x^{s, r}}\|_{L^\rho(\Omega)} \leq CT^\theta \|\phi_k - \phi_j\|_{HS(L^2; H^s)} \longrightarrow 0,$$

as $k, j \rightarrow \infty$. Namely, $\{\Psi_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^\rho(\Omega; L^q([0, T]; W^{s, r}(\mathbb{R}^d)))$. By the uniqueness of the limit, we conclude that $\{\Psi_k\}_{k \in \mathbb{N}}$ converges to Ψ in $L^\rho(\Omega; L^q([0, T]; W^{s, r}(\mathbb{R}^d)))$. In particular, we have

$$\|\|\Psi\|_{L_T^q W_x^{s, r}}\|_{L^\rho(\Omega)} \leq CT^\theta \|\phi\|_{HS(L^2; H^s)} < \infty.$$

This proves (ii). □

¹³In fact, the following estimate holds true:

$$\|\langle \nabla \rangle^s \Psi_k(t, x)\|_{L^\rho(\Omega)} \leq \|\langle \nabla \rangle^s \Psi_k(t, x)\|_{L^2(\Omega)}$$

Namely, there is no constant depending on $\rho \geq 2$. See [32, Theorem I.22]. For our purpose, however, the elementary argument in (2.3) suffices.

3. PROOF OF THEOREMS 1.1 AND 1.3

In this section, we present the proofs of our main results (Theorems 1.1 and 1.3). We first recall the following nonhomogeneous Strichartz estimate; let (q, r) and (\tilde{q}, \tilde{r}) be Schrödinger admissible. Then, we have

$$\left\| \int_0^t S(t-t')F(t')dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \quad (3.1)$$

where \tilde{q}' and \tilde{r}' denote the Hölder conjugates of \tilde{q} and \tilde{r} , respectively.

3.1. Proof of Theorem 1.1. Let $s_0, s \in \mathbb{R}$ to be specified later. Given $u_0 \in H^{s_0}(\mathbb{R}^d)$ and $\phi \in HS(L^2; H^s)$, we define $\Gamma = \Gamma_{u_0, \phi, \xi}$ by

$$\Gamma v(t) := S(t)u_0 + \int_0^t S(t-t')\mathcal{N}(v + \Psi)(t')dt'.$$

Then, we have the following nonlinear estimates.

Proposition 3.1. *Let d and p be as in Theorem 1.1. We set*

- (i.a) $s_0 \geq \frac{d}{2} - \frac{d}{p+1}$, $s_1 = 0$, $r = p + 1$ in the energy-subcritical case,
- (ii) $s_0 - 1 \geq s_1 > s_{\text{crit}} - 1$ and $r = \frac{2d}{d-2} - \delta$ for some small $\delta = \delta(s_1) > 0$ in the energy-(super)critical case.

Then, the following estimates hold for some $q \gg 1$:

$$\begin{aligned} \|\Gamma v\|_{C_T W_x^{s_1, r}} &\lesssim \|u_0\|_{H^{s_0}} + T^\theta \left(\|v\|_{C_T W_x^{s_1, r}}^p + \|\Psi\|_{L_T^q W_x^{s_1, r}}^p \right), \\ \|\Gamma v_1 - \Gamma v_2\|_{C_T W_x^{s_1, r}} &\lesssim T^\theta \left(\|v_1\|_{C_T W_x^{s_1, r}}^{p-1} + \|v_2\|_{C_T W_x^{s_1, r}}^{p-1} \right. \\ &\quad \left. + \|\Psi\|_{L_T^q W_x^{s_1, r}}^{p-1} \right) \|v_1 - v_2\|_{C_T W_x^{s_1, r}} \end{aligned} \quad (3.2)$$

for all $v, v_1, v_2 \in C([0, T]; W^{s_1, r}(\mathbb{R}^d))$ and $T > 0$. Moreover, we have $\Gamma v \in C([0, T]; H^{s_1}(\mathbb{R}^d))$ for all $v \in C([0, T]; W^{s_1, r}(\mathbb{R}^d))$, $\Psi \in L^q([0, T]; W^{s_1, r}(\mathbb{R}^d))$, and $T > 0$.

