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Abstract

We study generalised Navier–Stokes equations governing the motion of an electro-rheological fluid subject to stochastic perturbation. Stochastic effects are implemented through (i) random initial data, (ii) a forcing term in the momentum equation represented by a multiplicative white noise and (iii) a random character of the variable exponent $p = p(\omega, t, x)$ (as a result of a random electric field). We show the existence of a weak martingale solution provided the variable exponent satisfies $p \geq p^- > \frac{3n}{n+2}$ ($p^- > 1$ in two dimensions). Under additional assumptions we obtain also stochastically strong solutions.

1 Introduction

Electro-rheological fluids are special smart fluids which change their material properties due to the application of an electric field firstly observed by Winslow [32]. Since then a vast development in the chemical constitution of electro-rheological fluids has taken place and nowadays dramatic changes by a factor of 10^3 in 1ms in the viscosity are possible. This provides the opportunity for the gainful exploitation of this fact in technological applications for instance in clutches, shock absorbers, valves, actuators and exercise equipment.

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The simplest approach for the modelling of such suspensions is to treat them in a homogenised sense within the framework of continuum mechanics and in this respect, we restrict ourself to incompressible fluids with density $\varrho > 0$. The conservation of mass and the balance of linear momentum are given by

$$\begin{cases} \partial_t(\varrho\mathbf{v}) - \operatorname{div}\mathbf{S} = -\operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v}) - \nabla\pi + \varrho\mathbf{f} + \mathbf{f}_e & \text{in } Q, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } Q, \end{cases} \tag{1.1}$$

where $Q = (0, T) \times \mathcal{O}$ denotes the parabolic cylinder ($\mathcal{O}$ is a bounded domain in $\mathbb{R}^n$, $n = 2, 3$) and $\otimes$ is the tensor product in $\mathbb{R}^n$ (that is we have $\mathbf{a}\otimes\mathbf{b} = \mathbf{a}\mathbf{b}^T$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$). Here $\mathbf{v} : Q \to \mathbb{R}^n$ is the velocity field, $\pi : Q \to \mathbb{R}$ the pressure, $\mathbf{S} : Q \to \mathbb{R}^{n\times n}$ the viscous stress tensor whereas $\mathbf{f} : Q \to \mathbb{R}^n$ is the external mechanical body force and $\mathbf{f}_e : Q \to \mathbb{R}^n$ the electromagnetic force. The material properties of an electro-rheological fluid—according to Rajagopal and Růžička [23,24]—are described by the relation

$$\mathbf{S} = \alpha_{21}\big((1 + |\boldsymbol{\varepsilon}(\mathbf{v})|^2)^{\frac{p-1}{2}} - 1\big)\mathbf{E}\otimes\mathbf{E} + (\alpha_{31} + \alpha_{33}|\mathbf{E}|^2)(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|^2)^{\frac{p-2}{2}}\boldsymbol{\varepsilon}(\mathbf{v})$$
$$+ \alpha_{51}(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|^2)^{\frac{p-2}{2}}(\mathbf{E}\otimes\boldsymbol{\varepsilon}(\mathbf{v})\mathbf{E} + \mathbf{E}\otimes\boldsymbol{\varepsilon}(\mathbf{v})\mathbf{E}). \tag{1.2}$$

Here $\mathbf{E} : Q \to \mathbb{R}^n$ is the electric field (which is a solution to the quasi-static Maxwell equations and is not influenced by the motion of the fluid), $\boldsymbol{\varepsilon}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$ the symmetric gradient of the velocity field and α_{ij} are material constants. The exponent $p = p(|\mathbf{E}|^2)$ depends on the strength of the electric field (and hence on time and space) and satisfies in Q

$$1 < p^- \leq p \leq p^+ < \infty. \tag{1.3}$$

In the mathematical literature about electro-rheological fluids (starting with [26]) it is common to study the constitutive law

$$\mathbf{S} = \mu(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2}\boldsymbol{\varepsilon}(\mathbf{v}), \quad \mu > 0, \tag{1.4}$$

which contains the same mathematical difficulties as (1.2) but simplifies the calculations. Essentially, there are two parts in the model where randomness can occur:

- The electromagnetic force is mainly influenced by the gradient of the electric field $\mathbf{E}$ and the electric polarization $\mathbf{P}$ such that $\mathbf{f}_e = [\nabla\mathbf{E}]\mathbf{P}$. All missing quantities which are neglected here (for instance magnetic field and magnetic polarization) can be summarized in some random perturbation. In addition, it can incorporate physical uncertainties and turbulence in the fluid motion.
- The exponent p depends on the strength of the electric field which is a solution to Maxwell's equation, the latter having been widely studied in literature. Randomness naturally appears in the Maxwell equation (see, for instance [7,16] for stochastic Maxwell equations), and the randomness in the Maxwell equation transfers to randomness in the exponent in the model (1.6). In conclusion, the assumption of a random exponent is very reasonable and required by applications.

Second, it is not possible to give an explicit formula for the exponent p. Its dependence on the electric field has to be determined via physical experiments. Hence some (random) derivation from the "real" exponent is expected.

In this respect, the aim of this paper is to give a rigorous analysis of the following stochastic model for electro-rheological fluids (without loss of generality we assume that $\varrho = 1$ and $\mathbf{f}_e = 0$)

$$\begin{cases} d\mathbf{v} = \operatorname{div} \mathbf{S} \, dt - \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) \, dt - \nabla \pi \, dt + \mathbf{f} \, dt + \Phi(\mathbf{v}) dW & \text{in } Q, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } Q, \\ \mathbf{v}(0) = \mathbf{v}_0 & \text{in } \mathcal{O}, \end{cases} \quad (1.5)$$

with $\mathbf{S}$ given by

$$\mathbf{S} = \mu(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2}\boldsymbol{\varepsilon}(\mathbf{v}), \quad \mu > 0. \quad (1.6)$$

We suppose that the electric field $\mathbf{E}$ is given and that $p = p(\omega, t, x)$ satisfies (1.3). The quantity W denotes a cylindrical Wiener process with values in some Hilbert space and Φ is nonlinear in $\mathbf{v}$ with linear growth, cp. Sect. 2.2 for further details.

In the general three-dimensional case, regularity and uniqueness of solutions to (1.5)–(1.6) is a longstanding open problem (already in the deterministic situation) even if $p \equiv 2$, leading to the classical Navier–Stokes equations for Newtonian fluids. Consequently, the solution is understood weakly in space–time (in the sense of distributions) and also weakly in the probabilistic sense (i.e., the underlying probability space is part of the solution). This concept of stochastically weak solutions already appears on the level of stochastic ODEs if uniqueness is not available.

As far as stochastic PDEs are concerned, a milestone was the existence of martingale solutions to the stochastic Navier–Stokes equation [(1.5)–(1.6) with $p \equiv 2$] by Flandoli–Gatarek [13]. Today there exists an abundant amount of literature concerning the dynamics of incompressible Newtonian fluids driven by stochastic forcing. We refer to the lecture notes by Flandoli [12], the monograph of Kuksin and Shyrikian [20], the survey by Romito [25] as well as the references cited therein for a recent overview. Much less is known if other fluid types are concerned. Just very recently, an analysis of non-Newtonian fluids (see [4,30,33]) and compressible fluids (see [6,28]) subject to stochastic forcing started.

The analysis the system (1.5)–(1.6) brings a completely new aspect into play: a random variable exponent. As a consequence, solutions are located in a random function space generated by the a priori information

$$\mathbb{E}\left[\int_Q |\boldsymbol{\varepsilon}(\mathbf{v})(\omega, t, x)|^{p(\omega,t,x)} \, dx \, dt\right] < \infty.$$

So, we have

$$\boldsymbol{\varepsilon}(\mathbf{v})(\omega, \cdot) \in \mathrm{L}^{p(\omega,\cdot)}(Q) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

where

$$L^{p(\cdot)}(G) = \left\{ f \in L^1(G) : \int_G |f(y)|^{p(y)} \, \mathrm{d}y < \infty \right\} \tag{1.7}$$

for $G \subset \mathbb{R}^m$ and $p : G \to [1, \infty)$ measurable. Variable exponent Lebesgue spaces (and Sobolev spaces) as in (1.7) have been studied extensively over the last two decades motivated by the model for electro-rheological fluids from [23,24], and we refer to [11] for a comprehensive treatment. As far as stochastic problems are concerned, a first analysis of problems involving random variable exponents can be found in [31] (see also [3] for a previous result on a stochastic $p(t, x)$-Laplacian). In this work, the existence and uniqueness of weak solutions of a stochastic $p(\omega, t, x)$-Laplacian type equation is established by use of the variational approach, and problems connected to compactness and non-uniqueness as in the Navier-Stokes context do not occur.

In order to complete system (1.5) we have to describe the behaviour of the fluid at the boundary. The boundary conditions in real world applications are quite complicated and of substantial influence on the fluid motion. Nevertheless, our goal is to focus on the effect of a random variable exponent as well as stochastic perturbations imposed through stochastic volume forces. So, for a first analysis we consider *periodic* boundary conditions, where the physical domain is identified with the flat torus

$$\mathbb{T}^n = \left([0, 1] \Big|_{\{0,1\}} \right)^n.$$

The first main result of this paper is the existence of a weak martingale solution to (1.5)–(1.6) under periodic boundary conditions where the variable exponent p is Lipschitz continuous in x and satisfies

$$\inf_{\Omega \times Q} p > \frac{3n}{n+2}, \tag{1.8}$$

see Theorem 2.2 for the precise statement. This generalises the results from [30] to the case of variable exponents. As a consequence of the nature of martingale solutions we are not able to describe the variable exponent as a given function defined on $\Omega \times Q$. Instead, we rather describe a probability law on $C^0([0, T] \times \mathbb{T}^n)$.

Our approach is based on a finite-dimensional Galerkin approximation and a refined stochastic compactness method involving Skorokhod's representation theorem. Since the system (1.5)–(1.6) is nonlinear in the gradient of the velocity field we have to demonstrate its compactness first. This is achieved by fractional estimates for $\nabla \mathbf{v}$ inspired by the results from [21, Chap. 5], where deterministic problems with constant p are considered. Under more restrictive assumptions on the variable exponent p, we are able to show pathwise uniqueness of solutions. As a consequence, we obtain pathwise solutions (see Theorem 2.10) using the method by Gyöngy–Krylov [15]. Eventually, we are concerned with the existence of analytically strong solutions (see Definitions 2.6, 2.12), where Eq. (1.5)₁ holds almost everywhere in space. This is based on the existence of second derivatives of the velocity field. Because of the non-standard growth character of (1.6) this is much more involved than the situation with

constant p. By simply differentiating Eq. $(1.5)_1$ we are left with an a priori unbounded integral, cp. (5.9). This issue can be overcome by combining a parabolic interpolation as in [2] with an improved moment estimate, cp. Theorem 3.2. Consequently, we obtain weak (or even strong) pathwise solutions to (1.5)–(1.6), see Theorem 2.7 and Corollary 2.13.

The paper is organised as follows. In Sect. 2 we present the mathematical framework, the various solution concepts to (1.5)–(1.6) as well as the main results. In Sect. 3 we study the finite-dimensional approximation to (1.5)–(1.6) and derive uniform a priori estimates. Section 4 is dedicated to the existence of martingale solutions. Under more restrictive assumptions on the exponent p, we then show existence of stochastically strong solutions. In the final section we establish the existence of analytically strong solutions subject to suitable additional assumptions imposed on the data.

2 Framework and main results

2.1 Function space setup

In this section we briefly introduce the function spaces to be dealt with in the main part of the paper. Incorporating the periodic boundary conditions, all spatial function spaces are defined on the torus $\mathbb{T}^n$. Specifically, we define for $0 < \kappa < \infty$ and $1 < q < \infty$ the corresponding Bessel–Sobolev spaces by

$$\mathrm{W}^{\kappa,q}(\mathbb{T}^n) := \left\{ v \colon \mathbb{T}^n \to \mathbb{R}^n \colon \|v\|_{\kappa,q}^q := \sum_{k \in \mathbb{Z}} \langle k \rangle^{\kappa q} |c_k(v)|^q < \infty \right\},$$

$$\mathrm{W}_{\mathrm{div}}^{\kappa,q}(\mathbb{T}^n) := \mathrm{W}^{\kappa,q}(\mathbb{T}^n)^n \cap \{v \in \mathrm{L}^1(\mathbb{T}^n; \mathbb{R}^n) \colon \ \mathrm{div}(v) = 0$$

in the sense of distributions$\},$

where $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$ and $c_k(v)$ are the Fourier coefficients of $v \colon \mathbb{T}^n \to \mathbb{R}^n$ with respect to the standard Fourier basis $\{x \mapsto \exp(2\pi \mathrm{i} k \cdot x)\}_k$. Given a real Banach space $(X, \|\cdot\|)$, we moreover introduce the fractional Sobolev space $\mathrm{W}^{\kappa,q}(0, T; X)$ as the collection of all measurable $v \colon [0, T] \to X$ such that $v \in \mathrm{L}^q(0, T; X)$ (in the sense of Bochner integrability) and

$$[v]_{\kappa,q} := \int_0^T \int_0^T \frac{\|v(s) - v(t)\|_X^q}{|s - t|^{1 + \kappa q}} \, \mathrm{d}s \, \mathrm{d}t < \infty.$$

Let us note that the former space could be defined similarly by use of Fourier coefficients, however, we refrained from doing so to emphasize the non-periodicity with respect to time.

We continue with a brief introduction of variable exponent Lebesgue spaces and refer to [11] for a detailed exposition. For a given continuous function $p \colon Q \to [1, \infty)$ with $Q = (0, T) \times \mathbb{T}^n$ we define the variable exponent Lebesgue space $\mathrm{L}^{p(\cdot)}(Q)$ by

$$\mathrm{L}^{p(\cdot)}(Q) = \sup \left\{ f \in \mathrm{L}^1(Q) \colon \int_Q |f(t, y)|^{p(t,y)} \, \mathrm{d}y \, \mathrm{d}t < \infty \right\}.$$

It is a Banach space together with the Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf\left\{k \geq 0 : \int_Q \left|\frac{f(t, y)}{k}\right|^{p(t,y)} \, dy \, dt \leq 1\right\}. \tag{2.1}$$

For most of the interesting functional analytical properties of $L^{p(\cdot)}(Q)$ some mild regularity of p is needed. We say that a function $g : Q \to \mathbb{R}$ is log-Hölder continuous in Q if there exists a constant $c \geq 0$ such that

$$|g(X) - g(Y)| \leq \frac{c}{\log(e + 1/|X - Y|)},$$

for all $X \neq Y \in Q$. The smallest such constant c is the log-Hölder constant of g. We define $\mathcal{P}^{\log}(Q)$ to consist of those exponents $p \in L^1(Q)$ for which $\frac{1}{p} : Q \to (0, 1]$ is log-Hölder continuous. The norm $\|p\|_{\mathcal{P}^{\log}(Q)}$ is the log-Hölder constant of $1/p$. For $p \in \mathcal{P}^{\log}(Q)$ almost all properties of the classical Lebesgue spaces extend to $L^{p(\cdot)}(Q)$. In particular smooth functions are dense with respect to the norm given in (2.1).

Lastly, we shall sometimes surpress the target space and write, e.g., $W^{\kappa,q}(\mathbb{T}^n)$ instead of $W^{\kappa,q}(\mathbb{T}^n)^n$. However, no ambiguities will arise from this.

2.2 Probability setup

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathscr{F}_t) = (\mathscr{F}_t)_{t \geq 0}$ which is a nondecreasing family of sub-σ-fields of $\mathscr{F}$, i.e., $\mathscr{F}_s \subset \mathscr{F}_t$ for $0 \leq s \leq t \leq T$. We further assume that $(\mathscr{F}_t)_{t \geq 0}$ is right-continuous and $\mathscr{F}_0$ contains all the $\mathbb{P}$-negligible events in $\mathscr{F}$.

For a Banach space $(X, \|\cdot\|_X)$ and corresponding Borel σ-algebra $\mathfrak{B}(X)$, we denote by for $1 \leq p < \infty$ by $L^p(\Omega; X)$ the Banach space of all measurable functions $v : (\Omega, \mathscr{F}) \to (X, \mathfrak{B}(X))$ such that

$$\mathbb{E}\big[\|v\|_X^p\big] = \int_\Omega \|v\|_X^p \, d\mathbb{P} < \infty.$$

Let $\mathfrak{U}$ be a Hilbert space with orthonormal basis $(e_k)_{k \in \mathbb{N}}$ and let $L_2(\mathfrak{U}, L^2(\mathbb{T}^n))$ be the set of Hilbert–Schmidt operators from $\mathfrak{U}$ to $L^2(\mathbb{T}^n)$. Moreover, define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ as

$$\mathfrak{U}_0 := \left\{e = \sum_{k=1}^\infty \alpha_k e_k : \sum_{k=1}^\infty \frac{\alpha_k^2}{k^2} < \infty\right\},$$

$$\|e\|_{\mathfrak{U}_0}^2 := \sum_{k=1}^\infty \frac{\alpha_k^2}{k^2}, \quad e = \sum_{k=1}^\infty \alpha_k e_k. \tag{2.2}$$

Throughout the paper we consider a cylindrical $(\mathscr{F}_t)$-Wiener process $W = (W_t)_{t \geq 0}$ which has the form

$$W = \sum_{k \in \mathbb{N}} \beta_k e_k \tag{2.3}$$

with a sequence (β_k) of independent real valued $(\mathscr{F}_t)$-Wiener processes. The embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert–Schmidt and trajectories of W are $\mathbb{P}$-a.s. continuous with values in $\mathfrak{U}_0$ (see [8]). Now, for $\Psi \in L^2(\Omega; L^2(0, T; L_2(\mathfrak{U}, L^2(\mathbb{T}^n))))$ $(\mathscr{F}_t)$-progressively measurable[1] we have that $\int_0^t \Psi \, dW$ defines a $\mathbb{P}$-almost surely continuous $L^2(\mathbb{T}^n)$-valued $(\mathscr{F}_t)$-martingale (cp. [8] for stochastic calculus in infinite dimensions). Moreover, we can multiply with test-functions because

$$\int_{\mathbb{T}^n} \int_0^t \Psi \, dW \cdot \boldsymbol{\varphi} \, dx = \sum_{k=1}^\infty \int_0^t \int_{\mathbb{T}^n} \Psi e_k \cdot \boldsymbol{\varphi} \, dx \, d\beta_k, \quad \boldsymbol{\varphi} \in L^2(\mathbb{T}^n),$$

is well-defined.

In the SPDEs appearing in this paper we consider a noise coefficient $\Phi(\mathbf{v})$ (depending on the solution $\mathbf{v}$) with values in $L_2(\mathfrak{U}, L^2(\mathbb{T}^n))$. We suppose the following linear growth assumptions on Φ: For each $\mathbf{z} \in L^2(\mathbb{T}^n)$ there is a mapping $\Phi(\mathbf{z}) : \mathfrak{U} \to L^2(\mathbb{T}^n)$ defined by $\Phi(\mathbf{z})e_k = g_k(\mathbf{z}(\cdot))$. In particular, we suppose that $g_k \in C^1(\mathbb{R}^n)$ and the following conditions for some $L \geq 0$

$$\sum_{k \in \mathbb{N}} |g_k(\boldsymbol{\xi})|^2 \leq L(1 + |\boldsymbol{\xi}|^2), \quad \sum_{k \in \mathbb{N}} |\nabla g_k(\boldsymbol{\xi})|^2 \leq L, \quad \boldsymbol{\xi} \in \mathbb{R}^n. \tag{2.4}$$

2.3 Martingale solutions

Now we are in position to give a precise formulation of the meaning of a martingale solutions. We start with a weak martingale solution. This solution is weak on both senses. Derivatives have to be understood in the sense of distributions (weak in the PDE-sense) and the underlying probability space is not a priori given but is part of the problem (weak in the probabilistic sense). Accordingly, the initial condition is given as a Borel probability measure on $L^2_{\mathrm{div}}(\mathbb{T}^n)$. The same applies for the forcing $\mathbf{f}$ which will be given as a Borel probability measure on $L^2(Q)$. As usual the moments of data measured via

$$C_r(\Lambda_0, \Lambda_{\mathbf{f}}) = \int_{L^2_{\mathrm{div}}(\mathbb{T}^n)} \|\mathbf{u}\|_{L^2(\mathbb{T}^n)}^{2r} \, d\Lambda_0(\mathbf{u}) + \int_{L^2(Q)} \|\mathbf{g}\|_{L^2(Q)}^{2r} \, d\Lambda_{\mathbf{f}}(\mathbf{g})$$

for $r \geq 1$ transfer to the solution. Solutions as described above are called martingale solutions due to the connection to the so-called Stroock–Varadhan martingale problem (see, e.g., [19, Chap. 5.4]).