Once we prove Proposition 3.1, Theorem 1.1 (i.a) and (ii) follow from a standard contraction argument with Lemma 2.1 (with $s = s_{\text{crit}} - 1 + \varepsilon$ for Theorem 1.1 (ii)).

Proof. We first consider the energy-subcritical case (i.a). By Sobolev's inequality $H^{s_0}(\mathbb{R}^d) \subset L^{p+1}(\mathbb{R}^d)$ and the dispersive estimate (1.9), we have

$$\begin{aligned} \|\Gamma v\|_{C_T L_x^{p+1}} &\lesssim \|S(t)u_0\|_{C_T H^{s_0}} + \sup_{t \in [0, T]} \int_0^t \frac{1}{|t-t'|^{\frac{d}{2} - \frac{d}{p+1}}} \|\mathcal{N}(v + \Psi)(t')\|_{L_x^{\frac{p+1}{p}}} dt' \\ &\lesssim \|u_0\|_{H^{s_0}} + T^\theta \|v + \Psi\|_{L_T^q L_x^{p+1}}^p \\ &\lesssim \|u_0\|_{H^{s_0}} + T^\theta \left(\|v\|_{C_T L_x^{p+1}}^p + \|\Psi\|_{L_T^q L_x^{p+1}}^p \right) \end{aligned} \quad (3.3)$$

for some $q \gg 1$ and small $\theta > 0$, provided that $s_0 \geq \frac{d}{2} - \frac{d}{p+1}$ and $\frac{d}{2} - \frac{d}{p+1} < 1$, namely $p < 1 + \frac{4}{d-2}$. When p is an odd integer, a similar computation yields the following difference estimate:

$$\begin{aligned} \|\Gamma v_1 - \Gamma v_2\|_{C_T L_x^{p+1}} &\lesssim T^\theta \|\mathcal{N}(v_1 + \Psi) - \mathcal{N}(v_2 + \Psi)\|_{L_T^q L_x^{\frac{p+1}{p}}} \\ &\lesssim T^\theta \left(\|v_1\|_{C_T L_x^{p+1}}^{p-1} + \|v_2\|_{C_T L_x^{p+1}}^{p-1} + \|\Psi\|_{L_T^q L_x^{p+1}}^{p-1} \right) \|v_1 - v_2\|_{C_T L_x^{p+1}}. \end{aligned} \quad (3.4)$$

Next, we consider the case when $p > 1$ is not an odd integer. By the mean value theorem, we have

$$\begin{aligned} \mathcal{N}(v_1 + \Psi) - \mathcal{N}(v_2 + \Psi) &= \int_0^1 \left\{ \partial_z \mathcal{N}(v_2 + \Psi + \theta(v_1 - v_2))(v_1 - v_2) \right. \\ &\quad \left. + \partial_{\bar{z}} \mathcal{N}(v_2 + \Psi + \theta(v_1 - v_2))(\overline{v_1 - v_2}) \right\} d\theta. \end{aligned} \quad (3.5)$$

With $\mathcal{N}(z) = i|z|^{p-1}z$, we have

$$\partial_z \mathcal{N}(z) = i \frac{p+1}{2} |z|^{p-1} \quad \text{and} \quad \partial_{\bar{z}} \mathcal{N}(z) = i \frac{p-1}{2} |z|^{p-1} \frac{z^2}{|z|^2}. \quad (3.6)$$

Then, by repeating the computation above with (3.5) and (3.6), we obtain (3.4). Given $p+1 < \frac{2d}{d-2}$, let $(\tilde{q}, p+1)$ be Schrödinger admissible. Then, it follows from (3.1) that

$$\begin{aligned} \|\Gamma v - S(t)u_0\|_{C_T L_x^2} &\lesssim \|\mathcal{N}(v + \Psi)\|_{L_T^{\tilde{q}'} L_x^{\frac{p+1}{p}}} \\ &\lesssim T^\theta \left(\|v\|_{C_T L_x^{p+1}}^p + \|\Psi\|_{L_T^q L_x^{p+1}}^p \right) \end{aligned}$$

for some $q \gg 1$ and small $\theta > 0$. Hence, we conclude that $\Gamma v \in C([0, T]; L^2(\mathbb{R}^d))$ for all $v \in C([0, T]; L^{p+1}(\mathbb{R}^d))$ and $T > 0$.