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[1] We understand progressive measurability for non-continuous processes in the sense of random distributions introduced in [5, Sect. 2.2].

Definition 2.1 (*Weak martingale solution*) Let Λ be a Borel probability law on $L^2_{\text{div}}(\mathbb{T}^n) \times L^2(Q) \times C^0([0,T] \times \mathbb{T}^n)$ with marginals Λ_0, $\Lambda_{\mathbf{f}}$, Λ_p. Then a quintuple

$$\big((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \mathbf{v}, \mathbf{f}, p, W\big)$$

is called a *weak martingale solution* to (1.5)–(1.6) with the initial datum Λ_0, right-hand-side $\Lambda_{\mathbf{f}}$ and exponent Λ_p provided

(a) $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration,
(b) W is an $(\mathscr{F}_t)$-cylindrical Wiener process,
(c) $\mathbf{f} \in L^2(\Omega, \mathcal{F}, \mathbb{P}; L^2(Q))$ is $(\mathscr{F}_t)$-progressively measurable,
(d) $p \in L^2(\Omega, \mathcal{F}, \mathbb{P}; C^0([0,T] \times \mathbb{T}^n))$ is $(\mathscr{F}_t)$-progressively measurable,
(e) the velocity field satisfies $\mathbf{v} \in C_w([0,T]; L^2(\mathbb{T}^n))$, $\boldsymbol{\varepsilon}(\mathbf{v}) \in L^{p(\cdot)}(Q)$, $\mathbb{P}$-a.s. and is $(\mathscr{F}_t)$-progressively measurable,
(f) we have $\Lambda = \mathbb{P} \circ (\mathbf{v}(0), \mathbf{f}, p)^{-1}$,
(g) for all $\boldsymbol{\varphi} \in C^\infty_{\text{div}}(\mathbb{T}^n)$ and all $t \in [0,T]$ there holds $\mathbb{P}$-a.s.[2]

$$\int_{\mathbb{T}^n} \mathbf{v}(t) \cdot \boldsymbol{\varphi} \, dx + \int_0^t \int_{\mathbb{T}^n} \mu(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2} \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \, dx \, d\sigma$$

$$- \int_0^t \int_{\mathbb{T}^n} \mathbf{v} \otimes \mathbf{v} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \, dx \, d\sigma$$

$$= \int_{\mathbb{T}^n} \mathbf{v}(0) \cdot \boldsymbol{\varphi} \, dx + \int_{\mathbb{T}^n} \int_0^t \mathbf{f} \cdot \boldsymbol{\varphi} \, dx \, d\sigma$$

$$+ \int_{\mathbb{T}^n} \int_0^t \Phi(\mathbf{v}) \, dW \cdot \boldsymbol{\varphi} \, dx.$$

We obtain the following result.

Theorem 2.2 (Weak martingale solution) *Suppose that*

$$\int_{L^2_{\text{div}}(\mathbb{T}^n)} \|\mathbf{u}\|^2_{W^{1,2}(\mathbb{T}^n)} \, d\Lambda_0(\mathbf{u}) < \infty,$$

$$\int_{L^2(Q)} \|\mathbf{g}\|^2_{L^2(0,T;W^{1,2}(\mathbb{T}^n))} \, d\Lambda_{\mathbf{f}}(\mathbf{g}) < \infty, \tag{2.5}$$

as well as $C_r(\Lambda_0, \Lambda_{\mathbf{f}}) < \infty$ for all $1 \le r < \infty$. Moreover, assume that

$$\Lambda_p\{h \in \mathcal{P}^{\log}(Q) : p^- \le h \le p^+, \ \|h\|_\infty + \|\nabla h\|_\infty \le c_p\} = 1, \tag{2.6}$$

where $c_p < \infty$ and

$$\frac{3n}{n+2} < p^- \le p^+ < \frac{n+2}{n} p^- \tag{2.7}$$

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[2] By : we denote the inner product between matrices, that is $\mathbf{A} : \mathbf{B} = \sum_{ij} A_{ij} B_{ij}$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$.

for some deterministic constants p^- and p^+. Additionally, suppose that

$$\int_{C^0([0,T]\times\mathbb{T}^n)} \|h\|_{\mathcal{P}^{\log}(Q)}\, \mathrm{d}\Lambda_p(h) < \infty. \tag{2.8}$$

Finally, assume that Φ satisfies (2.4). Then there is a weak martingale solution to (1.5)–(1.6) in the sense of Definition 2.1. We have the energy estimate

$$\mathbb{E}\left[\sup_{t\in(0,T)} \int_{\mathbb{T}^n} |\mathbf{v}(t)|^2\, \mathrm{d}x + \int_Q |\boldsymbol{\varepsilon}(\mathbf{v})|^{p(\cdot)}\, \mathrm{d}x\, \mathrm{d}t\right]^r \le c\big(1 + C_r(\Lambda_0, \Lambda_{\mathbf{f}})\big). \tag{2.9}$$

for any $r \ge 1$.

Remark 2.3 Let us explain the assumptions on upper and lower bound on p in (2.7).

- The lower bound is the same as in the case of constant p from [30] in the two and three dimensional case (we do not consider higher dimensions as they are not of physical interest).
- It will become clear from the proof of Theorem (2.2) that the assumption (2.7) can be relaxed to

$$\frac{3n-4}{n} < p^- \le p^+ < np^- + 4 \tag{2.10}$$

provided $p^- \ge 2$ (where the lower bound is redundant for $n = 2, 3$). We decided for the version in (2.7) as it is physically meaningful that p^- is as low as possible whereas non-Newtonian fluids with growth-exponent higher than $p = 3$ are not known (the case $p = 3$ refers to the the classical Smagorinsky model [29]).

Remark 2.4 By slightly refining our estimates it is possible to weaken the assumption in (2.6) from a deterministic constant c_p to a random variable c_p with arbitrary high moments. This seems more realistic in view of the random character of the exponent.

Remark 2.5 In contrast to the deterministic case we need assumptions between p^- and p^+ to balance our estimates. In the deterministic case this can be avoided by localizing the problem and arguing on a small parabolic cube where p^- and p^+ are arbitrary close (recall that p is continuous). This is not possible here because of the random character of p.

The method we are using in the proof of Theorem 2.2 originates from [21, Chap. 5], where the deterministic problem with constant p is studied. The key idea is to analyse fractional derivatives of the velocity gradient. This method is only very powerful in the case of periodic boundary conditions, where a test with $\Delta\mathbf{v}^N$ ($\mathbf{v}^N$ is the Galerkin approximation of the velocity field) is possible. The situation in the two-dimensional situation is much better than the 3D case as we have

$$\int_{\mathbb{T}^n} \mathbf{v}^N \otimes \mathbf{v}^N : \nabla\mathbf{v}^N\, \mathrm{d}x = 0.$$

Due to this we can expect solutions which are strong in PDE sense. Before we give a proper definition we have to introduce the pressure function (as we need a formulation which holds a.e. in space without test-functions).

Assume that $((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \mathbf{v}, \mathbf{f}, p, W)$ is a weak martingale solution to (1.5)–(1.6) in the sense of Definition 2.1. In particular, we have $\mathbb{P}$-a.s.

$$\int_{\mathbb{T}^n} \mathbf{v}(t) \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^n} \mu(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2} \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \, \mathrm{d}x \, \mathrm{d}\sigma$$
$$- \int_0^t \int_{\mathbb{T}^n} \mathbf{v} \otimes \mathbf{v} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \, \mathrm{d}x \, \mathrm{d}\sigma$$
$$= \int_{\mathbb{T}^n} \mathbf{v}(0) \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \int_{\mathbb{T}^n} \int_0^t \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}\sigma$$
$$+ \int_{\mathbb{T}^n} \int_0^t \Phi(\mathbf{v}) \, \mathrm{d}W \cdot \boldsymbol{\varphi} \, \mathrm{d}x$$

for all $\boldsymbol{\varphi} \in C_{\mathrm{div}}^\infty(\mathbb{T}^n)$ and all $t \in [0, T]$. Now, for $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^n)$ we can insert $\boldsymbol{\varphi} - \nabla \Delta^{-1} \operatorname{div} \boldsymbol{\varphi}$ and obtain

$$\int_{\mathbb{T}^n} \mathbf{v}(t) \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^n} \mu(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2} \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \, \mathrm{d}x \, \mathrm{d}\sigma$$
$$- \int_0^t \int_{\mathbb{T}^n} \mathbf{v} \otimes \mathbf{v} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \, \mathrm{d}x \, \mathrm{d}\sigma$$
$$= \int_{\mathbb{T}^n} \mathbf{v}(0) \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \int_0^t \int_{\mathbb{T}^n} \pi_{\mathrm{det}} \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}\sigma$$
$$+ \int_{\mathbb{T}^n} \int_0^t \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}\sigma$$
$$+ \int_{\mathbb{T}^n} \int_0^t \Phi(\mathbf{v}) \, \mathrm{d}W \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \int_{\mathbb{T}^n} \int_0^t \Phi^\pi \, \mathrm{d}W \cdot \boldsymbol{\varphi} \, \mathrm{d}x, \tag{2.11}$$

where

$$\pi_{\mathrm{det}} = \pi_{\mathrm{det}}^1 + \pi_{\mathrm{det}}^2 + \pi_{\mathrm{det}}^3,$$
$$\pi_{\mathrm{det}}^1 = \Delta^{-1} \operatorname{div} \operatorname{div} \left(\mu(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2} \boldsymbol{\varepsilon}(\mathbf{v}) \right),$$
$$\pi_{\mathrm{det}}^2 = -\Delta^{-1} \operatorname{div} \operatorname{div} \left(\mathbf{v} \otimes \mathbf{v} \right),$$
$$\pi_{\mathrm{det}}^3 = \Delta^{-1} \operatorname{div} \mathbf{f},$$
$$\Phi^\pi = -\nabla \Delta^{-1} \operatorname{div} \Phi(\mathbf{v}).$$

This corresponds to the stochastic pressure decomposition introduced in [4, Chap. 3]. However, the situation with periodic boundary conditions we are considering here is much easier as the harmonic component of the pressure disappears. From a strong solution (in the PDE-sense) we expect that (2.11) holds without the use of the test-functions, i.e. we have

$$\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \left[\operatorname{div}\left(\mu(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2}\boldsymbol{\varepsilon}(\mathbf{v}) \right) - \operatorname{div}\left(\mathbf{v} \otimes \mathbf{v} \right) - \nabla\pi_{\det} + \mathbf{f} \right] d\sigma$$

$$+ \int_0^t \left[\Phi(\mathbf{v}) + \Phi^\pi \right] dW$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. We remark that already under the assumptions of Theorem 2.2 we have enough spatial regularity to define $\operatorname{div}\left(\mathbf{v} \otimes \mathbf{v} \right)$ as an L^1-function (in fact $p^- \geq \frac{2n+2}{n+2}$ is required). So, the critical point is whether second derivatives of $\mathbf{v}$ exists and $\operatorname{div}\left((\kappa + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p-2}\boldsymbol{\varepsilon}(\mathbf{v}) \right)$ is an L^1-function. The required regularity of the pressure terms follows immediately from this and continuity properties of Δ^{-1} on Lebesgue and Sobolev spaces. Let us finally mention that regularity of $\mathbf{v}$ is usually measured via the nonlinear function $\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}))$, where

$$\mathbf{F}_p(\omega, t, x, \boldsymbol{\eta}) = (1 + |\boldsymbol{\eta}|)^{\frac{p(\omega, t, x)-2}{2}}\boldsymbol{\eta}, \quad \boldsymbol{\eta} \in \mathbb{R}^{n \times n}.$$

Now we are ready to define a strong martingale solution.

Definition 2.6 (*Strong martingale solution*) Let Λ be a Borel probability law on $L^2_{\operatorname{div}}(\mathbb{T}^n) \times L^2(Q) \times C^0([0, T] \times \mathbb{T}^n)$ with marginals $\Lambda_0, \Lambda_{\mathbf{f}}, \Lambda_p$. Then a quintuple

$$\left((\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}), \mathbf{v}, \mathbf{f}, p, W \right)$$

is called a *strong martingale solution* to (1.5)–(1.6) with the initial datum Λ_0, right-hand-side $\Lambda_{\mathbf{f}}$ and exponent Λ_p provided it is a weak martingale solution in the sense of Definition 2.1 and the following holds.

(a) We have $\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v})) \in L^2(0, T; W^{1,2}(\mathbb{T}^n))$ $\mathbb{P}$-a.s.,
(b) there are $\pi_{\det}$ and Φ^π $(\mathscr{F}_t)$-progressively measurable such that $\pi_{\det} \in L^1(Q)$ and $\Phi^\pi \in L^2(0, T; L_2(\mathfrak{U}; L^2(\mathbb{T}^n)))$ $\mathbb{P}$-a.s. as well as

$$\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \left[\operatorname{div}\left(\mu(1 + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2}\boldsymbol{\varepsilon}(\mathbf{v}) \right) - \operatorname{div}\left(\mathbf{v} \otimes \mathbf{v} \right) - \nabla\pi_{\det} + \mathbf{f} \right] d\sigma$$

$$+ \int_0^t \left[\Phi(\mathbf{v}) + \Phi^\pi \right] dW \tag{2.12}$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$.

Theorem 2.7 (Strong martingale solution) *Let the assumptions of Theorem 2.2 be satisfied. Suppose that either we have*

(i) $n = 2$ *and* $1 < p^- \leq p^+ < 4$ *or;*
(ii) $n = 3$ *and* $\frac{11}{5} < p^- \leq p^+ \leq p^- + \frac{4}{5}$.

Then there is a strong martingale solution to (1.5)–(1.6) *in the sense of Definition* 2.6. *We have the estimate*

$$\mathbb{E}\left[\sup_{t\in(0,T)}\int_{\mathbb{T}^n}|\nabla\mathbf{v}(t)|^2\,\mathrm{d}x + \int_Q|\nabla\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}))|^2\,\mathrm{d}x\,\mathrm{d}t\right] \le c(\Lambda_0,\Lambda_{\mathbf{f}}). \qquad (2.13)$$

Remark 2.8 • We remark that the most interesting situation for physical applications is when p can vary between 1 and 2 as assumed in part (i) of Theorem 2.7. This refers to a range between a Newtonian fluid ($p = 2$) and a plastic material (p close to 1) which has been observed in experiments on electro-rheological fluids.

• Similar to (2.7) (ii) it is also possible to gain a result in two dimensions if $p^+ \ge 4$. In this case the assumption reads as $p^+ < p^- + 1$. However this situation is outside the range of physical interest and we leave the details to the reader.

2.4 Stochastically strong solutions

We are now concerned with the question whether a solution to (1.5)–(1.6) can be constructed on a given probability space and with a given initial velocity $\mathbf{v}_0$ (which is a a random variable rather than a probability law). This goes hand in hand with the question of unique solvability and holds already on the level of stochastic ODEs (see, e.g., [19, Chap. 5]). We start with a formulation which is weak in the PDE-sense.

Definition 2.9 (*Weak stochastically strong solution*) Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration and let W be an $(\mathscr{F}_t)$-cylindrical Wiener process. Let $\mathbf{v}_0$ be an $\mathrm{L}^2(\mathbb{T}^n)$-valued $\mathscr{F}_0$-measurable random variable. Let $\mathbf{f}$ and p be $(\mathscr{F}_t)$-progressively measurable processes such that $\mathbf{f} \in \mathrm{L}^2(Q)$ and $p \in C^0([0, T] \times \mathbb{T}^n)$ with $p \ge 1$ $\mathbb{P}$-a.s. A function $\mathbf{v}$ is called a weak stochastically strong solution solution to (1.5)–(1.6) provided

(a) the velocity field satisfies $\mathbf{v} \in C_w([0, T]; \mathrm{L}^2(\mathbb{T}^n))$, $\boldsymbol{\varepsilon}(\mathbf{v}) \in \mathrm{L}^{p(\cdot)}(Q)$, $\mathbb{P}$-a.s. and is $(\mathscr{F}_t)$-progressively measurable,
(b) we have $\mathbf{v}(0) = \mathbf{v}_0$ $\mathbb{P}$-a.s.,
(c) for all $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^n)$ and all $t \in [0, T]$ there holds $\mathbb{P}$-a.s.

$$\int_{\mathbb{T}^n}\mathbf{v}(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^n}\mu(1+|\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2}\boldsymbol{\varepsilon}(\mathbf{v}):\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}\sigma$$

$$-\int_0^t\int_{\mathbb{T}^n}\mathbf{v}\otimes\mathbf{v}:\boldsymbol{\varepsilon}(\boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}\sigma$$

$$=\int_{\mathbb{T}^n}\mathbf{v}(0)\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_{\mathbb{T}^n}\int_0^t\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\sigma$$

$$+\int_{\mathbb{T}^n}\int_0^t\Phi(\mathbf{v})\,\mathrm{d}W\cdot\boldsymbol{\varphi}\,\mathrm{d}x.$$

We obtain the following result (recall Remark 2.3 for the assumptions on p in (2.16) below).

Theorem 2.10 (Weak stochastically strong solution) *Let* $\mathbf{v}_0$ *be an* $L^2(\mathbb{T}^n)$*-valued* $\mathscr{F}_0$*-measurable random variable. Let* $\mathbf{f}$ *and* p *be* $(\mathscr{F}_t)$*-progressively measurable processes such that* $\mathbf{f} \in L^2(Q)$ *and* $p \in C^0([0, T] \times \mathbb{T}^n)$ $\mathbb{P}$*-a.s. Suppose that*

$$\mathbb{E}\|\mathbf{v}_0\|_{L^2(\mathbb{T}^n)}^{2r} < \infty, \quad \mathbb{E}\|\mathbf{f}\|_{L^2(Q)}^{2r} < \infty, \quad \mathbb{E}\|p\|_{\mathcal{P}^{\log}(Q)} < \infty. \qquad (2.14)$$

for all $1 \leq r < \infty$ *as well as*

$$\mathbb{E}\|\mathbf{v}_0\|_{W^{1,2}(\mathbb{T}^n)}^2 < \infty, \quad \mathbb{E}\|\mathbf{f}\|_{L^2(0,T;W^{1,2}(\mathbb{T}^n))}^2 < \infty. \qquad (2.15)$$

Moreover, assume that we have $\mathbb{P}$*-a.s.*

$$p^- \leq p \leq p^+, \quad \|p\|_\infty + \|\nabla p\|_\infty \leq c_p,$$

where $c_p < \infty$ *and*

$$\frac{n+2}{2} \leq p^- \leq p^+ < np^- + 4. \qquad (2.16)$$

Finally, assume that Φ *satisfies* (2.4)*. Then there is a weak stochastically strong solution to* (1.5)–(1.6) *in the sense of Definition* 2.9*. We have the energy estimate*

$$\mathbb{E}\left[\sup_{t \in (0,T)} \int_{\mathbb{T}^n} |\mathbf{v}(t)|^2 \, dx + \int_Q |\boldsymbol{\varepsilon}(\mathbf{v})|^{p(\cdot)} \, dx \, dt\right]^r$$

$$\leq c\, \mathbb{E}\left[\int_{\mathbb{T}^n} |\mathbf{v}_0|^2 \, dx + \int_Q |\mathbf{f}|^2 \, dx \, dt\right]^r. \qquad (2.17)$$

Remark 2.11 As in the deterministic case (see [10,21]) the assumptions on p^- yielding uniqueness are rather restrictive. The same bounds are needed in Theorem 2.10 for the existence of stochastically strong solutions.

Having a look at Definitions 2.6 and 2.9 we can expect strong stochastically strong solutions if the assumptions of Theorems 2.7 and 2.10 are satisfied. These solutions are strong in both senses.

Definition 2.12 (*Strong stochastically strong solution*) Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration and let W be an $(\mathscr{F}_t)$-cylindrical Wiener process. Let $\mathbf{v}_0$ be an $L^2(\mathbb{T}^n)$-valued $\mathscr{F}_0$-measurable random variable. Let $\mathbf{f}$ and p be $(\mathscr{F}_t)$-progressively measurable processes such that $\mathbf{f} \in L^2(Q)$ and $p \in C^0([0, T] \times \mathbb{T}^n)$ with $p \geq 1$ $\mathbb{P}$-a.s. A function $\mathbf{v}$ is called a strong stochastically strong solution to (1.5)–(1.6) provided it is a weak stochastically strong solution in the sense of Definition 2.9 and the following holds.

(a) We have $\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v})) \in L^2(0, T; W^{1,2}(\mathbb{T}^n))$ $\mathbb{P}$-a.s.,

(b) there are π_{det} and Φ^π $(\mathscr{F}_t)$-progressively measurable such that
$\pi_{\text{det}} \in L^1(0, T; W^{1,1}(\mathbb{T}^n))$ and $\Phi^\pi \in L^2(0, T; L_2(\mathfrak{U}; L^2(\mathbb{T}^n)))$ $\mathbb{P}$-a.s. as well as

$$
\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \Big[\text{div} \Big(\mu (1 + |\boldsymbol{\varepsilon}(\mathbf{v})|)^{p(\cdot)-2} \boldsymbol{\varepsilon}(\mathbf{v}) \Big) - \text{div} \left(\mathbf{v} \otimes \mathbf{v} \right)
$$
$$
- \nabla \pi_{\text{det}} + \mathbf{f} \Big] d\sigma + \int_0^t \left[\Phi(\mathbf{v}) + \Phi^\pi \right] dW \qquad (2.18)
$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$.

By combining the ideas of the proofs of Theorems 2.7 and 2.10 we obtain the following corollary (see end of Sect. 5 for the proof).

Corollary 2.13 *Let the assumptions of Theorem 2.7 be satisfied. Suppose in addition that $p^- \geq \frac{n+2}{2}$. Then there is a strong stochastically strong solution to (1.5)–(1.6) in the sense of Definition 2.12.*

3 Galerkin approximation

Our approach is a stochastic variant of the usual Galerkin ansatz, thereby reducing the problem of interest to an stochastic ordinary differential equation. In this respect, we firstly record the following fundamental fact on eigenvector expansions for the Stokes operator, the proof of which can be found in the appendix of [21]:

Lemma 3.1 *There is a sequence $(\lambda_k) \subset \mathbb{R}$ and a sequence of functions $(\mathbf{w}_k) \subset W^{1,2}_{\text{div}}(\mathbb{T}^n)$ such that the following hold:*

(a) *For each $k \in \mathbb{N}$, $\mathbf{w}_k$ is an eigenvector to the eigenvalue λ_k of the Stokes-operator in the sense that*

$$
\langle \mathbf{w}_k, \boldsymbol{\varphi} \rangle_{W^{1,2}(\mathbb{T}^n)} = \lambda_k \int_{\mathbb{T}^n} \mathbf{w}_k \cdot \boldsymbol{\varphi} \, dx \quad \textit{for all } \boldsymbol{\varphi} \in W^{1,2}_{\text{div}}(\mathbb{T}^n),
$$

(b) $\int_{\mathbb{T}^n} \mathbf{w}_k \cdot \mathbf{w}_m \, dx = \delta_{km}$ *for all $k, m \in \mathbb{N}$,*
(c) $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ *and $\lambda_k \to \infty$,*
(d) $\langle \frac{\mathbf{w}_k}{\sqrt{\lambda_k}}, \frac{\mathbf{w}_m}{\sqrt{\lambda_m}} \rangle_{W^{1,2}(\mathbb{T}^n)} = \delta_{km}$ *for all $k, m \in \mathbb{N}$,*
(e) $(\lambda_k^{-1/2} \mathbf{w}_k)$ *is a Hilbert space basis of $W^{1,2}_{\text{div}}(\mathbb{T}^n)$.*

We consider the Skorokhod representation of the law $\Lambda \otimes \Gamma$, where Γ is the law of a cylindrical Wiener process on $\mathfrak{U}$. We obtain a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a random variable $(\mathbf{v}_0, \mathbf{f}, p)$ with law Λ, as well as a cylindrical Wiener process $W = \sum_k \beta_k e_k$. Finally, we set

$$\mathscr{F}_t := \sigma\Big(\sigma[\mathbf{v}_0] \cup \sigma[p|_{[0,t]}] \cup \sigma[\mathbf{f}|_{[0,t]}]$$

$$\cup \bigcup_{k=1}^{\infty} \sigma[W_k|_{[0,t]}] \cup \{N \in \mathscr{F} : \mathbb{P}(N) = 0\}\Big), \quad t \in [0, T].$$

Our objective for the rest of the section is to establish the existence of solutions $\mathbf{v}^N$ of the system (1.1) in the particular form

$$\mathbf{v}^N = \sum_{k=1}^{N} c_k^N \mathbf{w}_k = \mathbf{C}^N \cdot \boldsymbol{\omega}^N, \quad \boldsymbol{\omega}^N = (\mathbf{w}_1, \dots, \mathbf{w}_N), \tag{3.1}$$

where $\mathbf{C}^N = (c_i^N) : \Omega \times (0, T) \to \mathbb{R}^N$. Our aim is hereafter to solve ($k = 1, \dots, N$)

$$\int_{\mathbb{T}^n} d\mathbf{v}^N \cdot \mathbf{w}_k \, dx + \int_{\mathbb{T}^n} \mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)) : \boldsymbol{\varepsilon}(\mathbf{w}_k) \, dx \, dt$$

$$= \int_{\mathbb{T}^n} \mathbf{v}^N \otimes \mathbf{v}^N : \nabla \mathbf{w}_k \, dx \, dt \tag{3.2}$$

$$+ \int_{\mathbb{T}^n} \mathbf{f} \cdot \mathbf{w}_k \, dx \, dt + \int_{\mathbb{T}^n} \Phi(\mathbf{v}^N) \, dW^N \cdot \mathbf{w}_k \, dx,$$

$$\mathbf{v}^N(0) = \mathcal{P}^N \mathbf{v}_0.$$

with

$$\mathbf{S}_p(\omega, t, x, \boldsymbol{\eta}) = \mu(1 + |\boldsymbol{\eta}|)^{p(\omega,t,x)-2}\boldsymbol{\eta}.$$

Here $\mathcal{P}^N : L^2_{\text{div}}(\mathbb{T}^n) \to \mathcal{X}_N := \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_N\}$ is the orthogonal projection, i.e.

$$\mathcal{P}^N \mathbf{u} = \sum_{k=1}^{N} \langle \mathbf{u}, \mathbf{w}_k \rangle_{L^2} \mathbf{w}_k.$$

The equation above is to be understood $\mathbb{P}$ a.s. and for a.e. t and we set

$$W^N = \sum_{k=1}^{N} \mathbf{e}_k \beta_k = \mathbf{e}^N \cdot \boldsymbol{\beta}^N.$$

It is equivalent to solving

$$\begin{cases} d\mathbf{C}^N &= \big[\boldsymbol{\mu}(t, \mathbf{C}^N)\big] \, dt + \boldsymbol{\Sigma}(\mathbf{C}^N) \, d\boldsymbol{\beta}_t^N \\ \mathbf{C}^N(0) &= \mathbf{C}_0 \end{cases} \tag{3.3}$$

with the abbreviations

$$\boldsymbol{\mu}(\mathbf{C}^N) = \left(- \int_{\mathbb{T}^n} \mathbf{S}_p(\cdot, \mathbf{C}^N \cdot \boldsymbol{\varepsilon}(\mathbf{w}^N)) : \boldsymbol{\varepsilon}(\mathbf{w}_k) \, \mathrm{d}x \right.$$
$$+ \int_{\mathbb{T}^n} (\mathbf{C}^N \cdot \mathbf{w}^N) \otimes (\mathbf{C}^N \cdot \mathbf{w}^N) : \nabla \mathbf{w}_k \, \mathrm{d}x \Bigg)_{k=1}^N$$
$$+ \left(\int_{\mathbb{T}^n} \mathbf{f}(t) \cdot \mathbf{w}_k \, \mathrm{d}x \right)_{k=1}^N,$$
$$\boldsymbol{\Sigma}(\mathbf{C}^N) = \left(\int_{\mathbb{T}}^n \Phi(\mathbf{C}^N \cdot W^N)\mathbf{e}_l \cdot \mathbf{w}_k \, \mathrm{d}x \right)_{k,l=1}^N,$$
$$\mathbf{C}_0 = \left(\langle \mathbf{v}_0, \mathbf{w}_k \rangle_{\mathrm{L}^2(\mathbb{T}^n)} \right)_{k=1}^N.$$

We apply the results from [22, Theorem 3.1.1]. In the following we will check the assumptions. We have by the monotonicity of $\mathbf{S}_p$

$$\left(\boldsymbol{\mu}(t, \mathbf{C}^N) - \boldsymbol{\mu}(t, \tilde{\mathbf{C}}^N) \right) \cdot \left(\mathbf{C}^N - \tilde{\mathbf{C}}^N \right)$$
$$= - \int_{\mathbb{T}^n} \left(\mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)) - \mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\tilde{\mathbf{v}}^N)) \right) : \left(\boldsymbol{\varepsilon}(\mathbf{v}^N) - \boldsymbol{\varepsilon}(\tilde{\mathbf{v}}^N) \right) \mathrm{d}x$$
$$+ \int_{\mathbb{T}^n} \left(\mathbf{v}^N \otimes \mathbf{v}^N - \tilde{\mathbf{v}}^N \otimes \tilde{\mathbf{v}}^N \right) : \left(\boldsymbol{\varepsilon}(\mathbf{v}^N) - \boldsymbol{\varepsilon}(\tilde{\mathbf{v}}^N) \right) \mathrm{d}x$$
$$\leq \int_{\mathbb{T}^n} \left(\mathbf{v}^N \otimes \mathbf{v}^N - \tilde{\mathbf{v}}^N \otimes \tilde{\mathbf{v}}^N \right) : \left(\boldsymbol{\varepsilon}(\mathbf{v}^N) - \boldsymbol{\varepsilon}(\tilde{\mathbf{v}}^N) \right) \mathrm{d}x.$$

If $|\mathbf{C}^N| \leq R$ and $|\tilde{\mathbf{C}}^N| \leq R$ there holds

$$\left(\boldsymbol{\mu}(t, \mathbf{C}^N) - \boldsymbol{\mu}(t, \tilde{\mathbf{C}}^N) \right) \cdot \left(\mathbf{C}^N - \tilde{\mathbf{C}}^N \right) \leq c(R, N) |\mathbf{C}^N - \tilde{\mathbf{C}}^N|^2.$$

Here we took into account boundedness of $\mathbf{w}_k$ and $\nabla \mathbf{w}_k$. This implies weak monotonicity in the sense of [22, (3.1.3)] using Lipschitz continuity $\boldsymbol{\Sigma}$ in $\mathbf{C}^N$, cp. (2.4). On account of $\int_{\mathbb{T}^n} \mathbf{v}^N \otimes \mathbf{v}^N : \boldsymbol{\varepsilon}(\mathbf{v}^N) \, \mathrm{d}x = 0$ there holds further

$$\boldsymbol{\mu}(t, \mathbf{C}^N) \cdot \mathbf{C}^N = - \int_{\mathbb{T}^n} \mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)) : (\boldsymbol{\varepsilon}(\mathbf{v}^N) \, \mathrm{d}x$$
$$+ \int_{\mathbb{T}^n} \mathbf{f}(t) \cdot \mathbf{v}^N \, \mathrm{d}x \leq c \left(1 + \|\mathbf{f}(t)\|_2 \|\mathbf{v}^N\|_2 \right)$$
$$\leq (1 + \|\mathbf{f}(t)\|_2)(1 + \|\mathbf{v}^N\|^2) \leq c \left(1 + \|\mathbf{f}(t)\|_2 \right)(1 + |\mathbf{C}^N|^2).$$

So we have using the linear growth of $\boldsymbol{\Sigma}$ which follows from 2.4

$$\boldsymbol{\mu}(\mathbf{C}^N) \cdot \mathbf{C}^N + |\boldsymbol{\Sigma}(\mathbf{C}^N)|^2 \leq c(+\|\mathbf{v}^N\|_2^2)\left(1 + |\mathbf{C}^N|^2 \right).$$

As the integral $\int_0^T (1 + \|\mathbf{f}(t)\|_2) \, dt$ is finite $\mathbb{P}$-a.s. this yields weak coercivity in the sense of [22, (3.1.4)]. We obtain a unique strong solution $\mathbf{C}^N \in L^2(\Omega; C[0, T])$ to the SDE (3.3).

We obtain the following a priori estimate.

Theorem 3.2 *Assume* (1.6) *with* $p : \Omega \times Q \to (1, \infty)$, (2.4) *and for some* $r \geq 1$

$$\int_{L_{\mathrm{div}}^2(\mathbb{T}^n)} \|\mathbf{u}\|_{L^2(\mathbb{T}^n)}^{2r} \, d\Lambda_0(\mathbf{u}) < \infty, \quad \int_{L^2(Q)} \|\mathbf{g}\|_{L^2(Q)}^{2r} \, d\Lambda_{\mathbf{f}}(\mathbf{g}) < \infty. \tag{3.4}$$

Then there holds uniformly in N

$$\mathbb{E}\left[\sup_{t \in (0,T)} \int_{\mathbb{T}^n} |\mathbf{v}^N(t)|^2 \, dx + \int_Q |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)} \, dx \, dt \right]^r \leq C_r(\Lambda_0, \Lambda_{\mathbf{f}}),$$

$$C_r(\Lambda_0, \Lambda_{\mathbf{f}}) = c \left(1 + \int_{L_{\mathrm{div}}^2(\mathbb{T}^n)} \|\mathbf{u}\|_{L^2(\mathbb{T}^n)}^{2r} \, d\Lambda_0(\mathbf{u}) + \int_{L^2(Q)} \|\mathbf{g}\|_{L^2(Q)}^{2r} \, d\Lambda_{\mathbf{f}}(\mathbf{g}) \right),$$

$$\tag{3.5}$$

provided $C_r(\Lambda_0, \Lambda_{\mathbf{f}})$ *is finite.*

Proof We apply Itô's formula to the function $f(\mathbf{C}) = \frac{1}{2} |\mathbf{C}|^2$ which shows

$$\frac{1}{2} \|\mathbf{v}^N(t)\|_{L^2(\mathbb{T}^n)}^2 = \frac{1}{2} \|\mathbf{C}^N(0)\|_{L^2(\mathbb{T}^n)}^2 + \sum_{k=1}^N \int_0^t c_k^N \, d(c_k^N) + \frac{1}{2} \sum_{k=1}^N \int_0^t d\langle\langle c_k^N \rangle\rangle$$

$$= \frac{1}{2} \|\mathcal{P}^N \mathbf{v}_0\|_{L^2(\mathbb{T}^n)}^2 - \int_0^t \int_{\mathbb{T}^n} \mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)) : \boldsymbol{\varepsilon}(\mathbf{v}^N) \, dx \, d\sigma$$

$$+ \int_0^t \int_{\mathbb{T}^n} \mathbf{f} \cdot \mathbf{v}^N \, dx \, d\sigma + \int_{\mathbb{T}^n} \int_0^t \mathbf{v}^N \cdot \Phi(\mathbf{v}^N) \, d\,W^N \, dx$$

$$+ \frac{1}{2} \int_{\mathbb{T}^n} \int_0^t d\left\langle\!\!\left\langle \int_0^{\cdot} \Phi(\mathbf{v}^N) \, d\,W^N \right\rangle\!\!\right\rangle dx.$$

Here we used $d\mathbf{v}^N = \sum_{k=1}^N dc_k^N \mathbf{w}_k$, $\int_{\mathbb{T}^n} \mathbf{v}^N \otimes \mathbf{v}^N : \nabla \mathbf{v}^N \, dx = 0$ and property (ii) of the base $(\mathbf{w}_k)$. Now we can follow, taking the r-th power, and the supremum, building expectations and using (1.6) that

$$\mathbb{E}\left[\sup_{(0,T)} \int_{\mathbb{T}^n} |\mathbf{v}^N(t)|^2 \, dx + \int_0^T \int_{\mathbb{T}^n} |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)} \, dx \, d\sigma \right]^r$$

$$\leq c \, \mathbb{E}\left[1 + \|\mathbf{v}_0\|_{L^2(\mathbb{T}^n)}^2 + J_1(T) + \sup_{(0,T)} J_2(t) + J_3(T) \right]^r.$$

Here we abbreviated

$$J_1(t) = \int_0^t \int_{\mathbb{T}^n} |\mathbf{f}| |\mathbf{v}^N| \, dx \, d\sigma,$$

$$J_2(t) = \int_{\mathbb{T}^n} \int_0^t \mathbf{v}^N \cdot \Phi(\mathbf{v}^N) \, dW^N \, dx,$$

$$J_3(t) = \int_{\mathbb{T}^n} \int_0^t d\left\langle\!\!\left\langle \int_0^\cdot \Phi(\mathbf{v}^N) \, dW^N \right\rangle\!\!\right\rangle dx.$$

We obviously have

$$J_1 \le \int_0^t \int_{\mathbb{T}^n} |\mathbf{f}|^2 \, dx \, d\sigma + \int_0^t \int_{\mathbb{T}^n} |\mathbf{v}^N|^2 \, dx \, d\sigma.$$

Straightforward calculations show on account of (2.4)

$$\mathbb{E}[J_3]^r = \mathbb{E}\left[\sum_{k=1}^N \int_0^t \left(\int_{\mathbb{T}^n} \Phi(\mathbf{v}^N) \mathbf{e}_k \, dx \right)^2 d\sigma \right]^r$$

$$\le \mathbb{E}\left[\sum_{k=1}^\infty \int_0^t \int_{\mathbb{T}^n} |g_k(\mathbf{v}^N)|^2 \, dx \, d\sigma \right]^r$$

$$\le c\, \mathbb{E}\left[1 + \int_0^t \int_{\mathbb{T}^n} |\mathbf{v}^N|^2 \, dx \, d\sigma \right]^r.$$

On account of Burgholder–Davis–Gundi inequality, Young's inequality and (2.4) we gain

$$\mathbb{E}\left[\sup_{t \in (0,T)} |J_2(t)| \right]^r = \mathbb{E}\left[\sup_{t \in (0,T)} \left| \int_0^t \int_{\mathbb{T}^n} \mathbf{v}^N \cdot \Phi(\mathbf{v}^N) \, dx \, dW^N \right| \right]^r$$

$$= \mathbb{E}\left[\sup_{t \in (0,T)} \left| \int_0^t \sum_{k=1}^N \int_{\mathbb{T}^n} \mathbf{v}^N \cdot g_k(\mathbf{v}^N) \, dx \, d\beta_k \right| \right]^r$$

$$\le c\, \mathbb{E}\left[\int_0^T \sum_{k=1}^N \left(\int_{\mathbb{T}^n} \mathbf{v}^N \cdot g_k(\mathbf{v}^N) \, dx \right)^2 dt \right]^{\frac{1}{2}}$$

$$\le c\, \mathbb{E}\left[\left(\int_0^T \left(\sum_{k=1}^\infty \int_{\mathbb{T}^n} |\mathbf{v}^N|^2 \, dx \int_{\mathbb{T}^n} |g_k(\mathbf{v}^N)|^2 \, dx \right) dt \right)^{\frac{r}{2}} \right]$$

$$\le c\, \mathbb{E}\left[1 + \int_0^T \left(\int_G |\mathbf{v}^N|^2 \, dx \right)^2 dt \right]^{\frac{r}{2}}$$

$$\le \delta\, \mathbb{E}\left[\sup_{t \in (0,T)} \int_G |\mathbf{v}^N|^2 \, dx \right]^r + c(\delta)\, \mathbb{E}\left[1 + \int_0^T \int_G |\mathbf{v}^N|^2 \, dx \, dt \right]^r,$$

where $\delta > 0$ is arbitrary. This finally proves the claim by Gronwall's lemma for δ sufficiently small using $\Lambda_0 = \mathbb{P} \circ \mathbf{v}_0^{-1}$ and $\Lambda_\mathbf{f} = \mathbb{P} \circ \mathbf{f}^{-1}$. $\qquad \square$

4 Analytically weak solutions

This section is devoted to the proof of Theorems 2.2 and 2.10. In view of compactness, our main concern is the derivation of fractional estimates for $\nabla \mathbf{v}^N$. Based on this we are able to apply the stochastic compactness method employing Skorokhod's theorem to pass to the limit in the Galerkin approximation from the previous section.