Next, we consider the energy-(super)critical case (ii): $p \geq 1 + \frac{4}{d-2}$ when $d \geq 3$. Since $s_0 - 1 \geq s_1 > s_{\text{crit}} - 1 = \frac{d-2}{2} - \frac{2}{p-1}$, we can choose $\delta > 0$ sufficiently small such that

$$H^{s_0}(\mathbb{R}^d) \subset W^{s_1, r}(\mathbb{R}^d) \subset L^{\frac{(p-1)r}{r-2}}(\mathbb{R}^d), \quad (3.7)$$

where $r = \frac{2d}{d-2} - \delta$. Since p is an odd integer, the nonlinearity $\mathcal{N}(u)$ is algebraic and hence we can apply the fractional Leibniz rule. Then, proceeding with the dispersive estimate (1.9), the fractional Leibniz rule, and (3.7), we have

$$\begin{aligned} \|\Gamma v\|_{C_T W_x^{s_1, r}} &\lesssim \|S(t)u_0\|_{C_T H^{s_0}} + T^\theta \|\langle \nabla \rangle^{s_1} \mathcal{N}(v + \Psi)\|_{L_T^{\frac{q}{p}} L_x^{r'}} \\ &\lesssim \|u_0\|_{H^{s_0}} + T^\theta \|\langle \nabla \rangle^{s_1} (v + \Psi)\|_{L_T^q L_x^r} \|v + \Psi\|_{L_T^q L_x^{\frac{(p-1)r}{r-2}}}^{p-1} \\ &\lesssim \|u_0\|_{H^{s_0}} + T^\theta \|v + \Psi\|_{L_T^q W_x^{s_1, r}}^p \\ &\lesssim \|u_0\|_{H^{s_0}} + T^\theta \left(\|v\|_{C_T W_x^{s_1, r}}^p + \|\Psi\|_{L_T^q W_x^{s_1, r}}^p \right) \end{aligned} \quad (3.8)$$

for some $q \gg 1$ and small $\theta > 0$. The difference estimate (3.2) follows in a similar manner. Given $r = \frac{2d}{d-2} - \delta$, let (\tilde{q}, r) be Schrödinger admissible. Then, proceeding as in (3.8) with (3.1), we have

$$\begin{aligned} \|\Gamma v - S(t)u_0\|_{C_T H_x^{s_1}} &\lesssim \|\langle \nabla \rangle^{s_1} \mathcal{N}(v + \Psi)\|_{L_T^{\tilde{q}'} L_x^{r'}} \\ &\lesssim T^\theta \left(\|v\|_{C_T W_x^{s_1, r}}^p + \|\Psi\|_{L_T^q W_x^{s_1, r}}^p \right) \end{aligned}$$

for some $q \gg 1$ and small $\theta > 0$. This shows $\Gamma v \in C([0, T]; H^{s_1}(\mathbb{R}^d))$ for all $v \in C([0, T]; W^{s_1, r}(\mathbb{R}^d))$ and $T > 0$. \square

Similarly, Theorem 1.1 (i.b) follows from the following proposition. In order to control the linear solution at a lower regularity, we apply the Strichartz estimate (1.7).

Proposition 3.2. *Let d and p be as in Theorem 1.1 (i.b) and $s_0 > s_{\text{crit}}$. Then, the following estimates hold for some $q \gg 1$:*

$$\begin{aligned} \|\Gamma v\|_{L_T^q L_x^{p+1}} &\lesssim \|u_0\|_{H^{s_0}} + T^\theta \left(\|v\|_{L_T^q L_x^{p+1}}^p + \|\Psi\|_{L_T^q L_x^{p+1}}^p \right), \\ \|\Gamma v_1 - \Gamma v_2\|_{L_T^q L_x^{p+1}} &\lesssim T^\theta \left(\|v_1\|_{L_T^q L_x^{p+1}}^{p-1} + \|v_2\|_{L_T^q L_x^{p+1}}^{p-1} + \|\Psi\|_{L_T^q L_x^{p+1}}^{p-1} \right) \|v_1 - v_2\|_{L_T^q L_x^{p+1}} \end{aligned} \quad (3.9)$$

for all $v, v_1, v_2 \in L^q([0, T]; L^{p+1}(\mathbb{R}^d))$ and $T > 0$. Moreover, we have $\Gamma v \in C([0, T]; L^2(\mathbb{R}^d))$ for all $v \in L^q([0, T]; L^{p+1}(\mathbb{R}^d))$ and $T > 0$.