4.1 Fractional differentiability

To set up fractional estimates in a convenient manner, we introduce the concave function for $\theta \geq 0$

$$g(\theta) = g_\lambda(\theta) := \begin{cases} \frac{1}{1-\lambda}(1+\theta)^{1-\lambda}, & \lambda \neq 1 \\ \ln(1+\theta), & \lambda = 1 \end{cases}$$

for

$$\lambda = \frac{2(\overline{q}-p^-)}{np^- - \overline{q}n + 4},$$

where $\overline{q} = \max\{3, p^+ + \varrho\}$ with $\varrho > 0$ arbitrarily small. The additional power ϱ arises from the elementary inequality

$$\ln(1+|\boldsymbol{\xi}|) \leq c_\varrho(1+|\boldsymbol{\xi}|^\varrho) \quad \boldsymbol{\xi} \in \mathbb{R}^{n \times n}. \tag{4.1}$$

Note that the denominator in the definition of λ is positive as long as

$$p^- > \frac{\overline{q}n - 4}{n}. \tag{4.2}$$

Similar to [30, Sect. 3] we have the following theorem.

Theorem 4.1 *Suppose that*

$$\int_{L^2_{\mathrm{div}}(\mathbb{T}^n)} \|\mathbf{u}\|^2_{W^{1,2}(\mathbb{T}^n)} \, d\Lambda_0(\mathbf{u}) < \infty,$$

$$\int_{L^2(Q)} \|\mathbf{g}\|^2_{L^2(0,T;W^{1,2}(\mathbb{T}^n))} \, d\Lambda_{\mathbf{f}}(\mathbf{g}) < \infty. \tag{4.3}$$

Moreover, assume that $\mathbb{P}$-a.s. $p \in C^0([0, T] \times \mathbb{T}^n)$ such that $\mathbb{P}$-a.s. we have

$$1 < p^- \leq p \leq p^+, \quad \|\nabla p\|_\infty \leq c_p, \tag{4.4}$$

where $c_p < \infty$ and that (4.2) holds. Finally, assume that Φ satisfies (2.4). Then we have

a) *If $p^- \geq 2$ then there holds uniformly in N:*

$$\mathbb{E}\left[\int_0^T \frac{\|\nabla^2 \mathbf{v}^N(t)\|_2^2}{(1 + \|\nabla \mathbf{v}^N(t)\|_2^2)^\lambda}\, dt\right] \leq C_1(\Lambda_0, \Lambda_\mathbf{f}).$$

a) *If $p^- < 2$ then there holds uniformly in N:*

$$\mathbb{E}\left[\int_0^T \frac{\|\nabla^2 \mathbf{v}^N(t)\|_{p^-}^2}{(1 + \|\nabla \mathbf{v}^N(t)\|_2^2)^\lambda (1 + \|\nabla \mathbf{v}^N(t)\|_{p^-})^{2-p^-}}\, dt\right] \leq C_1(\Lambda_0, \Lambda_\mathbf{f}).$$

Proof We start with the evolution of $\|\nabla \mathbf{v}^N(t)\|_{L^2(\mathbb{T}^n)}^2$. Applying Itô's formula to the mapping $\mathbf{C} \mapsto \|\nabla \mathbf{v}\|_2^2$, where $\mathbf{C} = (c^1, \ldots, c^N)$ and $\mathbf{v}$ are related through $\mathbf{v} = \sum_{k=1}^N c_k \mathbf{w}_k$. We obtain

$$\frac{1}{2}\|\nabla \mathbf{v}^N(t)\|_{L^2(\mathbb{T}^n)}^2 = \frac{1}{2}\|\nabla \mathcal{P}^N \mathbf{v}_0\|_{L^2(\mathbb{T}^n)}^2$$
$$- \int_0^t \int_{\mathbb{T}^n} D_\xi \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))(\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N), \partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N))\, dx\, d\sigma$$
$$- \int_0^t \int_{\mathbb{T}^n} D_x \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)) : \partial_\gamma \nabla \mathbf{v}^N\, dx\, d\sigma + \int_0^t \int_{\mathbb{T}^n} \operatorname{div}\left(\mathbf{v}^N \otimes \mathbf{v}^N\right) : \Delta \mathbf{v}^N\, dx\, d\sigma$$
$$+ \int_0^t \int_{\mathbb{T}^n} \partial_\gamma \mathbf{v}^N \cdot \partial_\gamma \left(\Phi(\mathbf{v}^N)\, dW\right) dx + \frac{1}{2}\int_{\mathbb{T}^n} \int_0^t d\left\langle\!\left\langle \int_0^{\cdot} \partial_\gamma\left(\Phi(\mathbf{v}^N)\, dW\right)\right\rangle\!\right\rangle dx,$$

where the sum is taken over all $\gamma \in \{1, \ldots, n\}$. Now we apply Itô's formula to the mapping $\mathbf{C}^N \mapsto g_\lambda(\|\nabla \mathbf{v}\|_2^2)$ and obtain

$$g_\lambda(\|\nabla \mathbf{v}^N(t)\|_2^2) = g_\lambda(\|\nabla \mathbf{v}^N(0)\|_2^2) + \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} d\|\nabla \mathbf{v}^N\|_2^2$$
$$- \frac{\lambda}{2}\int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^{\lambda+1}} d\langle\!\langle\|\nabla \mathbf{v}^N\|_2^2\rangle\!\rangle,$$

where we have

$$\int_0^t \frac{2}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} d\|\nabla \mathbf{v}^N\|_2^2$$
$$= -\int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} D_\xi \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))(\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N), \partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N))\, dx\, d\sigma$$
$$- \int_0^t \frac{2}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} D_x \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)) : \partial_\gamma \nabla \mathbf{v}^N\, dx\, d\sigma$$
$$+ \int_0^t \frac{2}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} \operatorname{div}\left(\mathbf{v}^N \otimes \mathbf{v}^N\right) : \Delta \mathbf{v}^N\, dx\, d\sigma$$

$$+ \int_0^t \frac{2}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} \partial_\gamma \mathbf{v}^N \cdot \partial_\gamma \left(\Phi(\mathbf{v}^N) \, \mathrm{d}W \right) \mathrm{d}x$$

$$+ \int_0^t \int_{\mathbb{T}^n} \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \mathrm{d}\left\langle\!\left\langle \int_0^{\cdot} \partial_\gamma (\Phi(\mathbf{v}^N) \, \mathrm{d}W) \right\rangle\!\right\rangle \mathrm{d}x$$

$$= -J_1 - J_2 + J_3 + J_4 + J_5.$$

Moreover, there holds

$$-\frac{\lambda}{2} \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^{\lambda+1}} \mathrm{d}\left\langle\!\left\langle \|\nabla \mathbf{v}^N\|_2^2 \right\rangle\!\right\rangle \le 0.$$

$\mathbb{P}$-a.s. such that this term can be neglected. We start with the lower estimate

$$J_1 \ge c \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p(\cdot)-2} |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, \mathrm{d}x \, \mathrm{d}\sigma$$

$$\ge c \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p^- - 2} |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, \mathrm{d}x \, \mathrm{d}\sigma.$$

All other terms will be estimate form above. By Young's inequality we obtain using (4.1)

$$J_2 \le c \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} \ln(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p(\cdot)-1} |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)| \, \mathrm{d}x \, \mathrm{d}\sigma$$

$$\le \kappa \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p(\cdot)-2} |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, \mathrm{d}x \, \mathrm{d}\sigma$$

$$+ c(\kappa) \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} \left(1 + |\nabla \mathbf{v}^N|^{\overline{q}}\right) \mathrm{d}x \, \mathrm{d}t,$$

where $\kappa > 0$ is arbitrary. For κ small enough we will be able to absorb the corresponding term in J_1. Moreover, we have

$$J_3 \le \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} |\nabla \mathbf{v}^N|^3 \, \mathrm{d}x \, \mathrm{d}\sigma$$

$$\le \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} \left(1 + |\nabla \mathbf{v}^N|^{\overline{q}}\right) \mathrm{d}x \, \mathrm{d}\sigma$$

using integration by parts. Finally, we obtain from (2.4)

$$J_5 = \sum_k \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \left(\int_{\mathbb{T}^n} \nabla g_k(\mathbf{v}^N) \, \mathrm{d}x \right)^2 \mathrm{d}t$$

$$\le \sum_k \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} |\nabla g_k(\mathbf{v}^N)|^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$\le c \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} |\nabla \mathbf{v}^N|^2 \, dx \, dt$$

$$\le c \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} \left(1 + |\nabla \mathbf{v}^N|^{\overline{q}}\right) dx \, dt.$$

Applying expectations (note that $\mathbb{E}[J_4] = 0$) and choosing κ small enough we end up with

$$\mathbb{E} g_\lambda (\|\nabla \mathbf{v}^N(t)\|_2^2) + \mathbb{E} \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p(\cdot)-2} |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, dx \, d\sigma$$

$$\le c \,\mathbb{E} \left[g_\lambda (\|\nabla \mathbf{v}^N(0)\|_2^2) + \int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} \left(1 + |\nabla \mathbf{v}^N|^{\overline{q}}\right) dx \, dt \right]. \quad (4.5)$$

The last term on the right-hand side cannot be controlled so far. In order to suitably bound $\|\nabla \mathbf{v}^N\|_{\overline{q}}^{\overline{q}}$, let $2 > q \ge n(\overline{q} - p^-)/\overline{q}$, existence of which follows from (4.2) and $\overline{q} > 2$, and put

$$\alpha := \frac{p^-(np^- + 2q - \overline{q}n)}{2(np^- + \overline{q}q - \overline{q}n)} \quad \text{so that} \quad 1 - \alpha = \frac{(\overline{q} - p^-)(np^- + 2q - 2n)}{2(np^- + \overline{q}q - \overline{q}n)} \quad (4.6)$$

so that, in particular, $np^-/(n-q) \ge \overline{q}$. By Lyapunov's interpolation inequality, we obtain

$$\|\nabla \mathbf{v}^N\|_{\overline{q}} \le \|\nabla \mathbf{v}^N\|_2^{\theta_1} \|\nabla \mathbf{v}^N\|_{np^-/(n-q)}^{\theta_2}$$
$$\|\nabla \mathbf{v}^N\|_{\overline{q}} \le \|\nabla \mathbf{v}^N\|_{p^-}^{\theta_3} \|\nabla \mathbf{v}^N\|_{np^-/(n-q)}^{\theta_4}, \quad (4.7)$$

where

$$\theta_1 := \frac{2(np^- + \overline{q}q - \overline{q}n)}{\overline{q}(np^- + 2q - 2n)}, \quad \theta_2 := \frac{(\overline{q} - 2)np^-}{\overline{q}(np^- + 2q - 2n)}, \quad \theta_3 := \frac{np^- + \overline{q}q - \overline{q}n}{\overline{q}q},$$
$$\theta_4 := \frac{n(\overline{q} - p^-)}{\overline{q}q}.$$

We then obtain

$$\|\nabla \mathbf{v}^N\|_{\overline{q}}^{\overline{q}} = \|\nabla \mathbf{v}^N\|_{\overline{q}}^{\overline{q}(1-\alpha)} \|\nabla \mathbf{v}^N\|_{\overline{q}}^{\overline{q}\alpha}$$
$$\le \|\nabla \mathbf{v}^N\|_2^{\overline{q}(1-\alpha)\theta_1} \|\nabla \mathbf{v}^N\|_{np^-/(n-q)}^{\overline{q}(1-\alpha)\theta_2 + \overline{q}\alpha\theta_4} (1 + \|\nabla \mathbf{v}^N\|_{p^-})^{\overline{q}\alpha\theta_3} \quad (4.8)$$
$$= \|\nabla \mathbf{v}^N\|_2^{2q_1} (1 + \|\nabla \mathbf{v}^N\|_{p^-})^{q_2} (\|\nabla \mathbf{v}^N\|_{np^-/(n-q)})^{q_3} = (*),$$

where q_1, q_2, q_3 are defined in the obvious manner. To estimate $(*)$, we note that for $\mathbb{P} \otimes \mathscr{L}^1$-a.e. $(\omega, t) \in \Omega \times [0, T]$ there holds by Korn's inequality.

$$\|\nabla \mathbf{u}(\omega, t, \cdot)\|_{\frac{np^-}{n-q}} \le c \|\boldsymbol{\varepsilon}(\mathbf{u}(\omega, t, \cdot))\|_{\frac{np^-}{n-q}}$$

Next we claim that there exists a constant $C > 0$ independent of $N \in \mathbb{N}$ such that

$$\|\nabla \mathbf{v}^N\|_{np^-/(n-q)} \leq C \left(\int_{\mathbb{T}^n} D_{\boldsymbol{\xi}} \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))(\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N), \partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)) \, \mathrm{d}x \right)^{\frac{q}{2p^-}} \times \tag{4.9}$$
$$\times \left(1 + \|\nabla \mathbf{v}^N\|_{p^-} \right)^{\frac{2-q}{2}}$$

holds $\mathbb{P}$—a.e. in Ω. The estimate (4.9) is a consequence of the interpolation of $L^{\frac{np^-}{n-q}}(\mathbb{T}^n)$ between $L^{p^-}(\mathbb{T}^n)$ and $L^{\frac{np^-}{n-2}}(\mathbb{T}^n)$, Sobolev's embedding $W^{1,2}(\mathbb{T}^n) \hookrightarrow L^{\frac{2n}{n-2}}(\mathbb{T}^n)$ (if $n = 2$ we have to replace $\frac{n}{n-2}$ by an arbitrary finite exponent) and the inequality

$$\left| \nabla (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{\frac{p^-}{2}} \right|^2 \leq c(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{\frac{p^- - 2}{2}} |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2$$
$$\leq c D_{\boldsymbol{\xi}} \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))(\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N), \partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)).$$

Using (4.9), we further estimate (4.8) by use of Young's inequality for any $r > 1$ and $\kappa > 0$

$$(*) \leq C \|\nabla \mathbf{v}^N\|_2^{\overline{q}(1-\alpha)\theta_1} \left(1 + \|\nabla \mathbf{v}^N\|_{p^-} \right)^{\frac{2-q}{2}(\overline{q}(1-\alpha)\theta_2 + \overline{q}\alpha\theta_4) + \overline{q}\alpha\theta_3}$$
$$\times \left(\int_{\mathbb{T}^n} D_{\boldsymbol{\xi}} \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))(\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N), \partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)) \, \mathrm{d}x \right)^{\frac{q}{2p^-}(\overline{q}(1-\alpha)\theta_2 + \overline{q}\alpha\theta_4)}$$
$$\leq C(\kappa, r) \left(\|\nabla \mathbf{v}^N\|_2^{\overline{q}(1-\alpha)\theta_1} \left(1 + \|\nabla \mathbf{v}^N\|_{p^-} \right)^{\frac{2-q}{2}(\overline{q}(1-\alpha)\theta_2 + \overline{q}\alpha\theta_4) + \overline{q}\alpha\theta_3} \right)^{\frac{r}{r-1}}$$
$$+ \kappa \left(\int_{\mathbb{T}^n} D_{\boldsymbol{\xi}} \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))(\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N), \partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)) \, \mathrm{d}x \right)^{\frac{q}{2p^-}(\overline{q}(1-\alpha)\theta_2 + \overline{q}\alpha\theta_4)r}.$$

To determine the relevant parameters, we shall now require

$$\frac{q}{2p^-}(\overline{q}(1-\alpha)\theta_2 + \overline{q}\alpha\theta_4)r = 1,$$
$$\left(\frac{2-q}{2}(\overline{q}(1-\alpha)\theta_2 + \overline{q}\alpha\theta_4) + \overline{q}\alpha\theta_3 \right) \frac{r}{r-1} = p^-. \tag{4.10}$$

Indeed (4.10) is satisfied indeed provided α is defined by (4.6) and we have

$$r = \frac{4}{\overline{q}n - np^-} \quad \text{und} \quad r' = \frac{4}{np^- - \overline{q}n + 4}. \tag{4.11}$$

On the other hand, this implies

$$\overline{q}(1-\alpha)\theta_1 r' = \frac{4(\overline{q} - p^-)}{np^- - \overline{q}n + 4}.$$

We obtain

$$(*) \leq C(\kappa, r)\|\nabla \mathbf{v}^N\|_2^{\overline{q}(1-\alpha)\theta_1 \frac{r}{r-1}}\left(1 + \|\nabla \mathbf{v}^N\|_{p^-}\right)^{p^-}$$
$$+ \kappa \int_{\mathbb{T}^n} D_{\boldsymbol{\xi}}\mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))(\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N), \partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)) \, \mathrm{d}x.$$

Inserting this into (4.5), choosing κ small enough can recalling the definition of λ yields by Korn's inequality

$$\mathbb{E}g_\lambda(\|\nabla \mathbf{v}^N(t)\|_2^2) + \mathbb{E}\int_0^t \frac{1}{(1 + \|\nabla \mathbf{v}^N\|_2^2)^\lambda} \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p(\cdot)-2}|\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, \mathrm{d}x \, \mathrm{d}\sigma$$
$$\leq c\,\mathbb{E}\left[g_\lambda(\|\nabla \mathbf{v}_0\|_2^2) + \int_0^t \int_{\mathbb{T}^n} \left(1 + |\nabla \mathbf{v}^N|^{p^-}\right) \mathrm{d}x \, \mathrm{d}t\right]$$
$$\leq c\,\mathbb{E}\left[g_\lambda(\|\nabla \mathbf{v}_0\|_2^2) + \int_0^t \int_{\mathbb{T}^n} \left(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p^-}\right) \mathrm{d}x \, \mathrm{d}t\right]$$
$$\leq c\,\mathbb{E}\left[g_\lambda(\|\nabla \mathbf{v}_0\|_2^2) + \int_0^t \int_{\mathbb{T}^n} \left(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)}\right) \mathrm{d}x \, \mathrm{d}t\right] \tag{4.12}$$

where the right-hand side is uniformly bounded by $C_1(\Lambda_0, \Lambda_{\mathbf{f}})$, cp. Theorem 3.2. If $p^- \geq 2$ the claim follows directly by Korn's inequality. If $p^- < 2$ we estimate using again Korn's inequality

$$\|\nabla^2 \mathbf{v}^N(t)\|_{p^-}^2 \leq c\left(\int_{\mathbb{T}^n} |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p^-} \, \mathrm{d}x\right)^{\frac{2}{p^-}}$$
$$= c\left(\int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p^- \frac{p^- - 2}{2}}|\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p^-}(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p^- \frac{2-p^-}{2}} \, \mathrm{d}x\right)^{\frac{2}{p^-}}$$
$$\leq c\int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p^- - 2}|\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, \mathrm{d}x\left(\int_{\mathbb{T}^n} (1 + |\nabla \mathbf{v}^N|)^{p^-} \, \mathrm{d}x\right)^{\frac{2-p^-}{p^-}}.$$

So, the claim follows again from (4.12) and $p^- \leq p$. $\qquad \square$

Corollary 4.2 *Let assumptions of Theorem 4.1 be satisfied. Assume in addition that $p^- > \frac{\overline{q}n}{n+2}$ if $p^- < 2$. Then for any $\overline{p} < \min\{p^-, \frac{2n}{n-2}\}$ there is $\beta > 0$ such that*

$$\mathbb{E}\left[\int_0^T \|\nabla \mathbf{v}^N\|_{\beta,\overline{p}}^{\overline{p}} \, \mathrm{d}t\right] \leq C_1(\lambda_0, \Lambda_{\mathbf{f}})$$

uniformly in N.