Proof. Given (d, p) , fix $q \geq 2$ such that $\frac{1}{q} + 1 = \left(\frac{d}{2} - \frac{d}{p+1} + \varepsilon\right) + \frac{p}{q}$ for some small $\varepsilon > 0$. Furthermore, let $r \geq 2$ such that (q, r) is Schrödinger admissible. Then, by Sobolev's inequality, we have

$$W^{s_0, r}(\mathbb{R}^d) \subset L^{p+1}(\mathbb{R}^d). \quad (3.10)$$

By proceeding as in (3.3) with (3.10), the dispersive estimate (1.9), the Strichartz estimate (1.7), and Young's inequality, we have

$$\begin{aligned} \|\Gamma v\|_{L_T^q L_x^{p+1}} &\lesssim \|S(t)u_0\|_{L_T^q W^{s_0, r}} + \left\| \int_0^t \frac{1}{|t-t'|^{\frac{d}{2} - \frac{d}{p+1}}} \|\mathcal{N}(v + \Psi)(t')\|_{L_x^{\frac{p+1}{p}}} dt' \right\|_{L_T^q} \\ &\lesssim \|u_0\|_{H^{s_0}} + T^\theta \left(\|v\|_{L_T^q L_x^{p+1}}^p + \|\Psi\|_{L_T^q L_x^{p+1}}^p \right) \end{aligned} \quad (3.11)$$

for some $\theta > 0$. The difference estimate (3.9) follows in a similar manner. Given $p+1 < \frac{2d}{d-2}$, let $(\tilde{q}, p+1)$ be Schrödinger admissible. Then, from the mass-supercritical condition: $p > 1 + \frac{4}{d}$, we see that $q > p\tilde{q}$. Hence, it follows from (3.1) that

$$\begin{aligned} \|\Gamma v - S(t)u_0\|_{C_T L_x^2} &\lesssim \|\mathcal{N}(v + \Psi)\|_{L_T^{\tilde{q}} L_x^{\frac{p+1}{p}}} \\ &\lesssim T^\theta \left(\|v\|_{L_T^q L_x^{p+1}}^p + \|\Psi\|_{L_T^q L_x^{p+1}}^p \right) \end{aligned}$$

for some small $\theta > 0$. □

Remark 3.3. In (3.11), it is possible to apply Hardy-Littlewood-Sobolev's inequality instead of Young's inequality since $q < \infty$. Namely, proceeding with $\varepsilon = 0$, Hardy-Littlewood-Sobolev's inequality gives

$$\|\Gamma v\|_{L_T^q L_x^{p+1}} \lesssim \left(\|S(t)u_0\|_{L_T^q W^{s_0, r}} + \|\Psi\|_{L_T^q L_x^{p+1}}^p \right) + \|v\|_{L_T^q L_x^{p+1}}^p.$$

The difference estimate (3.9) also holds without the T^θ -factor. Then, we can carry out a contraction argument by making $T = T_\omega > 0$ sufficiently small such that

$$\|S(t)u_0\|_{L_T^q W^{s_0, r}} + \|\Psi\|_{L_T^q L_x^{p+1}}^p \ll 1,$$

as in the mass-critical local well-posedness theory for NLS (1.4), and prove local well-posedness of (1.1) even when $s_0 = s_{\text{crit}}$. As the argument is standard, we omit details. Note that we measure the stochastic convolution Ψ only with the subcritical L_x^{p+1} -norm.

3.2. Proof of Theorem 1.3. Given u_0 on \mathbb{R}^d , let u_0^ω be its Wiener randomization defined in (1.14). Then, we define $\tilde{\Gamma} = \tilde{\Gamma}_{u_0^\omega, \phi, \xi}$ by

$$\tilde{\Gamma}v(t) := \int_0^t S(t-t')\mathcal{N}(v + \tilde{\Psi})(t')dt',$$

where $\tilde{\Psi}$ is the stochastic convolution defined in (1.15) such that $\tilde{\Psi}|_{t=0} = u_0^\omega$. Then, by proceeding as in the proof of Proposition 3.1, we obtain the following nonlinear estimates.

Proposition 3.4. *Let d and p be as in Theorem 1.1. We set*

- (i) $s = 0$ and $r = p + 1$ in the energy-subcritical case,
- (ii) $s > s_{\text{crit}} - 1$ and $r = \frac{2d}{d-2} - \delta$ for some small $\delta = \delta(s) > 0$ in the energy-(super)critical case.

Then, the following estimates hold for some $q \gg 1$:

$$\begin{aligned} \|\tilde{\Gamma}v\|_{C_T W_x^{s,r}} &\lesssim T^\theta \left(\|v\|_{C_T W_x^{s,r}}^p + \|\tilde{\Psi}\|_{L_T^q W_x^{s,r}}^p \right), \\ \|\tilde{\Gamma}v_1 - \tilde{\Gamma}v_2\|_{C_T W_x^{s,r}} &\lesssim T^\theta \left(\|v_1\|_{C_T W_x^{s,r}}^{p-1} + \|v_2\|_{C_T W_x^{s,r}}^{p-1} + \|\tilde{\Psi}\|_{L_T^q W_x^{s,r}}^{p-1} \right) \|v_1 - v_2\|_{C_T W_x^{s,r}} \end{aligned}$$

for all $v, v_1, v_2 \in C([0, T]; W^{s,r}(\mathbb{R}^d))$.

Next, let us state the following probabilistic Strichartz estimates. See [1] for the proof.

Lemma 3.5. *Let $s \in \mathbb{R}$. Given u_0 on $H^s(\mathbb{R}^d)$, let u_0^ω be its randomization defined in (1.14), satisfying (1.13). Then, the following statements hold almost surely:*

- (i) $S(t)u_0^\omega \in C(\mathbb{R}; H^s(\mathbb{R}^d))$,
- (ii) Given finite $q, r \geq 2$, we have $S(t)u_0^\omega \in L^q([0, T]; W^{s,r}(\mathbb{R}^d))$ for any $T > 0$.

In particular, Lemmas 2.1 and 3.5 state that the new stochastic convolution $\tilde{\Psi} = S(t)u_0^\omega + \Psi$ also satisfies the conclusion of Lemma 2.1. Therefore, together with this observation, Proposition 3.4 implies Theorem 1.3.

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REFERENCES

- [1] Á. Bényi, T. Oh, O. Pocovnicu, *Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS*, Excursions in Harmonic Analysis, Volume 4, 3-25, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York, 2015.
- [2] Á. Bényi, T. Oh, O. Pocovnicu, *On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on \mathbb{R}^d , $d \geq 3$* , Trans. Amer. Math. Soc. Ser. B 2 (2015), 1–50.
- [3] Á. Bényi, T. Oh, O. Pocovnicu, *Higher order expansions for the probabilistic local Cauchy theory of the cubic nonlinear Schrödinger equation on \mathbb{R}^3* , to appear in Trans. Amer. Math. Soc.
- [4] Á. Bényi, T. Oh, O. Pocovnicu, *On the probabilistic Cauchy theory for nonlinear dispersive PDEs*, to appear in Landscapes of Time-Frequency Analysis, Appl. Numer. Harmon. Anal.
- [5] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. 3 (1993), 107–156.
- [6] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. 176 (1996), no. 2, 421–445.
- [7] J. Brereton, *Almost sure local well-posedness for the supercritical quintic NLS*, Tunisian J. Math. 1 (2019), no. 3, 427–453.
- [8] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. I. Local theory*, Invent. Math. 173 (2008), no. 3, 449–475.