Proof If $p^- < 2$ we set (recall that $p^- > \frac{\overline{q}n}{n+2}$)

$$\beta = \frac{((n+2)p^- - \overline{q}n)p^-}{2((n+5)p^- - \overline{q}n - (p^-)^2)} \in \left(0, \frac{1}{2}\right)$$

and obtain

$$
\mathbb{E}\left[\int_0^T \|\nabla^2 \mathbf{v}^N(t)\|_{p^-}^{2\beta}\right] = \mathbb{E}\left[\int_0^T \left((1 + \|\nabla \mathbf{v}^N(t)\|_2^2)^\lambda (1 + \|\nabla \mathbf{v}^N(t)\|_{p^-})^{2-p^-}\right)^\beta \right.
$$
$$
\times \left(\frac{\|\nabla^2 \mathbf{v}^N(t)\|_{p^-}^2}{(1 + \|\nabla \mathbf{v}^N(t)\|_2^2)^\lambda (1 + \|\nabla \mathbf{v}^N(t)\|_{p^-})^{2-p^-}}\right)^\beta dt\right]
$$
$$
\leq \mathbb{E}\left[\int_0^T \left((1 + \|\nabla \mathbf{v}^N(t)\|_2^2)^{\frac{\lambda\beta}{1-\beta}} (1 + \|\nabla \mathbf{v}^N(t)\|_{p^-})^{(2-p^-)\frac{\beta}{1-\beta}} dt\right]^{1-\beta}
$$
$$
\times \mathbb{E}\left[\int_0^T \frac{\|\nabla^2 \mathbf{v}^N(t)\|_{p^-}^2}{(1 + \|\nabla \mathbf{v}^N(t)\|_2^2)^\lambda (1 + \|\nabla \mathbf{v}^N(t)\|_{p^-})^{2-p^-}} dt\right]^\beta
$$
$$
\leq C_1(\lambda_0, \Lambda_{\mathbf{f}})^\beta \mathbb{E}\left[I_1 + I_2\right]^{1-\beta} \tag{4.13}
$$

where

$$
I_1 = \int_0^T (1 + \|\nabla \mathbf{v}^N(t)\|_{p^-})^{(2-p^-)\frac{\beta}{1-\beta}} dt,
$$
$$
I_2 = \int_0^T \|\nabla \mathbf{v}^N(t)\|_2^{\frac{2\lambda\beta}{1-\beta}} (1 + \|\nabla \mathbf{v}^N(t)\|_{p^-})^{(2-p^-)\frac{\beta}{1-\beta}} dt.
$$

We can estimate I_1 by

$$
\mathbb{E}[I_1] \leq c\, \mathbb{E}\int_0^T \int_{\mathbb{T}^n} \left(1 + |\nabla \mathbf{v}^N(t)|^{p^-}\right) dx\, dt
$$
$$
\leq c\, \mathbb{E}\int_0^T \left(1 + |\nabla \mathbf{v}^N(t)|^{p(\cdot)}\right) dx\, dt \leq C_1(\lambda_0, \Lambda_{\mathbf{f}}) \tag{4.14}
$$

using $(2 - p^-)\frac{\beta}{1-\beta} \leq p^-$ and Theorem 3.2. For I_2 we use the interpolation interpolation inequality

$$
\|v\|_2 \leq \|v\|_{p^-}^{\frac{(n+2)p^- - 2n}{2p^-}} \|v\|_{\frac{np^-}{n-p^-}}^{\frac{n(2-p^-)}{2p^-}},
$$

which holds for $p^- \in (\frac{2n}{n+2}, 2)$, and the continuous embedding

$$
W^{2, p^-}(\mathbb{T}^n) \hookrightarrow W^{1, \frac{np^-}{n-p^-}}(\mathbb{T}^n).
$$

As a consequence of Theorem 3.2 (setting $\delta = \frac{2p^-}{n(2-p^-)}\frac{1-\beta}{\lambda}$) we can estimate I_2 by

$$
\mathbb{E}[I_2] \leq c\, \mathbb{E}\int_0^T \|\nabla^2 \mathbf{v}^N(t)\|_{p^-}^{\frac{n(2-p^-)}{p^-}\frac{\lambda\beta}{1-\beta}} (1 + \|\nabla \mathbf{v}^N(t)\|_{p^-})^{\left[(2-p^-) + \frac{(n+2)p^- - 2n}{p^-}\lambda\right]\frac{\beta}{1-\beta}} dt
$$

$$\leq c\,\mathbb{E}\left(\int_0^T \|\nabla^2\mathbf{v}^N(t)\|_{p^-}^{2\beta}\right)^{\frac{1}{\delta}}\left(\int_0^T \left(1+\|\nabla\mathbf{v}^N(t)\|_{p^-}\right)^{p^-}dt\right)^{\frac{1}{\delta'}}$$

$$\leq \kappa\,\mathbb{E}\int_0^T \|\nabla^2\mathbf{v}^N(t)\|_{p^-}^{2\beta}\,dt + c(\kappa)\,\mathbb{E}\int_0^T\int_{\mathbb{T}^n}\left(1+|\nabla\mathbf{v}^N|^{p^-}\right)dx\,dt$$

$$\leq \kappa\,\mathbb{E}\int_0^T \|\nabla^2\mathbf{v}^N(t)\|_{p^-}^{2\beta}\,dt + c(\kappa)\,\mathbb{E}\int_0^T\left(1+|\nabla\mathbf{v}^N|^{p(\cdot)}\right)dx\,dt$$

$$\leq \kappa\,\mathbb{E}\int_0^T \|\nabla^2\mathbf{v}^N(t)\|_{p^-}^{2\beta} + C_1(\Lambda_0,\Lambda_{\mathbf{f}}), \tag{4.15}$$

where $\kappa > 0$ is arbitrary. Combining (4.13)–(4.15) and choosing κ small enough we have shown

$$\mathbb{E}\left[\int_0^T \|\nabla^2\mathbf{v}^N\|_{p^-}^{2\beta}\,dt\right] \leq C_1(\Lambda_0,\Lambda_{\mathbf{f}}). \tag{4.16}$$

In order to proceed we use the interpolation inequality

$$\|v\|_{1+\sigma,p^-} \leq \|v\|_{1,p^-}^{1-\sigma}\|v\|_{2,p^-}^{\sigma}$$

for $\sigma = \frac{2\beta(p^--\overline{p})}{\overline{p}(p^--2\beta)}$. We obtain

$$\mathbb{E}\int_0^T \|\mathbf{v}^N\|_{1+\sigma,p^-}^{\overline{p}}\,dt \leq \mathbb{E}\int_0^T \|\mathbf{v}^N\|_{1,p^-}^{(1-\sigma)\overline{p}}\|\mathbf{v}^N\|_{2,p^-}^{\sigma\overline{p}}\,dt$$

$$\leq \left(\mathbb{E}\int_0^T \|\mathbf{v}^N\|_{1,p^-}^{p^-}\,dt\right)^{\frac{(1-\sigma)\overline{p}}{p^-}}\left(\mathbb{E}\int_0^T \|\mathbf{v}^N\|_{2,p^-}^{2\beta}\,dt\right)^{1-\frac{(1-\sigma)\overline{p}}{p^-}} \leq C_1(\Lambda_0,\Lambda_{\mathbf{f}})$$

as a consequence of Theorems 3.2 and (4.17).

If $p^- \geq 2$ estimate (4.17) can be shown much easier. Indeed, we have by Theorems 3.2 and 4.1

$$\mathbb{E}\left[\int_0^T \|\nabla^2\mathbf{v}^N\|_2^{2\beta}\,dt\right] = \mathbb{E}\left[\int_0^T (1+\|\nabla\mathbf{v}^N(t)\|_2^2)^{\lambda\beta}\frac{\|\nabla^2\mathbf{v}^N(t)\|_2^{2\beta}}{(1+\|\nabla\mathbf{v}^N(t)\|_2^2)^{\lambda\beta}}\,dt\right]$$

$$\leq \left[\mathbb{E}\int_0^T \frac{\|\nabla^2\mathbf{v}^N(t)\|_2^2}{(1+\|\nabla\mathbf{v}^N(t)\|_2^2)^{\lambda}}\,dt\right]^{\beta}\left[\mathbb{E}\int_0^T (1+\|\nabla\mathbf{v}^N(t)\|_2^2)^{\frac{p^-}{2}}\,dt\right]^{1-\beta}$$

$$\leq C_1(\Lambda_0,\Lambda_{\mathbf{f}}). \tag{4.17}$$

In order to proceed we use the interpolation inequality

$$\|v\|_{1+\sigma,\overline{p}} \leq \|v\|_{1,\overline{p}}^{1-\frac{\sigma}{s}}\|v\|_{1+s,\overline{p}}^{\frac{\sigma}{s}}$$

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which holds for any $0 < \sigma < s$. Combining this with the embedding (recall that $\overline{p} < \frac{2n}{n-2}$)

$$W^{2,2}(\mathbb{T}^n) \hookrightarrow W^{1+s,\overline{p}}(\mathbb{T}^n), \quad s = \frac{2n - (n-2)\overline{p}}{2\overline{p}},$$

we obtain for $\sigma = s\frac{2\beta(p^- - \overline{p})}{\overline{p}(p^- - 2\beta)}$

$$\mathbb{E}\int_0^T \|\mathbf{v}^N\|_{1+\sigma,\overline{p}}^{\overline{p}} \, dt \leq \mathbb{E}\int_0^T \|\mathbf{v}^N\|_{1,\overline{p}}^{\overline{p}(1-\frac{\sigma}{s})} \|\mathbf{v}^N\|_{2,2}^{\frac{\sigma\overline{p}}{s}} \, dt$$

$$\leq \left(\mathbb{E}\int_0^T \|\nabla\mathbf{v}^N(t)\|_{\overline{p}}^{p^-} \, dt\right)^{\frac{\overline{p}}{p^-}(1-\frac{\sigma}{s})}$$

$$\left(\mathbb{E}\int_0^T \|\nabla^2\mathbf{v}^N(t)\|_2^{2\beta} \, dt\right)^{1-\frac{\overline{p}}{p^-}(1-\frac{\sigma}{s})}.$$

The claim follows again from Theorem 3.2 combined with Korn's inequality (recall that $\overline{p} < p^-$) and (4.17). $\qquad\square$

4.2 Compactness

Before we can apply the stochastic compactness method we need to gain some information concerning the time regularity of $\mathbf{v}^N$. We go back to the system (3.2) and see that for any $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^n)^n$ there holds

$$\int_{\mathbb{T}^n} d\mathbf{v}^N \cdot \mathcal{P}_\ell^N \boldsymbol{\varphi} \, dx + \int_{\mathbb{T}^n} \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)) : \boldsymbol{\varepsilon}(\mathcal{P}_\ell^N \boldsymbol{\varphi}) \, dx \, dt$$

$$= \int_{\mathbb{T}^n} \mathbf{v}^N \otimes \mathbf{v}^N : \nabla\mathcal{P}_\ell^N \boldsymbol{\varphi} \, dx \, dt \qquad (4.18)$$

$$+ \int_{\mathbb{T}^n} \mathbf{f} \cdot \mathcal{P}_\ell^N \boldsymbol{\varphi} \, dx \, dt + \int_{\mathbb{T}^n} \Phi(\mathbf{v}^N) \, d W^N \cdot \mathcal{P}_\ell^N \boldsymbol{\varphi} \, dx.$$

Here $\mathcal{P}_\ell^N$ denotes the orthogonal projection on $\mathcal{X}_N$ with respect to the $W^{\ell,2}(\mathbb{T}^n)$ inner product, where ℓ is chosen such that $W^{\ell,2}_{\mathrm{div}}(\mathbb{T}^n) \hookrightarrow W^{1,\infty}_{\mathrm{div}}(\mathbb{T}^n)$. We now define for $t \in [0,T]$ the functionals $\mathcal{H}_N(t, \cdot)$ on $C^\infty_{\mathrm{div}}(\mathbb{T}^n)$ by

$$\mathcal{H}_N(t, \boldsymbol{\varphi}) := -\int_0^t \int_{\mathbb{T}^n} \mathbb{H}^N : \nabla\mathcal{P}^N \boldsymbol{\varphi} \, dx \, d\sigma, \quad \boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^n), \qquad (4.19)$$

where for $N \in \mathbb{N}$

$$\mathbb{H}^N := -\mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)) + \mathbf{v}^N \otimes \mathbf{v}^N - \nabla\Delta^{-1}\mathbf{f}, \qquad (4.20)$$

so that by Theorem 3.2 and the hypotheses collected in Definition 2.1, there holds

$$\mathbb{H}^N \in L^{p_0}(\Omega, \mathscr{F}, \mathbb{P}; L^{p_0}(0, T; L^{p_0}(\mathbb{T}^n))) \tag{4.21}$$

uniformly in $N \in \mathbb{N}$ for some $p_0 > 1$. Here, Δ^{-1} is the solution operator of the Poisson problem on the torus as has been recalled in Sect. 2.1. Now we claim that

$$\sup_{N \in \mathbb{N}} \mathbb{E}\left[\|\mathcal{H}^N\|_{W^{1,p_0}([0,T]; W_{\mathrm{div}}^{-\ell, p_0}(\mathbb{T}^n))} \right] < \infty. \tag{4.22}$$

Recall that $\ell \in \mathbb{N}$ is chosen so large such that $W_{\mathrm{div}}^{\ell,2}(\mathbb{T}^n)^n \hookrightarrow W_{\mathrm{div}}^{1,\infty}(\mathbb{T}^n)^n$. To see (4.22), note that

$$
\begin{aligned}
\left\| \frac{\mathrm{d}}{\mathrm{d} t} \mathcal{H}^N(t, \cdot) \right\|_{L^{p_0}(0, T; W_{\mathrm{div}}^{-\ell, p_0}(\mathbb{T}^n)))} &= \left\| \sup_{\|\boldsymbol{\varphi}\|_{\ell, p_0'} \leq 1} \frac{\mathrm{d}}{\mathrm{d} t} \mathcal{H}^N(t, \boldsymbol{\varphi}) \right\|_{L^{p_0}(0, T)} \\
&= \left\| \sup_{\|\boldsymbol{\varphi}\|_{\ell, p_0'} \leq 1} \int_{\mathbb{T}^n} \mathbb{H}^N : \nabla \mathcal{P}_\ell^N \boldsymbol{\varphi} \, \mathrm{d} x \right\|_{L^{p_0}(0, T)} \\
&\leq \left\| \sup_{\|\boldsymbol{\varphi}\|_{\ell, p_0'} \leq 1} \|\mathbb{H}^N(t, \cdot)\|_{L^{p_0}} \|\nabla \mathcal{P}_\ell^N \boldsymbol{\varphi}\|_{L^{p_0'}} \right\|_{L^{p_0}(0, T)} \\
&\leq C \left(\int_0^T \|\mathbb{H}^N(t, \cdot)\|_{L^{p_0}}^{p_0} \, \mathrm{d} \sigma \right)^{\frac{1}{p_0}}.
\end{aligned}
$$

In consequence, raising the previous inequality to the p_0-th power and taking expectations in conjunction with (4.21) gives (4.22). On the other hand, we have for all $0 \leq s < t \leq T$

$$
\begin{aligned}
\mathbb{E}\left[\left\| \int_0^t \Phi(\mathbf{v}^N) \, \mathrm{d} W_\sigma^N - \int_0^s \Phi(\mathbf{v}^N) \, \mathrm{d} W^N \right\|_{L^2(\mathbb{T}^n)}^\theta \right] &= \mathbb{E}\left[\left\| \int_s^t \Phi(\mathbf{v}^N) \, \mathrm{d} W^N \right\|_{L^2(\mathbb{T}^n)}^\theta \right] \\
&= \mathbb{E}\left[\left\| \int_s^t \sum_{k=1}^\infty \Phi(\mathbf{v}^N) e_k \, \mathrm{d} \beta_k^N \right\|_{L^2(\mathbb{T}^n)}^\theta \right] \\
&\leq \mathbb{E}\left[\left\| \int_s^t \sum_{k=1}^\infty g_k(\mathbf{v}^N) \, \mathrm{d} \beta_k^N \right\|_{L^2(\mathbb{T}^n)}^{2 \cdot \frac{\theta}{2}} \right] \\
&= \mathbb{E}\left[\left(\int_s^t \sum_{k=1}^\infty \|g_k(\mathbf{v}^N)\|_{L^2(\mathbb{T}^n)}^2 \, \mathrm{d} \sigma \right)^{\frac{\theta}{2}} \right] \\
&\stackrel{(2.4)}{\leq} C\mathbb{E}\left[\left(\int_s^t (1 + \|\mathbf{v}^N\|_{L^2(\mathbb{T}^n)}^2) \, \mathrm{d} \sigma \right)^{\frac{\theta}{2}} \right]
\end{aligned}
$$

$$= C|t - s|^{\frac{\theta}{2}} \left(\mathbb{E} \left[\sup_{t \in (0,T)} (1 + \|\mathbf{v}^N\|_{\mathrm{L}^2(\mathbb{T}^n)}^2) \right] \right)^{\frac{\theta}{2}}$$

$$\overset{(3.5)}{\leq} C|t - s|^{\frac{\theta}{2}}.$$

At this point we are in position to apply the Kolmogorov continuity criterion to conclude that there exists $0 < \kappa < 1$ such that

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| \int_0^\cdot \Phi(\mathbf{v}^N) \, d\, W^N \right\|_{C^\kappa([0,T];\mathrm{L}^2(\mathbb{T}^n))} \right] < \infty. \tag{4.23}$$

Let us note that since $W_{\mathrm{div}}^{\ell,p_0'}(\mathbb{T}^n) \hookrightarrow W_{\mathrm{div}}^{1,2}(\mathbb{T}^n) \hookrightarrow \mathrm{L}^2(\mathbb{T}^n)$ and $1 < p_0 < \infty$ we have $\mathrm{L}^2(\mathbb{T}^n) \hookrightarrow W^{-\ell,p_0}(\mathbb{T}^n)$. Hence (4.23) implies that $\mathbb{E}[\|\int_0^\cdot \Phi(\mathbf{v}^N) \, d\, W^N\|_{C^\kappa([0,T];W^{-\ell,p_0}(\mathbb{T}^n))}]$ is uniformly bounded in N. Combining this with (4.22), a straightforward interpolation argument yields some $0 < \mu < 1$ such that

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[\|\mathbf{v}^N\|_{C^\mu([0,T];W_{\mathrm{div}}^{-\ell,p_0}(\mathbb{T}^n)^n)} \right] < \infty. \tag{4.24}$$

In view of compactness, let us now define the path space

$$\mathcal{X} := \mathcal{X}_\mathbf{v} \otimes \mathcal{X}_p \otimes \mathcal{X}_\mathbf{f} \otimes \mathcal{X}_W, \tag{4.25}$$

where

$$\mathcal{X}_\mathbf{v} := C([0, T]; W_{\mathrm{div}}^{-\ell,p_0}(\mathbb{T}^n)^n) \cap L^{\overline{p}}(0, T; W_{\mathrm{div}}^{1,\overline{p}}(\mathbb{T}^n)),$$
$$\mathcal{X}_p := C^0([0, T] \times \mathbb{T}^n),$$
$$\mathcal{X}_\mathbf{f} := L^2(0, T; W^{1,2}(\mathbb{T}^n)),$$
$$\mathcal{X}_W := C([0, T]; \mathfrak{U}_0).$$

Here $\overline{p}$ is some fixed but arbitrary number in $\left(1, \min\{p^-, \frac{2n}{n-2}\}\right)$. We obtain the following.

Proposition 4.3 *The set* $\{\mathcal{L}[\mathbf{v}^N, p, \mathbf{f}, W]; \ N \in \mathbb{N}\}$ *is tight on* $\mathcal{X}$.

Proof By a fractional version of Aubin–Lions theorem (see [13, Theorem 5.1.22]) we have compactness of the embedding

$$C^\mu([0, T]; W_{\mathrm{div}}^{-\ell,p_0}(\mathbb{T}^n)) \cap L^{\overline{p}}(0, T; W_{\mathrm{div}}^{1+\beta,\overline{p}}(\mathbb{T}^n)^n)$$
$$\hookrightarrow\hookrightarrow L^{\overline{p}}(0, T; W_{\mathrm{div}}^{1,\overline{p}}(\mathbb{T}^n)^n). \tag{4.26}$$

On the other Arcelà-Ascoli's theorem yields

$$C^\mu([0, T]; W_{\mathrm{div}}^{-\ell,p_0}(\mathbb{T}^n)) \hookrightarrow\hookrightarrow C([0, T]; W_{\mathrm{div}}^{-\ell,p_0}(\mathbb{T}^n)).$$

So, we obtain tightness of $\mathcal{L}[\mathbf{v}^N]$ on $\mathcal{X}_{\mathbf{v}}$ from (4.24), Corollary 4.2 and Tschebyscheff's inequality. Tightness of the law of p on $\mathcal{X}_p$ follows by using (2.8) and the compact embedding

$$\mathcal{P}^{\log}(Q) \hookrightarrow\hookrightarrow C^0([0, T] \times \mathbb{T}^n).$$

The latter one is a simple consequence of Arcela–Ascoli's theorem. Finally the laws of $\mathbf{f}$ and W of on their corresponding path spaces are tight as being Radon measures on Polish spaces. □

Prokhorov's Theorem (see [17, Theorem 2.6]) implies that $\{\mathcal{L}[\mathbf{v}^N, p, \mathbf{f}, W]; N \in \mathbb{N}\}$ is also relatively weakly compact. This means we have a weakly convergent subsequence. Now we use Skorohod's representation theorem [17, Theorem 2.7] to infer the following result.