- [9] T. Cazenave, F. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s* , Nonlinear Anal. 14 (1990), no. 10, 807–836.
- [10] K. Cheung, R. Mosincat, *Stochastic nonlinear Schrödinger equations on tori*, to appear in Stoch. Partial Differ. Equ. Anal. Comput.
- [11] K. Cheung, O. Pocovnicu, *On the local well-posedness of the stochastic cubic nonlinear Schrödinger equation on \mathbb{R}^d , $d \geq 3$, with supercritical noise*, preprint.
- [12] M. Christ, J. Colliander, T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, arXiv:math/0311048 [math.AP].
- [13] M. Christ, M. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. 100 (1991), 87–109.
- [14] G. Da Prato, A. Debussche, *Two-dimensional Navier-Stokes equations driven by a space-time white noise*, J. Funct. Anal. 196 (2002), no. 1, 180–210.
- [15] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Second edition. Encyclopedia of Mathematics and its Applications, 152. Cambridge University Press, Cambridge, 2014. xviii+493 pp.
- [16] A. de Bouard, A. Debussche, *The stochastic nonlinear Schrödinger equation in H^1* , Stochastic Anal. Appl. 21 (2003), no. 1, 97–126.
- [17] B. Dodson, J. Lührmann, D. Mendelson, *Almost sure scattering for the 4D energy-critical defocusing nonlinear wave equation with radial data*, to appear in Amer. J. Math.
- [18] B. Dodson, J. Lührmann, D. Mendelson, *Almost sure local well-posedness and scattering for the 4D cubic nonlinear Schrödinger equation*, arXiv:1802.03795 [math.AP].
- [19] J. Ginibre, G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. 144 (1992), no. 1, 163–188.
- [20] M. Hadac, S. Herr, H. Koch, *Well-posedness and scattering for the KP-II equation in a critical space*, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 3, 917–941. *Erratum to “Well-posedness and scattering for the KP-II equation in a critical space”*, Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010), no. 3, 971–972.
- [21] S. Herr, D. Tataru, N. Tzvetkov, *Global well-posedness of the energy critical nonlinear Schrödinger equation with small initial data in $H^1(\mathbb{T}^3)$* , Duke Math. J. 159 (2011), 329–349.
- [22] T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. 46 (1987), no. 1, 113–129.
- [23] M. Keel, T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. 120 (1998), no. 5, 955–980.
- [24] N. Kishimoto, *A remark on norm inflation for nonlinear Schrödinger equations*, arXiv:1806.10066 [math.AP].
- [25] J. Lührmann, D. Mendelson, *Random data Cauchy theory for nonlinear wave equations of power-type on \mathbb{R}^3* , Comm. Partial Differential Equations 39 (2014), no. 12, 2262–2283.
- [26] H.P. McKean, *Statistical mechanics of nonlinear wave equations. IV. Cubic Schrödinger*, Comm. Math. Phys. 168 (1995), no. 3, 479–491. *Erratum: Statistical mechanics of nonlinear wave equations. IV. Cubic Schrödinger*, Comm. Math. Phys. 173 (1995), no. 3, 675.
- [27] T. Oh, *A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces*, Funkcial. Ekvac. 60 (2017), 259–277.
- [28] T. Oh, M. Okamoto, O. Pocovnicu, *On the probabilistic well-posedness of the nonlinear Schrödinger equations with non-algebraic nonlinearities*, arXiv:1708.01568 [math.AP].
- [29] T. Oh, O. Pocovnicu, *Probabilistic global well-posedness of the energy-critical defocusing quintic nonlinear wave equation on \mathbb{R}^3* , J. Math. Pures Appl. 105 (2016), 342–366.
- [30] O. Pocovnicu, *Probabilistic global well-posedness of the energy-critical defocusing cubic nonlinear wave equations on \mathbb{R}^4* , J. Eur. Math. Soc. (JEMS) 19 (2017), 2321–2375.
- [31] O. Pocovnicu, Y. Wang, *An L^p -theory for almost sure local well-posedness of the nonlinear Schrödinger equations*, C. R. Math. Acad. Sci. Paris. 356 (2018), no. 6, 637–643.
- [32] B. Simon, *The $P(\varphi)_2$ Euclidean (quantum) field theory*, Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1974. xx+392 pp.
- [33] R. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. 44 (1977), no. 3, 705–714.
- [34] Y. Tsutsumi, *L^2 -solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. 30 (1987), no. 1, 115–125.
- [35] N. Wiener, *Tauberian theorems*, Ann. of Math. 33 (1932), no. 1, 1–100.

- [36] K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys. 110 (1987), no. 3, 415–426.

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