Proposition 4.4 *There exists a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\mathcal{X}$-valued Borel measurable random variables $(\tilde{\mathbf{v}}^N, \tilde{p}^N, \tilde{\mathbf{f}}^N, \tilde{W}^N)$, $N \in \mathbb{N}$, and $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{f}}, \tilde{W})$ such that (up to a subsequence)*

(a) *the law of $(\tilde{\mathbf{v}}^N, \tilde{p}^N, \tilde{\mathbf{f}}^N, \tilde{W}^N)$ on $\mathcal{X}$ is given by $\mathcal{L}[\mathbf{v}^N, p, \mathbf{f}, W]$, $N \in \mathbb{N}$,*
(b) *the law of $(\tilde{\mathbf{v}}, \tilde{p}, \mathbf{f}, \tilde{W})$ on $\mathcal{X}$ is a Radon measure,*
(c) *$(\tilde{\mathbf{v}}^N, \tilde{p}^N, \tilde{\mathbf{f}}^N, \tilde{W}^N)$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{\mathbf{f}}, \tilde{W})$ in the topology of $\mathcal{X}$, i.e.*

$$
\begin{aligned}
\tilde{\mathbf{v}}^N &\to \tilde{\mathbf{v}} \text{ in } C([0, T]; W_{\text{div}}^{-\ell, p_0}(\mathbb{T}^n)) \ \tilde{\mathbb{P}}\text{-a.s.,} \\
\tilde{\mathbf{v}}^N &\to \tilde{\mathbf{v}} \text{ in } L^{\overline{p}}(0, T; W_{\text{div}}^{1, \overline{p}}(\mathbb{T}^n)) \ \tilde{\mathbb{P}}\text{-a.s.,} \\
\tilde{p}^N &\to \tilde{p} \text{ in } C^0([0, T] \times \mathbb{T}^n) \ \tilde{\mathbb{P}}\text{-a.s.,} \quad (4.27) \\
\tilde{\mathbf{f}}^N &\to \tilde{\mathbf{f}} \text{ in } L^2(0, T; W^{1,2}(\mathbb{T}^n)) \ \tilde{\mathbb{P}}\text{-a.s.,} \\
\tilde{W}^N &\to \tilde{W} \text{ in } C([0, T]; \mathfrak{U}_0) \ \tilde{\mathbb{P}}\text{-a.s.}
\end{aligned}
$$

4.3 Conclusion

The variables $\tilde{\mathbf{v}}$, $\tilde{p}$, $\tilde{\mathbf{f}}$, $\tilde{W}$ are progressively measurable with respect to their canonical filtration, namely,

$$
\begin{aligned}
\tilde{\mathcal{F}}_t := \sigma\Bigg(&\sigma[\tilde{\mathbf{v}}|_{[0,t]}] \cup \sigma[\tilde{p}|_{[0,t]}] \cup \sigma[\tilde{\mathbf{f}}|_{[0,t]}] \\
&\cup \bigcup_{k=1}^{\infty} \sigma[\tilde{W}_k|_{[0,t]}] \cup \{N \in \tilde{\mathcal{F}} : \tilde{\mathbb{P}}(N) = 0\} \Bigg), \quad t \in [0, T].
\end{aligned}
$$

In view of Lemma [5, Chap. 2, Lemma 2.1.35], the process $\tilde{W}$ is a cylindrical Wiener processes with respect to its canonical filtration. It follows from Corollary [5, Chap. 2, Corollary 2.1.36] that $\tilde{W}$ is a cylindrical Wiener process with respect to $(\tilde{\mathcal{F}}_t)_{t \geq 0}$.

Modifying slightly the proof, the result of [5, Chap. 2, Theorem 2.9.1] remains valid in the current setting. Hence, as a consequence of the equality of laws from Proposition 4.4, the approximate Eq. (3.2) is satisfied on the new probability space, i.e. we have

$$\int_{\mathbb{T}^n} \tilde{\mathbf{v}}^N \cdot \mathbf{w}_k \, dx + \int_0^t \int_{\mathbb{T}^n} \mu(1 + |\boldsymbol{\varepsilon}(\tilde{\mathbf{v}}^N)|)^{\tilde{p}^N(\cdot)-2} \boldsymbol{\varepsilon}(\tilde{\mathbf{v}}^N) : \boldsymbol{\varepsilon}(\mathbf{w}_k) \, dx \, dt$$

$$= \int_{\mathbb{T}^n} \tilde{\mathbf{v}}^N(0) \cdot \mathbf{w}_k \, dx + \int_0^t \int_{\mathbb{T}^n} \tilde{\mathbf{v}}^N \otimes \tilde{\mathbf{v}}^N : \nabla \mathbf{w}_k \, dx \, dt$$

$$+ \int_0^t \int_{\mathbb{T}^n} \tilde{\mathbf{f}}^N \cdot \mathbf{w}_k \, dx \, dt + \int_0^t \int_{\mathbb{T}^n} \Phi(\tilde{\mathbf{v}}^N) \, d\tilde{W}^N \cdot \mathbf{w}_k \, dx$$

$\tilde{\mathbb{P}}$-a.s. for all $t \in [0, T]$. Using the convergence from (4.27) it is easy to pass to the limit and we obtain

$$\int_{\mathbb{T}^n} \tilde{\mathbf{v}}(t) \cdot \boldsymbol{\varphi} \, dx + \int_0^t \int_{\mathbb{T}^n} \mu(1 + |\boldsymbol{\varepsilon}(\tilde{\mathbf{v}})|)^{\tilde{p}(\cdot)-2} \boldsymbol{\varepsilon}(\tilde{\mathbf{v}}) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \, dx \, d\sigma$$

$$= \int_{\mathbb{T}^n} \tilde{\mathbf{v}}(0) \cdot \boldsymbol{\varphi} \, dx + \int_0^t \int_{\mathbb{T}^n} \tilde{\mathbf{v}} \otimes \tilde{\mathbf{v}} : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \, dx \, d\sigma \qquad (4.28)$$

$$+ \int_{\mathbb{T}^n} \int_0^t \tilde{\mathbf{f}} \cdot \boldsymbol{\varphi} \, dx \, d\sigma + \int_{\mathbb{T}^n} \int_0^t \Phi(\tilde{\mathbf{v}}) \, d\tilde{W} \cdot \boldsymbol{\varphi} \, dx$$

for all $\boldsymbol{\varphi} \in C_{\text{div}}^\infty(\mathbb{T}^n)$ and all $t \in [0, T]$ $\tilde{\mathbb{P}}$-a.s. where, for the limit passage in the stochastic integral, we use [9, Lemma 2.1].

4.4 Stochastically strong solutions

Let us start by showing pathwise uniqueness.

Proposition 4.5 (Pathwise uniqueness) *Let the assumptions of Theorem 2.10 be valid. In particular, we suppose $p^- \geq \frac{n+2}{2}$. Let $\mathbf{v}^1, \mathbf{v}^2$ be two weak stochastically strong solutions to (1.5)–(1.6) in the sense of Definition 2.9 defined on the same stochastic basis with the same Wiener process W, the same forcing $\mathbf{f}$ and the same exponent p. If*

$$\mathbb{P}\left[\mathbf{v}^1(0) = \mathbf{v}^2(0)\right] = 1,$$

then

$$\mathbb{P}\left[\mathbf{v}^1(t) = \mathbf{v}^2(t), \text{ for all } t \in [0, T]\right] = 1.$$

Proof We set $\mathbf{w} = \mathbf{v}^1 - \mathbf{v}^2$ and apply Itô's formula to $\mathbf{w} \mapsto \frac{1}{2} \int_{\mathbb{T}^n} |\mathbf{w}|^2 \, dx$ (recall that by our assumptions on p^- the term $\int_{\mathbb{T}^n} \mathbf{v} \otimes \mathbf{v} : \nabla \mathbf{v} \, dx$ is well-defined). This procedure can be made rigorous by applying a regularization to the equation for $\mathbf{w}$. Eventually,

the standard one-dimensional Itô formula can be applied to $|\mathbf{w}_\varrho|^2$ pointwise in x, where ϱ is the regularization parameter. Smooth approximations converge in $L^p(Q)$ and $L^{p'}(Q)$ as we have $p \in \mathcal{P}^{\log}(Q)$ $\mathbb{P}$-a.s. by assumption, cf. [11, Theorem 9.1.7]. We obtain using $\mathbf{w}(0) = 0$

$$
\begin{aligned}
\frac{1}{2}\|\mathbf{w}(t)\|^2_{L^2(\mathbb{T}^n)} = &-\int_0^t \int_{\mathbb{T}^n} \left(\mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^1)) - \mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^2))\right) : \boldsymbol{\varepsilon}(\mathbf{v}^1 - \mathbf{v}^2) \, \mathrm{d}x \, \mathrm{d}\sigma \\
&+ \int_0^t \int \left((\nabla \mathbf{v}^1)\mathbf{v}^1 - (\nabla \mathbf{v}^2)\mathbf{v}^2\right) \cdot \mathbf{w} \, \mathrm{d}x \\
&+ \frac{1}{2} \int_{\mathbb{T}^n} \int_0^t \mathrm{d}\left\langle\!\!\left\langle \int_0^\cdot \left(\Phi(\mathbf{v}^1) - \Phi(\mathbf{v}^2)\right) \mathrm{d}W \right\rangle\!\!\right\rangle \mathrm{d}x \\
&+ \int_{\mathbb{T}^n} \int_0^t \mathbf{w} \cdot \left(\Phi(\mathbf{v}^1) - \Phi(\mathbf{v}^2)\right) \mathrm{d}W \, \mathrm{d}x.
\end{aligned}
$$

By monotonicity of $\mathbf{S}_p$ the first term on the right-hand side is non-negative and we have by Korn's inequality

$$
\int_{\mathbb{T}^n} \left(\mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^1)) - \mathbf{S}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^2))\right) : \boldsymbol{\varepsilon}(\mathbf{v}^1 - \mathbf{v}^2) \, \mathrm{d}x \geq \mu \|\boldsymbol{\varepsilon}(\mathbf{w})\|_2^2 \geq \frac{\mu}{c} \|\nabla \mathbf{w}\|_2^2
$$

as $p^- \geq 2$. The critical part is the term arising from the convective term. Here, we follow ideas of [21, Theorem 4.29] and write

$$
\begin{aligned}
\int_{\mathbb{T}^n} &\left((\nabla \mathbf{v}^1)\mathbf{v}^1 - (\nabla \mathbf{v}^2)\mathbf{v}^2\right) \cdot (\mathbf{v}^1 - \mathbf{v}^2) \, \mathrm{d}x \\
&= \int_{\mathbb{T}^n} (\nabla \mathbf{v}^1)\mathbf{w} \cdot \mathbf{w} \, \mathrm{d}x \leq \|\nabla \mathbf{v}^1\|_{p^-} \|\mathbf{w}\|_{\frac{2p^-}{p^- - 1}}^2.
\end{aligned}
$$

Now, we use the interpolation

$$
\|v\|_q \leq \|v\|_2^\alpha \|\nabla v\|_2^{1-\alpha}, \quad \alpha = \frac{2n - q(n-2)}{2q},
$$

valid for all $q \in [2, \frac{2n}{n-2}]$ if $n \geq 3$ and $q \in [2, \infty)$ if $n = 2$, cp. [21, Lemma 4.35]. Choosing $q = \frac{2p^-}{p^- - 1}$ we obtain

$$
\int_{\mathbb{T}^n} \left((\nabla \mathbf{v}^1)\mathbf{v}^1 - (\nabla \mathbf{v}^2)\mathbf{v}^2\right) \cdot (\mathbf{v}^1 - \mathbf{v}^2) \, \mathrm{d}x \leq \|\nabla \mathbf{v}^1\|_{p^-} \|\mathbf{w}\|_2^{\frac{2p^- - n}{p^-}} \|\nabla \mathbf{w}\|_2^{\frac{n}{p^-}}
$$

$$
\leq \mu \|\nabla \mathbf{w}\|_2^2 + c(\mu) \|\nabla \mathbf{v}^1\|_{p^-}^{\frac{2p^-}{2p^- - n}} \|\mathbf{w}\|_2^2
$$

using also Young's inequality. Finally, we estimate the correction term by

$$
\int_{\mathbb{T}^n} \int_0^t d\left\langle\!\left\langle \int_0^{\cdot} \left(\Phi(\mathbf{v}^1) - \Phi(\mathbf{v}^2)\right) dW \right\rangle\!\right\rangle dx = \sum_{k=1}^{\infty} \int_0^t \left(\int_{\mathbb{T}^n} \left(g_k(\mathbf{v}^1) - g_k(\mathbf{v}^2)\right) dx \right)^2 d\sigma
$$

$$
\leq \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}^n} \left| g_k(\mathbf{v}^1) - g_k(\mathbf{v}^2) \right|^2 dx \, d\sigma
$$

$$
\leq \int_0^t \int_{\mathbb{T}^n} |\mathbf{v}^1 - \mathbf{v}^2|^2 \, dx \, d\sigma
$$

using (2.4). Summarising, we obtain

$$
d \, \|\mathbf{w}\|_{L^2}^2 \leq c\left(\|\nabla \mathbf{v}^1\|_{p^-}^{\frac{2p^-}{2p^- - n}} + 1 \right) \|\mathbf{w}\|_2^2 \, dt
$$
$$
+ \int_{\mathbb{T}^n} \mathbf{w} \cdot \left(\Phi(\mathbf{v}^1) - \Phi(\mathbf{v}^2)\right) d \, W \, dx \tag{4.29}
$$

for some finite constant $c > 0$. We now define $G: \Omega \times [0, T] \to \mathbb{R}$ by

$$
G(\omega, t) := c\left(\|\nabla \mathbf{v}^1(\omega, t)\|_{p^-}^{\frac{2p^-}{2p^- - n}} + 1 \right)
$$

so that in particular $G \in L^1(0, T)$ for $\mathbb{P}$—a.e. $\omega \in \Omega$. This is a consequence of $\frac{2p^-}{2p^- - n} \leq p^-$ (which follows from the assumption $p^- \geq \frac{n+2}{n}$) and $\nabla \mathbf{v}^1 \in L^{p^-}(Q)$ $\mathbb{P}$-a.s. (which follows from $\boldsymbol{\varepsilon}(\mathbf{v}^1) \in L^{p(\cdot)}(Q)$ $\mathbb{P}$-a.s. and Korn's inequality). We then obtain by use of Itô's formula (similar to [27])

$$
d\left(e^{-\int_0^t G \, ds} \|\mathbf{w}\|_{L^2}^2 \right) = -G \, e^{-\int_0^t G \, ds} \|\mathbf{w}\|_{L^2}^2 \, d \, t + e^{-\int_0^t G \, ds} \, d \, \|\mathbf{w}\|_{L^2}^2
$$
$$
\overset{(4.29)}{\leq} e^{-\int_0^t G \, ds} \int_{\mathbb{T}^n} \mathbf{w} \cdot \left(\Phi(\mathbf{v}^1) - \Phi(\mathbf{v}^2)\right) d \, W \, dx \tag{4.30}
$$

by definition of G. Now we apply the expectation to both sides of the inequality and consequently obtain

$$
\mathbb{E}\left[e^{-\int_0^t G \, ds} \|\mathbf{w}\|_{L^2}^2 \right] = 0.
$$

Consequently we obtain $\mathbf{v}^1 = \mathbf{v}^2$ $\mathbb{P}$-a.s. and the proof of Proposition 4.5 is complete.
□

Based on the pathwise uniqueness we will employ the Gyöngy–Krylov characterization of convergence in probability introduced in [15]. It applies to situations when pathwise uniqueness and existence of a martingale solution are valid and allows to establish existence of a stochastically strong solution. We consider two sequences

(N_n), $(N_m) \subset \mathbb{N}$ diverging to infinity. Let $\mathbf{v}^n := \mathbf{v}^{N_n}$ and $\mathbf{v}^m := \mathbf{v}^{N_m}$. We consider the collection of joint laws of $(\mathbf{v}^n, \mathbf{v}^m, p, \mathbf{f}, W)$ on the extended path space

$$\mathcal{X}^J = \mathcal{X}_{\mathbf{v}}^2 \otimes \mathcal{X}_p \otimes \mathcal{X}_{\mathbf{f}} \otimes \mathcal{X}_W,$$

Similarly to Proposition 4.3 we obtain the following result.

Proposition 4.6 *The set*

$$\{\mathcal{L}[\mathbf{v}^n, \mathbf{v}^m, p, \mathbf{f}, W]; \ n, m \in \mathbb{N}\}$$

is tight on $\mathcal{X}^J$.

Let us take any subsequence $(\mathbf{v}^{n_k}, \mathbf{v}^{m_k}, p, \mathbf{f}, W)$. By the Skorokhod representation theorem we infer (for a further subsequence but without loss of generality we keep the same notation) the existence of a probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ with a sequence of random variables $(\hat{\mathbf{v}}^{n_k}, \check{\mathbf{v}}^{m_k}, \bar{p}_k, \bar{\mathbf{f}}_k, \bar{W}_k)$ converging almost surely in $\mathcal{X}^J$ to a random variable $(\hat{\mathbf{v}}, \check{\mathbf{v}}, \bar{p}, \bar{\mathbf{f}}, \bar{W})$. Moreover,

$$\mathcal{L}[\hat{\mathbf{v}}^{n_k}, \check{\mathbf{v}}^{m_k}, \bar{p}^k, \bar{\mathbf{f}}^k, \bar{W}^k] = \mathcal{L}[\mathbf{v}_{n_k}, \mathbf{v}_{m_k}, p, \mathbf{f}, W]$$

on $\mathcal{X}^J$ for all $k \in \mathbb{N}$. Observe that in particular, $\mathcal{L}[\mathbf{v}_{n_k}, \mathbf{v}_{m_k}, \bar{p}^k, \bar{\mathbf{f}}^k, \bar{W}^k]$ converges weakly to the measure $\mathcal{L}[\hat{\mathbf{v}}, \check{\mathbf{v}}, \bar{p}, \bar{\mathbf{f}}, \bar{W}]$. As in (4.28) we can show that $(\hat{\mathbf{v}}, \bar{p}, \bar{\mathbf{f}}, \bar{W})$ and $(\check{\mathbf{v}}, \bar{p}, \bar{\mathbf{f}}, \bar{W})$ are weak martingale solutions to (1.5)–(1.6) defined on the same stochastic basis $(\bar{\Omega}, \bar{\mathscr{F}}, (\bar{\mathscr{F}}_t), \bar{\mathbb{P}})$, where $(\bar{\mathscr{F}}_t)_{t \geq 0}$ is the $\bar{\mathbb{P}}$-augmented canonical filtration of $(\hat{\mathbf{v}}, \check{\mathbf{v}}, \bar{p}, \bar{\mathbf{f}}, \bar{W})$. We employ the pathwise uniqueness result from Proposition 4.5. Indeed, it follows from our assumptions on the approximate initial laws Λ_0 that $\hat{\mathbf{v}}(0) = \check{\mathbf{v}}(0) = 1$ $\bar{\mathbb{P}}$-a.s. Therefore, the solutions $\hat{\mathbf{v}}$ and $\check{\mathbf{v}}$ coincide $\bar{\mathbb{P}}$-a.s. and we have

$$\mathcal{L}[\hat{\mathbf{v}}, \check{\mathbf{v}}, \bar{W}]\Big((\mathbf{v}_1, \mathbf{v}_2, p, \mathbf{f}, W) \in \mathcal{X}^J : \ \mathbf{v}_1 = \mathbf{v}_2\Big) = \bar{\mathbb{P}}(\hat{\mathbf{v}} = \check{\mathbf{v}}) = 1.$$

Now, we have all in hand to apply the Gyöngy–Krylov theorem. It implies that the original sequence $\mathbf{v}^N$ defined on the initial probability space $(\Omega, \mathscr{F}, \mathbb{P})$ converges in probability in the topology of $\mathcal{X}_{\mathbf{v}}$ to the random variable $\mathbf{v}$. Therefore, we finally deduce that $\mathbf{v}$ is a weak stochastically strong solution to (1.5)–(1.6). □

5 Analytically strong solutions

5.1 A-priori bounds

In this section we establish the existence result, Theorem 2.2. We begin with a strengthening of the a-priori estimate given by Theorem 3.2. Note that we work under the additional assumption that either we have

(i) $n = 2$ and $1 < p^- \leq p^+ < 4$ or;

(ii) $n = 3$ and $\frac{11}{5} < p^- \leq p^+ \leq p^- + \frac{4}{5}$.

Theorem 5.1 *Let the assumptions of Theorem 2.7 be satisfied. Let $\mathbf{v}^N$ be the Galerkin approximation constructed in Sect. 3. Then there exists a constant $c > 0$ such that*

$$
\mathbb{E}\left[\sup_{t \in (0,T)} \int_{\mathbb{T}^n} |\nabla \mathbf{v}^N(t)|^2 \, dx + \int_Q |\nabla_{\boldsymbol{\xi}} \mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))|^2 \, dx \, dt \right]
$$
$$
\leq c \left(\int_{L^2_{\mathrm{div}}(\mathbb{T}^n)} \|\mathbf{u}\|^2_{L^2(\mathbb{T}^n)} \, d\Lambda_0(\mathbf{u}), \int_{L^2(Q)} \|\mathbf{g}\|^2_{L^2(0,T;W^{1,2}(\mathbb{T}^n))} \, d\Lambda_{\mathbf{f}}(\mathbf{g}) \right) \tag{5.1}
$$

uniformly in $N \in \mathbb{N}$.

Corollary 5.2 *Under the assumptions of Theorem 5.1 we have*

$$
\mathbb{E}\left[\int_Q |\nabla^2 \mathbf{v}^N|^{\min(p^-,2)} \, dx \, dt \right]
$$
$$
\leq c \left(\int_{L^2_{\mathrm{div}}(\mathbb{T}^n)} \|\mathbf{u}\|^{2r}_{L^2(\mathbb{T}^n)} \, d\Lambda_0(\mathbf{u}), \int_{L^2(Q)} \|\mathbf{g}\|^{2r}_{L^2(0,T;W^{1,2}(\mathbb{T}^n))} \, d\Lambda_{\mathbf{f}}(\mathbf{g}) \right) \tag{5.2}
$$

uniformly in $N \in \mathbb{N}$.

Proof of Corollary 5.2 If $p^- \geq 2$ the claim follows immediately from Theorem 5.1, the definition of $\mathbf{F}_p$ and Korn's inequality. So, let us assume that $p^- < 2$. By Korn's and Young's inequality we obtain

$$
\mathbb{E}\left[\int_Q |\nabla^2 \mathbf{v}^N|^{p^-} \, dx \, dt \right] \leq c\, \mathbb{E}\left[\int_Q |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p^-} \, dx \, dt \right]
$$
$$
\leq c\, \mathbb{E}\left[1 + \int_Q |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^p \, dx \, dt \right]
$$
$$
= c\, \mathbb{E}\left[1 + \int_Q (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p\frac{2-p}{2}} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p\frac{p-2}{2}} |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^p \, dx \, dt \right]
$$
$$
\leq c\, \mathbb{E}\left[\int_Q (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^p \, dx \, dt + \int_Q (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p-2} |\nabla \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, dx \, dt \right].
$$

Now, the first term is bounded by Theorem 3.2 and the second one by Theorem 5.1. Clearly, $c > 0$ does not depend on N, and hence the statement of the corollary follows. $\square$

Proof of Theorem 5.1 In a similar vein as for Theorem 3.2, the core of the proof consists in a suitable Itô-expansion. We hereafter apply Itô's formula to the function $f_\gamma(\mathbf{u}) := \frac{1}{2}\|\partial_\gamma \mathbf{u}\|^2_{L^2(\mathbb{T}^n)^n}$ (with $\gamma \in \{1, 2\}$ for $n = 2$ and $\gamma \in \{1, 2, 3\}$ for $n = 3$) and obtain

$$
\frac{1}{2}\|\partial_\gamma \mathbf{v}^N(t)\|^2_{L^2(\mathbb{T}^n)} = \frac{1}{2}\|\partial_\gamma \mathcal{P}^N \mathbf{v}_0\|^2_{L^2(\mathbb{T}^n)} + \int_0^t f'(\mathbf{v}^N) \, d\mathbf{v}^N_\sigma + \frac{1}{2}\int_0^t f''(\mathbf{v}^N) \, d\langle \mathbf{v}^N \rangle_\sigma
$$

$$= \frac{1}{2} \|\partial_\gamma \mathcal{P}^N \mathbf{v}_0\|^2_{L^2(\mathbb{T}^n)} + \int_{\mathbb{T}^n} \int_0^t \partial_\gamma \mathbf{v}^N \cdot \mathrm{d}\partial_\gamma \mathbf{v}^N_\sigma \, \mathrm{d}x$$

$$+ \frac{1}{2} \int_{\mathbb{T}^n} \int_0^t \mathrm{d} \left\langle\!\!\left\langle \int_0^\cdot \partial_\gamma \left(\Phi(\mathbf{v}^N)\, \mathrm{d}\mathbf{W}\right)\right\rangle\!\!\right\rangle_\sigma \mathrm{d}x$$

$$=: (I) + (II) + (III). \tag{5.3}$$

We consider the three integrals separately.

1. We begin with (I). By continuity of the projection, we record the estimate

$$\|\partial_\gamma \mathcal{P}^N \mathbf{v}_0\|^2_{L^2(\mathbb{T}^n)} \leq \|\mathcal{P}^N \mathbf{v}_0\|^2_{W^{1,2}(\mathbb{T}^n)} \leq C \|\mathbf{v}_0\|^2_{W^{1,2}(\mathbb{T}^n)}.$$

2. Deferring the estimation of (III) to the end of the proof, we turn to (II). Summing over γ, we find

$$(II) = -(II)_1 - (II)_2 + (II)_3 + (II)_4 + (II)_5,$$

$$(II)_1 := \int_0^t \int_{\mathbb{T}^n} D_\xi \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))(\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N), \partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)) \, \mathrm{d}x \, \mathrm{d}\sigma,$$

$$(II)_2 := \int_0^t \int_{\mathbb{T}^n} \partial_\gamma \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)) : \partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N) \, \mathrm{d}x \, \mathrm{d}\sigma,$$

$$(II)_3 := \int_0^t \int_{\mathbb{T}^n} \partial_\gamma \mathbf{v}^N \cdot \partial_\gamma \left(\Phi(\mathbf{v}^N)\, \mathrm{d}W_\sigma\right) \mathrm{d}x,$$

$$(II)_4 := \int_0^t \int_{\mathbb{T}^n} \partial_\gamma \mathbf{v}^N \cdot \partial_\gamma \mathbf{f} \, \mathrm{d}x \, \mathrm{d}\sigma,$$

$$(II)_5 := \int_{\mathbb{T}^n} \mathrm{div}(\mathbf{v}^N \otimes \mathbf{v}^N) \cdot \partial_\gamma^2 \mathbf{v}^N \, \mathrm{d}x.$$

Ad $(II)_1$. Using the assumptions for $\mathbf{S}$ in (1.6) we obtain

$$(II)_1 \geq \tilde{c} \int_0^t \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{\frac{p(\cdot)-2}{2}} |\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, \mathrm{d}x \, \mathrm{d}\sigma. \tag{5.4}$$

Ad $(II)_2$. We now turn to the second term $(II)_2$. By uniform Lipschitz continuity of $p(\omega, \cdot)$ we obtain

$$|\partial_\gamma \mathbf{S}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))| \leq c \ln(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{p(\cdot)-2} |\boldsymbol{\varepsilon}(\mathbf{v}^N)| \tag{5.5}$$

with an absolute constant $c > 0$ for all $N \in \mathbb{N}$. We find by virtue of Young's Inequality for arbitrary $\delta > 0$

$$(II)_2 \leq c \left(1 + \int_0^t \int_{\mathbb{T}^n} \ln(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)-1})|\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)| \, \mathrm{d}x \, \mathrm{d}\sigma\right)$$

$$\leq c(\delta) \left(1 + \int_0^t \int_{\mathbb{T}^n} \ln^2(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|)(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)}) \, \mathrm{d}x \, \mathrm{d}\sigma\right)$$

$$+ \delta \left(\int_0^t \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)-2}) |\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, \mathrm{d}x \, \mathrm{d}\sigma \right) = c(\delta)\mathbf{I}' + \delta \mathbf{II}'.$$

Choosing $\delta > 0$ sufficiently small, $\delta \mathbf{II}'$ may be absorbed into the left side of the overall inequality by the coercive estimation of $(II)_1$ (cp. (5.4)), and therefore it remains to give a suitable upper bound for $c(\delta)\mathbf{I}'$. It is easy to see that for every $2 < \mu < 3$ there exists a constant $C = C(\mu) > 0$ such that for all $t > 0$ there holds $t^2 \log^2(1 + t) \leq C(1 + t^\mu)$. Using the Gagliardo–Nirenberg interpolation inequality on the torus [14, Theorem 7.28], we obtain for $1 \leq q, r \leq \infty, 0 \leq \alpha \leq 1$ the implication

$$\frac{1}{p} = \left(\frac{1}{r} - \frac{1}{n} \right)\alpha + \frac{1-\alpha}{q} \implies \|u\|_{\mathrm{L}^p}$$
$$\leq C \|u\|_{\mathrm{W}^{1,r}}^\alpha \|u\|_{\mathrm{L}^q}^{1-\alpha} \quad \text{for } u \in (\mathrm{W}^{1,r} \cap \mathrm{L}^q)(\mathbb{T}^n), \tag{5.6}$$

where $C > 0$ only depends on q, r and n. Now set $p = \mu, q = 2$ and $r = 2$, so that the condition in (5.6) is satisfied with $\alpha = \frac{\mu-2}{\mu}$. Then we have $1 - \alpha = \frac{2}{\mu}$ and so by Young's inequality with $\delta > 0$ to be fixed later

$$\|v\|_{\mathrm{L}^\mu(\mathbb{T}^n)}^\mu \leq C \|v\|_{\mathrm{W}^{1,2}}^{\mu-2} \|v\|_{\mathrm{L}^2}^2$$
$$\leq C \left(\delta \|v\|_{\mathrm{W}^{1,2}}^2 + C_\delta \|v\|_{\mathrm{L}^2}^{\frac{4}{4-\mu}} \right)$$
$$= C \left(\delta \left(\|v\|_{\mathrm{L}^2}^2 + \|\nabla v\|_{\mathrm{L}^2}^2 \right) + C_\delta \|v\|_{\mathrm{L}^2}^{\frac{4}{4-\mu}} \right) \tag{5.7}$$

for every $v \in \mathrm{W}^{1,2}(\mathbb{T}^n)$.
This estimation is implicit in [10, Chap. Eq. (4.62)]. We apply the previous estimate to $v := (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{p(\cdot)/4}$ to find

$$\mathbf{I}' \leq C \int_0^t \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{\mu p(\cdot)/4} \, \mathrm{d}x \, \mathrm{d}\sigma$$
$$\leq C + C\delta \int_0^t \left(\|(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{p(\cdot)/4}\|_{\mathrm{L}^2}^2 + \|\nabla(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{p(\cdot)/4}\|_{\mathrm{L}^2}^2 \right) \mathrm{d}\sigma$$
$$+ C_\delta \int_0^t \|(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{p(\cdot)/4}\|_{\mathrm{L}^2}^{\frac{4}{4-\mu}} \mathrm{d}\sigma$$
$$\leq C + C\delta \int_{Q_T} |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)} \, \mathrm{d}x \, \mathrm{d}\sigma + C\delta \int_0^t \|\nabla(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{p(\cdot)/4}\|_{\mathrm{L}^2}^2 \, \mathrm{d}\sigma$$
$$+ C_\delta \int_0^t \|(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{p(\cdot)/4}\|_{\mathrm{L}^2}^{\frac{4}{4-\mu}} \, \mathrm{d}\sigma. \tag{5.8}$$

By (5.5), we obtain

$$\int_0^t \|\nabla(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{p(\cdot)/4}\|_{L^2}^2 \, d\sigma$$

$$\leq C \int_0^t \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{\frac{p(\cdot)}{2}} \ln^2 \left(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2\right) dx \, d\sigma$$

$$+ C \int_0^t \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{\frac{p(\cdot)-2}{2}} |\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, dx \, d\sigma.$$

So that, choosing $\delta > 0$ small enough and absorbing the first term of the right side of the previous inequality into $\mathbf{I}'$, we end up with (recall $2 < \mu < 3$ so that $4/(4 - \mu) \leq 4$),

$$\mathbf{I}' \leq C + C\delta \int_{Q_T} |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)} \, dx \, d\sigma$$

$$+ C\delta \int_0^t \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{\frac{p(\cdot)-2}{2}} |\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, dx \, d\sigma$$

$$+ C\delta \int_0^t \|(1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{p(\cdot)/4}\|_{L^2}^4 \, d\sigma$$

$$\leq C + C \int_{Q_T} |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)} \, dx \, d\sigma$$

$$+ C\delta \int_0^t \int_{\mathbb{T}^n} (1 + |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2)^{\frac{p(\cdot)-2}{2}} |\partial_\gamma \boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, dx \, d\sigma$$

$$+ C\delta \int_0^t \int_{\mathbb{T}^n} |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)\frac{2}{4-\mu}} \, dx \, d\sigma = \mathbf{I}_1' + \cdots + \mathbf{I}_4'.$$

The terms $\mathbf{I}_1'$ and $\mathbf{I}_2'$ are already in a convenient form. For δ small enough consequently may absorb $\mathbf{I}_3'$ into the right side of (5.4).

Ad (*II*)$_3$. We decompose

$$(II)_3 = \int_{\mathbb{T}^n} \int_0^t \partial_\gamma \mathbf{v}^N \cdot \partial_\gamma \left(\Phi(\mathbf{v}^N) e_k \, d\beta_k\right) dx$$

$$= \sum_k \int_{\mathbb{T}^n} \int_0^t \partial_\gamma \mathbf{v}^N \cdot \partial_\gamma \left(g_k(\mathbf{v}^N) \, d\beta_k\right) dx$$

$$= \sum_k \int_{\mathbb{T}^n} \int_0^t \nabla_\xi g_k(\mathbf{v}^N)(\partial_\gamma \mathbf{v}^N, \partial_\gamma \mathbf{v}^N) \, d\beta_k \, dx$$

$$+ \sum_k \int_{\mathbb{T}^n} \int_0^t \partial_\gamma \mathbf{v}^N \cdot \partial_\gamma g_k(\mathbf{v}^N) \, d\beta_k \, dx$$

$$= \int_{\mathbb{T}^n} \int_0^t \mathcal{G}^\xi(\partial_\gamma \mathbf{v}^N, \partial_\gamma \mathbf{v}^N) \, d\beta_k \, dx.$$

On account of assumption (2.4), Burkholder–Davis–Gundy inequality and Young's inequality we obtain for arbitrary $\delta > 0$

$$\mathbb{E}\left[\sup_{0<t<T} |(II)^1_3|\right] \leq \mathbb{E}\left[\sup_{0<t<T} \left| \int_0^t \sum_k \int_{\mathbb{T}^n} \nabla g_k(\mathbf{v}_N)(\partial_\gamma \mathbf{v}^N, \partial_\gamma \mathbf{v}^N) \, dx \, d\beta_k \right|\right]$$

$$\leq c \, \mathbb{E}\left[\int_0^T \left(\int_{\mathbb{T}^n} \nabla g_k(\mathbf{v}^N)(\partial_\gamma \mathbf{v}^N, \partial_\gamma \mathbf{v}^N) \, dx \right)^2 dt\right]^{\frac{1}{2}}$$

$$\leq c \, \mathbb{E}\left[\left(\int_0^T \left(\int_{\mathbb{T}^n} |\partial_\gamma \mathbf{v}^N|^2 \, dx \right)^2 dt \right)\right]^{\frac{1}{2}}$$

$$\leq \delta \, \mathbb{E}\left[\sup_{0<t<T} \int_{\mathbb{T}^n} |\partial_\gamma \mathbf{v}^N|^2 \, dx\right] + c(\delta) \, \mathbb{E}\left[\int_Q |\partial_\gamma \mathbf{v}^N|^2 \, dx \, dt\right].$$

Ad $(II)_4$. After we shall have passed to the supremum in the overall inequality, by Young's inequality we obtain for a finite constant $C_\delta > 0$

$$(II)_4 \leq C\delta \sup_{0<t<T} \int_{\mathbb{T}^n} |\partial_\gamma \mathbf{v}^N|^2 \, d\,x + C_\delta \int_0^t \int_{\mathbb{T}^n} |\nabla \mathbf{f}|^2 \, dx \, d\sigma.$$

We then may choose $\delta > 0$ so small such that $\delta \|\partial_\gamma \mathbf{v}^N\|^2_{L^2(Q)}$ can be absorbed into (5.3).

Ad (III). We have by (2.4)

$$(III) = \frac{1}{2} \int_{\mathbb{T}^n} \int_0^t d\left\langle\!\!\left\langle \int_0^\cdot \partial_\gamma \left(\Phi(\mathbf{v}^N) \, dW^N\right)\right\rangle\!\!\right\rangle_\sigma dx$$

$$\leq \frac{1}{2} \sum_k \int_{\mathbb{T}^n} \int_0^t d\left\langle\!\!\left\langle \int_0^\cdot \partial_\gamma \left(\Phi(\mathbf{v}^N)\mathbf{e}_k\right) d\beta_k\right\rangle\!\!\right\rangle_\sigma dx$$

$$\leq \frac{1}{2} \sum_k \int_0^t \int_{\mathbb{T}^n} \left|\nabla_\xi g_k(\cdot, \mathbf{v}^N) \cdot \partial_\gamma \mathbf{v}^N\right|^2 dx \, d\sigma$$

$$\leq c \int_0^t \int_{\mathbb{T}^n} |\partial_\gamma \mathbf{v}^N|^2 \, dx \, d\sigma + c \int_0^t \int_{\mathbb{T}^n} |\mathbf{v}^N|^2 \, dx \, d\sigma.$$

$\square$

5.2 The case $n = 2$

Ad $(II)_5$. The crucial impact of our assumption $n = 2$ is that $(II)_5 = 0$ which can be established by elementary calculations. Gathering estimates, we have shown

$$\mathbb{E}\left[\sup_{0<t<T} \int_{\mathbb{T}^n} |\nabla \mathbf{v}^N(t)|^2 \, dx + \int_Q |\nabla_\xi \mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))|^2 \, dx \, dt\right]$$

$$\leq c \, \mathbb{E}\left(1 + \int_{\mathbb{T}^n} \left(|\mathbf{v}_0|^2 + |\nabla \mathbf{v}_0|^2\right) dx + \mathbb{E}\int_Q \left(|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2\right) dx \, dt\right)$$

$$+ c\,\mathbb{E}\left(\int_Q |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)}\,\mathrm{d}x\,\mathrm{d}t + \int_Q |\mathbf{v}^N|^2\,\mathrm{d}x\,\mathrm{d}t + \int_Q |\nabla\mathbf{v}^N|^2\,\mathrm{d}x\,\mathrm{d}t\right)$$

$$+ c\,\mathbb{E}\int_Q \left(|\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{p(\cdot)}\right)^{\frac{q}{2}}\,\mathrm{d}x\,\mathrm{d}t,$$

where $q := \frac{4}{4-\mu}$.

The terms in the first line of the right hand side are bounded by assumption. The terms in the second line are bounded by the a priori estimates from Theorem 3.2 except of the last one. It can, however, be handled by Gronwall's lemma leading to

$$\mathbb{E}\left[\sup_{0<t<T}\int_{\mathbb{T}^n} |\nabla\mathbf{v}^N|^2\,\mathrm{d}x + \left[\int_Q |\nabla_{\boldsymbol{\xi}}\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))|^2\,\mathrm{d}x\,\mathrm{d}t\right]\right]$$

$$\leq c\,\mathbb{E}\left(1 + \int_Q |\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))|^q\,\mathrm{d}x\,\mathrm{d}t\right).$$

By Lipschitz continuity of p we obtain

$$|\nabla\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))| \leq |\nabla_{\boldsymbol{\xi}}\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))| + |\partial_\gamma\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))|$$

$$\leq |\nabla_{\boldsymbol{\xi}}\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))| + c\,\ln(1+|\boldsymbol{\varepsilon}(\mathbf{v}^N)|)(1+|\boldsymbol{\varepsilon}(\mathbf{v}^N)|)^{\frac{p(\cdot)}{2}}$$

$$\leq |\nabla_{\boldsymbol{\xi}}\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))| + c\left(|\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))|^{\frac{q}{2}} + 1\right)$$

such that

$$\mathbb{E}\left[\sup_{0<t<T}\int_{\mathbb{T}^n} |\nabla\mathbf{v}^N|^2\,\mathrm{d}x + \left[\int_Q |\nabla\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))|^2\,\mathrm{d}x\,\mathrm{d}t\right]\right]$$
$$\leq c\,\mathbb{E}\left(1 + \int_Q |\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))|^q\,\mathrm{d}x\,\mathrm{d}t\right). \tag{5.9}$$

Note that q can be chosen arbitrarily close to 2. The objective of the following is to find a suitable bound for the remaining integral on the right hand side.

By Korn's inequality, $\int_{\mathbb{T}^n} |\nabla\mathbf{v}^N|^2$ and $\int_{\mathbb{T}^n} |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2$ are equivalent. Using the elementary inequality $|\mathbf{F}_p(\cdot,\boldsymbol{\xi})|^\tau \leq c(|\boldsymbol{\xi}|^2+1)$ for $\tau = 4/p^+$ and Sobolev's embedding $W^{1,2}(\mathbb{T}^n) \hookrightarrow L^{2\sigma}(\mathbb{T}^n))$ (with $\sigma = \frac{n}{n-2}$ if $n \geq 3$ and σ arbitrary for $n = 2$) we deduce from (5.9) that

$$\mathbb{E}\left[\sup_{0<t<T}\int_{\mathbb{T}^n} |\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))|^\tau\,\mathrm{d}x + \left[\int_0^T \left(\int_{\mathbb{T}^n} |\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))|^{2\sigma}\,\mathrm{d}x\right)^{\frac{1}{\sigma}}\,\mathrm{d}t\right]\right]$$
$$\leq c\,\mathbb{E}\left(1 + \int_Q |\mathbf{F}_p(\cdot,\boldsymbol{\varepsilon}(\mathbf{v}^N))|^q\,\mathrm{d}x\,\mathrm{d}t\right). \tag{5.10}$$

In order to proceed, we use the interpolation (recall that $\tau > 1$ as $p^+ < 4$)

$$\left(L^\infty(0,T;L^\tau(\mathbb{T}^n)); L^2(0,T;L^{2\sigma}(\mathbb{T}^n))\right)_\Theta = L^r(0,T;L^r(\mathbb{T}^n)),$$

$$r = 2 + \tau - \frac{\tau}{\sigma}, \quad \Theta = 1 - \frac{2}{r},$$

and obtain for $\chi = \frac{2\tau}{2\Theta + \tau(1-\Theta)}$

$$\|v\|_r^\chi \leq \|v\|_{L_t^\infty L_x^\tau}^{\chi\Theta} \|v\|_{L_t^2 L_x^{2\sigma}}^{\chi(1-\Theta)} \leq \|v\|_{L_t^\infty L_x^\tau}^\tau + \|v\|_{L_t^2 L_x^{2\sigma}}^2. \tag{5.11}$$

Combining (5.10) and (5.11) yields

$$\mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^r}^\chi \leq c\left(1 + \mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^q}^q\right).$$

We continue with the interpolation

$$\left(L^r(Q); L^2(Q)\right)_\beta = L^q(Q), \quad \beta = \frac{r}{q}\frac{q-2}{r-2},$$

and obtain

$$\mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^q}^q \leq \mathbb{E}\left(\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^r}^{\beta q}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^2}^{(1-\beta)q}\right)$$

$$\leq \left(\mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^r}^{\beta q\gamma}\right)^{\frac{1}{\gamma}}\left(\mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^2}^{(1-\beta)q\gamma'}\right)^{\frac{1}{\gamma'}}$$

using also Hölder's inequality for $\gamma \in (1, \infty)$ arbitrary. By Theorem 3.2, the definition of $\mathbf{F}_p$ and the assumptions on the initial law we find that the second term is uniformly bounded for any choice of γ. So, we obtain

$$\mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^r}^\chi \leq c\,(1 + \mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^q}^q)$$

$$\leq c\left(1 + \mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^r}^{\beta q\gamma}\right)^{\frac{1}{\gamma}}. \tag{5.12}$$

If $\beta q < \chi$ (note that β can be made arbitrarily small if we choose q close enough to 2 and γ can be chosen arbitrarily close to 1), we finally obtain

$$\mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^r}^\chi \leq c$$

uniformly in N. By (5.12) this implies

$$\mathbb{E}\|\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))\|_{L_{t,x}^q}^q \leq c \tag{5.13}$$

uniformly. Inserting this into (5.9) yields the claim.

5.3 The case $n = 3$

If $n = 3$, the convective term does not vanish. We have to estimate it which is only possible under a restrictive assumption on p^-. We have

$$(II)_5 \leq \int_0^t \int_{\mathbb{T}^n} |\nabla \mathbf{v}^N|^3 \, dx \, d\sigma$$

such that we end up with

$$
\mathbb{E}\left[\sup_{0<t<T} \int_{\mathbb{T}^n} |\nabla \mathbf{v}^N|^2 \, dx + \left[\int_Q |\nabla \mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))|^2 \, dx \, dt \right] \right.
$$
$$
\left. \leq c \, \mathbb{E}\left(1 + \int_Q |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{\overline{q}} \, dx \, dt \right),
$$
(5.14)

where $\overline{q} = \max\{p^+ + \varrho, 3\}$ ($\varrho > 0$ is arbitrary) as a counterpart to (5.9). Using again Sobolev's embedding shows

$$
\mathbb{E}\left[\sup_{0<t<T} \int_{\mathbb{T}^n} |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, dx + \left[\int_0^T \left(\int_{\mathbb{T}^n} |\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))|^6 \, dx \right)^{\frac{1}{3}} dt \right] \right.
$$
$$
\left. \leq c \, \mathbb{E}\left(1 + \int_Q |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{\overline{q}} \, dx \, dt \right).
$$

We obtain finally

$$
\mathbb{E}\left[\sup_{0<t<T} \int_{\mathbb{T}^n} |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^2 \, dx + \left[\int_0^T \left(\int_{\mathbb{T}^n} |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{3p^-} \, dx \right)^{\frac{1}{3}} dt \right] \right.
$$
$$
\left. \leq c \, \mathbb{E}\left(1 + \int_Q |\boldsymbol{\varepsilon}(\mathbf{v}^N)|^{\overline{q}} \, dx \, dt \right).
$$
(5.15)

Now we use an interpolation which is quite similar to the two-dimensional case. However, the quantity of interest is now $\boldsymbol{\varepsilon}(\mathbf{v}^N)$ instead of $\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))$. Using the interpolation

$$
\left(L^\infty(0, T; L^2(\mathbb{T}^n)); L^{p^-}(0, T; L^{3p^-}(\mathbb{T}^n)) \right)_\Theta = L^r(0, T; L^r(\mathbb{T}^n)),
$$
$$
r = \frac{4}{3} + p^-, \quad \Theta = 1 - \frac{p^-}{r},
$$

we obtain for $\chi = \frac{3}{5} r$

$$
\|v\|_r^\chi \leq \|v\|_{L_t^\infty L_x^2}^{\chi \Theta} \|v\|_{L_t^{p^-} L_x^{3p^-}}^{\chi(1-\Theta)} \leq \|v\|_{L_t^\infty L_x^2}^2 + \|v\|_{L_t^{p^-} L_x^{3p^-}}^{p^-}
$$
(5.16)

such that

$$\mathbb{E}\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^r}^{\chi} \leq c\left(1 + \mathbb{E}\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^{\overline{q}}}^{\overline{q}}\right).$$

On account of the interpolation

$$\left(L^r(Q); L^{p^-}(Q)\right)_{\beta} = L^{\overline{q}}(Q), \quad \beta = \frac{r}{\overline{q}}\frac{\overline{q} - p^-}{r - p^-},$$

we gain similarly to the two-dimensional case

$$\mathbb{E}\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^{\overline{q}}}^{\overline{q}} \leq \mathbb{E}\left(\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^r}^{\beta\overline{q}}\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^{p^-}}^{(1-\beta)\overline{q}}\right)$$

$$\leq \left(\mathbb{E}\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^r}^{\beta\overline{q}\gamma}\right)^{\frac{1}{\gamma}}\left(\mathbb{E}\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^{p^-}}^{(1-\beta)\overline{q}\gamma'}\right)^{\frac{1}{\gamma'}}.$$

By Theorem 3.2 the second term is uniformly bounded and hence

$$\mathbb{E}\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^r}^{\chi} \leq c\left(1 + \mathbb{E}\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^{\overline{q}}}^{\overline{q}}\right) \leq c\left(1 + \mathbb{E}\|\boldsymbol{\varepsilon}(\mathbf{v}^N)\|_{L_{t,x}^r}^{\beta\overline{q}\gamma}\right)^{\frac{1}{\gamma}}. \tag{5.17}$$

Now we have to check that $\beta\overline{q} < \chi$. This is equivalent to $\overline{q} < p^- + \frac{4}{5}$ which follows from our assumption $\frac{11}{5} < p^- \leq p^+ \leq p^- + \frac{4}{5}$. So, the proof can be finished as before if we chose γ close enough to 1. □

5.4 Compactness

As in (4.24) we have again

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\|\mathbf{v}^N\|_{C^{\mu}([0,T];\mathrm{W}_{\mathrm{div}}^{-\ell,p_0}(\mathbb{T}^n)^n)}\right] < \infty \tag{5.18}$$

for certain $\mu > 0$, $\ell \in \mathbb{N}$ and $p_0 > 1$. In view of compactness, let us now define the path space

$$\mathcal{X} := \mathcal{X}_{\mathbf{v}} \otimes \mathcal{X}_{\mathbf{F}} \otimes \mathcal{X}_p \otimes \mathcal{X}_{\mathbf{f}} \otimes \mathcal{X}_W, \tag{5.19}$$

where[3]

$$\mathcal{X}_{\mathbf{v}} := C([0,T]; \mathrm{W}_{\mathrm{div}}^{-\ell,p_0}(\mathbb{T}^n)) \cap L^2(0,T; \mathrm{W}_{\mathrm{div}}^{1,2}(\mathbb{T}^n)),$$
$$\mathcal{X}_{\mathbf{F}} := \left(L^2(0,T; W^{1,2}(\mathbb{T}^n)), w\right),$$
$$\mathcal{X}_p := C^0([0,T] \times \mathbb{T}^n),$$

---

[3] (X, w) denotes a Banach space equipped with the weak topology.

$$\mathcal{X}_{\mathbf{f}} := L^2(0, T; W^{1,2}(\mathbb{T}^n)),$$
$$\mathcal{X}_W := C([0, T]; \mathfrak{U}_0).$$

We obtain the following.

Proposition 5.3 *The set* $\{\mathcal{L}[\mathbf{v}^N, \mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)), p, \mathbf{f}, W]; \ N \in \mathbb{N}\}$ *is tight on* $\mathcal{X}$.

Proof We recall an interpolation result of Aubin–Lions-type due to Amann [1] to conclude that

$$L^\infty(0, T; W^{1,2}_{\mathrm{div}}(\mathbb{T}^n)) \cap C^\mu([0, T]; W^{-\ell, p_0}_{\mathrm{div}}(\mathbb{T}^n)) \cap L^{p^-}(0, T; W^{1,p^-}_{\mathrm{div}}(\mathbb{T}^n))$$
$$\hookrightarrow\hookrightarrow L^2(0, T; W^{1,2}_{\mathrm{div}}(\mathbb{T}^n)). \tag{5.20}$$

On the other hand, Ascoli–Arzelá's theorem yields

$$C^\mu([0, T]; W^{-\ell, p_0}_{\mathrm{div}}(\mathbb{T}^n)) \hookrightarrow\hookrightarrow C([0, T]; W^{-\ell, p_0}_{\mathrm{div}}(\mathbb{T}^n)).$$

So, we obtain tightness of $\mathbf{v}^N$ on $\mathcal{X}_{\mathbf{v}}$ from (5.18), Theorems 3.2 and 5.1 and Tschebyscheff's inequality. Tightness of $\mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))$ on $\mathcal{X}_{\mathbf{F}}$ follows immediately from Theorem 3.2 and 5.1. Finally the laws of p, $\mathbf{f}$ and W are tight as before in Sect. 4.2. □

Accordingly, we apply Jakubowski's extension of Skorokhod's theorem (see [18]). We infer the following result.

Proposition 5.4 *There exists a complete probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *with* $\mathcal{X}$-*valued Borel measurable random variables* $(\tilde{\mathbf{v}}^N, \tilde{\mathbf{F}}^N, \tilde{p}^N, \tilde{\mathbf{f}}^N, \tilde{W}^N)$, $N \in \mathbb{N}$, *and* $(\tilde{\mathbf{v}}, \tilde{\mathbf{F}}, \tilde{p}, \tilde{\mathbf{f}}, \tilde{W})$ *such that (up to a subsequence)*
(a) *the law of* $(\tilde{\mathbf{v}}^N, \tilde{\mathbf{F}}^N, \tilde{p}^N, \tilde{\mathbf{f}}^N, \tilde{W}^N)$ *on* $\mathcal{X}$ *is given by* $\mathcal{L}[\mathbf{v}^N, \mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N)), p, \mathbf{f}, W]$, $N \in \mathbb{N}$,
(b) *the law of* $(\tilde{\mathbf{v}}, \tilde{\mathbf{F}}, \tilde{p}, \tilde{\mathbf{f}}, \tilde{W})$ *on* $\mathcal{X}$ *is a Radon measure,*
(c) $(\tilde{\mathbf{v}}^N, \tilde{\mathbf{F}}^N, \tilde{p}^N, \tilde{\mathbf{f}}^N, \tilde{W}^N)$ *converges* $\tilde{\mathbb{P}}$-*almost surely to* $(\tilde{\mathbf{v}}, \tilde{\mathbf{F}}, \tilde{p}, \tilde{\mathbf{f}}, \tilde{W})$ *in the topology of* $\mathcal{X}$, *i.e.*

$$\tilde{\mathbf{v}}^N \to \tilde{\mathbf{v}} \text{ in } C([0, T]; W^{-\ell, p_0}_{\mathrm{div}}(\mathbb{T}^n)) \ \tilde{\mathbb{P}}\text{-}a.s.,$$
$$\tilde{\mathbf{v}}^N \to \tilde{\mathbf{v}} \text{ in } L^2(0, T; W^{1,2}_{\mathrm{div}}(\mathbb{T}^n)) \ \tilde{\mathbb{P}}\text{-}a.s.,$$
$$\tilde{\mathbf{F}}^N \rightharpoonup \tilde{\mathbf{F}} \text{ in } L^2(0, T; W^{1,2}(\mathbb{T}^n)) \ \tilde{\mathbb{P}}\text{-}a.s.,$$
$$\tilde{p}^N \to \tilde{p} \text{ in } C^0([0, T] \times \mathbb{T}^n) \ \tilde{\mathbb{P}}\text{-}a.s.,$$
$$\tilde{\mathbf{f}}^N \to \tilde{\mathbf{f}} \text{ in } L^2(0, T; W^{1,2}(\mathbb{T}^n)) \ \tilde{\mathbb{P}}\text{-}a.s.,$$
$$\tilde{W}^N \to \tilde{W} \text{ in } C([0, T]; \mathfrak{U}_0) \ \tilde{\mathbb{P}}\text{-}a.s. \tag{5.21}$$

The equality of laws from Proposition 4.4 implies immediately that $\tilde{\mathbf{F}}^N = \mathbf{F}_{\tilde{p}^N}(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^N))$. Using the convergences from (5.21) we obtain

$$\tilde{\mathbf{F}} = \mathbf{F}_{\tilde{p}}(\cdot, \boldsymbol{\varepsilon}(\tilde{\mathbf{v}})). \tag{5.22}$$

Also, the uniform estimates from Theorems 3.2 and 5.1 continue to holds on the new probability space. The proof of Theorem 2.7 can now be completed as in Sect. 4.

5.5 Strong stochastically strong solutions

The existence of a strong pathwise solution follows now along the lines of the proof of Theorem 2.7 with some minor modifications. The most important change is that the classical Gyöngy-Krylov argument does not apply as the path space $\mathcal{X}$ is not Polish anymore due to the weak topology on $\mathcal{X}_{\mathbf{F}}$. A generalization which applies to the very general class of sub-Polish spaces (including Banach spaces with weak topologies) can be found in [5, Chap. 2, Theorem 2.10.3]. We consider the collection of joint laws of $(\mathbf{X}^n, \mathbf{X}^m, p, \mathbf{f}, W)$, where

$$\mathbf{X}^n = (\mathbf{v}^{N_n}, \mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^{N_n}))), \quad \mathbf{X}^m = (\mathbf{v}^{N_m}, \mathbf{F}_p(\cdot, \boldsymbol{\varepsilon}(\mathbf{v}^{N_m}))),$$

on the extended path space

$$\mathcal{X}^J = (\mathcal{X}_{\mathbf{v}} \times \mathcal{X}_{\mathbf{F}})^2 \otimes \mathcal{X}_p \otimes \mathcal{X}_{\mathbf{f}} \times \mathcal{X}_W.$$

As in Proposition 4.6 we obtain tightness of the set

$$\{\mathcal{L}[\mathbf{X}^n, \mathbf{X}^m, p, \mathbf{f}, W]; \ n, m \in \mathbb{N}\}$$

on $\mathcal{X}^J$. Let $(\mathbf{X}^{n_k}, \mathbf{X}^{m_k}, p, \mathbf{f}, W)$ be an arbitrary subsequence. By the Jakubowski–Skorokhod theorem [18] we infer (for a further subsequence but without loss of generality we keep the same notation) the existence of a probability space $(\bar{\Omega}, \bar{\bar{\mathscr{F}}}, \bar{\mathbb{P}})$ with a sequence of random variables $(\hat{\mathbf{X}}^{n_k}, \check{\mathbf{X}}^{m_k}, \bar{p}_k, \bar{\mathbf{f}}_k, \bar{W}_k)$ with

$$\hat{\mathbf{X}}^{n_k} = (\hat{\mathbf{v}}^{n_k\cdot}, \hat{\mathbf{F}}^{n_k}), \quad k \in \mathbb{N},$$
$$\hat{\mathbf{X}}^{m_k} = (\check{\mathbf{v}}^{m_k\cdot}, \check{\mathbf{F}}^{m_k}), \quad k \in \mathbb{N},$$

converging almost surely in $\mathcal{X}^J$ to a random variable $(\hat{\mathbf{X}}, \check{\mathbf{X}}, \bar{p}, \bar{\mathbf{f}}, \bar{W})$. with

$$\hat{\mathbf{X}} = (\hat{\mathbf{v}}, \hat{\mathbf{F}}), \quad \check{\mathbf{X}} = (\hat{\mathbf{v}}, \hat{\mathbf{F}}).$$

As before in (5.22) it follows that

$$\hat{\mathbf{F}} = \mathbf{F}_{\bar{p}}(\cdot, \boldsymbol{\varepsilon}(\hat{\mathbf{v}})), \quad \check{\mathbf{F}} = \mathbf{F}_{\bar{p}}(\cdot, \boldsymbol{\varepsilon}(\check{\mathbf{v}})). \tag{5.23}$$

As in (4.28) we can show that $(\hat{\mathbf{v}}, \bar{p}, \bar{\mathbf{f}}, \bar{W})$ and $(\check{\mathbf{v}}, \bar{p}, \bar{\mathbf{f}}, \bar{W})$ are weak martingale solutions to (1.5)–(1.6) defined on the same stochastic basis $(\bar{\Omega}, \bar{\bar{\mathscr{F}}}, (\bar{\mathscr{F}}_t), \bar{\mathbb{P}})$. We apply the pathwise uniqueness result from Proposition 4.5 to conclude

$$\mathcal{L}[\hat{\mathbf{X}}, \check{\mathbf{X}}, \bar{W}]\Big((\mathbf{X}_1, \mathbf{X}_2, p, \mathbf{f}, W) \in \mathcal{X}^J : \ \mathbf{X}_1 = \mathbf{X}_2\Big)$$

$$= \bar{\mathbb{P}}\Big((\hat{\mathbf{v}}, \hat{\mathbf{F}}) = (\check{\mathbf{v}}, \check{\mathbf{F}}) \Big) = \bar{\mathbb{P}}(\hat{\mathbf{v}} = \check{\mathbf{v}}) = 1.$$

Now, [5, Chap. 2, Theorem 2.10.3] implies that the original sequence $\mathbf{v}^N$ defined on the initial probability space converges in probability in the topology of $\mathcal{X}_{\mathbf{v}}$ to the random variable $\mathbf{v}$. Therefore, we finally deduce that $\mathbf{v}$ is a strong stochastically strong solution to (1.5)–(1.6). Note that the pressure terms can be recovered as in (2.11) (see the explanations below (2.11) for the regularity of the pressure terms). The proof of Corollary 2.13 is hereby complete. □

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